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Optimal Production and Maintenance Control Under a Time Variant Demand

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In this paper, optimal production and maintenance planning of a flexible manufacturing system under a time variant demand is considered. There is a preventive maintenance plan to reduce the failure rate of the machine. It is assumed that the failure rate of the machine is a function of its age and its maintenance rate. It is, also, assumed that the demand of the manufacturing product is time dependent and its rate depends on the level of the advertisement on that product. The objective is to maximize the expected discounted total profit of the firm over an infinite time horizon. To solve this optimization problem, first, an optimal control is characterized by a set of Hamilton-Jacobi-Bellman partial differential equations. Then, since this set of equations cannot be solved analytically, this stochastic optimal control model is approximated by a deterministic optimal control problem. By solving this new deterministic problem under practical assumptions, a set of suboptimal controls can be found.

INTRODUCTION

The simultaneous planning of production and maintenance in a flexible manufacturing system is considered in this research. The system considered is composed of one machine producing a single product. The machine is failure prone and there is a preventive maintenance plan to reduce its failure rate. The probability of failure of the machine is supposed to be dependent on its age. The preventive maintenance actions restore the age of the machine to a lower level. The failure rate of the machine is assumed to be an increasing function of its age and a decreasing function of its maintenance rate.

Recently, there have been many efforts made to use stochastic optimal control techniques in the production planning of manufacturing systems. In most of them, the demand of the product or products is assumed to be constant over time. In this research, it is assumed that the demand of the manufacturing product is time dependent and that its rate depends on the level of advertisement on that product. The assumption of a time variant demand is more realistic and makes the results of this research more practical.

The objective is to determine the production rate and the maintenance rule of the machine, as well as the advertisement cost rate of the product, in order to maximize the expected discounted total profit of the firm over an infinite time horizon. In writing the performance criterion, it is assumed that the revenue of the firm is equal to the price of the product time demand. The total cost consists of the cost of the product surplus, defined as the discrepancy between total cumulative production and total cumulative demand, the cost of the repair activity after failure, the cost of the maintenance activity and the cost of the advertisement. It is also assumed that repair is more costly than preventive maintenance.

For solving the stochastic optimal control problem of this paper, first, the necessary conditions are written as a set of partial differential equations. Then, not being able to solve these equations in a closed form, it is possible either to solve them numerically or to propose an approximation procedure for finding a near optimal solution. The second method is chosen, i.e. proposing a deterministic optimal control problem whose solution approximates the stochastic optimal control. Then, this deterministic problem is solved to characterize a suboptimal solution.

The rest of this paper contains the following sections. First, a literature review in the general area of this research is presented. Then, the problem statement containing the mathematical model of the problem is discussed and the optimal control is char-

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acterized by dynamic programming equations. After that, the approximate suboptimal control is proposed and, then the case study of the paper is presented. Finally, the conclusions of the paper are mentioned.

LITERATURE REVIEW

An interesting feature of many automated production systems is that they can be considered as deterministic systems, as long as no machine breakdowns or stoppages occur. Therefore, these systems fall into the category of "piecewise deterministic processes", according to the terminology of Davis [1]. A class of systems closely related to those considered were defined previously by Sworder [2] and Rishel [3,4] and called "systems with jump Markov disturbances". Olsder and Suri [5] were the first ones to recognize this fact when they proposed a stochastic control model based on Rishel's formalism for the planning of production in a flexible manufacturing system. In their model, each machine is subject to random failure, according to a homogeneous Markov process.

The research of production planning using stochastic control techniques has drawn much attention lately. Akella and Kumar [6] formulated a one-machine one-part-type production problem as a stochastic optimal control problem, in which the part demand is assumed to be a constant, the state of the machine is assumed to be a two-state continuous-time Markov chain and the objective function is a discounted inventory/shortage cost over an infinite time horizon. It was shown that optimal control is given by a single threshold inventory level. Bielecki and Kumar, then, treated a long-run average cost [7] and an optimal hedging point policy was obtained.

According to this policy, at any point in time, the control guides the production surplus towards a nonnegative level, depending on the capacity state in place. This capacity state specific level is known as the corresponding hedging point. The idea behind this policy is that some nonnegative production surplus should be maintained at times of excess capacity to hedge against future capacity shortages [8].

Although the structure of the optimal policy is known, analytic solutions of the optimal controller exist only for single-part-type systems. For multiple-part-type systems, one has to resort either to discretization-based numerical techniques, which are practical for problems of a small size, or approximation techniques [9] that exploit the structure of the optimal policy and are practical for larger problems. One of these approximation policies is introduced in [10], which is parameterized over a finite set of parameters. These parameters define quadratic functional approximations to the value functions characterizing the optimal policy [11]. Derivative estimates of the objective function, with respect to these parameters, are obtained via infinitesimal perturbation analyses and are used to drive a stochastic approximation algorithm for parameter optimization.

Extensions to the model of Akella and Kumar are considered by Perkins and Srikant [12], in which they incorporated a multiple part-type in the model, and by Liberopoulos and Hu [13], where they studied the structural properties of the hedging point policies. In these papers, the objective is to minimize an expected cost with approximate choices of the production control variables.

Boukas and Yang [14] extended Akella and Kumar's model, to allow the simultaneous planning of production and maintenance in a flexible manufacturing system. Their system is composed of a single machine, which produces a given commodity. The machine is subject to some random failures. The probability of machine failure is supposed to be an increasing function of its age. The commodity demand rate is assumed to be constant. The objective is meeting the demand while minimizing the discounted inventory and maintenance cost. Under some appropriate conditions, they established similar results to the ones given by Akella and Kumar.

Sharifnia [15] showed how the optimal hedging point, in the case of one part-type multiple machine-states, can be calculated. The problem of the complete evaluation of the optimal production policy in multiple part-type multiple machine-states is difficult, because it requires solving either systems of partial differential equations or large dynamic programming problems that easily run into the "curse of dimensionality". Therefore, several approximation procedures have been proposed to obtain near-optimal controllers [16,17].

In most of the manufacturing flow control models considered, it has been assumed that the machine failure rates are independent of the production rates and are constant as long as the system is in one of its discrete capacity states, or, in other words, the underlying Markov chain is homogeneous. In reality, however, this assumption is often violated and the failure rate of a machine usually depends on many factors, for example, the age of the machine and the instantaneous rate of production. In most cases, it is reasonable to assume that if a machine works at a faster rate, it is more likely to fail. Very few studies have been done for systems with operation dependent failure rates.

Boukas and Haurie [18] considered a system which has two machines with age-dependent failure rates and where preventive maintenance is a decision option. They used a numerical method to evaluate the optimal control policy and showed that, in their context, optimal hedging surfaces can be defined to represent optimal production policies. Hu and Xiang [19] derived some structural properties of the optimal control for a system with multiple machine-states and age-dependent failure rates. They showed that the closer to the zero capacity state the system is, the larger the hedging point should be. Rishel [20] studied a machine-wear model, in which the wear rate (the machine failure rate) depends on the machine operating speed (the production rate). Particularly, he considered the case in which the wear rate is a quadratic function of the production rate. Hu, Vakili and Yu [21] considered a one machine one part-type system with operation dependent failure rates. They assumed that the failure rate depends on the instantaneous rate of production. Since one of the desirable features of the hedging point policy is the simplicity of the policy and its ease of implementation, they answered the following question: Under what failure rate functions are hedging point policies optimal? They derived both necessary and sufficient conditions for the optimality of a hedging point policy.

Sethi and Zhang [22] formulated a continuous-time production and setup scheduling model. Using the theory of viscosity solutions of Hamilton-Jacobi-Bellman equations, they were able to establish optimality conditions. However, a closed form optimal solution in these cases is an impossible task to accomplish. In order to be able to use the optimality theory on real time production control, numerical methods for the model developed in [22] seem to be the only feasible approach [23].

The aforementioned papers dealt with infinite-horizon cost functions, whereas a finite-horizontal counter part was considered in Zhang and Yin [24], in which the corresponding optimal control was obtained in terms of the time-dependent turnpike sets under "traceability" conditions. In addition, it was demonstrated that as T goes to infinity, the infinite-horizon results of Akella and Kumar are recovered from those obtained with finite-horizon costs. In connection with robust control, Boukas, Yang and Zhang [25] considered a minimax production planning model with Markov capacity process and deterministic, but unknown demand rate and optimal control with a discounted cost criterion was obtained.

A common feature in these papers is that the optimal control policies are a threshold type, which is attractive in application because of the simple structure of hedging. It reveals much insight, such as dependence on various parameters of the system under consideration. Furthermore, it can be used to treat more complex systems via hierarchical decompositions and hierarchical production planning methods (see, for example, [26]) by approximating a complex system with a simpler limit system and by constructing asymptotic optimal control leading to near optimality.

Due to the complexity of manufacturing systems, traditionally, marketing decision making and other decision related areas, such as productions, are often treated separately. Clearly, a marketing model with the addition of production is more realistic and useful from a practical point of view. In this connection, Abad [27] proposed a decentralized marketing-production planning model and solved the problem by applying Pontryagin's maximum principle. Sethi and Zhang [28] considered a marketing-production model, in which the demand is assumed to be a Markov decision process. The main focus of [24] is a reduction of dimensionality of the underlying problem via a hierarchical control approach (see, also, Sogomonian and Tang [29] for another model concerning interfaces of marketing and production and Yin and Zhang [30] for further work on singularly perturbed Markov chains).

In a recent paper [31], Zhang, Yin and Boukas considered a marketing-production planning model. Using a stochastic control formulation, the demand rate is modeled as a finite-state continuous-time Markov chain. Their objective is to choose the optimal strategy (including the choice of production and advertising rate), so that the overall expected profit is maximized. Under reasonable conditions, they derived the closed-form optimal control. An interesting and important observation is that the optimal market-production policy is of the hedging-point type and the hedging point depends on the amount of marginal revenue.

PROBLEM STATEMENT

As mentioned earlier, the model in this paper for production and maintenance planning in a flexible manufacturing system consists of a single workstation producing a one part-type through a single operation. The system considered has a state comprising, both, a continuous and a discrete component. This production system has continuous state variables x, a and z, corresponding to the cumulative production surplus of parts, the machine age and the demand rate of the part-type, respectively. Let $u(t)$ be the production rate of the workstation at time t. The state equation of the surplus is given by:

$$\dot{x}(t) = u(t) - z(t), \quad x(0) = x_0, \tag{1}$$

where x_0 is a given initial surplus value. The surplus change rate is equal to the difference between the workstation production rate and the part-type demand rate at time t.

It is assumed that the aging of the workstation at time t is an increasing function of its production rate and a decreasing function of its maintenance rate. Let $v(t)$ be the maintenance intensity of the workstation at time t. Thus, the cumulative age of the workstation is the solution of the following differential equation:

$$\dot{a}(t) = f\left(u(t), v(t)\right), \quad a(T) = 0, \quad t > T, \tag{2}$$

where T is the last restart time of the workstation and $a(T) = 0$ implies that a repair, or a preventive maintenance job, restores the cumulative age to a zero value, since the aging rate of the workstation is a function of its production rate and its maintenance intensity at time t.

The demand rate of the part-type is assumed to be time variant and is denoted by $z(t)$ at time t. The dynamic equation of the demand is also a non-homogeneous differential equation with the input variable $w(t)$, which is taken to be the cost rate of advertisement at time t. The initial value of the part-type demand at time 0 is the given constant, z_0. The equation is:

$$\dot{z}(t) = c_0 z(t) + c_1 w(t), \quad z(0) = z_0, \tag{3}$$

where c_0 and c_1 are constant values, since the part-type demand change rate is taken to be a linear function of the demand itself and the advertisement cost at time t.

The discrete part of the state vector represents the system operational mode. Let $E = \{1, 2, 3\}$. The operational mode of the workstation at time t is given by the random variable, $\xi(t)$, with value in E. This mode indicates if the workstation is operational, $\xi(t) = 1$, in repair $\xi(t) = 2$, or in maintenance $\xi(t) = 3$, at time t. $\lambda_{\alpha\beta}(a, v)$ is called the transition rate from state $\alpha \in E$ to state $\beta \in E$ for the workstation at time t, where:

$$\lambda_{12}(a(t)) = \lim \left\{ \frac{1}{dt}[p(\xi(t + dt) = 2/\xi(t) = 1)] \right\}, \tag{4}$$

$$\lambda_{21}(a(t)) = \lim \left\{ \frac{1}{dt}[p(\xi(t + dt) = 1/\xi(t) = 2)] \right\}, \tag{5}$$

$$\lambda_{13}(v(t)) = v(t) = \lim \left\{ \frac{1}{dt}[p(\xi(t + dt) = 3/\xi(t) = 1)] \right\}, \tag{6}$$

$$\lambda_{31}(a(t)) = \lim \left\{ \frac{1}{dt}[p(\xi(t + dt) = 1/\xi(t) = 3)] \right\}, \tag{7}$$

as $dt \to 0$ and where λ_{23} and λ_{32} are equal to zero, because there is no transition between the repair mode and the maintenance mode of the workstation and vice versa.

Equation 4 implies that failures occur as a Poisson process and the failure rate of the workstation depends on its age. Equation 5 implies that the repair duration for a failed workstation is an exponential random variable whose mean also depends on the age of the workstation. Equation 6 defines a transition rate from the mode "operational" to the mode "preventive maintenance". This will be a control variable, which is denoted by $v(t)$. The inverse of this control variable represents the expected delay between a call for the technician and his arrival. This modeling of preventive maintenance is more realistic than the preventive repair or replacement of a component as an impulsive control. Equation 7 implies that the duration of a preventive maintenance job on the workstation may also depend on its age. The common characteristic of the operational time, the repair time and the maintenance time of the workstation is their memory-less property and, because of this property, the exponential distribution is the suitable distribution to model these time intervals. So, their corresponding stochastic processes will be of the Poisson type. From the rate functions (Equations 4 to 7), one can easily deduce the transition rates, $q_{\alpha\beta}(a, v)$, for the process $\xi(t)$, where α and β are in E.

From the initial condition of Equation 2 at a jump time, τ, for the process $\xi(t)$, one defines a reset function, $\phi(a, \xi) : \Re_+ \times E \to \Re_+$, by the following relation:

$$\phi(a, \xi) = \begin{cases} 0, \text{if } \xi(\tau^+) = 1 & \text{and } \xi(\tau^-) \neq 1, \\ a(\tau^-), & \text{otherwise} \end{cases}$$

This function describes the age discontinuity, which may occur at a jump time of the operating state of a machine.

The variables x, a, z and ξ are the state variables of the system. $y = (x, a, z) \in \Re^3$ is called the continuous part of the state. The variables u, v and w are the control variables. The complete control vector will be denoted by $\theta = (u, v, w)$. Also, the set of all admissible controls will be called $\Theta(\beta)$, which depends on the operational state of the system and is a compact set.

First, for being able to solve the dynamic programming partial differential equation analytically, the cost rate function is restricted not to depend on the production rate and the maintenance rate of the workstation. So, the cost rate function, $\varphi^\beta(a, x, u, v)$, is taken as a function of the surplus variable and the operational mode of the system, as follows:

$$\varphi^\beta(a, x, u, v) = h(x) + c^\beta, \quad \forall \beta \in E,$$

in which:

$$c^\beta = c_2 I(\xi(t) = 2) + c_3 I(\xi(t) = 3),$$

where:

$I(\cdot)$ = the indicator function of the corresponding set,
c_2 = the cost rate of the repair,
c_3 = the cost rate of the maintenance, with $c_2 > c_3$.

Let (Ω, F) be a measurable space and $\{F_t; t \geq 0\}$ an increasing class of sigma-fields representing the history of the (a, x, ξ) process. A sample value, ω, corresponds to an a-trajectory, having a finite set of discontinuities on any finite interval, an x-trajectory, which is continuous and a sequence of ξ-values without accumulation points. The set, Γ, of admissible control policies is a family of F_t-adapted processes with values in $\Theta(\beta)$. The class Γ is such that, for a given β, the mapping $\gamma(.,.,\beta) : (a, x) \rightarrow \Theta(\beta)$ is piecewise continuously differentiable. Thus, with each control policy, $\gamma \in \Gamma$ is associated with a probability measure, P_γ, on (Ω, F), such that the process, (a, x, ξ), is well defined. An admissible control policy is a set of feedback controls, each one corresponding to a different operating state of the system. Each feedback determines the production level, the preventive maintenance intensity and the advertisement cost as a function of the surplus level, machine age and product demand.

The objective of this paper is to find, in Γ, a control policy, γ^*, which maximizes the expected discounted total profit of the firm for each initial condition (a_0, x_0, β) over an infinite horizon, or, equivalently, to minimize the negative profit, as follows:

$$J^\beta(x, a, z, u, v, w) = E\left\{\int_0^\infty e^{-\rho t}[\varphi^{\xi(t)}(a(t), x(t), u(t), v(t)) + w(t) - \pi z(t)]dt\right\},$$

$$x(0) = x_0, \quad a(0) = a_0, \quad z(0) = z_0, \quad \xi(0) = \beta, \qquad (8)$$

where ρ is a positive discount rate, π is the revenue per unit sale and $\xi(t), a(t), x(t), u(t), v(t)$ are the stochastic processes defined by the control policy, γ, and the initial conditions. As seen in Equation 8, the negative profit function is equal to the integral of the discounted cost rate function, plus advertisement cost, minus the revenue of the manufacturing firm, over an infinite horizon. The value function, or equivalently the optimal cost function, is denoted by $V(\beta, a, x)$. This optimal cost function is the solution of the dynamic programming partial differential equation and characterizes the optimal control variables and the optimal trajectory of the problem.

DYNAMIC PROGRAMMING EQUATION

This optimal control problem belongs to the class of problems considered by Rishel. Under appropriate assumptions of smoothness for the control, the following set of Hamilton-Jacobi-Bellman partial differential equations characterizes an optimal control:

$$\rho V(\beta, a, x) = \min\left\{\varphi^\beta(a, x, u, v) + \frac{\partial}{\partial a}V(\beta, a, x)f(u, v) + \frac{\partial}{\partial x}V(\beta, a, x)(u(t) - z(t)) + \sum_{\alpha \in E} q_{\beta\alpha}(a, v)[V(\alpha, \phi(a, \alpha), x) - V(\beta, a, x)]\right\}, \quad \forall \beta \in E,$$

$$\text{over} \quad \theta \in \Theta(\beta). \qquad (9)$$

The cost rate function, $\varphi^\beta(a, x, u, v)$, has been defined as a function of the state variables only. Therefore, it can be deduced from the dynamic programming Equation 9 that the control, u, will be chosen to minimize the trajectory derivative, $(\frac{d}{dt})V(\beta, a(t), x(t))$.

The controls v and w, corresponding to the preventive maintenance actions and the advertisement cost, respectively, are determined independently from u. When the failure rates are not age dependent, the production rate will be determined according to a so-called hedging point policy. This means that, for each operational mode, where the demand rate can be met by a feasible production rate, the surplus trajectory will tend to reach as rapidly as possible to a steady state called the hedging point that corresponds to the minimum of the value function. In the case considered here, since the age of the machine is always increasing when it is used, there cannot be a steady state and, thus, the concept of a hedging point is not directly relevant anymore. However, one can define a related concept, which is defined by the mappings $x = \tilde{x}^\beta(a), \beta \in E$ where:

$$\min_x V(\beta, a, x) = V(\beta, a, \tilde{x}^\beta(a)) \qquad (10)$$

The production rate will determine the age and surplus trajectories. They will be chosen so as to reach, as rapidly as possible, the mapping (Equation 10) corresponding to the current operational mode and, once on the mapping, the trajectory will be maintained on it if there is enough controlability, as long as the mode remains the same. Thus, the mapping (Equation 10) will be a convenient and concise way to represent the optimal production policy.

However, it is more practical to define the cost rate function, $\varphi^\beta(a, x, u, v)$, as a function of the state variable, x, and the control variables, u and v. In such a case, there is no analytic solution to the partial differential Equation 9. So, the approach of researchers in literature, to find any kind of solution to these equations, is made in one of the following two alternative ways: They may either solve the equations numerically, or use an approximation procedure to find a near optimal solution. In the next section, such an approximation procedure is proposed to find a suboptimal control.

AN APPROXIMATE SUBOPTIMAL CONTROL

The author's approach to find a suboptimal control is to approximate this stochastic optimal control problem by an independent deterministic optimal control problem in each operational mode of the system. The solution of this deterministic problem gives a suboptimal control in the corresponding operational mode. So, this suboptimal control consists of three such deterministic sets of solution for the three different operational modes of the system. When the operational mode of the system jumps from one mode to another, the control variables also jump from one set of solution to another, correspondingly.

To start writing the necessary conditions, one, first, defines the Hamiltonian function in each operational mode:

$$\begin{aligned} &H^{\beta}(x,a,z,u,v,w,p_1,p_2,p_3,t)\\ &\quad= e^{-\rho t}[\varphi^{\xi(t)}(a(t),x(t),u(t),v(t))+w(t)-\pi z(t)]\\ &\quad+p_1(t)(u(t)-z(t))+p_2(t)f(u(t),v(t))+p_3(t)\\ &\quad\ (c_0 z(t)+c_1 w(t)),\\ &\forall\beta\in E, \end{aligned} \tag{11}$$

when $p_1(t), p_2(t)$ and $p_3(t)$ are the co-state variables. Then, the necessary conditions for a suboptimal control in this mode are:

$$\dot{x}(t)=u(t)-z(t), \qquad x(0)=x_0,$$

$$\dot{a}(t)=f(u(t),v(t)), \qquad a(0)=a_0,$$

$$\dot{z}(t)=c_0 z(t)+c_1 w(t), \quad z(0)=z_0,$$

$$\dot{p}_1(t)=-\frac{\partial H^{\beta}}{\partial x}=-e^{-\rho t}\frac{\partial\varphi^{\xi(t)}}{\partial x},$$

$$\dot{p}_2(t)=-\frac{\partial H^{\beta}}{\partial a}=-e^{-\rho t}\frac{\partial\varphi^{\xi(t)}}{\partial a},$$

$$\dot{p}_3(t)=-\frac{\partial H^{\beta}}{\partial z}=\pi e^{-\rho t}+p_1(t)-c_0 p_3(t),$$

$$\frac{\partial H^{\beta}}{\partial u}=e^{-\rho t}\frac{\partial\varphi^{\xi(t)}}{\partial u}+p_1(t)+p_2(t)\frac{\partial f}{\partial u}=0,$$

$$\frac{\partial H^{\beta}}{\partial v}=e^{-\rho t}\frac{\partial\varphi^{\xi(t)}}{\partial v}+p_2(t)\frac{\partial f}{\partial v}=0,$$

$$\frac{\partial H^{\beta}}{\partial w}=e^{-\rho t}+c_1 p_3(t)=0, \quad \forall\beta\in E. \tag{12}$$

To solve the necessary conditions (Equation 12), one should assume specific forms for the aging rate function, $f(u,v)$, and for the cost rate function, $\varphi^{\beta}(a,x,u,v)$. In the next section, these specific forms are assumed to be able to solve the case study of this paper.

A CASE STUDY

In order to make sure that a solution to the necessary conditions (Equation 12) exists, this problem should be converted to a linear-quadratic optimal control problem, by assuming some linear dynamic equations and a quadratic performance criterion function. So, a linear form is assumed for the aging rate function, f, and a quadratic form for the cost rate function, φ^{β}. These assumptions are realistic, since the aging of the workstation is an increasing function of its production rate, u, and a decreasing function of its maintenance rate, v. Also, the cost rate function, φ^{β}, is an increasing function of the cumulative surplus, x, the production rate, u, and the maintenance rate, v. Thus, these functions are taken to be:

$$f(u,v)=b_1 u-b_2 v, \quad f\geq 0, \quad b_1>0, \quad b_2>0,$$

$$\varphi^{\beta}(a,x,u,v)=h(x)+g(u)+l(v)+c^{\beta}, \quad \forall\beta\in E,$$

when:

$$h(x)=dx^2, \qquad d>0,$$

$$g(u)=eu^2, \qquad e>0,$$

$$l(v)=fv^2, \qquad f>0,$$

and:

$$c^{\beta}=c_2 I(\xi(t)=2)+c_3 I(\xi(t)=3), c_2>c_3.$$

The increasing and decreasing forms of the functions f and φ^{β}, with respect to their variables, imply the positive sign of the constants b_1, b_2, d, e and f in the above definitions. Then, the following set of conditions can be derived from the necessary conditions

(Equation 12):

$$\dot{x}(t) = u(t) - z(t), \qquad x(0) = x_0,$$

$$\dot{a}(t) = b_1 u(t) - b_2 v(t), \quad a(0) = a_0,$$

$$\dot{z}(t) = c_0 z(t) + c_1 w(t), \quad z(0) = z_0,$$

$$\dot{p}_1(t) = -2dx(t)e^{-\rho t},$$

$$\dot{p}_2(t) = 0,$$

$$\dot{p}_3(t) = \pi e^{-\rho t} + p_1(t) - c_0 p_3(t),$$

$$2eu(t)e^{-\rho t} + p_1(t) + b_1 p_2(t) = 0,$$

$$2fv(t)e^{-\rho t} - b_2 p_2(t) = 0,$$

$$e^{-\rho t} + c_1 p_3(t) = 0. \tag{13}$$

Now, the conditions (Equation 13) can be solved analytically using direct integration to find the following set of solutions:

$$p_1(t) = \frac{\rho - c_1\pi - c_0}{c_1} e^{-\rho t},$$

$$p_2(t) = p_2 = \frac{c_0 + c_1\pi - \rho}{c_1 b_1} - \frac{2ez_0}{b_1},$$

$$p_3(t) = -\frac{1}{c_1} e^{-\rho t},$$

$$x(t) = \frac{\rho^2 - c_1\pi\rho - c_0\rho}{2c_1 d},$$

$$a(t) = \frac{b_1(c_0 + c_1\pi - \rho)}{2ec_1} t - \left(\frac{b_1^2 p_2}{2e\rho} + \frac{b_2^2 p_2}{2f\rho}\right) e^{\rho t} + \frac{b_1^2 p_2}{2e\rho} + \frac{b_2^2 p_2}{2f\rho} + a_0,$$

$$z(t) = \frac{c_0 + c_1\pi - \rho}{2ec_1} - \frac{b_1 p_2}{2e} e^{\rho t},$$

$$u(t) = \frac{c_0 + c_1\pi - \rho}{2ec_1} - \frac{b_1 p_2}{2e} e^{\rho t},$$

$$v(t) = \frac{b_2 p_2}{2f} e^{\rho t},$$

$$w(t) = \left(\frac{c_0 b_1 p_2 - b_1 p_2 \rho}{2c_1 e}\right) e^{\rho t} - \frac{c_0^2 + c_0 c_1 \pi \rho}{2ec_1^2}. \tag{14}$$

The set of suboptimal solutions (Equations 14), which refers to all of the above nine equations, gives us the co-states, the state variables and the control variables of the system as a function of time, t, and the parameters of the system. By suboptimal solution, one means that this set of solutions is the best that one can get under the proposed approximation procedure for each operational mode of the system, i.e. if the value of the control variables is kept at the levels provided in Equations 14, then the state variables will remain at the levels presented in these equations.

CONCLUSIONS

The simultaneous planning of production and maintenance in a flexible manufacturing system is considered in this paper, which is different from previous research in this area in two separate ways. First, the failure rate of the machine is supposed to be a function of its age. Second, it is assumed that the demand of the manufacturing product is time dependent and that its rate depends on the level of advertisement on that product. These assumptions are more realistic and make the results of this research more practical.

In the process of finding a solution to the problem, first, an optimal control was characterized by a set of Hamilton-Jacobi-Bellman partial differential equations. Then, it was realized that under practical assumptions, this set of equations cannot be solved analytically. Thus, to find a suboptimal control, the original stochastic optimal control model was approximated by a deterministic optimal control problem. Then, this deterministic optimal control problem was solved under reasonable assumptions and a set of suboptimal solutions was found.

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