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Research Note

Non-Symmetrical Plane Contact of a Wedge Indenter

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Different parameters of the asymmetric contact problem between an elastic wedge and a half-plane have been introduced in this paper. These parameters include the distribution of contact pressure and length of contact zones due to frictionless normal loading. For each parameter, the results have been compared with the results of the symmetric problem and numerical solution, which show excellent agreement. The method of approach is a completely analytical method based on singular integral equations. In this method, the boundary conditions of the problem are stated as some singular integrals, and distribution of the contact pressure is specified. Then, with use of the equilibrium equations and the consistency conditions of the singular integral solution, the lengths of the contact zones are specified.

INTRODUCTION

Contact is one of the principal methods of applying loads to the surfaces of deformable bodies. The resulting regions of stress concentration are often the most critical regions in the body. Contact is characterized by unilateral inequalities, describing the physical impossibility of tensile contact tractions (except under special circumstances) and of material interpenetration. Additional inequalities and/or non-linearities are introduced when friction laws are taken into account. These complex boundary conditions can lead to problems with the existence and uniqueness of a quasi-static solution and to a lack of convergence of numerical algorithms. In frictional problems, there can also be a lack of stability, leading to stick-slip motion and frictional vibrations.

Classical elastic contact problems for the wedge and cone are attractive analytically because they have a relatively simple form of solution and the interior state of stress induced may also be found easily, including the effects of sliding frictional shear tractions. The principal applications of this geometry are to understand the stress state induced beneath asperities on rough surfaces, the contacts arising in certain fretting fatigue experiments and the loading imposed by stylus instruments, such as surface profilometers. In particular, the use of indenters with a linear profile is attractive for fretting fatigue experiments, as this permits a wide range of size of incomplete contacts to be obtained and, thus, facilitates a control of the size effect, i.e. the different fretting fatigue behavior of the material under geometrically identical contacts of varying size.

The problem of contact between two similar cylinders in the absence of friction was first studied by Hertz [1]. The results of Hertz's work are still considered as the basis for many practical designs. However, the friction between the contacting parts plays a significant role in creating surface traction. Hence, it is necessary to consider friction when the growth of surface cracks is the dominating parameter for life prediction. This was first formulated by Cattaneou using two similar spheres [2]. It should be noted that the friction does not affect the peak stresses at the contact area [3]. Similar studies can be found on contacting bodies with different geometries [4-8].

The complexity of the geometries of the contacting bodies, loading conditions, material properties and environmental effects, etc. makes preferable the use of numerical methods, like the FEM (Finite Element Method) in analyzing some problems. The existing FEM studies are divided into two different groups. The first group is involved with integral equations obtained from the weighted residual methods and variational inequalities [9-11]. However, the reported studies in

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the second group are focused on the application of the developed algorithms to practical problems [12-14]. It must be mentioned that this group includes solving particular problems that cannot be generalized and, for some of the studies the discrepancy between the FEM results and the Hertz solution is enormous [13].

The problem of the normal indentation of dissimilar elastic bodies was first considered in detail by Spence [15,16]. The Spence solution concentrates on a rigid flat-ended punch and uses an iterative scheme to develop the coupled solution. Using the same method, Nowell et al. developed a solution for the contact of dissimilar elastic cylinders [17,18].

The majority of contact solutions encountered in the literature assume symmetrical profiles and symmetrical indentation. This greatly simplifies the solution of related contact problems. However, nonsymmetrical cases may be of considerable interest in many engineering applications. Even with nominally symmetrical contacts, there is the possible effect of a relative rotation, as a result of an applied moment, or of undesired geometrical asymmetry. Non-symmetrical contact problems may occur in situations like hardness testing and surface roughness measurement. In this paper, the effects of asymmetry on contact pressure distribution and contact lengths are considered. The method used is completely analytical, based on singular integral equations [19,20]. In this method, the boundary conditions are expressed as singular integrals and, then, by solving them, the contact pressure distribution and lengths are calculated.

FORMULATION AND PROBLEM SOLUTION

A plane contact problem containing an elastic half-space and an elastic wedge indenter is considered (Figure 1) [5]. A vertical force is exerted on the tilted wedge, pushing it against the half space. It is assumed that the external angles at the apex of the indenter (ϕ_1, ϕ_2) are small. This permits the use of an elasticity formulation appropriate to a semi-infinite body and this, in turn, facilitates a closed form solution. Also, it is assumed that either of the contact surfaces is lubricated or that the contacting bodies are elastically similar to prevent the development of interfacial shearing tractions. Even if neither of these conditions is fulfilled, their influence on the contact pressure distribution is likely to be very small.

It is well known that in the case of plane contact, the basic singular integral equation relating the relative surface vertical displacements of the two bodies, $h(x)$, to the contact pressure distribution, $p(x)$, is given by:

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int_L \frac{p(\eta)}{x-\eta}d\eta, \tag{1}$$

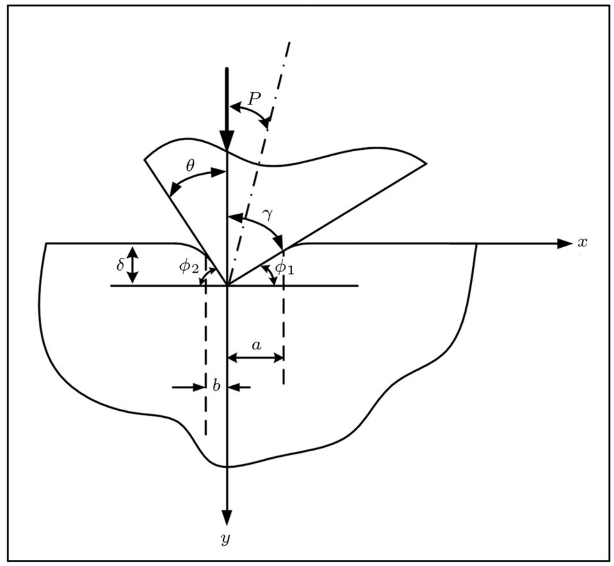

Figure 1. Tilted wedge in contact with half-plane.

where the composite compliance, A, is given by:

$$A = \frac{K_1+1}{4\mu_1} + \frac{K_2+1}{4\mu_2}. \tag{2}$$

In which, $K_i = 3-4\nu_i$ in plane strain, $K_i = \frac{3-\nu_i}{1+\nu_i}$ in plane stress and L indicates the contact region. Also, ν_i and μ_i are the Poisson ratio and the shear modulus of body $i (i = 1, 2)$, respectively. From Figure 1, it can be seen that due to the unsymmetrical nature of the contact phenomena, the contact regions on both sides of the y-axis are not the same. So, Equation 1 can be written as:

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int_{-b}^{a} \frac{p(\eta)}{x-\eta}d\eta, \tag{3}$$

where a and b are the lengths of the contact area along the $x-$axis. The relative surface vertical displacements of the contacting bodies, $h(x)$, can be defined with a unique function using the singularity function:

$$h(x) = \delta - \phi_1 < x > -\phi_2 < -x >, \tag{4}$$

where the singularity function is defined, in general, as:

$$< x >^n = \begin{cases} x^n; & x > 0 \\ 0; & x < 0 \end{cases}. \tag{5}$$

Substituting for $h(x)$ from Equation 4 into Equation 3, yields:

$$\frac{1}{A}[-\phi_1 < x >^0 + \phi_2 < -x >^0] = \frac{1}{\pi}\int_{-b}^{a} \frac{p(\eta)}{x-\eta}d\eta. \tag{6}$$

The following linear mappings can be defined to normalize the variables in Equation 6:

$$\eta = \frac{a+b}{2}r + \frac{a-b}{2} = \alpha r + \beta,$$

$$x = \frac{a+b}{2}s + \frac{a-b}{2} = \alpha s + \beta, \tag{7}$$

where:

$$\alpha=\frac{a+b}{2},\quad \beta=\frac{a-b}{2},\tag{8}$$

and $-1\leq r,s\leq 1$.

Substitution of Equations 7 into Equation 6 leads to:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(r)}{s-r}dr=\frac{-\phi_1<\alpha s+\beta>^0+\phi_2<-\alpha s-\beta>^0}{A}.\tag{9}$$

Noticing that the pressure distribution is bounded in the considered incomplete contact problem, the singular integral Equation 9 can be solved to be:

$$p(s)=-\frac{\sqrt{1-s^2}}{\pi A}\int_{-1}^{1}\frac{\phi_1<\alpha r+\beta>^0-\phi_2<-\alpha r-\beta>^0}{(r-s)\sqrt{1-r^2}}dr,\tag{10}$$

provided that the following relation holds:

$$\int_{-1}^{1}\frac{-\phi_1<\alpha s+\beta>^0+\phi_2<-\alpha s-\beta>^0}{A\sqrt{1-s^2}}ds=0.\tag{11}$$

Note that:

$$<\alpha r+\beta>^0=\begin{cases}1; & \alpha r+\beta>0 \quad \text{or} \quad r>-\frac{\beta}{\alpha}\\ 0; & \alpha r+\beta<0 \quad \text{or} \quad r<-\frac{\beta}{\alpha}\end{cases},$$

$$<-\alpha r-\beta>^0=\begin{cases}1; & -\alpha r-\beta>0 \quad \text{or} \quad r<-\frac{\beta}{\alpha}\\ 0; & -\alpha r-\beta<0 \quad \text{or} \quad r>-\frac{\beta}{\alpha}\end{cases}.\tag{12}$$

Equation 10 can be integrated to yield:

$$p(s)=-\frac{\phi_1+\phi_2}{\pi A}\ln\left|\frac{\alpha+\beta s+\sqrt{\alpha^2-\beta^2}\sqrt{1-s^2}}{\alpha s+\beta}\right|.\tag{13}$$

In the special case of symmetrical contact problem ($\phi_1=\phi_2=\phi$), one should have $a=b$ and, therefore, $\alpha=a$ and $\xi=0$. Hence, the pressure distribution can be simplified to give:

$$p(s)=\frac{2\phi}{\pi A}\cosh^{-1}\left(\frac{1}{|s|}\right).\tag{14}$$

This is exactly the same relation obtained by Truman et al. for the symmetrical contact of a wedge indenter [5].

Substituting for s, $s=\frac{x-\beta}{\alpha}$ into Equation 13 leads to:

$$p(x)=-\frac{\phi_1+\phi_2}{\pi A}\ln\left|\frac{\alpha^2-\beta^2+\beta x+\sqrt{\alpha^2-\beta^2}\sqrt{\alpha^2-(x-\beta)^2}}{\alpha x}\right|.\tag{15}$$

Integrating the consistency Equation 11 leads to:

$$\frac{\beta}{\alpha}=\sin\left(\frac{\pi}{2}\frac{\phi_2-\phi_1}{\phi_2+\phi_1}\right).\tag{16}$$

Substitution of Equation 16 into Equation 13 results in:

$$p(s)=-\frac{\phi_1+\phi_2}{\pi A}\ln\left|\frac{1+s\sin\psi+\cos\psi\sqrt{1-s^2}}{s+\sin\psi}\right|,\tag{17}$$

where:

$$\psi=\frac{\pi}{2}\frac{\phi_2-\phi_1}{\phi_2+\phi_1}.\tag{18}$$

It is interesting to note that the pressure distribution given by Equation 17 is independent of the lengths of the contact zones, a and b (or α and β).

The dimensionless contact pressure distribution is plotted in Figure 2, as a function of nondimensionalized parameter s. Two sets of combinations of angles are considered, which are $\phi_1+\phi_2=4$ and $\phi_1+\phi_2=16$. It is seen from the figure that, in all cases, the pressure distribution has singularity at a point which corresponds to the wedge apex. Also, it is observed that the greater the sum of the external angles at the apex, the greater the pressure magnitude and the severer the pressure singularity at the apex.

To find the lengths of the contact zone, the following relation should be used:

$$P=\int_{-b}^{a}p(x)dx.\tag{19}$$

In fact, this relation indicates the resultant of the contact forces due to pressure distribution. Normalizing Equation 19 with the aid of Equations 7 and, then,

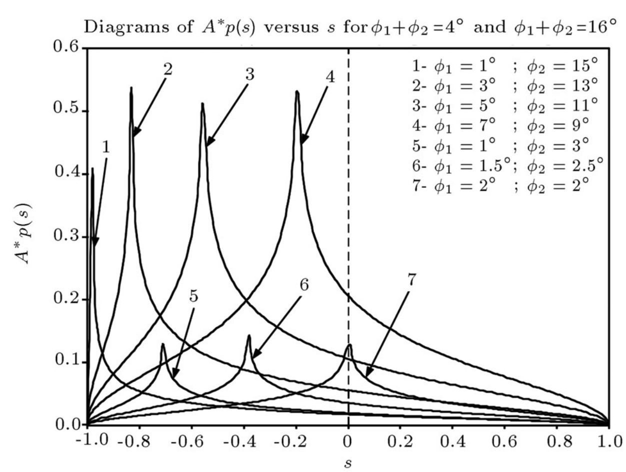

Figure 2. The dimensionless contact pressure distributions versus nondimensionalized parameter s.

using Equation 17, the following equation is obtained:

$$-\frac{\pi AP}{\phi_1+\phi_2}=\alpha\int_{-1}^{1}\ln\left|\frac{s\sin\psi+\cos\psi\sqrt{1-s^2}}{s+\sin\psi}\right|ds$$

$$=\alpha\left[\int_{-1}^{1}\ln\left|s\sin\psi+\sqrt{1-s^2}\cos\psi\right|ds\right.$$

$$\left.-\int_{-1}^{1}\ln\left|s+\sin\psi\right|ds\right]. \tag{20}$$

Using the change of variable $s=\cos u$ in the first integral, the following equation is derived:

$$\alpha=\frac{AP}{(\phi_1+\phi_2)\cos\psi}. \tag{21}$$

Then, using Equation 16, one may easily obtain:

$$\beta=\frac{AP}{\phi_1+\phi_2}\tan\psi. \tag{22}$$

The lengths of the contact zone can be obtained using Equation 8:

$$a=\frac{AP}{\phi_1+\phi_2}\frac{1+\sin\psi}{\cos\psi},$$

$$b=-\frac{AP}{\phi_1+\phi_2}\frac{1-\sin\psi}{\cos\psi}. \tag{23}$$

Examination of the above equations, for the special case of symmetrical contact of a wedge and a half-space, yields:

$$a=\frac{AP}{2\phi},\qquad b=-\frac{AP}{2\phi}. \tag{24}$$

These results are the same as those published by Truman et al. for the symmetrical case [5].

It is worth mentioning that when $\phi_1\to 0$ and $\psi\to\frac{\pi}{2}$ and taking limits, then:

$$a\to\infty,\qquad b\to 0. \tag{25}$$

This means that when ϕ is small, the corresponding contact surface lies fully on the half-space and the other surface ceases to contact with the half-space, as was expected in a physical sense. A similar fact holds true when $\phi_2\to 0$. These are shown in Figure 3.

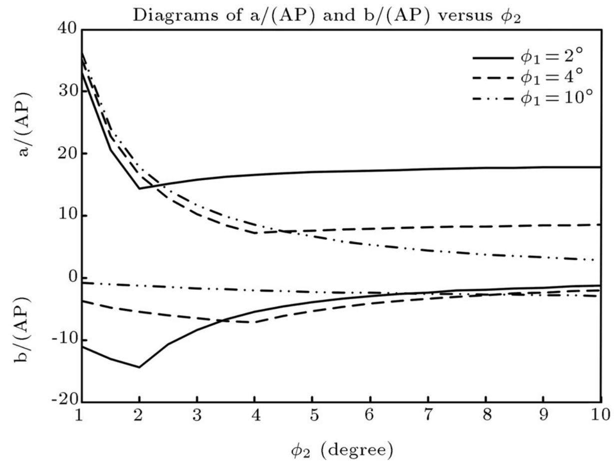

Figure 3. The dimensionless contact lengths distributions versus ϕ_2 at some constant ϕ_1.

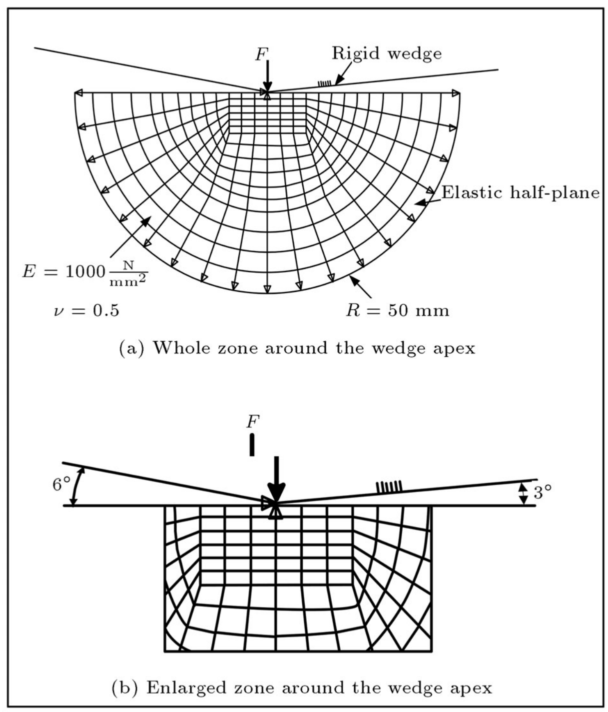

Figure 4. Finite element model of the contact problem.

COMPARISON WITH FEM

A finite element model of the specific geometry is constructed (Figure 4) and is then analyzed with ANSYS 7.0. Rigid Elements TARGE169 are used to represent the rigid wedge with $\phi_1=3°$ and $\phi_2=6°$. A circular region of the half-plane is modeled with 2-D elements PLANE42 and is fixed radially at $R=50$ mm. The x-axis of the half-plane represents the contact surface and consists of CONTA171 elements, which are associated with the target elements. The half-plane has a modulus of Elasticity of $E=1000\frac{N}{\text{mm}^2}$ and Poisson ratio of $\nu=0.4999\approx 0.5$. The normal force is applied on the wedge and is equal to $100,000\frac{N}{\text{m}}$.

A comparison of the normal pressure distribution from FEM and the analytical results is shown in Figure 5. The difference between both methods is very small. These small errors are due to the approximate nature of the FE analysis. The figures

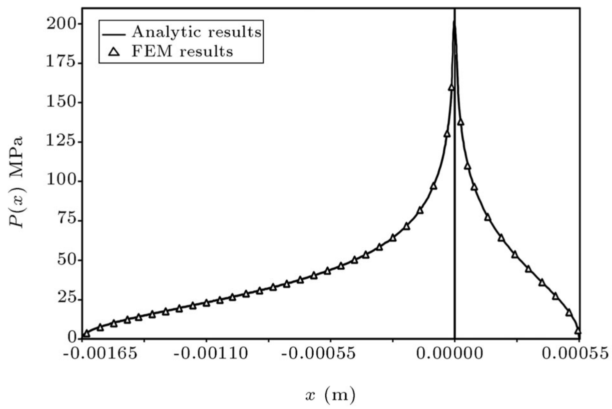

Figure 5. Contact pressure distribution versus x for the case $\phi_1 = 3°$ and $\phi_2 = 6°$.

also verify another result of the analytical solution, i.e., the lengths of the contact areas. This is because of the unsymmetrical pressure distribution, which implies that either of the regions on the left and right of the line $x = 0$, corresponds to one of the contact areas and, hence, indicates the length of the contact area.

CONCLUSION

A general model for the problem of contact between a tilted wedge and a half-plane is developed. The method of solution is completely analytical and is based on Cauchy's principal values for the singular integral equations. In the case of elastically similar materials, this model was solved for pressure distribution and contact lengths, of which the results are in full agreement with the previous solutions for the symmetric case. The effect of asymmetry on contact pressure distribution and contact length diminishes with decreasing the tilt angle and vice versa. Finally, the results are compared with the FE solution and it is shown that there is good agreement between these methods.

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