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Research Note

Solution of Three-Dimensional Line-of-Sight Guidance with a Moving Tracker

S.H. Jalali-Naini* and V. Esfahanian[1]

The closed-form solution of a three-dimensional line-of-sight guidance with a moving tracker is derived for an ideal case, in which a pursuer is always on the instantaneous line between the target tracker and the target. The solution can be applied to both surface-to-air and air-to-surface applications. Some significant characteristics, such as intercept time, cumulative velocity increment, initial condition for interception, and the effect of acceleration limit, are also obtained and discussed. In addition, the equivalent effective navigation ratio for the line-of-sight guidance is introduced. Finally, solutions for a special case of maneuvering target are presented.

INTRODUCTION

In Line-Of-Sight (LOS) guidance, a pursuer maneuvers so as to be on the instantaneous LOS between the target tracker and the target. If the pursuer is always on the tracker-target LOS, then it will surely hit the target. This guidance law is also called 3-point guidance [1-9].

The differential equation of the pursuer range, with respect to its angular position for LOS trajectory, is not integrable, even for a constant-speed pursuer and nonmaneuvering targets [4]. Locke gave a 10-term series solution for this problem [1]. The solution can also be described in terms of elliptic integrals [5]. Jalali-Naini and Esfahanian presented the closed-form solution of the LOS trajectory for nonmaneuvering targets, assuming the total pursuer acceleration to be equal to the required acceleration in the direction normal to the LOS [10,11]. This solution was extended to solve a modified LOS guidance [12]. A general differential equation was then introduced by Shoucri to model the two-dimensional (2-D) trajectory of a pursuer for various guidance laws against maneuvering targets [13]. An approximate solution of the 3-D LOS guidance, for a pursuer with an arbitrarily time-varying velocity against maneuvering targets, was presented in [14], in which the pursuer commanded acceleration to be in the direction normal to the pursuer velocity.

*. *Corresponding Author, Aerospace Research Institute, P.O. Box 15875-3885, Tehran 14666, I.R. Iran.*

1. *Department of Mechanical Engineering, University of Tehran, Tehran 14395, I.R. Iran.*

In this study, the solution of the 3-D LOS trajectory for nonmaneuvering targets is derived for a moving tracker. Here, the total pursuer acceleration is also assumed to be equal to the required acceleration in the direction normal to the LOS. This solution is also valid for air-to-surface applications, whereas the previous solutions are only suitable for a stationary tracker. In addition, the analytical solution for a special case of maneuvering target is presented. Note that the minimum acceleration which must be applied in order to keep the pursuer on the LOS is termed "required acceleration".

EQUATIONS OF RELATIVE MOTION

The inertial O_{Ixyz} Cartesian coordinate system and the nonrotating $Oxyz$ Cartesian coordinates with the origin fixed at the moving target-tracker (point O), are shown in Figure 1. These two coordinate systems are coincident at firing instant. In the figure, M and T denote the pursuer and the target, respectively and r_p is the distance of particle P (pursuer M or target T) from the tracker, O. Consider the spherical coordinates (r, β, ε) with origin at the target-tracker, in which β and ε are azimuth and elevation angles, respectively. Let $(\mathbf{e}_r, \mathbf{e}_\beta, \mathbf{e}_\varepsilon)$ be the unit vectors along the spherical coordinate axes.

The position, velocity and acceleration vectors of the target tracker and particle P, with respect to the inertial reference, are $\mathbf{R}_{\mathrm{O}}, \mathbf{V}_{\mathrm{O}}, \mathbf{A}_{\mathrm{O}}$ and $\mathbf{R}_p, \mathbf{V}_p, \mathbf{A}_p$, respectively. One can write the relative position vector; $\mathbf{r}_p = \mathbf{R}_p - \mathbf{R}_{\mathrm{O}}$, velocity vector; $\mathbf{v}_p = \mathbf{V}_p - \mathbf{V}_{\mathrm{O}}$ and acceleration vector; $\mathbf{a}_p = \mathbf{A}_p - \mathbf{A}_{\mathrm{O}}$, for particle P.

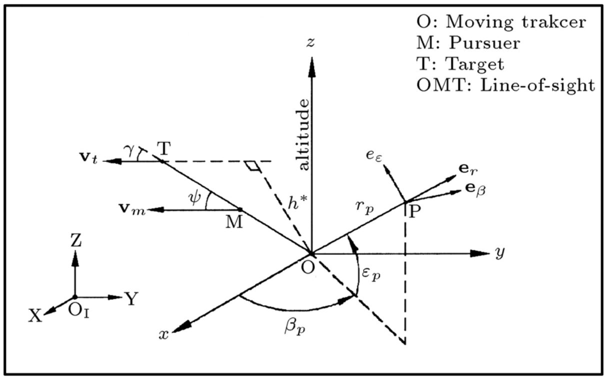

Figure 1. Pursuer-target engagement geometry.

Therefore, one has:

$$\mathbf{r}_p = r_p\mathbf{e}_r, \tag{1a}$$

$$\mathbf{v}_p = \dot{r}_p\mathbf{e}_r + r_p\dot{\beta}_p\cos\varepsilon_p\mathbf{e}_\beta + r_p\dot{\varepsilon}_p\mathbf{e}_\varepsilon. \tag{1b}$$

The three relative acceleration components of particle P ($a_p^r, a_p^\beta, a_p^\varepsilon$) are as follows:

$$a_p^r = \ddot{r}_p - r_p\dot{\varepsilon}_p^2 - r_p\dot{\beta}_p^2\cos^2\varepsilon_p, \tag{2a}$$

$$a_p^\beta = r_p\ddot{\beta}_p\cos\varepsilon_p + 2\dot{r}_p\dot{\beta}_p\cos\varepsilon_p - 2r_p\dot{\beta}_p\dot{\varepsilon}_p\sin\varepsilon_p, \tag{2b}$$

$$a_p^\varepsilon = r_p\ddot{\varepsilon}_p + 2\dot{r}_p\dot{\varepsilon}_p + r_p\dot{\beta}_p^2\sin\varepsilon_p\cos\varepsilon_p. \tag{2c}$$

The preceding relations can be expressed in the form of:

$$\mathbf{v}_p = \dot{r}_p\mathbf{e}_r + \mathbf{\Omega}_p \times \mathbf{r}_p, \tag{3a}$$

$$\mathbf{a}_p = \ddot{r}_p\mathbf{e}_r + 2\dot{r}_p(\mathbf{\Omega}_p\times\mathbf{e}_r) + \dot{\mathbf{\Omega}}_p\times\mathbf{r}_p + \mathbf{\Omega}_p\times(\mathbf{\Omega}_p\times\mathbf{r}_p), \tag{3b}$$

where $\mathbf{\Omega}_p$ is the angular velocity vector of $\mathbf{r}_p$ and is given by:

$$\mathbf{\Omega}_p = (\mathbf{r}_p \times \mathbf{v}_p)/r_p^2, \tag{4a}$$

$$\Omega_p = \sqrt{\dot{\varepsilon}_p^2 + \dot{\beta}_p^2\cos^2\varepsilon_p}, \tag{4b}$$

It should be noted that Equations 3 are in terms of $\mathbf{\Omega}_p$ instead of the angular velocity vector of the moving spherical coordinates $\boldsymbol{\omega}_p = \omega_p^r\mathbf{e}_r + \mathbf{\Omega}_p$, in which $\omega_p^r = \dot{\beta}_p\sin\varepsilon_p$ [12]. The reason is simple. Since:

$$\dot{\boldsymbol{\omega}}_p \times \mathbf{e}_r = -\omega_p^r\mathbf{\Omega} + \dot{\mathbf{\Omega}}_p \times \mathbf{e}_r, \tag{5a}$$

$$\boldsymbol{\omega}_p \times (\boldsymbol{\omega}_p \times \mathbf{e}_r) = \omega_p^r\mathbf{\Omega} + \mathbf{\Omega}_p \times (\mathbf{\Omega}_p \times \mathbf{e}_r), \tag{5b}$$

one has:

$$\dot{\boldsymbol{\omega}}_p\times\mathbf{e}_r + \boldsymbol{\omega}_p\times(\boldsymbol{\omega}_p\times\mathbf{e}_r) = \dot{\mathbf{\Omega}}_p\times\mathbf{e}_r + \mathbf{\Omega}_p\times(\mathbf{\Omega}_p\times\mathbf{e}_r). \tag{6}$$

The angular momentum of unit mass for the relative motion ($\mathbf{H}_p = \mathbf{r}_p \times \mathbf{v}_p$) can be written as:

$$\mathbf{H}_p = \mathrm{H}_p\mathbf{e}_h = r_p^2(-\dot{\varepsilon}_p\mathbf{e}_\beta + \dot{\beta}_p\cos\varepsilon_p\mathbf{e}_\varepsilon), \tag{7}$$

where $\mathbf{e}_h$ is the unit vector of $\mathbf{H}_p$ or $\mathbf{\Omega}_p(\mathbf{\Omega}_p = \Omega_p\mathbf{e}_h)$. The other unit vector can be defined as $\mathbf{e}_h^\perp = \mathbf{e}_h \times \mathbf{e}_r$; therefore [15];

$$\mathbf{e}_h^\perp = (r_p^2/\mathrm{H}_p)(\dot{\beta}_p\cos\varepsilon_p\mathbf{e}_\beta + \dot{\varepsilon}_p\mathbf{e}_\varepsilon). \tag{8}$$

By the preceding definitions, Equations 3 can be expressed in the following form:

$$\mathbf{v}_p = \dot{r}_p\mathbf{e}_r + r_p\Omega_p\mathbf{e}_h^\perp, \tag{9a}$$

$$\mathbf{a}_p = (\ddot{r}_p - r_p\Omega_p^2)\mathbf{e}_r + 2\dot{r}_p\Omega_p\mathbf{e}_h^\perp + \dot{\mathbf{\Omega}}_p \times \mathbf{r}_p. \tag{9b}$$

The set of unit vectors $(\mathbf{e}_r, \mathbf{e}_h^\perp, \mathbf{e}_h)$ constitutes new moving coordinates, which are called h coordinates [15]. The relative acceleration vector can be expanded in this coordinate system with the components of $(a_p^r, a_p^\perp, a_p^h)$:

$$\mathbf{a}_p = a_p^r\mathbf{e}_r + a_p^\perp\mathbf{e}_h^\perp + a_p^h\mathbf{e}_h. \tag{10}$$

In terms of the h coordinates, Equations 2 can be rewritten as [16]:

$$\ddot{r}_p - r_p\Omega_p^2 = a_p^r, \tag{11a}$$

$$r_p\dot{\Omega}_p + 2\dot{r}_p\Omega_p = a_p^\perp, \tag{11b}$$

$$\dot{\mathbf{e}}_h = -\frac{a_p^h}{r_p\Omega_p}\mathbf{e}_h^\perp. \tag{11c}$$

Equations for a Nonmaneuvering Target and Tracker

Consider a case in which the target and its tracker do not maneuver. In other words, their velocity vectors and the target relative velocity, $\mathbf{v}_t$, remain constant. Therefore, the angle between the target relative velocity and the LOS, $\gamma = \cot^{-1}\frac{\dot{r}_t}{r_t\Omega_t}$, decreases with time where the subscript "t" denotes the target (see Figure 1). In addition, h^* is defined to be the nearest distance from the tracker to the line along the target relative velocity vector.

The unit angular momentum for the target relative motion remains constant for a nonmaneuvering target and tracker, that is:

$$\mathrm{H}_t = r_t\mathrm{v}_t\sin\gamma = r_{t_0}\mathrm{v}_t\sin\gamma_0, \tag{12}$$

where the subscript "0" describes the initial value. Therefore:

$$r_t\sin\gamma = r_{t_0}\sin\gamma_0 = h^*. \tag{13}$$

It can be seen that h^* is equal to $\mathrm{H}_t/\mathrm{v}_t$. From Figure 1, one can write:

$$\dot{r}_t = \mathrm{v}_t \cos\gamma, \tag{14a}$$

$$r_t \Omega_t = \mathrm{v}_t \sin\gamma, \tag{14b}$$

where Ω_t is the tracker-target LOS rate. By differentiating Equation 13 with respect to time and using Equation 14a, one obtains:

$$\dot{\gamma} = -\frac{\mathrm{v}_t \sin^2\gamma}{h^*} = -\frac{\mathrm{v}_t \sin\gamma}{r_t}, \tag{15}$$

Comparing Equations 14b and 15, one can conclude the relation $\Omega_t = -\dot{\gamma}$. The other useful relations can then be found as:

$$\ddot{\gamma} = 2\dot{\gamma}^2 \cot\gamma, \tag{16}$$

$$\cot\gamma = \cot\gamma_0 + (\mathrm{v}_t/h^*)t. \tag{17}$$

The relations for $(r_t, \beta_t, \varepsilon_t)$ and their time derivatives are simply derived in the moving Cartesian coordinates (x, y, z). For instance, one has:

$$r_t^2 = r_{t_0}^2 + 2(r_{t_0}\dot{r}_{t_0})t + \mathrm{v}_t^2 t^2, \tag{18}$$

$$\tan\beta_t = y_t/x_t = (y_{t_0} + \dot{y}_{t_0}t)/(x_{t_0} + \dot{x}_{t_0}t), \tag{19}$$

$$\sin\varepsilon_t = z_t/r_t = \frac{z_{t_0} + \dot{z}_{t_0}t}{\sqrt{r_{t_0}^2 + 2r_{t_0}\dot{r}_{t_0}t + \mathrm{v}_t^2 t^2}}. \tag{20}$$

Using Equation 19 gives an expression for the current time as:

$$t = (x_{t_0}\tan\beta_t - y_{t_0})/(\dot{y}_{t_0} - \dot{x}_{t_0}\tan\beta_t), \tag{21}$$

and by substituting that into Equation 18 or 20, one can eliminate the current time, t, from the equations. Therefore, an explicit relation between r_t and β_t or ε_t and β_t can be obtained. Similar procedure can be applied to obtain relations between the variables $(\dot{r}_t, \dot{\beta}_t, \dot{\varepsilon}_t)$ and β_t.

Assume that the target relative velocity is in the negative y direction and $x_{t_0}, z_{t_0} \neq 0$. One can obtain simple relations for this case as:

$$r_t = (r_{t_0}\sin\varepsilon_{t_0})/\sin\varepsilon_t, \tag{22}$$

$$\tan\varepsilon_t = (\tan\varepsilon_{t_0}/\cos\beta_{t_0})\cos\beta_t, \tag{23}$$

$$\sin\gamma = (\sin\gamma_0/\sin\varepsilon_{t_0})\sin\varepsilon_t = \sqrt{1 - \cos^2\varepsilon_t \sin^2\beta_t}, \tag{24}$$

and for angular rates, one arrives at:

$$\dot{\beta}_t/\dot{\beta}_{t_0} = \cos^2\beta_t/\cos^2\beta_{t_0}, \tag{25}$$

$$\dot{\varepsilon}_t/\dot{\varepsilon}_{t_0} = (\sin^2\varepsilon_t \sin\beta_t)/(\sin^2\varepsilon_{t_0}\sin\beta_{t_0}). \tag{26}$$

EQUATIONS OF LOS GUIDANCE

The basic guidance law in 3-point guidance is $\beta_m(t) = \beta_t(t)$ and $\varepsilon_m(t) = \varepsilon_t(t)$, where the subscript "$m$", denotes the pursuer. An ideal case is assumed, in which the pursuer is always on the line between the target tracker and the target without any error. If the pursuer is initially fired at the target from the target tracker and maneuvers according to the following relations:

$$a_m^\beta = r_m\ddot{\beta}_t\cos\varepsilon_m + 2\dot{r}_m\dot{\beta}_t\cos\varepsilon_m - 2r_m\dot{\beta}_t\dot{\varepsilon}_m\sin\varepsilon_m, \tag{27a}$$

$$a_m^\varepsilon = r_m\ddot{\varepsilon}_t + 2\dot{r}_m\dot{\varepsilon}_t + r_m\dot{\beta}_t^2\sin\varepsilon_m\cos\varepsilon_m, \tag{27b}$$

then, it will always remain on the tracker-target LOS. The proof of Equations 27 can be observed as follows. By the definition $\rho_m = r_m\cos\varepsilon_m$, Equation 27a can be rewritten as $a_m^\beta = \rho_m\ddot{\beta}_t + 2\dot{\rho}_m\dot{\beta}_t$; therefore, one has:

$$\rho_m\frac{d}{dt}(\dot{\beta}_m - \dot{\beta}_t) + 2\dot{\rho}_m(\dot{\beta}_m - \dot{\beta}_t) = 0. \tag{28}$$

The preceding relation is a linear first-order differential equation which has a solution in the form of $\rho_m^2(\dot{\beta}_m - \dot{\beta}_t) =$ constant. Thus, one has $\dot{\beta}_m = \dot{\beta}_t$. Using Equation 27b, one can also obtain $r_m^2(\dot{\varepsilon}_m - \dot{\varepsilon}_t) =$ constant. Hence, one has $\beta_m(t) = \beta_t(t)$ and $\varepsilon_m(t) = \varepsilon_t(t)$. Hereafter, the subscript "m" or "t" will be dropped for β and ε for convenience. It should be noted that $a_m^r = 0$ is assumed. The value of a_m^r does not cause any deviation from the LOS.

The objective of LOS guidance may be stated by the vector product $\mathbf{r}_m \times \mathbf{r}_t = \mathbf{0}$ for $r_m \le r_t$ and $\mathbf{r}_m \cdot \mathbf{r}_t > 0$ [5]. By differentiating the preceding relation with respect to time, one has:

$$\mathbf{r}_m \times \mathbf{v}_t = \mathbf{r}_t \times \mathbf{v}_m. \tag{29}$$

The scalar form of Equation 29 can be written as:

$$\sin\psi = \frac{\mathrm{v}_t r_m}{\mathrm{v}_m r_t}\sin\gamma, \tag{30}$$

where ψ is the angle between the pursuer relative velocity and the LOS. By differentiating Equation 29 with respect to time, one arrives at:

$$\mathbf{a}_m \times \mathbf{e}_r = \frac{2\mathbf{v}_t \times \mathbf{v}_m}{r_t} + \frac{r_m}{r_t}(\mathbf{a}_t \times \mathbf{e}_r). \tag{31}$$

With the definition $\mathbf{a}_{\mathrm{plos}} = \mathbf{a} - (\mathbf{a}.\mathbf{e}_r)\mathbf{e}_r$, one can obtain:

$$\mathbf{a}_{m_{\mathrm{plos}}} = 2(\dot{r}_m - \frac{r_m}{r_t}\dot{r}_t)\Omega\mathbf{e}_h^\perp + \frac{r_m}{r_t}\mathbf{a}_{t_{\mathrm{plos}}}. \tag{32}$$

The pursuer relative acceleration can then be found as:

$$\mathbf{a}_m = (\ddot{r}_m - r_m\Omega^2)\mathbf{e}_r + 2(\dot{r}_m - \frac{r_m}{r_t}\dot{r}_t)\Omega\mathbf{e}_h^\perp + \frac{r_m}{r_t}\mathbf{a}_{t_{\mathrm{plos}}} \tag{33}$$

or:

$$\mathbf{a}_m = (\ddot{r}_m - r_m\Omega^2)\mathbf{e}_r + (r_m\dot{\Omega} + 2\dot{r}_m\Omega)\mathbf{e}_h^{\perp} + \frac{r_m}{r_t}a_t^h\mathbf{e}_h. \tag{34}$$

The pursuer relative velocity vector can be expressed in the form of $\mathbf{v}_m = c_1\mathbf{e}_r + c_2\mathbf{v}_t$, in which $c_1 = \dot{r}_m - r_m\dot{r}_t/r_t$ and $c_2 = r_m/r_t$. This means that the vectors $\mathbf{e}_r, \mathbf{v}_t$ and $\mathbf{v}_m$ are coplanar.

SOLUTION OF LOS GUIDANCE FOR A NONMANEUVERING TARGET AND TRACKER

Consider that the target and its tracker move with constant velocity vectors. The total pursuer acceleration is also assumed to be equal to the required acceleration in the direction normal to the LOS, therefore, $a_m^r = 0$, which becomes:

$$\ddot{r}_m - r_m\dot{\gamma}^2 = 0. \tag{35}$$

By using:

$$\dot{r}_m = \dot{\gamma}\frac{dr_m}{d\gamma}, \tag{36}$$

$$\ddot{r}_m = \ddot{\gamma}\frac{dr_m}{d\gamma} + \dot{\gamma}^2\frac{d^2r_m}{d\gamma^2}, \tag{37}$$

and with the change of independent variable t to γ, one may rewrite the differential Equation 35 in the form of

$$\ddot{\gamma}\frac{dr_m}{d\gamma} + \dot{\gamma}^2\left(\frac{d^2r_m}{d\gamma^2} - r_m\right) = 0. \tag{38}$$

Using Equation 16, the preceding differential equation simplifies to :

$$\frac{d^2r_m}{d\gamma^2} + 2\cot\gamma\frac{dr_m}{d\gamma} - r_m = 0. \tag{39}$$

By changing the variable $u = r_m\sin\gamma$, one can rewrite Equation 39 as:

$$\frac{1}{\sin\gamma}\frac{d^2u}{d\gamma^2} = 0, \tag{40}$$

which has a solution in the following form:

$$r_m = \frac{A_1\gamma + A_2}{\sin\gamma}, \tag{41}$$

where A_1 and A_2 are integrating constants and can be determined from the initial conditions. From Figure 1, one may write:

$$\dot{r}_m = \mathrm{v}_m\cos\psi, \tag{42a}$$

$$r_m\Omega = \mathrm{v}_m\sin\psi, \tag{42b}$$

In LOS guidance, the pursuer is initially fired at the target from the tracker ($r_{m_0} = 0$). By using Equations 42, one can conclude $\psi_0 = 0$ and $\dot{r}_{m_0} = \mathrm{v}_{m_0}$; therefore,

$$r_m = \frac{nr_{t_0}(\gamma_0 - \gamma)}{\sin\gamma} \quad \text{for} \quad \gamma_0 \neq 0, \pi, \tag{43}$$

where n is the ratio of initial relative velocity of the pursuer to the target relative velocity ($\mathrm{v}_{m_0}/\mathrm{v}_t$). By differentiating Equation 43 with respect to time, one obtains:

$$\dot{r}_m = (\mathrm{v}_{m_0}/\sin\gamma_0)[\sin\gamma + (\gamma_0 - \gamma)\cos\gamma], \tag{44}$$

One can also obtain the pursuer acceleration as:

$$a_m = \frac{2\mathrm{v}_{m_0}\mathrm{v}_t}{h^*\sin\gamma_0}\sin^3\gamma = \frac{2h^*\mathrm{v}_{m_0}\mathrm{v}_t r_{t_0}}{r_t^3}. \tag{45}$$

Dividing Equation 42a by Equation 42b yields $r_m\cot\psi = -dr_m/d\gamma$; therefore:

$$\tan\psi = \frac{\gamma_0 - \gamma}{1 + (\gamma_0 - \gamma)\cot\gamma}. \tag{46}$$

By using the relation $\mathrm{v}_m^2 = \dot{r}_m^2 + r_m^2\Omega^2$ and after some manipulation, one derives:

$$\mathrm{v}_m^2 = \frac{\mathrm{v}_{m_0}^2}{\sin^2\gamma_0}[(\gamma_0 - \gamma)^2 + \sin^2\gamma + (\gamma_0 - \gamma)\sin 2\gamma]. \tag{47}$$

One may also write:

$$V_m\sin\Psi = |\mathbf{V}_m \times \mathbf{e}_r|, \tag{48a}$$

$$V_m\cos\Psi = \mathbf{V}_m.\mathbf{e}_r, \tag{48b}$$

in which Ψ, the pursuer velocity-to-beam angle, is the angle between the pursuer velocity vector and the LOS. Therefore:

$$\tan\Psi = \frac{\sqrt{V_{\mathrm{O}}^{h^2} + (V_{\mathrm{O}}^{\perp} + r_m\Omega)^2}}{\dot{r}_m + V_{\mathrm{O}}^r}, \tag{49}$$

where V_{O}^r, $V_{\mathrm{O}}^{\perp}$ and V_{O}^h are the components of the tracker velocity in h coordinates.

The pursuer and target positions are equal at the collision instant, therefore, by equating r_m and r_t from Equations 13 and 43, one arrives at:

$$\gamma_f = \gamma_0 - (1/n)\sin\gamma_0, \tag{50}$$

where the subscript "f" denotes the final value. Using Equation 17, intercepting time can then be found as:

$$t_f = (h^*/\mathrm{v}_t)(\cot\gamma_f - \cot\gamma_0). \tag{51}$$

Using Equations 13 and 43, for interception of the target, one must have:

$$n > \sin\gamma_0/\gamma_0 \quad \text{for} \quad \gamma_0 \neq 0, \pi. \tag{52}$$

The cumulative velocity increment is defined as:

$$\Delta V = \int_0^{t_f} |\mathbf{A}_m| dt, \tag{53}$$

in which $\mathbf{A}_m$ is the pursuer acceleration. Therefore, by using Equation 45 and for a nonmaneuvering tracker ($\mathbf{A}_m = \mathbf{a}_m$), one arrives at:

$$\Delta V = (2\mathrm{v}_{m_0}/\sin\gamma_0)(\cos\gamma_f - \cos\gamma_0). \tag{54}$$

The pursuer acceleration in the spherical coordinates can be expressed as:

$$\mathbf{a}_m = a_m \mathbf{e}_h^{\perp} = \frac{2\mathrm{v}_{m_0}\sin\gamma}{\sin\gamma_0}(\dot\beta\cos\varepsilon\ \mathbf{e}_\beta + \dot\varepsilon\ \mathbf{e}_\varepsilon). \tag{55}$$

An important parameter in a command to LOS system is the integral of the component of angular velocity vector of the moving spherical coordinates along the LOS, that is:

$$\Gamma = \int_0^t \dot\beta \sin\varepsilon\ dt. \tag{56}$$

In a case where the target relative velocity is in the negative y direction, the two preceding relations simplify to:

$$\mathbf{a}_m = \frac{2\mathrm{v}_{m_0}\mathrm{v}_t}{r_{t_0}\sin^2\varepsilon_0}(-\cos\beta\mathbf{e}_\beta + \sin\varepsilon\sin\beta\mathbf{e}_\varepsilon)\sin^2\varepsilon, \tag{57}$$

$$\Gamma = \mathrm{sgn}(x_{t_0}z_{t_0})\left[\sin^{-1}\left(\frac{\sin\beta}{c}\right) - \sin^{-1}\left(\frac{\sin\beta_0}{c}\right)\right], \tag{58}$$

where $\mathrm{sgn}(\cdot)$ is the sign function and c is:

$$c = \sqrt{1 + \cos^2\beta_0\cot^2\varepsilon_0}. \tag{59}$$

Equation 58 is valid for $x_{t_0}, z_{t_0} \neq 0$, but for $x_{t_0} = 0$ or $z_{t_0} = 0$, one has $\Gamma = 0$. When $x_{t_0} = z_{t_0} = 0$ and $y_{t_0} > 0$, the target will be intercepted before it passes through the tracker.

In a special case, when target and tracker velocity vectors are equal ($\mathbf{V}_t = \mathbf{V}_{\mathrm{O}}$), the pursuer moves on a straight course with respect to an inertial reference. It means $\mathbf{V}_m =$ const. and $\mathbf{a}_m = 0$. Therefore, one has $\Psi = \Psi_0$ and $r_m = \mathrm{v}_{m_0}t$.

Analysis of LOS Guidance for a Stationary Tracker

The three-point guidance performance, unlike Proportional Navigation (PN), is highly dependent on target speed. In 3-point guidance, pursuer acceleration is proportional to target speed. The pursuer acceleration increases with time for an approaching target ($\dot r_t < 0$) and decreases for a receding target ($\dot r_t > 0$). The maximum pursuer acceleration occurs when the target is at the nearest distance of its trajectory to the target tracker.

In 3-point guidance, the ratio of pursuer initial velocity to target velocity must be greater than $\sin\gamma_0/\gamma_0$, in order for the pursuer to have the capability of intercepting the target.

In order to compare 3-point guidance and true PN (TPN), let the pursuer be fired at the target and then guided under the TPN guidance law. The question is what the effective navigation ratio should be so that the pursuer will follow the LOS trajectory. For this purpose, one can equate Equation 45 and the TPN commanded acceleration. Therefore, one can find the equivalent effective navigation ratio, N'_{eq}, as:

$$N'_{\mathrm{eq}} = \frac{a_m}{V_c\Omega} = \frac{2V_{m_0}V_t\sin^3\gamma}{h^*\sin\gamma_0 V_c\Omega}, \tag{60}$$

where V_c is the pursuer-target closing velocity. By using the relation $\Omega = (V_t/h^*)\sin^2\gamma$, one has:

$$N'_{\mathrm{eq}} = \frac{2V_{m_0}\sin\gamma}{V_c\sin\gamma_0}. \tag{61}$$

When the pursuer is on the tracker-target LOS, the closing velocity can be obtained from the simple relation $V_c = \dot r_m - \dot r_t$. Therefore, one can arrive at the following relation for the equivalent navigation ratio:

$$N'_{\mathrm{eq}} = 2/[1 - (\gamma - \gamma_f)\cot\gamma]. \tag{62}$$

The pursuer follows the LOS trajectory if it is initially fired at the target and guided under the TPN guidance law with the preceding equivalent navigation ratio. The value of N'_{eq} for an approaching target is less than 2, whereas, for a receding target, it is greater than 2 and equal to 2 at the collision instant for both approaching and receding targets.

LOS guidance may be used for midcourse guidance followed by PN. The previous solution can, thus, be applied to a mixed guidance strategy, i.e., LOS guidance in midcourse and TPN in the terminal phase. The equivalent navigation ratio is a useful concept for switching between the two guidance methods.

In practice, the pursuer maneuvering acceleration cannot exceed a limit, which is called saturation acceleration (a_{sat}). Because the pursuer can ideally keep itself on the LOS without any error, one must have $a_m \le a_{\text{sat}}$ or:

$$r_{t_0} \ge \frac{2V_{m_0}V_t}{a_{\text{sat}}}\begin{cases}\sin\gamma_0 & \gamma_0 \le \frac{\pi}{2}\\ 1/\sin^2\gamma_0 & \gamma_0 > \frac{\pi}{2}, \gamma_f < \frac{\pi}{2}\\ \frac{\sin^3(\gamma_0 - \frac{1}{n}\sin\gamma_0)}{\sin^2\gamma_0} & \gamma_0 > \frac{\pi}{2}, \gamma_f \ge \frac{\pi}{2}\end{cases}. \tag{63}$$

Consider a case where the target velocity is in the negative y axis ($\sin\gamma = \sqrt{1-\cos^2\varepsilon\sin^2\beta}$). Figure 2 presents the loci of target initial positions, in which a pursuer can keep itself on the LOS until interception occurs due to the acceleration limit. The region shown in the figure is produced by rotating the 2-D area in yz plane around the y axis using $a_{\text{sat}} = 30$ g and the pursuer initial velocity and the target velocity are 400 and 200 m/s, respectively. Because of the symmetry of this region, with respect to yz plane, only for $-\pi/2 \le \beta \le \pi/2$ is shown.

Analysis of LOS Guidance for Stationary Targets

Consider the target to be stationary ($\mathbf{V}_t = \mathbf{0}$), which yields $\mathbf{v}_t = -\mathbf{V}_{\text{O}}$. Therefore, the relations in the previous section can be simplified as:

$$V_m \sin\Psi = \left(1 - \frac{r_m}{r_t}\right) V_{\text{O}} \sin\gamma, \tag{64a}$$

$$V_m \cos\Psi = \dot{r}_m - V_{\text{O}}\cos\gamma. \tag{64b}$$

Dividing Equation 64b by Equation 64a results in:

$$\cot\Psi = \frac{n}{\sin\gamma_0 - n(\gamma_0 - \gamma)} - \cot\gamma, \tag{65}$$

where $n = \text{v}_{m_0}/V_{\text{O}}$. Suppose that a helicopter has a constant velocity and moves at a constant altitude of 500 m. When its distance from a surface target is 5000 m, it fires a missile at the target. The missile initial velocity, with respect to the helicopter, is 150 m/s. For the 2-D engagement, one has $h^* = 500$ m. The angle between the missile velocity and the LOS versus γ is shown in Figure 3 for two cases, $V_{\text{O}} =$ 150 m/s (Case 1) and 300 m/s (Case 2), respectively. As can be seen, the final value of Ψ becomes zero.

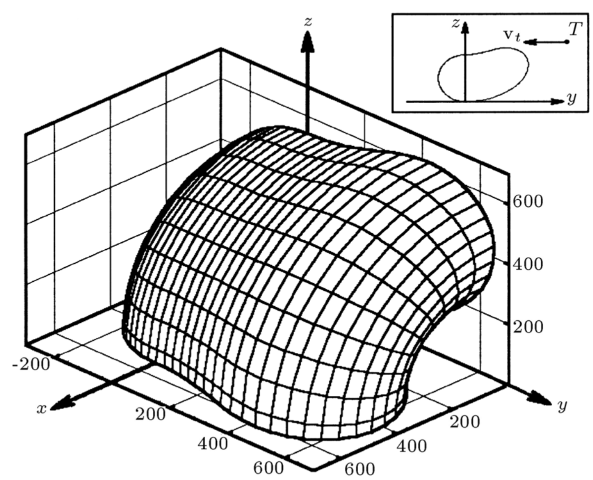

Figure 2. The region in which the pursuer can keep itself on the LOS due to the acceleration limit (Equation 63) (all the units are in meters).

SOLUTION FOR A SPECIAL CASE OF MANEUVERING TARGET

Consider a special case where a constant-speed maneuvering target moves on the surface of a sphere with radius ρ and origin fixed at the stationary target tracker. Therefore, one has $r_t = \rho, \Omega = \text{v}_t/\rho$ and $\mathbf{a}_t = -\rho\Omega^2\mathbf{e}_r + a_t^h\mathbf{e}_h$, in which a_t^h depends on the target motion. Also, $\eta = \cot^{-1}(-\rho\Omega^2/a_t^h)$ is defined as the angle between the target acceleration and the LOS.

One may express the pursuer velocity in terms of $\mathbf{e}_r$ and $\mathbf{V}_t$, as follows:

$$\mathbf{V}_m = \dot{r}_m\mathbf{e}_r + (r_m/r_{t_0})\mathbf{V}_t. \tag{66}$$

As can be seen, the vectors $\mathbf{e}_r$, $\mathbf{V}_t$, and $\mathbf{V}_m$ are coplanar, although the engagement is not planar.

The pursuer acceleration in Equation 34 can be simplified as:

$$\mathbf{a}_m = (\ddot{r}_m - r_m\Omega^2)\mathbf{e}_r + 2\dot{r}_m\Omega\mathbf{e}_h^{\perp} + (r_m/\rho)a_t^h\mathbf{e}_h. \tag{67}$$

Equation 67 implies that the pursuer acceleration is not in the plane containing $(\mathbf{e}_r, \mathbf{V}_t, \mathbf{V}_m)$. The pursuer acceleration has a component in the direction of $\mathbf{e}_h$. For this maneuvering target, simple but useful analytical solutions are available for the two following cases:

1. The pursuer acceleration being equal to the required acceleration;

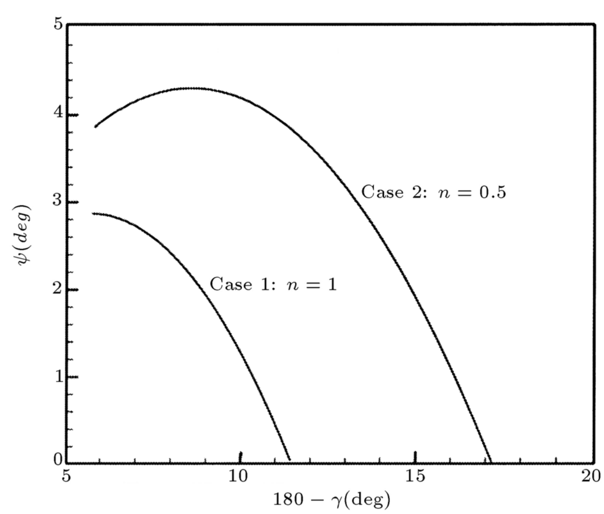

Figure 3. The pursuer velocity-to-beam angle versus γ.

2. A constant-speed pursuer.

When a target circles around the target tracker with a constant radius at a constant altitude, which is a special case of moving on the surface of a sphere, one has $\varepsilon = \varepsilon_0, \dot{\beta} = \dot{\beta}_0$ and $a_t^h = \rho\Omega^2 \tan\varepsilon_0$ for $\dot{\beta}_0 > 0$. This case was studied for a constant-speed pursuer in [5] and with the assumption that the pursuer acceleration is equal to the required acceleration in [11].

Case 1: Pursuer Acceleration is Equal to the Required Acceleration

Consider the pursuer is initially fired at the target from the tracker and then maneuvers according to Equations 27, in order to remain on the tracker-target LOS. With the assumption that the pursuer acceleration is equal to the required acceleration, Equation 11a reduces to $\ddot{r}_m - r_m\Omega^2 = 0$. Therefore, one can derive the following solutions as:

$$r_m = (V_{m_0}/\Omega)\sinh(\Omega t), \tag{68}$$

$$V_m = V_{m_0}\sqrt{\cosh(2\Omega t)}, \tag{69}$$

$$\tan\Psi = \tanh(\Omega t). \tag{70}$$

The pursuer acceleration is also obtained as:

$$a_m = V_{m_0}\Omega\sqrt{4 + (4 + \tan^2\eta)\sinh^2(\Omega t)}. \tag{71}$$

The pursuer and target positions are equal at the collision instant, therefore,

$$\sinh(\Omega t_f) = 1/n. \tag{72}$$

In this case, one must have $n > 0$ for intercepting the target. The final values for the pursuer velocity, acceleration and velocity-to-beam angle can then be found as:

$$V_{m_f} = V_{m_0}\sqrt{1 + (2/n^2)}, \tag{73}$$

$$a_{m_f}/a_{t_f} = \sqrt{1 + (4n^2 + 3)\cos^2\eta_f}, \tag{74}$$

$$\tan\Psi_f = 1/\sqrt{n^2 + 1}. \tag{75}$$

Case 2: Constant-Speed Pursuer

Consider that the pursuer, with a constant speed, is initially fired at the target from the target tracker and then maneuvers according to Equations 27. Note, for this case, $a_m^r \neq 0$. In other words, the pursuer acceleration is in the direction normal to its velocity vector and the component of the pursuer acceleration normal to the LOS must be equal to the required acceleration. The solution of $r_m(t)$ can be found by rearrangement and integrating the relation $\dot{r}_m^2 + r_m^2\Omega^2 = V_m^2$, with respect to time, that is:

$$r_m = (V_m/\Omega)\sin(\Omega t). \tag{76}$$

Comparing Equations 42b and 76, the pursuer velocity-to-beam angle can be found as:

$$\Psi = \Omega t. \tag{77}$$

The pursuer acceleration is also obtained as:

$$a_m = V_m\Omega\sqrt{4 + \tan^2\eta\sin^2(\Omega t)}. \tag{78}$$

The intercept time is then calculated by:

$$\sin(\Omega t_f) = 1/n. \tag{79}$$

The preceding relation implies that the intercept condition is $n \geq 1$. The final values for the pursuer acceleration and velocity-to-beam angle can also be obtained as:

$$a_{m_f}/a_{t_f} = \sqrt{1 + (4n^2 - 1)\cos^2\eta_f}, \tag{80}$$

$$\Psi_f = \sin^{-1}(1/n). \tag{81}$$

When a target circles around the stationary tracker, one has $\eta = \pi - \varepsilon_0$ for $\dot{\beta}_0 > 0$.

CONCLUSIONS

The 3-D equations of LOS guidance with a moving tracker are presented for maneuvering targets. Then, the closed-form solution of the 3-D LOS trajectory of a pursuer for a moving tracker and nonmaneuvering targets is derived with the assumption that the total pursuer acceleration is equal to the required acceleration in the direction normal to the LOS. In this study, the pursuer is always on the line between the target tracker and the target without any error. The present solution can be used in both surface-to-air and air-to-surface applications. In addition, some significant characteristics, such as total flight time, cumulative velocity increment, initial conditions for interception and the effects of acceleration limit, are obtained and discussed. The equivalent effective navigation ratio for the LOS guidance is also derived for comparison with TPN guidance law. Finally, the solutions for a special case of a maneuvering target, in which its trajectory is on the surface of a sphere with origin at its tracker, are presented for the two cases with different assumptions, namely, a constant-speed pursuer and the pursuer acceleration to be equal to the required acceleration.

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