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Surface Balance Laws of Linear and Angular Momenta and Cauchy's Stress Theorem

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Using Helmholtz's decomposition theorem, the laws of balance of linear and angular momenta are restated as surface integrals over the closed surface of an arbitrary subregion in a continuum. Newton's law of action and reaction and Cauchy's theorem for stress and couple-stress are proved as corollaries of these surface balance laws.

INTRODUCTION

Usually, balance laws in continuum physics are written as volume integrals because, when the integrand is continuous, so that the localization theorem may be used, field quantities, which are pointwise, can be derived. In this article, the laws of balance of linear and angular momenta are considered and restatements of them are obtained as surface integrals over the boundary of any subregion in the continuum. These restated balance laws are called surface balance laws. These surface balance laws are applied to some specific sequences of compact subsets of the continuum to prove Newton's law of action and reaction and Cauchy's theorem for stress and couple-stress.

This paper is structured as follows. First, the precise assumptions and applications of Helmholtz's representation theorem, are reviewed. Integral forms of balance of linear and angular momenta in a micropolar continuum are considered and the corresponding surface balance laws are obtained using Helmholtz's representation theorem for the total body force vector. By applying surface linear and angular momenta balance laws to a specific sequence of boxes, Newton's law of action and reaction for stress and couple-stress vectors is proved. The assumptions, significance and history of Cauchy's stress theorem for stress and couple-stress, are discussed. Finally, this theorem is proved using surface balance laws.

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HELMHOLTZ'S REPRESENTATION THEOREM

Helmholtz's decomposition theorem [1] (or the Stokes-Helmholtz's decomposition theorem [2]) states that any continuously differentiable vector function, with some decay conditions at infinity can be decomposed into a divergence-free and a curl-free vector (weak version) or into the gradient of a scalar function plus the curl of another vector function (strong version), i.e.:

$$\mathbf{F} = \nabla\Phi + \nabla\times\mathbf{\Psi} \quad \text{in} \quad \Omega, \tag{1}$$

where Φ is the scalar potential and Ψ the vector potential. This is also known as the fundamental theorem of vector analysis. If there exists a vector field $\mathbf{G}\in C^2(\Omega)$ such that:

$$\nabla^2\mathbf{G} = \mathbf{F} \quad \text{in} \quad \Omega, \tag{2}$$

then:

$$\mathbf{F} = \nabla\nabla.\mathbf{G} - \nabla\times\nabla\times\mathbf{G}. \tag{3}$$

Thus, $\mathbf{F}$ has the desired decomposition (Equation 1) with:

$$\Phi = \nabla.\mathbf{G}, \quad \mathbf{\Psi} = -\nabla\times\mathbf{G}. \tag{4}$$

Therefore, $\mathbf{F}$ has the representation of Equation 1 if the vector Poisson Equation 2 has a solution. When Ω is bounded regular [3] and $\mathbf{F}\in C^0(\overline{\Omega})\cap C^1(\Omega)$, $\mathbf{G}$ is expressed by:

$$\mathbf{G}(\mathbf{r}) = -\frac{1}{4\pi}\int_{\Omega(t)}\frac{\mathbf{F}(\mathbf{r}')dV'}{|\mathbf{r}-\mathbf{r}'|}, \tag{5}$$

and this proves Helmholtz's theorem [4,5]. Actually, this theorem holds for a larger space of functions. It can be proved [6] that Helmholtz's theorem is true for all L^2 functions ($L^2(\Omega)$ is the vector space of all square Lebesgue integrable functions in Ω). Applications of this theorem in elasticity can be found in [7-9].

SURFACE BALANCE LAWS OF LINEAR AND ANGULAR MOMENTA

Consider a continuum $\mathcal{R}$ in the reference configuration. The law of balance of linear momentum in the continuum can be written as [10]:

$$\int_{\partial B(t)} \boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t)dS + \int_{B(t)} \rho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)dV = \int_{B(t)} \rho(\mathbf{x},t)\mathbf{a}(\mathbf{x},t)dV, \tag{6}$$

where B is any nice subregion in $\mathcal{R}$ and ∂B is the boundary of B, and $\boldsymbol{\sigma}, \rho, \mathbf{b}$ and $\mathbf{a}$ are the stress vector, density, body force vector and acceleration vector, respectively. Also, $\mathbf{x}$ and $\mathbf{n}$ are, respectively, position and unit normal vectors. Note that, here, according to Cauchy's postulate, it is assumed that the stress vector acting at point $\mathbf{x}$ at time t on an oriented surface depends only on $\mathbf{x}, \mathbf{n}$ and t (see [11]). Equation 6 can be rewritten as:

$$\int_{\partial B(t)} \boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t)dS + \int_{B(t)} \mathbf{F}(\mathbf{x},t)dV = 0, \tag{7}$$

where:

$$\mathbf{F}(\mathbf{x},t) = \rho(\mathbf{x},t)[\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)]. \tag{8}$$

Here, instead of converting the surface integral in Equation 6 to a volume integral, the volume integrals are converted to surface integrals. Assuming that $\mathbf{b}(\mathbf{x},t)$ and $\mathbf{a}(\mathbf{x},t)$ are L^2 functions of $\mathbf{x}$, one can use Equation 1; therefore:

$$\int_{B(t)} \mathbf{F}(\mathbf{x},t)dV = \int_{B(t)} [\nabla\Phi(\mathbf{x},t) + \nabla\times\boldsymbol{\Psi}(\mathbf{x},t)]dV = \int_{B(t)} \nabla\Phi(\mathbf{x},t)dV + \int_{B(t)} \nabla\times\boldsymbol{\Psi}(\mathbf{x},t)dV. \tag{9}$$

Using the gradient and curl theorems, it is obtained that:

$$\int_{B(t)} \nabla\Phi(\mathbf{x},t)dV + \int_{B(t)} \nabla\times\boldsymbol{\Psi}(\mathbf{x},t)]dV = \int_{\partial B(t)} \mathbf{n}(\mathbf{x},t)\Phi(\mathbf{x},t)dS + \int_{\partial B(t)} \mathbf{n}(\mathbf{x},t)\times\boldsymbol{\Psi}(\mathbf{x},t)dS. \tag{10}$$

Substituting Equation 10 into Equation 7 yields:

$$\int_{\partial B(t)} [\boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t) + \mathbf{n}\Phi(\mathbf{x},t) + \mathbf{n}\times\boldsymbol{\Psi}(\mathbf{x},t)]dS = 0. \tag{11}$$

This holds for any closed surface, ∂B, in the continuum. It is seen that the integrand of this restated balance law depends on the unit normal vector in addition to the position vector and time. For obtaining the equations of motion of the continuum at a point $\mathbf{x}$, an arbitrary subregion is shrunk to the point arbitrarily; i.e., for any subregion, the only restriction is that the point $\mathbf{x}$ belongs to the subregion. Consider a sequence $\{B_n\}_{n=1}^{\infty}$ of compact subregions with the following properties:

$$\begin{array}{lll} \text{(i)} & \mathbf{x} \in B_n & \forall\, \mathbf{n} \in | \\ \text{(ii)} & B_{n+1} \subset B_n & \forall\, \mathbf{n} \in | \\ \text{(iii)} & \lim\limits_{n\to\infty} \mathrm{vol}(B_n) = 0 & , \end{array} \tag{12}$$

where $\mathrm{vol}(B_n)$ is the volume (3-measure) of the subregion B_n. The balance of linear momentum as a volume integral reads:

$$\int_{B_n(t)} \{\nabla.\mathbf{S}(\mathbf{x},t) + \rho(\mathbf{x},t)[\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)]\}\, dV = 0 \quad \forall\, n \in |, \tag{13}$$

where $\mathbf{S}$ is Cauchy's stress tensor. From Equations 12 and 13 one has:

$$\lim_{n\to\infty} \int_{B_n(t)} \{\nabla.\mathbf{S}(\mathbf{x},t) + \rho(\mathbf{x},t)[\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)]\}\, dV = 0. \tag{14}$$

Using localization theorem [10], it is found that:

$$\nabla.\mathbf{S}(\mathbf{x},t) + \rho(\mathbf{x},t)[\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)] = 0. \tag{15}$$

For the restatement of balance of linear momentum (Equation 11), one cannot use the localization theorem, because of the explicit dependence of the integrand on the unit normal vector, $\mathbf{n}$.

The law of balance of angular momentum for a micropolar continuum states that [12,13]:

$$\int_{\partial B(t)} [\mathbf{r}(\mathbf{x},t)\times\boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t) + \mathbf{m}(\mathbf{x},\mathbf{n},t)]dS + \int_{B(t)} \rho(\mathbf{x},t)[\mathbf{r}(\mathbf{x},t)\times\mathbf{b}(\mathbf{x},t) + \mathbf{c}(\mathbf{x},t)]dV = \int_{B(t)} \rho(\mathbf{x},t)[\mathbf{r}(\mathbf{x},t)\times\mathbf{a}(\mathbf{x},t) + \mathbf{i}(\mathbf{x},t)]dV, \tag{16}$$

where $\mathbf{m}$ is the couple-stress vector, $\mathbf{c}$ is the body-couple vector, $\mathbf{l}$ is the spin angular momentum and $\mathbf{i}$ is its material time derivative [13]. The above equation can be rewritten as:

$$\int_{\partial B(t)} [\mathbf{r}(\mathbf{x},t) \times \boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t) + \mathbf{m}(\mathbf{x},\mathbf{n},t)]dS + \int_{B(t)} \rho(\mathbf{x},t)[\mathbf{r}(\mathbf{x},t) \times (\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)) + \mathbf{c}(\mathbf{x},t) - \mathbf{i}(\mathbf{x},t)]dV = 0. \quad (17)$$

Assuming that $\rho(\mathbf{x},t)[\mathbf{r}(\mathbf{x},t) \times (\mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)) + \mathbf{c}(\mathbf{x},t) - \mathbf{i}(\mathbf{x},t)]$ is L^2, one can use Helmholtz's representation theorem to write:

$$\rho(\mathbf{x},t)[\mathbf{r}(\mathbf{x},t)\times(\mathbf{b}(\mathbf{x},t)-\mathbf{a}(\mathbf{x},t))+\mathbf{c}(\mathbf{x},t)-\mathbf{i}(\mathbf{x},t)] = \nabla\Phi(\mathbf{x},t) + \nabla \times \boldsymbol{\Psi}(\mathbf{x},t). \quad (18)$$

Hence:

$$\int_{\partial B(t)} [\mathbf{r}(\mathbf{x},t) \times \boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t) + \mathbf{m}(\mathbf{x},\mathbf{n},t) + \mathbf{n}\Phi(\mathbf{x},t) + \mathbf{n} \times \boldsymbol{\Psi}(\mathbf{x},t)]dS = 0. \quad (19)$$

This is a restatement of the balance of angular momentum as a surface integral over any closed surface in the continuum.

NEWTON'S LAW OF ACTION AND REACTION (CAUCHY'S POSTULATE ON THE TRACTION VECTOR)

Here, a proof of Cauchy's postulate on the traction vector is presented using the surface balance of linear momentum. Consider the box subregion, $B_{m,n}$, shown in Figure 1. Points $\mathbf{x}$ and $\mathbf{x}'$ lie on S_1 and S_2, respectively. Applying the restated balance of linear momentum on $B_{m,n}$, it is obtained that:

$$\mathbf{L}_{mn} = \int_{\partial B(t)} [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]dS = \sum_{i=1}^{6} \int_{\partial B_i(t)} [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]dS. \quad (20)$$

Now, consider a double sequence of box subregions, $\{B_{m,n}\}_{m,n=1}^{\infty}$, with the following properties:

(i) $\mathbf{x} \in B_{m,n} \quad \forall\, m,n \in |$

(ii) $B_{m,n+1} \subset B_{m,n} \quad \forall\, m,n \in |$

(iii) $\lim_{n\to\infty} \text{height}(B_{m,n}) = \lim_{m\to\infty} \text{width}(B_{m,n}) = 0. \quad (21)$

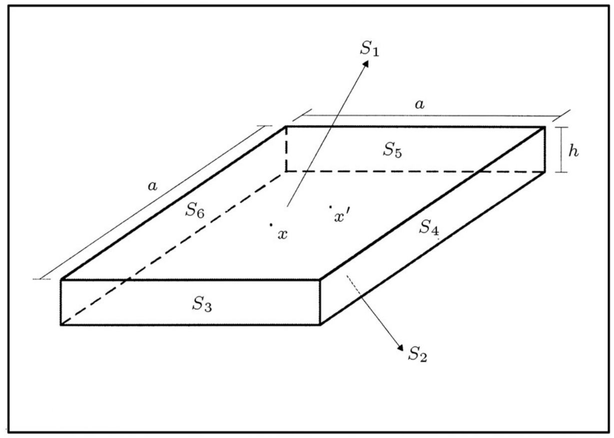

Figure 1. A box subregion $B_{m,n}$ for proving Cauchy's postulate on the traction vector.

Consider a limit process consisting of two steps: (1) $n \to \infty (h \to 0)$ and (2) $m \to \infty (a \to 0)$. When $n \to \infty$ one has:

$$\lim_{n\to\infty} \mathbf{L}_{mn} \equiv \mathbf{L}_m = \int_{S_1(t)} [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]\, dS + \int_{S_2(t)} [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]\, dS \cong [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},\mathbf{n}_1)} + [\boldsymbol{\sigma} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{n}_1)} = 0, \quad (22)$$

where $\mathbf{n}_1$ and $-\mathbf{n}_1$ are normals of the surfaces S_1 and S_2, respectively (see Figure 2a). Note that use has been made of the continuity of the integrand. As $m \to \infty, \mathbf{x} = \mathbf{x}'$ (see Figure 2b). Thus:

$$\big[\boldsymbol{\sigma}(\mathbf{x},\mathbf{n}_1,t) + \mathbf{n}_1\Phi(\mathbf{x},t) + \mathbf{n}_1 \times \boldsymbol{\Psi}(\mathbf{x},t)\big] + [\boldsymbol{\sigma}(\mathbf{x},-\mathbf{n}_1,t) - \mathbf{n}_1\Phi(\mathbf{x},t) - \mathbf{n}_1 \times \boldsymbol{\Psi}(\mathbf{x},t)] = 0. \quad (23)$$

Therefore:

$$\boldsymbol{\sigma}(\mathbf{x},-\mathbf{n}_1,t) = -\boldsymbol{\sigma}(\mathbf{x},\mathbf{n}_1,t). \quad (24)$$

A similar analysis, using the surface angular momentum balance (Equation 19), yields:

$$\mathbf{m}(\mathbf{x},-\mathbf{n}_1,t) = -\mathbf{m}(\mathbf{x},\mathbf{n}_1,t). \quad (25)$$

CAUCHY'S THEOREM FOR STRESS AND COUPLE-STRESS

Cauchy's stress theorem states that the stress vector on each oriented surface at any point in a continuum is a linear function of the unit normal to the surface. This

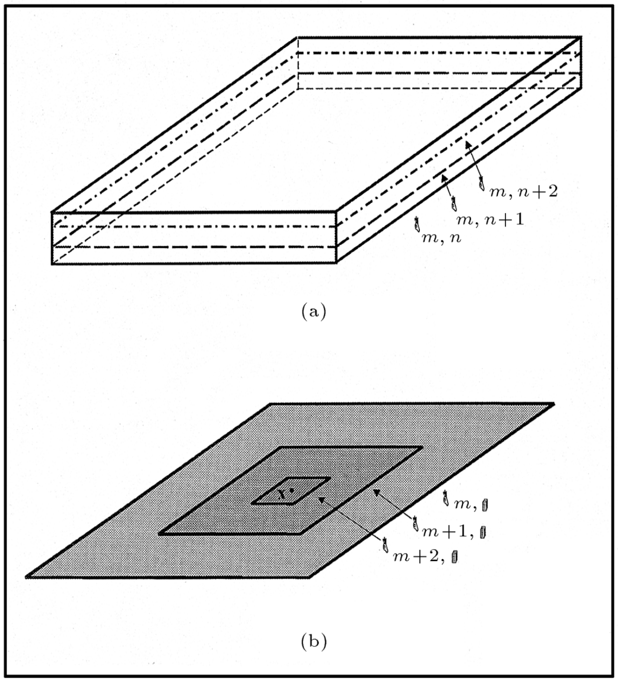

Figure 2. (a) Three consecutive members of the sequence $\{B_{m,n}\}$ when m is fixed and n is increasing; (b) Three consecutive members of the sequence when n is infinity and m is increasing.

is known as the most important theorem in continuum mechanics and guarantees the existence of the stress tensor. Cauchy [14] proved his theorem in 1823 by considering the balance of linear momentum for a tetrahedron. His proof was based on the assumption of the continuity of the stress and body force vector. However, some researchers offered proof of the theorem under weaker conditions. Gurtin et al. [15] showed that the theorem remains true under much weaker hypotheses. They proved that the theorem is true almost everywhere (The set in which the theorem is not valid has Lebesgue volume measure zero.), if the stress vector and the body force are integrable over the continuum volume (see also [16]). Fosdick and Virga [17] presented a variational proof of Cauchy's stress theorem. Segev and Rodnay [18] (see also [19] and [20]) generalized Cauchy's theorem on general differentiable manifolds.

The classical proof of Cauchy's stress theorem is to invoke Newton's second law (or the balance of linear momentum) for a tetrahedron and then shrink the tetrahedron to a point. In this way, inertia and body force effects disappear because they are of higher orders. Here, a proof of Cauchy's stress theorem is presented using the surface linear momentum balance. Applying the surface balance law to a special sequence of tetrahedrons yields Cauchy's stress theorem.

For the restated balance law (Equation 11), a sequence of specific subregions is considered, namely tetrahedrons. Consider a sequence $\{B_k\}_{k=1}^{\infty}$ of tetrahedrons with the properties (as in Equation 12). It is also assumed that three edges of each tetrahedron are along the coordinate axes. The origin (point $\mathbf{x}$) belongs to the tetrahedron and the unit normal to the oblique face is $\mathbf{n}$. Three members of this sequence are shown in Figure 3. Because $\boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t)$ is assumed to be continuous in $\mathbf{x}$, for large enough k, Equation 11 can be rewritten as:

$$\begin{aligned}\int_{\partial B(t)} &[\boldsymbol{\sigma}(\mathbf{x},\mathbf{n},t)+\mathbf{n}\Phi(\mathbf{x},t)+\mathbf{n}\times\boldsymbol{\Psi}(\mathbf{x},t)]\,ds\\ &\cong[\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}]_{(\mathbf{x},\mathbf{n},t)}\,\Delta S\\ &+[\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_1,t)}\,\Delta S_1\\ &+[\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_2,t)}\,\Delta S_2\\ &+[\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_3,t)}\,\Delta S_3.\end{aligned}\tag{26}$$

Again, the continuity of the integrand has been used. Therefore, in the limit, when $k\to\infty$, one has:

$$\begin{aligned}&\Big[\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}\Big]\Delta S-\Big[\boldsymbol{\sigma}^{(1)}+\mathbf{e}_1\Phi+\mathbf{e}_1\times\boldsymbol{\Psi}\Big]\,\Delta S_1\\ &\quad-\Big[\boldsymbol{\sigma}^{(2)}+\mathbf{e}_2\Phi+\mathbf{e}_2\times\boldsymbol{\Psi}\Big]\Delta S_2\\ &\quad-\Big[\boldsymbol{\sigma}^{(3)}+\mathbf{e}_3\Phi+\mathbf{e}_3\times\boldsymbol{\Psi}\Big]\Delta S_3=0.\end{aligned}\tag{27}$$

Substituting $\Delta S_j=n_j\Delta S$ into Equation 27 yields:

$$\begin{aligned}\boldsymbol{\sigma}+\mathbf{n}\Phi+\mathbf{n}\times\boldsymbol{\Psi}&-(n_1\boldsymbol{\sigma}^{(1)}+n_2\boldsymbol{\sigma}^{(2)}+n_3\boldsymbol{\sigma}^{(3)})\\ &-(n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3)\Phi\\ &-(n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3)\times\boldsymbol{\Psi}=0.\end{aligned}\tag{28}$$

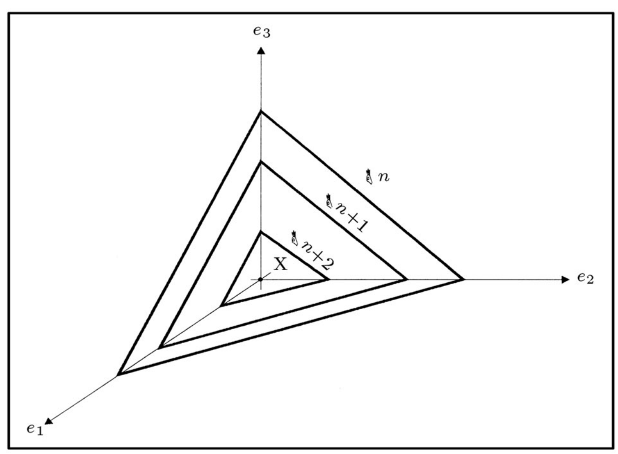

Figure 3. Three consecutive members of a sequence of tetrahedrons that approach a point $\mathbf{x}$ and all have unit normal vector $\mathbf{n}$ on the oblique face.

Thus:

$$\boldsymbol{\sigma} = n_i \boldsymbol{\sigma}^{(i)} = n_i S_{ji} \mathbf{e}_j \quad \Rightarrow \quad \boldsymbol{\sigma}_i = S_{ji} n_j, \tag{29}$$

where S_{ij} are components of the stress tensor. It is observed that when $k \to \infty$, the effects of scalar and vector potentials vanish. Hence, there is no need to have their explicit expressions in terms of the characteristics of body and inertia force vectors.

Similarly, the restated balance of angular momentum, Equation 19, can be considered for the same sequence of tetrahedrons. In the limit $k \to \infty$, one has:

$$\begin{aligned} &[\mathbf{r} \times \boldsymbol{\sigma} + \mathbf{m} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},\mathbf{n},t)} \Delta S \\ &+ [\mathbf{r} \times \boldsymbol{\sigma} + \mathbf{m} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_1,t)} \Delta S_1 \\ &+ [\mathbf{r} \times \boldsymbol{\sigma} + \mathbf{m} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_2,t)} \Delta S_2 \\ &+ [\mathbf{r} \times \boldsymbol{\sigma} + \mathbf{m} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi}]_{(\mathbf{x},-\mathbf{e}_3,t)} \Delta S_3 = 0, \end{aligned} \tag{30}$$

or:

$$\begin{aligned} &\mathbf{r} \times \boldsymbol{\sigma} + \mathbf{m} + \mathbf{n}\Phi + \mathbf{n} \times \boldsymbol{\Psi} \\ &- (\mathbf{r} \times \boldsymbol{\sigma}^{(1)} + \mathbf{m}^{(1)} + \mathbf{e}_1\Phi + \mathbf{e}_1 \times \boldsymbol{\Psi}) n_1 \\ &- (\mathbf{r} \times \boldsymbol{\sigma}^{(2)} + \mathbf{m}^{(2)} + \mathbf{e}_2\Phi + \mathbf{e}_2 \times \boldsymbol{\Psi}) n_2 \\ &- (\mathbf{r} \times \boldsymbol{\sigma}^{(3)} + \mathbf{m}^{(3)} + \mathbf{e}_3\Phi + \mathbf{e}_3 \times \boldsymbol{\Psi}) n_3 = 0. \end{aligned} \tag{31}$$

Thus,

$$\mathbf{r} \times (\boldsymbol{\sigma} - n_i \boldsymbol{\sigma}^{(i)}) + (\mathbf{m} - n_i \mathbf{m}^{(i)}) = 0. \tag{32}$$

Finally, from Equations 29 and 32, it is obtained that:

$$\mathbf{m} = n_i \mathbf{m}^{(i)} = n_i \mu_{ji} \mathbf{e}_j \quad \Rightarrow \quad m_i = \mu_{ji} n_j, \tag{33}$$

where μ_{ij} are components of the couple-stress tensor [12].

CONCLUSIONS

This article presents restatements of the laws of balance of linear and angular momenta as surface integrals over the closed boundary of any subregion in the continuum. For this restatement, Helmholtz's representation theorem is used. Newton's law of action and reaction and Cauchy's theorem for stress and couple-stress are proved as corollaries of these surface balance laws.

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