Scientia Iranica (2024) 31(5), 388–416

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
https://scientiairanica.sharif.edu

Cubic bipolar fuzzy aggregation operator with priority degree with multi-criteria decision-making

N. Jamil* **and M. Riaz**

Department of Mathematics, University of the Punjab, Lahore, Pakistan.

Received 1 August 2022; received in revised form 16 November 2022; accepted 3 December 2023

KEYWORDS
Fuzzy set;
Cubic Bipolar Fuzzy Set (CBFS);
Aggregation operator;
Priority degrees;
Multi-Criteria Decision Making (MCDM).

Abstract. Cubic Bipolar Fuzzy Numbers (CBFN) are useful for real-world ambiguous data. Prioritised Multi-Criteria Decision Making (MCDMs) use priority degrees. Aggregation Operators (AOs) result from tight priority levels and priority degrees. Thus, "cubic bipolar fuzzy prioritised averaging operator with priority degrees (CBFPDA)" and "cubic bipolar fuzzy prioritised averaging operator with priority degrees (CBFPGD)" are proposed CBFNs prioritised operators. The comparative studies are made and comparison analysis verifies the validity of the proposed method. The comparison study shows that the approach works. Comparing the current method to others emphasises its superiority over current operators. Priorities affect object ranking and information fusion. Discussing a 3PRLP optimisation problem's practical implementation is a secondary goal. The recommended 3PRLP reference is evaluated numerically. The best strategy is selected and compared.

1. Introduction

Decision Making (DM) is a vital occurrence in order to choose the best option from available options. However, due to inadequate data and inherent human judgments, this process entails ambiguous and hazy information. Classical techniques are unable to determine the best option in the face of ambiguity for these reasons. Zadeh [1,2] established the notion of Fuzzy Set (FS) to solve such serious challenges, and it has been successfully applied to a wide variety of real-life problems. An Intuitionistic Fuzzy Set (IFS) gives a membership grade $\mu \in [0,1]$ and a non-membership grade $\nu \in [0,1]$ to each object in the universe [3,4]. Some extensions of FSs which are necessary to understand the notion of Cubic Bipolar CBFS are given in Table 1. Many researchers have employed these models successfully in recent decades. All of these models were created in response to the necessity to deal with uncertainty in real-world problems.

Researchers appreciate data Aggregation Operators (AOs). Fuzzy number and interval data improve this model. This model has the most ratings, inaccuracy, and bipolarity. Joy and grief, drug effects and side effects, commodity sweetness and sourness, hopefulness and hopelessness, etc. can be shown by

*. *Corresponding author.*
E-mail addresses: nimra.jamilphd@gmail.com (N. Jamil); mriaz.math@pu.edu.pk (M. Riaz)

To cite this article:
N. Jamil and M. Riaz "Cubic bipolar fuzzy aggregation operator with priority degree with multi-criteria decision-making", *Scientia Iranica* (2024) **31**(5), pp. 388-416
DOI: 10.24200/sci.2023.60821.7003

Table 1. Some extensions of fuzzy sets.

Fuzzy models	Researchers	Constraints
Fuzzy set (FS)	Zadeh [1]	Membership values
Interval-Valued Fuzzy Set (IVFS)	Zadeh [2]	Interval grading
Intuitionistic Fuzzy Set (IFS)	Atanassov and Stoeva [3,4]	$\mu + \nu \leq 1$
Pythagorean Fuzzy Set (PFS)	Yager [13,14]	$\mu^2 + \nu^2 \leq 1$
Fermatean Fuzzy Set (FFS)	Senapati and Yager [15,16]	$\mu^3 + \nu^3 \leq 1$
q-rung Orthopair Fuzzy Set (q-ROFS)	Yager [17]	$\mu^q + \nu^q \leq 1$, $q \geq 1$
Bipolar Fuzzy Set (BFS)	Zhang [18,19]	Positive grading $\mu^+ \in [0, 1]$ and negative grading $\mu^- \in [-1, 0]$
Cubic Set (CS)	Jun et al. [20]	Interval and fuzzy grading
Cubic Bipolar Fuzzy Set (CBFS)	Riaz and Tehrim [21]	Hybrid model of CS and BFS

Table 2. Some basic Aggregation Operators (AOs).

Aggregation operators	Fuzzy models	Researchers	References
Ordered weighted averaging AO	Crisp	Yager (1988)	[22]
Geometric AO	IFS	Xu and Yager (2006)	[23]
AO	IFS	Xu (2007)	[24]
Geometric Einstein AO	IFS	Wang and Liu (2012)	[25]
Hamacher AO	BFS	Wei et al. (2017)	[26]
Bonferroni mean AO	Cubic IFS	Kaur and Garg (2018)	[27]
Dombi AO	Neutrosophic cubic sets	Shi and Ye (2018)	[28]
Dombi AO	Pythagorean	Akram et al. (2019)	[29]
Cubic fuzzy AO	Pythagorean	Khan et al. (2019)	[30]
Priority degree AO	q-rung orthopair FS	Riaz, Fareed, Shakeel et al. (2021)	[31]
Prioritized AO	q-rung orthopair FS	Riaz et al. (2020)	[32]
Prioritized weighted AO	Complex spherical	Akram et al. (2021)	[33]
Prioritized AOs with priority degree	Complex intuitionistic	Garg and Rani (2021)	[34]
Prioritized AO with priority degree	q-rung orthopair	Riaz et al. (2021)	[35]
Einstein prioritized AO	Linear Diophantine	Farid et al. (2022)	[36]
Einstein prioritized AO	Single-valued neutrosophic	Farid et al. (2022)	[37]
Prioritized interactive AO	q-rung orthopair	Farid and Riaz (2022)	[38]

a Bipolar Fuzzy Set (BFS). They maintain social order. Strategic decisions are subjective and two-sided. Several authors have reported bipolar fuzzy judgements using different methods. Most Multi-Criteria Decision Making (MCDM) problems require quantitative data aggregation. Data aggregation and fusion underpin machine learning, pattern recognition, image processing, and information processing. Information gathered forms an opinion. Crisp integer-based data processing cannot mimic human cognition. These strategies help DMs draw unclear conclusions from incomplete information. DMs need theories to understand ambiguous data values and adapt their DM requirements to the context-pattern recognition or human cognition to handle real-world ambiguous and fuzzy situations. Riaz and Jamil introduced cubic bipolar fuzzy topology in 2022 [5] and also utilize it in MCDM technique. AOs for IFSs proposed by Xu et al. [6,7] incorporate averaging and geometric operators. Many experts have made significant contributions to FS extensions, some important and most relevant are mentioned in Table 2.

The main contributions of the manuscript are as follows:

- New AOs with priority degrees are proposed, named the Cubic Bipolar Fuzzy Average operator with Priority Degree (CBFAPD) operator and the Cubic Bipolar Fuzzy Geometric operator with Priority Degree (CBFGPD) operator;
- Certain properties of proposed operators are investigated, including idempotency, boundary, and monotonicity;
- A practical application of MCDM under uncertainty is illustrated using the suggested operators for third party reverse logistic application;
- A numerical example is illustrated to discuss the scientific nature of the proposed MCDM approach to demonstrate its rationality, symmetry, and superiority.

The body of the article is organized as follows: Section 2 focuses on the fundamentals of CBFS, along with their score function, accuracy function, and essential aggregation functions. The article concludes by showcasing some of the developed cubic bipolar fuzzy averaging AOs with priority degrees in Sections 3 and 4 introduces cubic bipolar fuzzy geometric AOs with priority degree. We discuss the MCDM strategy as it relates to the selected operators in Section 5. In Section 6, we present a case study of third party reverse logistic providers alongside a numerical illustration. Section 7 provides the pros and cons of CBFPDA and the comparison analysis of the proposed method is given in Section 8. In Section 9, we present the foremost endings of this research.

2. Some fundamental notions

In this section, we review some rudiments of Cubic Bipolar Fuzzy Sets (CBFSs) and Cubic Bipolar Fuzzy Numbers (CBFNs), in addition to the operational laws that govern these concepts, such as inclusion, intersection, union, sum, product, scalar multiplication, and exponents under $P(R)$-order. We continue our discussion on the concepts of score functions and accuracy functions for the purpose of partial ordering and ranking CBFNs.

Definition 1[8]. Let V be a non-empty set. A CBFS $\mathscr{C}$ in V is defined as follows:

$$\mathscr{C} = \{\langle\chi, \mathscr{P}=[\mathscr{P}_l,\mathscr{P}_u],\ \mathscr{N}=[\mathscr{N}_l,\mathscr{N}_u],\lambda,\mu\rangle|\chi\in V\},$$

where $[\mathscr{P}_l, \mathscr{P}_u] \subseteq [0,1]$ and $[\mathscr{N}_l,\mathscr{N}_u]\subseteq [-1,0]$,$\lambda : V \to [0,1]$ and $\mu : V \to [-1,0]$.

Definition 2[8]. Let $\mathscr{C}_1=\langle\chi,\ \mathscr{P}_1,\ \mathscr{N}_1,\ \lambda_1,\ \mu_1\rangle$ and $\mathscr{C}_2=\langle\chi,\ \mathscr{P}_2,\ \mathscr{N}_2,\ \lambda_2,\ \mu_2\rangle$ be two CBFSs. Then:

$$\mathscr{C}_1 \bigoplus_P \mathscr{C}_2 =\{\langle\chi,[\mathscr{P}_{1l}+\mathscr{P}_{2l}-\mathscr{P}_{1l}*\mathscr{P}_{2l},\mathscr{P}_{1u} + \mathscr{P}_{2u}-\mathscr{P}_{1u}*\mathscr{P}_{2u}],$$
$$[-\mathscr{N}_{1l}*\mathscr{N}_{2l},-\mathscr{N}_{1u}*\mathscr{N}_{2u}],$$
$$\lambda_1+\lambda_2-\lambda_1*\lambda_2,-\mu_1*\mu_2\rangle|\chi\in V\}.$$

Definition 3[8]. Let $\mathscr{C}_1 = \langle\chi,\ \mathscr{P}_1,\mathscr{N}_1,\lambda_1,\mu_1\rangle$ and $\mathscr{C}_2=\langle\chi,\ \mathscr{P}_2,\mathscr{N}_2,\lambda_2,\mu_2\rangle$ be two CBFSs. Then,

$$\mathscr{C}_1 \bigotimes_P \mathscr{C}_2 =\{\langle\chi,[\mathscr{P}_{1l}*\mathscr{P}_{2l},\mathscr{P}_{1u}*\mathscr{P}_{2u}],$$
$$[-(-\mathscr{N}_{1l}-\mathscr{N}_{2l}+\mathscr{N}_{1l}*\mathscr{N}_{2l}),$$
$$-(-\mathscr{N}_{1u}-\mathscr{N}_{2u}+\mathscr{N}_{1u}*\mathscr{N}_{2u})],$$
$$\lambda_1*\lambda_2,-(-\mu_1-\mu_2-\mu_1*\mu_2)\rangle|\chi\in V\}.$$

Definition 4[8]. Let $\mathscr{C} = \langle\chi,\mathscr{P},\mathscr{N},\lambda,\mu\rangle$ be a CBFS and α >0, then α-scalar product is expressed as:

$$\mathscr{C}^\alpha =\Big\{\langle\chi,[(\mathscr{P}_l)^\alpha,(\mathscr{P}_u)^\alpha],[-(1-(1-\mathscr{N}_l)^\alpha),$$
$$-(1-(1-\mathscr{N}_u)^\alpha)],1-(1-\lambda)^\alpha,-(-\mu)^\alpha|\ \chi\in V\Big\}.$$

Definition 5[8]. Let $\mathscr{C} = \langle\chi,\mathscr{P},\mathscr{N},\lambda,\mu\rangle$ be a CBFS and α >0 then α-scalar product is defined as:

$$\alpha*\mathscr{C} =\Big\{\langle\chi,[1-(1-\mathscr{P}_l)^\alpha,\ 1-(1-\mathscr{P}_u)^\alpha],$$
$$[-(-\mathscr{N}_l)^\alpha,-(-\mathscr{N}_u)^\alpha],(\lambda)^\alpha,$$
$$-(1-(1-\mu)^\alpha)\rangle|\chi\in V\Big\}.$$

Definition 6[8]. Let $\mathscr{C} = \langle\chi,\mathscr{P},\mathscr{N},\lambda,\mu\rangle$ be a CBFS then it's complement is defined as:

$$\mathscr{C}^c = \Big\{\langle\chi,\mathscr{P}^c,\mathscr{N}^c,1-\lambda,1-\mu\rangle|\chi\in V\Big\}.$$

2.1. Score functions and accuracy functions

Now, we will define score functions and accuracy functions under $P(R)$-order which will help to order the $CBFNs$. The score functions are often used to rank FSs in Multi-Attribute Decision Making (MADM).

Definition 7[8]. For a $CBFS\ \mathscr{C} = \langle\chi,\mathscr{P},\mathscr{N},\lambda,\mu\rangle$, the P-order score function for $CBFS$ is defined as:

$$S_P(\mathscr{C}) = \frac{[\mathscr{P}_l+\mathscr{P}_u]+[\mathscr{N}_l+\mathscr{N}_u]-\lambda-\mu}{6},$$

where $S_P(\mathscr{C}_1) \in [-1,1]$:

- If $S_P(\mathscr{C}_1) \le S_P(\mathscr{C}_2)$ then $\mathscr{C}_1 \le \mathscr{C}_2$,
- If $S_P(\mathscr{C}_1) = S_P(\mathscr{C}_2)$ then $\mathscr{P}_1 = \mathscr{P}_2; \mathscr{C}_1 = \mathscr{C}_2$.

Definition 8.[8] For a CBFS $\mathscr{C} = \langle \chi, \mathscr{P}, \mathscr{N}, \lambda, \mu \rangle$, the R-order score function for CBFS is defined as:

$$S_R(\mathscr{C}) = \frac{[\mathscr{P}_l + \mathscr{P}_u] + [\mathscr{N}_l + \mathscr{N}_u] + \lambda + \mu}{6},$$

where $S_Q(\mathscr{C}_1) \in [-1, 1]$:

- If $S_R(\mathscr{C}_1) \leq S_Q(\mathscr{C}_2)$ then $\mathscr{C}_1 \leq \mathscr{C}_2$,
- If $S_R(\mathscr{C}_1) = S_Q(\mathscr{C}_2)$ then $\mathscr{P}_1 = \mathscr{P}_2; \mathscr{C}_1 = \mathscr{C}_2$.

Definition 9[8]. For a CBFS $\mathscr{C} = \langle \chi, \mathscr{P}, \mathscr{N}, \lambda, \mu \rangle$, the accuracy function for CBFS is defined as:

$$\mathscr{A}(\mathscr{C}) = \frac{[\mathscr{P}_l + \mathscr{P}_u] + [\mathscr{N}_l + \mathscr{N}_u] + \lambda - \mu}{6},$$

where $\mathscr{A}(\mathscr{C}_1) \in [-1, 1]$:

- If $\mathscr{A}(\mathscr{C}_1) \leq \mathscr{A}(\mathscr{C}_2)$ then $\mathscr{C}_1 \leq \mathscr{C}_2$,
- If $\mathscr{A}(\mathscr{C}_1) = \mathscr{A}(\mathscr{C}_2)$ then $\mathscr{C}_1 = \mathscr{C}_2$.

It's important to remember that $S \in [-1, 1]$. To enable the subsequent research, we design an innovative score function $S(\mathscr{C}) = \frac{3 + \mathscr{P}_l + \mathscr{P}_u + \mathscr{N}_l + \mathscr{N}_u + \lambda + \mu}{6}$. We can see that the score function lies between 0 and 1.

Example 1[8]. Consider two CBFNs $\mathscr{C}_1$ and $\mathscr{C}_2$ as:

$$\mathscr{C}_1 = \langle [0.35, 0.65], [-0.98, -0.34], 0.40, -0.63 \rangle,$$

$$\mathscr{C}_2 = \langle [0.25, 0.75], [-0.92, -0.40], 0.35, -0.77 \rangle,$$

and value of scalar is $k = 3$. Calculate union, intersection, ring sum, ring product, scalar power and scalar product under $P(R)$-order.

1. $\mathscr{C}_1 \cup_P \mathscr{C}_2 = \langle [0.25, 0.75], [-0.92, -0, 40], 0.40, -0.77 \rangle$,
2. $\mathscr{C}_1 \cap_P \mathscr{C}_2 = \langle [0.35, 0.65], [-0.98, -0.34], 0.35, -0.63 \rangle$,
3. $\mathscr{C}_1 \oplus_P \mathscr{C}_2 = \langle [0.5125, 0.9125], [-0.9016, -0.1360], 0.61, -0.4851 \rangle$,
4. $\mathscr{C}_1 \otimes_P \mathscr{C}_2 = \langle [0.0875, 0.4875], [-0.9984, -0.6040], 0.14, -0.9149 \rangle$,
5. $\mathscr{C}_1^3 = \langle [0.0429, 0.2746], [-0.9995, -0.7125], 0.0640, -0.9493 \rangle$ (under P-order),
6. $3 * \mathscr{C}_2 = \langle [0.5781, 0, 9844], [-0.7787, -0.0640], 0.7254, -0.4565 \rangle$ (under P-order),
7. $\mathscr{C}_1^3 = \langle [0.0429, 0.2746], [-0.9995, -0.7125], 0.7254, -0.4565 \rangle$ (under R-order),
8. $3 * \mathscr{C}_2 = \langle [0.5781, 0, 9844], [-0.7787, -0.0640], 0.0640, -0.9493 \rangle$ (under R-order).

2.2. Cubic bipolar fuzzy AOs

In the present section, we introduce CBF AOs and CBF weighted AOs.

Definition 10. *P*-order CBFG operator: Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be a collection of CBF Elements (CBFEs), then the CBFG operator is a mapping $\mathscr{M}$: $\mathscr{C}^n \rightarrow \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) = \left\langle [\prod_{k=1}^n (\mathscr{P}_{l_k}), \prod_{k=1}^n (\mathscr{P}_{u_k})], [-(1 - \prod_{k=1}^n (1 - \mathscr{N}_{l_k})), -(1 - \prod_{k=1}^n (1 - \mathscr{N}_{u_k}))], \prod_{k=1}^n (\lambda_k), -(1 - \prod_{k=1}^n (1 - \mu_k)) \right\rangle.$$

Definition 11. *P*-order CBFGW operator: Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of CBFEs and $W = [w_1, w_2, ..., w_n]^T$ be the weight vector, where $\sum_{k=1}^n w_k = 1$ then CBFGWA operator is a mapping $\mathscr{M} : \mathscr{C}^n \rightarrow \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) = \left\langle [\prod_{k=1}^n (\mathscr{P}_{l_k})^{w_k}, \prod_{k=1}^n (\mathscr{P}_{u_k})^{w_k}], [-(1 - \prod_{k=1}^n (1 - (\mathscr{N}_{l_k}))^{w_k}), -(1 - \prod_{k=1}^n (1 - (\mathscr{N}_{u_k}))^{w_k})], \prod_{k=1}^n (\lambda_k)^{w_k}, -(1 - \prod_{k=1}^n (1 - \mu_k)^{w_k}) \right\rangle.$$

Definition 12. *P*-order CBFA operator: Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of CBFEs then CBFG operator is a mapping $\mathscr{M}$:$\mathscr{C}^n \rightarrow \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \left\langle [1 - \prod_{k=1}^n (1 - \mathscr{P}_{l_k}), 1 - \prod_{k=1}^n (1 - \mathscr{P}_{u_k})], [-\prod_{k=1}^n (\mathscr{N}_{l_k}), -\prod_{k=1}^n (\mathscr{N}_{u_k})], 1 - \prod_{k=1}^n (1 - \lambda_k), -\prod_{k=1}^n (\mu_k) \right\rangle.$$

Definition 13. *P*-order CBFAW operator: Let $\mathscr{C}_k = \left\langle [\mathscr{P}_{l_k}, \mathscr{P}_{u_k}], [\mathscr{N}_{l_k}, \mathscr{N}_{u_k}], \lambda_k, \mu_k \right\rangle$ be collection of

CBFEs and $W=[w_1,w_2,...,w_n]^T$ be the weight vector, where $\sum_{k=1}^{n} w_k = 1$ then CBFAW operator is a mapping $\mathscr{M}:\mathscr{C}^n \rightarrow \mathscr{C}$ which we calculate under P-order as follows:

$$CBFG_P(\mathscr{C}_1,\mathscr{C}_2,...,\mathscr{C}_k)=\Big\langle\big[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{w_k},\ 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{w_k}\big],\big[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{w_k},-\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{w_k}\big],\ 1-\prod_{k=1}^{n}(1-\lambda_k)^{w_k},-\prod_{k=1}^{n}(\mu_k)^{w_k}\Big\rangle.$$

Example 2. Consider three CBFNs:

$$\mathscr{C}_1=\Big\langle\big[0.25,0.53\big],\big[-0.67,-0.31\big],0.37,-0.43\Big\rangle,$$

$$\mathscr{C}_2=\Big\langle\big[0.37,0.65\big],\big[-0.71,-0.39\big],0.43,-0.65\Big\rangle,$$

and:

$$\mathscr{C}_3=\Big\langle\big[0.53,0.87\big],\big[-0.83,-0.43\big],0.65,-0.67\Big\rangle.$$

Calculate CBF geometric AOs and arithmetics AOs under $P(R)$-order. Also calculate CBF weighted geometric AOs and weighted arithmetics AOs using weights $W=\{0.3,0.3,0.4\}$ under $P(R)$-order.

Solution By using definitions mentioned above, we have:

1. CBFGA under P-order: $\Big\langle[0.0490,0.2997],[-0.9837,-0.7601],0.1034,-0.9342\Big\rangle$,
2. CBFGWA under P-order: $\Big\langle[0.3798,0.6870],[-0.7565,-0.3840],0.4849,-0.6043\Big\rangle$,
3. CBFAA under P-order: $\Big\langle[0.7779,0.9786],[-0.3948,-0.0520],0.8743,-0.1879\Big\rangle$,
4. CBFAWA under P-order: $\Big\langle[0.4096,0.7427],[-0.7427,0.3785],0.5167,-0.5812\Big\rangle$.

3. Cubic bipolar fuzzy averaging AOs with priority degrees

Definition 14. Assume that $\mathscr{C}_j=\Big\langle\big[\mathscr{P}_{l_j},\mathscr{P}_{u_j}\big],\big[\mathscr{N}_{l_j},\mathscr{N}_{u_j}\big],\lambda_j,\mu_j\Big\rangle$ is the collection of CBFNs. A

CBFPDA operator is defined by the mapping $\Lambda^n\rightarrow\Lambda$ is expressed as:

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)=r_1{}^{d_1}\mathscr{C}_1\oplus r_2{}^{d_2}\mathscr{C}_2\oplus\cdots\oplus r_n{}^{d_n}\mathscr{C}_n, \quad (1)$$

where $r_i{}^{d_i}=\frac{\mathfrak{T}_1^{d_i}}{\sum_{k=1}^{n}\mathfrak{T}_k^{d_k}}$ and $\mathfrak{T}_j=\prod_{k=1}^{n-1}\big(S(\mathscr{C}_k)\big)^{d_k}$; $(j=1,2,...,n)$ and $\mathfrak{T}_1=1$.

Theorem 1. *Assume that* $\mathscr{C}_j=\Big\langle\big[\mathscr{P}_{l_j},\mathscr{P}_{u_j}\big],\big[\mathscr{N}_{l_j},\mathscr{N}_{u_j}\big],\lambda_j,\mu_j\Big\rangle$ *is the collection of CBFNs. A CBFPDA operator is defined by the mapping* $\Lambda^n\rightarrow\Lambda$ *is expressed as:*

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)=r_1{}^{d_1}\mathscr{C}_1\oplus r_2{}^{d_2}\mathscr{C}_2\oplus\cdots\oplus r_n{}^{d_n}\mathscr{C}_n$$
$$=\Big\langle\big[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k^d},1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k^d}\big],\big[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k^d},-\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k^d}\big],\prod_{k=1}^{n}\lambda_k{}^{r_k^d},-\big(1-\prod_{k=1}^{n}(1-\mu_k)^{r_k^d}\big)\Big\rangle.$$

Proof. To prove this theorem, we will use mathematical induction:

$$r_1{}^{d_1}\mathscr{C}_1=\Big\langle\big[1-(1-\mathscr{P}_{l_1})^{r_1{}^{d_1}},1-(1-\mathscr{P}_{u_1})^{r_1{}^{d_1}}\big],\big[-(\mathscr{N}_{l_1})^{r_1{}^{d_1}},-(\mathscr{N}_{u_1})^{r_1{}^{d_1}}\big],\lambda_1{}^{r_1{}^{d_1}},-\big(1-(1-\mu_1)^{r_1{}^{d_1}}\big)\Big\rangle,$$

$$r_2{}^{d_2}\mathscr{C}_2=\Big\langle\big[1-(1-\mathscr{P}_{l_2})^{r_2{}^{d_2}},1-(1-\mathscr{P}_{u_2})^{r_2{}^{d_2}}\big],\big[-(\mathscr{N}_{l_2})^{r_2{}^{d_2}},-(\mathscr{N}_{u_2})^{r_2{}^{d_2}}\big],\lambda_2{}^{r_2{}^{d_2}},-\big(1-(1-\mu_2)^{r_2{}^{d_2}}\big)\Big\rangle,$$

$$r_1{}^{d_1}\mathscr{C}_1 \bigoplus r_2{}^{d_2}\mathscr{C}_2 = \Big\langle \big[1-(1-\mathscr{P}_{l_1})^{r_1{}^{d_1}}, 1-(1-\mathscr{P}_{u_1})^{r_1{}^{d_1}}\big], \big[-(\mathscr{N}_{l_1})^{r_1{}^{d_1}}, -(\mathscr{N}_{u_1})^{r_1{}^{d_1}}\big], \lambda_1{}^{r_1{}^{d_1}}, -\big(1-(1-\mu_1)^{r_1{}^{d_1}}\big)\Big\rangle \bigoplus \Big\langle \big[1-(1-\mathscr{P}_{l_2})^{r_2{}^{d_2}}, 1-(1-\mathscr{P}_{u_2})^{r_2{}^{d_2}}\big], \big[-(\mathscr{N}_{l_2})^{r_2{}^{d_2}}, -(\mathscr{N}_{u_2})^{r_2{}^{d_2}}\big], \lambda_2{}^{r_2{}^{d_2}}, -\big(1-(1-\mu_2)^{r_2{}^{d_2}}\big)\Big\rangle,$$

$$r_1{}^{d_1}\mathscr{C}_1 \bigoplus r_2{}^{d_2}\mathscr{C}_2 = \Big\langle \big[1-(1-\mathscr{P}_{l_1})^{r_1{}^{d_1}} * (1-\mathscr{P}_{l_2})^{r_2 d_2}, 1-(1-\mathscr{P}_{u_1})^{r_1{}^{d_1}} * (1-\mathscr{P}_{u_2})^{r_2{}^{d_2}}\big], \big[-(\mathscr{N}_{l_1}^{r_1{}^{d_1}} * \mathscr{N}_{l_2}^{r_2{}^{d_2}}), -(\mathscr{N}_{u_1}^{r_1 d_1} * \mathscr{N}_{u_2}^{r_2 d_2})\big], \lambda_1{}^{r_1{}^{d_1}} * \lambda_2{}^{r_2{}^{d_2}}, -\big(1-(1-\mu_1)^{r_1{}^{d_1}} * (1-\mu_2)^{r_2{}^{d_2}}\big)\Big\rangle$$

$$= \Big\langle \big[1-\prod_{k=1}^{2}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}}, 1-\prod_{k=1}^{2}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\big], \big[-\prod_{k=1}^{2}(\mathscr{N}_{l_k})^{r_k{}^{d_k}}, -\prod_{k=1}^{2}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\big], \lambda_k{}^{r_k{}^{d_k}}, 1-\prod_{k=1}^{2}(\mathscr{N}_{l_k})^{r_k{}^{d_k}}\big(1-\prod_{k=1}^{2}\mu_k\big)^{r_k{}^{d_k}}\Big\rangle,$$

which shows that Eq. (1) is true for $n = 2$, now let Eq. (1) holds for $n = k$, i.e.,

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) = \Big\langle \big[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}}, 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\big], \big[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k{}^{d_k}}, -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\big], \prod_{k=1}^{n}\lambda_k{}^{r_k{}^{d_k}}, -\big(1-\prod_{k=1}^{n}(1-\mu_k)^{r_k{}^{d_k}}\big)\Big\rangle. \quad (2)$$

Now, we will show the Eq. (1) holds for $n = k + 1$, by using the CBFS operational laws:

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_{k+1}) = CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_k) \bigoplus \mathscr{C}_{k+1}$$

$$= \Big\langle \big[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}}, 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\big], \big[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k{}^{d_k}}, -\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\big], \prod_{k=1}^{n}\lambda_k{}^{r_k{}^{d_k}}, -\big(1-\prod_{k=1}^{n}(1-\mu_k)^{r_k{}^{d_k}}\big)\Big\rangle$$

$$\bigoplus \Big\langle \big[1-(1-\mathscr{P}_{l_{k+1}})^{r_{k+1}{}^{d_{k+1}}}, 1-(1-\mathscr{P}_{u_{k+1}})^{r_{k+1}{}^{d_{k+1}}}\big], \big[-(\mathscr{N}_{l_{k+1}})^{r_{k+1}{}^{d_{k+1}}}, -(\mathscr{N}_{u_{k+1}})^{r_{k+1}{}^{d_{k+1}}}\big], \lambda_{k+1}{}^{r_{k+1}{}^{d_{k+1}}}, -\big(1-(1-\mu_{k+1})^{r_{k+1}{}^{d_{k+1}}}\big)\Big\rangle$$

$$= \Big\langle \big[1-\prod_{k=1}^{n+1}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}}, 1-\prod_{k=1}^{n+1}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\big], \big[-\prod_{k=1}^{n+1}(\mathscr{N}_{l_k})^{r_k{}^{d_k}}, -\prod_{k=1}^{n+1}(\mathscr{N}_{u_k})^{r_k{}^{d_k}}\big], \prod_{k=1}^{n+1}\lambda_k{}^{r_k{}^{d_k}}, -\big(1-\prod_{k=1}^{n+1}(1-\mu_k)^{r_k{}^{d_k}}\big)\Big\rangle.$$

This proves that $n = k + 1$, Eq. (1) holds, then:

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \Big\langle \big[1-\prod_{k=1}^{n}(1-\mathscr{P}_{l_k})^{r_k{}^{d_k}}, 1-\prod_{k=1}^{n}(1-\mathscr{P}_{u_k})^{r_k{}^{d_k}}\big], \big[-\prod_{k=1}^{n}(\mathscr{N}_{l_k})^{r_k{}^{d_k}},$$

$$-\prod_{k=1}^{n}(\mathscr{N}_{u_k})^{{r_k}^{d_k}}\big],\prod_{k=1}^{n}{\lambda_k}^{{r_k}^{d_k}},$$

$$-\big(1-\prod_{k=1}^{n}(1-\mu_k)^{{r_k}^{d_k}}\big)\Big\rangle. \quad \square \qquad (3)$$

Eexample 3. Consider four $CBFNs$ $\mathscr{C}_1$,$\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_4$ as:

$$\mathscr{C}_1=\Big\langle\big[0.7391, 0.8756\big], \big[-0.7659, -0.4631\big], 0.7929, -0.5745\Big\rangle,$$

$$\mathscr{C}_2=\Big\langle\big[0.9431, 0.9996\big], \big[-0.3743, -0.1329\big], 0.9567, -0.2729\Big\rangle,$$

$$\mathscr{C}_3=\Big\langle\big[0.1457, 0.9192\big], \big[-0.7954, -0.2343\big], 0.7351, -0.5827\Big\rangle,$$

$$\mathscr{C}_4=\Big\langle\big[0.5299, 0, 8153\big], \big[-0.8137, -0.7143\big], 0.6979, -0.7799\Big\rangle.$$

Calculate cubic bipolar fuzzy AO with priority degree $d=(4,1,1)$.

Solution Firstly we will calculate score values and of each CBFN:

$$S(\mathscr{C}_1)=0.6007; \quad S(\mathscr{C}_2)=0.8532; \quad S(\mathscr{C}_3)=0.5312;$$

$$S(\mathscr{C}_4)=0.4557,$$

$$\mathfrak{T}_1=1.0000; \quad \mathfrak{T}_2=0.6007; \quad \mathfrak{T}_3=0.5125;$$

$$\mathfrak{T}_4=0.2722,$$

$$r_1^{d_1}=0.4192; \quad r_2^{d_2}=0.2561; \quad r_3^{d_3}=0.2185;$$

$$r_4^{d_4}=0.1160.$$

By using Eq. (1), we have:

$$CBDAPD(\mathscr{C}_1,\mathscr{C}_2,\mathscr{C}_3,\mathscr{C}_4)=\Big\langle\big[0.7581, 0.9733\big], \big[-0.6458, -0.3025\big], 0.8045, -1.0000\Big\rangle.$$

Here, we have some essential elements amongst CBF-PDA's operator.

Theorem 2. **(Idempotency:)** *Assume that* $\mathscr{C}_j=\Big\langle\big[\mathscr{P}_{l_j},\ \mathscr{P}_{u_j}\big],\ \big[\mathscr{N}_{l_j},\ \mathscr{N}_{u_j}\big],\ \lambda_j,\ \mu_j\Big\rangle$ *is the collection of* $CBFNs$*. A CBFPDA operator is defined by the mapping* $\Lambda^n\rightarrow\Lambda$ *is expressed as:*

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)$$

$$={r_1}^{d_1}\mathscr{C}_1\oplus{r_2}^{d_2}\mathscr{C}_2\oplus\cdots\oplus{r_n}^{d_n}\mathscr{C}_n, \qquad (4)$$

where ${r_i}^{d_i}=\frac{{\mathfrak{T}_1}^{d_i}}{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}$ *and* $\mathfrak{T}_j=\prod_{k=1}^{n-1}\big(S(\mathscr{C}_k)\big)^{d_k}$*;* $(j=1,2,...,n)$ *and* $\mathfrak{T}_1=1$ *and* $S(\mathscr{C}_k)$ *is the score function of kth CBFN. If* $\mathscr{C}_j=\mathscr{C}$ $\forall j$ *then* $CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)=\mathscr{C}$*.*

Proof. Consider the Eq. (1):

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)=$$

$${r_1}^{d_1}\mathscr{C}_1\oplus{r_2}^{d_2}\mathscr{C}_2\oplus\cdots\oplus{r_n}^{d_n}\mathscr{C}_n,$$

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)={r_1}^{d_1}\mathscr{C}\oplus{r_2}^{d_2}\mathscr{C}$$

$$\oplus\cdots\oplus{r_n}^{d_n}\mathscr{C}$$

$$=\frac{{\mathfrak{T}_1}^{d_1}}{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}\mathscr{C}$$

$$\oplus\frac{{\mathfrak{T}_2}^{d_2}}{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}\mathscr{C}$$

$$\oplus\cdots\oplus\frac{{\mathfrak{T}_n}^{d_n}}{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}\mathscr{C}$$

$$=\left(\frac{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}{\sum_{k=1}^{n}{\mathfrak{T}_k}^{d_k}}\mathscr{C}\right)$$

$$=\mathbb{1}\mathscr{C}$$

$$=\mathscr{C}. \qquad \square$$

Therem 3. (Monotonicity): *Consider that*

$$\mathscr{C}_j=\Big\langle\big[\mathscr{P}_{l_j},\mathscr{P}_{u_j}\big],\big[\mathscr{N}_{l_j},\mathscr{N}_{u_j}\big],\lambda_j,\mu_j\Big\rangle,$$

and:

$$\mathscr{C}_j^*=\Big\langle\big[\mathscr{P}_{l_j}^*,\mathscr{P}_{u_j}^*\big],\big[\mathscr{N}_{l_j}^*,\mathscr{N}_{u_j}^*\big],\lambda_j^*,\mu_j^*\Big\rangle$$

are the families of $CBFNs$*, where* $\mathfrak{T}_j=\prod_{k=1}^{j-1}\big(S(\mathscr{C}_k)\big)^{d_k}$ *and* $\mathfrak{T}_j^*=\prod_{k=1}^{j-1}\big(S(\mathscr{C}_k^*)\big)^{d_k}$*;* $(j=2,3,...,n)$*,* $\mathfrak{T}_1=1=\mathfrak{T}_1^*$*.*

$$CBFPDA(\mathscr{C}_1,\mathscr{C}_2,\cdots,\mathscr{C}_n)$$

$$\leq CBFPDA(\mathscr{C}_1^*,\mathscr{C}_2^*,\cdots,\mathscr{C}_n^*)$$

Proof. Consider the elements of CBFNs and develop relation between them as shown in Box I.

By combining all above generated inequalities, we have:

$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) \leq CBFPDA(\mathscr{C}_1^*, \mathscr{C}_2^*, ..., \mathscr{C}_n^*)$.

Theorem 4. (Boundedness): *Consider that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \right\rangle$ *is the family of CBFNs and* $\mathscr{C}^- = \min_j(\mathscr{C}_j)$ *and* $\mathscr{C}^+ = \max_j(\mathscr{C}_j)$ *then* $\mathscr{C}^- \leq \mathscr{C}_j \leq \mathscr{C}^+$.

Proof.

$$\min_j(\mathscr{P}_{l_j}) \leq \mathscr{P}_{l_j} \leq \max_j(\mathscr{P}_{l_j})$$

$$\Rightarrow \min_j(1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \geq (1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \geq \max_j(1 - \mathscr{P}_{l_j})^{r_j^{d_j}}$$

$$\Rightarrow \prod_{j=1}^{k} \min_j(1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \geq \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \geq \prod_{j=1}^{k} \max_j(1 - \mathscr{P}_{l_j})^{r_j^{d_j}}$$

$$\Rightarrow \min_j \left(1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j^{d_j}}\right) \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \leq \max_j \left(1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j^{d_j}}\right). \tag{5}$$

Similarly,

$$\min_j\left(1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j})^{r_j^{d_j}}\right) \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j})^{r_j^{d_j}} \leq \max_j \left(1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j})^{r_j^{d_j}}\right), \tag{6}$$

$$\min_j \mathscr{N}_{l_j} \geq \mathscr{N}_{l_j} \geq \max_j \mathscr{N}_{l_j}, - \min_j \left(\prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j^{d_j}} \right) \geq - \prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j^{d_j}} \geq - \max_j \left(\prod_{j=1}^{k}(\mathscr{N}_{l_j})^{r_j^{d_j}} \right). \tag{7}$$

Similarly,

$$- \min_j \left(\prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j^{d_j}} \right) \geq - \prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j^{d_j}} \geq - \max_j \left(\prod_{j=1}^{k}(\mathscr{N}_{u_j})^{r_j^{d_j}} \right), \tag{8}$$

$$\min_j (\lambda_j) \leq \lambda_j \leq \max_j (\lambda_j) \min_j \left(\lambda_j^{r_j^{d_j}}\right) \leq \lambda_j^{r_j^{d_j}} \leq \max_j \left(\lambda_j^{r_j^{d_j}}\right),$$

$$\min_j \left(\prod_{j=1}^{k} \lambda_j^{r_j^{d_j}} \right) \leq \prod_{j=1}^{k} \lambda_j^{r_j^{d_j}} \leq \max_j \left(\prod_{j=1}^{k} \lambda_j^{r_j^{d_j}} \right). \tag{9}$$

By combining Eqs. (5)–(9), we have:

$$\min_j \mathscr{C}_j \leq \mathscr{C}_j \leq \max_j \mathscr{C}_j. \qquad \square$$

Corollary 1. *Consider* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \right\rangle$ *is the assemblage of largest CBFNs i.e.,*

If $\mathscr{P}_{l_j} \leq \mathscr{P}_{l_j}^*$;	If $\Rightarrow \mathscr{P}_{u_j} \leq \mathscr{P}_{u_j}^*$
$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j})^{r_j^{d_j}} \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{l_j}^*)^{r_j^{d_j}}$;	$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j})^{r_j^{d_j}} \leq 1 - \prod_{j=1}^{k}(1 - \mathscr{P}_{u_j}^*)^{r_j^{d_j}}$
If $\mathscr{N}_{l_j} \geq \mathscr{N}_{l_j}^*$	If $\mathscr{N}_{u_j} \geq \mathscr{N}_{u_j}^*$

If $\lambda_j \leq \lambda_j^* \Rightarrow \prod_{j=1}^{k} (\lambda_j)^{r_j^{d_j}} \leq \prod_{j=1}^{k} (\lambda_j^*)^{r_j^{d_j}}$
If $\mu_j \geq \mu_j^*$
$\Rightarrow (1 - \mu_j)^{r_j^{d_j}} \geq (1 - \mu_j^*)^{r_j^{d_j}}$
$\Rightarrow 1 - \prod_{j=1}^{k}(1 - \mu_j)^{r_j^{d_j}} \geq 1 - \prod_{j=1}^{k}(1 - \mu_j^*)^{r_j^{d_j}}$

Box I

$\mathscr{C}_j = \langle [1,1],[-1,-1],\ 1,-1 \rangle$ *for all* j*, then:*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \langle [1,1], [-1,-1], 1, -1 \rangle.$$

Proof. The proof of Corollary 1 similar to the Theorem 2.

Corollary 2. *Consider* $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}],\ [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ *is the assemblage of smallest CBFNs i.e.,* $\mathscr{C}_j = \langle [0,0],\ [0,0],\ 0, 0 \rangle$ *for all* j*, then*:

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_k) = \langle [0,0], [0,0], 0, 0 \rangle.$$

Proof. Here, $\mathscr{C}_j = \langle [0,0],[0,0], 0, 0 \rangle$ then by the definition of the score function, we have: $S(\mathscr{C}_j) = 0$. Since, ${r_i}^{d_i} = \frac{{\mathfrak{T}_1}^{d_i}}{\sum_{k=1}^{n} {\mathfrak{T}_k}^{d_k}}$ and $\mathfrak{T}_j = \prod_{k=1}^{n-1} \big(S(\mathscr{C}_k)\big)^{d_k}$; $(j = 1, 2, ..., n)$ and $\mathfrak{T}_1 = 1$ and $\mathfrak{T}_j = 0$ for $j = 2, 3, ..., n$.

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n)$$
$$= {r_1}^{d_1}\mathscr{C}_1 \oplus {r_2}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus {r_n}^{d_n}\mathscr{C}_n$$
$$= 1 \cdot \mathscr{C}_1 \oplus 0 \cdot \mathscr{C}_2 \oplus \cdots \oplus 0 \cdot \mathscr{C}_n,$$
$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \mathscr{C}_1. \qquad \square$$

Theorem 5. *Consider* $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ *and* $\beta_j = \Big\langle [\mathscr{A}_{l_j}, \mathscr{A}_{u_j}], [\mathscr{B}_{l_j}, \mathscr{B}_{u_j}], \omega_j, \eta_j \Big\rangle$ *are two collection of CBFNs, if* $r > 0$ *and* $\beta = \Big\langle [\mathscr{A}_l, \mathscr{A}_u], [\mathscr{B}_l, \mathscr{B}_u], \omega, \eta \Big\rangle$ *is a CBFN, then:*

1. $CBFPDA(\mathscr{C}_1 \bigoplus \beta, \mathscr{C}_2 \bigoplus \beta, \cdots, \mathscr{C}_n \bigoplus \beta)$
$= CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \bigoplus \beta,$
2. $CBFPDA(r\mathscr{C}_1, r\mathscr{C}_2, \cdots, r\mathscr{C}_n)$
$= rCBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n),$
3. $CBFPDA(\mathscr{C}_1 \bigoplus \beta_1, \mathscr{C}_2 \bigoplus \beta_2, \cdots, \mathscr{C}_n \bigoplus \beta_n)$
$= CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \bigoplus CBFPDA(\beta_1, \beta_2, \cdots, \beta_n),$
4. $CBFPDA(r\mathscr{C}_1 \bigoplus \beta, r\mathscr{C}_2 \bigoplus \beta, \cdots, r\mathscr{C}_n \bigoplus \beta)$
$= rCBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \bigoplus \beta.$

Proof. This is trivial by definition.

Property 1. Assume that $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ is the collection of $CBFNs$, then we have $\lim_{(d_1, d_2, \cdots, d_{n-1})} \to (1, 1, \cdots, 1) CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = CBFW(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n)$.

Proof. Given that $(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)$ from this we have:

$${r_j}^{d_j} = \prod_{k=1}^{j-1} \Big(S(\mathscr{C}_k)\Big)^{d_k} \longrightarrow \prod_{k=1}^{j-1} \Big(S(\mathscr{C}_k)\Big) {r_j}^{d_j} = r_j.$$

By this, we obtained some equation are shown in Box II.

Property 2. Assume that $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ is the collection of $CBFNs$ and $S(\mathscr{C}_j) \neq 0 \forall j$, then we have: $\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0, 0, \cdots, 0)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \frac{1}{k}(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n)$.

Proof. Given that $(d_1, d_2, \cdots, d_{n-1}) \to (0, 0, \cdots, 0)$ by applying the limit we have, $\big(S(C_j)\big)^{d_j} = 1 \forall j$ and ${r_j}^{d_j} = \frac{1}{k}$.

$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0, 0, \cdots, 0)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n)$$
$$= \lim_{(d_1, d_2, \cdots, d_{n-1}) \to (0, 0, \cdots, 0)} \Big({r_1}^{d_1}\mathscr{C}_1 \oplus {r_2}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus {r_n}^{d_n}\mathscr{C}_n\Big)$$

$${r_j}^{d_j} \to r_j$$
$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = {r_1}^{d_1}\mathscr{C}_1 \oplus {r_2}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus {r_n}^{d_n}\mathscr{C}_n$$
$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)} \Big({r_1}^{d_1}\mathscr{C}_1 \oplus {r_2}^{d_2}\mathscr{C}_2 \oplus \cdots \oplus {r_n}^{d_n}\mathscr{C}_n\Big)$$
$$\lim_{(d_1, d_2, \cdots, d_{n-1}) \to (1, 1, \cdots, 1)} CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = r_1\mathscr{C}_1 \oplus r_2\mathscr{C}_2 \oplus \cdots \oplus r_n\mathscr{C}_n$$
$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = CBFW(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) \qquad \square$$

Box II

$$= \frac{1}{k}\mathscr{C}_1 \oplus \frac{1}{k}\mathscr{C}_2 \oplus \cdots \oplus \frac{1}{k}\mathscr{C}_n$$

$$= \frac{1}{k}\big(\mathscr{C}_1 \oplus \mathscr{C}_2 \oplus \cdots \oplus \mathscr{C}_k\big). \qquad \square$$

Hence proved.

Property 3. Assume that $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j},\ \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ is the collection of CBFNs and $S(\mathscr{C}_j) \neq 0$ or $S(\mathscr{C}_j) \neq 1\ \forall j$, then we have $\lim_{d_1 \to +\infty}$ CBFPDA $(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) = \mathscr{C}_1$.

Proof. By applying the limit $d_1 \to +\infty$ for each $g = 2, 3, \cdots k$, we have:

$$\mathfrak{T}_j = \lim_{d_1 \to +\infty} \prod_{k=1}^{j-1} \big(S(\mathscr{C}_k)\big)^{d_k}$$

$$= \big(S(\mathscr{C}_1)\big)^{\infty} \cdot \big(S(\mathscr{C}_2)\big)^{d_2} \cdot \big(S(\mathscr{C}_3)\big)^{d_3} \cdots \big(S(\mathscr{C}_{k-1})\big)^{d_{k-1}}$$

$$= 0; \quad \text{as} \quad 0 < S(\mathscr{C}_{k-1}) < 1,$$

$$\sum \mathfrak{T}_j^{(d)} = \mathfrak{T}_1, {r_1}^{d_1} = \frac{\mathfrak{T}_1^{(d)}}{\sum_j \mathfrak{T}_j^{(d)}} = 1,$$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \mathscr{C}_1.$$

Example 4. Consider four CBFNs $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$ listed below:

$$\mathscr{C}_1 = \langle [0.35, 0.65], [-0.75, -0.25], 0.45, -0.65 \rangle,$$

$$\mathscr{C}_2 = \langle [0.15, 0.55], [-0.71, -0.37], 0.40, -0.60 \rangle,$$

$$\mathscr{C}_3 = \langle [0.25, 0.75], [-0.65, -0.35], 0.50, -0.55 \rangle,$$

$$\mathscr{C}_4 = \langle [0.50, 0.90], [-0.55, -0.25], 0.75, -0.40 \rangle.$$

Now we will calculate the score functions: $S(\mathscr{C}_1) = 0.4667$, $S(\mathscr{C}_2) = 0.4033$, $S(\mathscr{C}_3) = 0.4917$, $S(\mathscr{C}_4) = 0.6583$ and $T_1 = 1$ as follows:

1. For $(d_1, d_2, d_3) = (1, 1, 1)$ $T_2 = 0.4667$; $T_3 = 0.1882$, $T_4 = 0.0925$ $\sum_j T_j = 1.7474$ $\mathfrak{T}_1 = 0.5723$, $\mathfrak{T}_2 = 0.2671$, $\mathfrak{T}_3 = 0.1077$, $\mathfrak{T}_4 = 0.0529$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3006, 0.6622], [-0.7160, -0.2878], 0.4531, -0.6166 \rangle;$$

2. For $(d_1, d_2, d_3) = (6, 1, 1)$ $T_2 = 0.0103$; $T_3 = 0.0042$, $T_4 = 0.0020$ $\sum_j T_j = 1.0165$ $\mathfrak{T}_1 = 0.9838$, $\mathfrak{T}_2 = 0.0101$, $\mathfrak{T}_3 = 0.0041$, $\mathfrak{T}_4 = 0.0020$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3482, 0.6505], [-0.74687, -0.2513], 0.4501, -0.6488 \rangle;$$

3. For $(d_1, d_2, d_3) = (8, 1, 1)$ $T_2 = 0.0023$; $T_3 = 0.0009$, $T_4 = 0.0004$ $\sum_j T_j = 1.0036$ $\mathfrak{T}_1 = 0.9964$, $\mathfrak{T}_2 = 0.0023$, $\mathfrak{T}_3 = 0.0009$, $\mathfrak{T}_4 = 0.0004$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3496, 0.6501], [-0.74697, -0.2503], 0.4500, -0.6498 \rangle;$$

4. For $(d_1, d_2, d_3) = (10, 1, 1)$ $T_2 = 0.0005$;$T_3 = 0.0002$,$T_4 = 0.0001$ $\sum_j T_j = 1.0008$ $\mathfrak{T}_1 = 0.9992$,$\mathfrak{T}_2 = 0.0005$,$\mathfrak{T}_3 = 0.0002$,$\mathfrak{T}_4 = 0.0001$

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \langle [0.3500, 0.6500], [-0.7500, -0.2500], 0.4500, -0.6500 \rangle.$$

Hence proved as $d_1 \to \infty$:

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4) = \mathscr{C}_1.$$

4. Cubic bipolar fuzzy geometric operator with priority degree

Definition 15. Assume that $\mathscr{C}_j = \Big\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \Big\rangle$ is a collection of CBFNs and CBFPDG: $M^n \to M$ is a mapping defined as:

$$CBFPDG(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n)$$

$$= {\mathscr{C}_1}^{{r_1}^{d_1}} \bigotimes {\mathscr{C}_2}^{{r_2}^{d_2}} \bigotimes \cdots {\mathscr{C}_n}^{{r_n}^{d_n}},$$

where $\mathfrak{T}_j = \prod_{k=1}^{j-1} \Big(S(\mathscr{C}_k)\Big)^{d_k}$ and $\mathfrak{T}_1 = 1$ such that $S(\mathscr{C}_k)$ is the score function of the kth CBFN.

The CBFPDG operator is explained in the theorem mentioned below whose proof follows the CBFN's operational laws.

Theorem 6. *Let* $\mathscr{C}_k = \Big\langle [\mathscr{P}_{l_k},\ \mathscr{P}_{u_k}], [\mathscr{N}_{l_k},\ \mathscr{N}_{u_k}], \lambda_k, \mu_k \Big\rangle$ *be a collection of CBFNs, then we can find CBFPDG by the mapping*:

$$CBFPDG(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) =$$

$$\Big\langle \Big[\prod_{j=1}^{n} (\mathscr{P}_{l_j})^{{r_j}^{d_j}}, \prod_{j=1}^{n} (\mathscr{P}_{u_j})^{{r_j}^{d_j}} \Big],$$

$$\Big[-\Big(1 - \prod_{j=1}^{n} (1 - \mathscr{N}_{l_j})^{{r_j}^{d_j}}\Big),$$

$$-\left(1-\prod_{j=1}^{n}(1-\mathscr{N}_{u_j})^{{r_j}^{d_j}}\right)\Big],$$

$$1-\prod_{j=1}^{n}(1-\lambda_j)^{{r_j}^{d_j}},-\prod_{j=1}^{n}(\mu_j)^{{r_j}^{d_j}}\Big\rangle.$$

Proof. Proof is similar to Theorem (1.) ☐

Theorem 7. (Idempotency): *Assume that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}],\ [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j,\ \mu_j \right\rangle$ *is a collection of CBFNs. A CBFPDA operator that is defined by the mapping* $\Lambda^n \to \Lambda$ *is expressed as:*

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = {r_1}^{d_1}\mathscr{C}_1 \bigotimes {r_2}^{d_2}\mathscr{C}_2 \bigotimes \cdots \bigotimes {r_n}^{d_n}\mathscr{C}_n, \quad (10)$$

where ${r_i}^{d_i} = \frac{{\mathfrak{T}_1}^{d_i}}{\sum_{k=1}^{n} {\mathfrak{T}_k}^{d_k}}$ *and* $\mathfrak{T}_j = \prod_{k=1}^{n-1}\left(S(\mathscr{C}_k)\right)^{d_k}$*;* $(j = 1, 2, ..., n)$ *and* $\mathfrak{T}_1 = 1$ *and* $S(\mathscr{C}_k)$ *is the score function of kth CBFN. If* $\mathscr{C}_j - \mathscr{C}\forall j$*, then* $CBFPDA(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_n) = \mathscr{C}$.

Proof. Proof is similar to Theorem 2. ☐

Theorem 8. (Monotonicity): *Consider that:*

$$\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j}, \mathscr{P}_{u_j}], [\mathscr{N}_{l_j}, \mathscr{N}_{u_j}], \lambda_j, \mu_j \right\rangle,$$

and

$$\mathscr{C}_j^* = \left\langle [\mathscr{P}_{l_j}^*, \mathscr{P}_{u_j}^*], [\mathscr{N}_{l_j}^*, \mathscr{N}_{u_j}^*], \lambda_j^*, \mu_j^* \right\rangle,$$

are families of CBFNs, where $\mathfrak{T}_j = \prod_{k=1}^{j-1}\left(S(\mathscr{C}_k)\right)^{d_k}$ *and* $\mathfrak{T}_j^* = \prod_{k=1}^{j-1}\left(S(\mathscr{C}_k^*)\right)^{d_k}$*;* $(j = 2, 3, ..., n)$*,* $\mathfrak{T}_1 = 1 = \mathfrak{T}_1^*$.

$$CBFPDA(\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_n) \leq CBFPDA(\mathscr{C}_1^*, \mathscr{C}_2^*, ..., \mathscr{C}_n^*).$$

Proof. Proof is similar to Theorem 3. ☐

Theorem 9. (Boundedness): *Consider that* $\mathscr{C}_j = \left\langle [\mathscr{P}_{l_j},\ \mathscr{P}_{u_j}],\ [\mathscr{N}_{l_j},\ \mathscr{N}_{u_j}],\ \lambda_j,\ \mu_j \right\rangle$ *is a family of* $CBFNs$ *and* $\mathscr{C}^- = \min_j(\mathscr{C}_j)$ *and* $\mathscr{C}^+ = \max_j(\mathscr{C}_j)$ *then* $\mathscr{C}^- \leq \mathscr{C}_j \leq \mathscr{C}^+$.

Proof. Proof is similar to Theorem (4.) ☐

5. Methodology for MCDM using profounded AOs

Let $\mathbb{M} = \{M_1, M_2, \cdots, M_n\}$ be a collection of alternatives and $\mathbb{C} = \{C_1, C_2, \cdots, C_m\}$ be the assemblage of criterions. Priorities are assigned between the criterions. $C_i \geq_{d_k} C_j$ indicates criteria C_i is superior to criteria C_j with degree d_k. Consider $\mathbb{D} = \{D_1, D_2, \cdots, D_l\}$ as a set of decision makers. Priorities are assigned between the DMs provided by strict priority orientation, $D_1 \succ_{d_1} D_2 \succ_{d_2} \succ \cdots \succ_{d_{l-1}} D_l$. DMs give a matrix according to their own opinions and viewpoints $\mathbb{D}^l = \left(\mathscr{C}_{ij}^l\right)_{n\times m}$ for the alternative M_i and criteria C_j by the D_l decision maker.

The suggested operators will be implemented to the MCDM, which will require the preceding steps.

Algorithm

Now we discuss the steps wise procedure of the proposed method:

Step 1: Obtain the decision matrix $\mathbb{D}^l = (\mathscr{P}_{ij})_{m\times n}$, where all entries of the matrix are CBFNs assigned by the standpoints of the decision makers.

Indicators of cost (τ_c) and benefit (τ_b) are the two types of criterion described in the decision matrix. If all indicators are of the same type, then normalisation is not necessary; however, in MCGDM, there may be two distinct criteria types. As a result of applying the normalisation formula presented in Eq. (11), the matrix is modified to become the transforming response matrix, with the notation. $\mathbb{D}^l = (\mathscr{Q}_{ij})_{m\times n}$:

$$(\mathscr{Q}_{ij})_{m\times n} = \begin{cases} (\mathscr{P}_{ij})^c, j \in \tau_c \\ (\mathscr{P}_{ij}), j \in \tau_b \end{cases} \quad (11)$$

Step 2: Using equations, combine all of the independent CBF decision matrices into one combined evaluation matrix of the alternatives using one of the provided AOs (Eqs. (1) and (2)).

Step 3: Aggregate the CBFNs for each alternative using CBFAPD (or CBFGPD) operator.

Step 4: Calculate the score values of all accumulative CBFNs alternatives assessments.

Step 5: Rank all score values of alternatives and choose the highest one as the best alternative.

Pictorial structure of the algorithm is viewed in Figure 1.

6. Case study

In this section, an algorithm for solving the MCDM problem in a cubic bipolar environment is proposed.

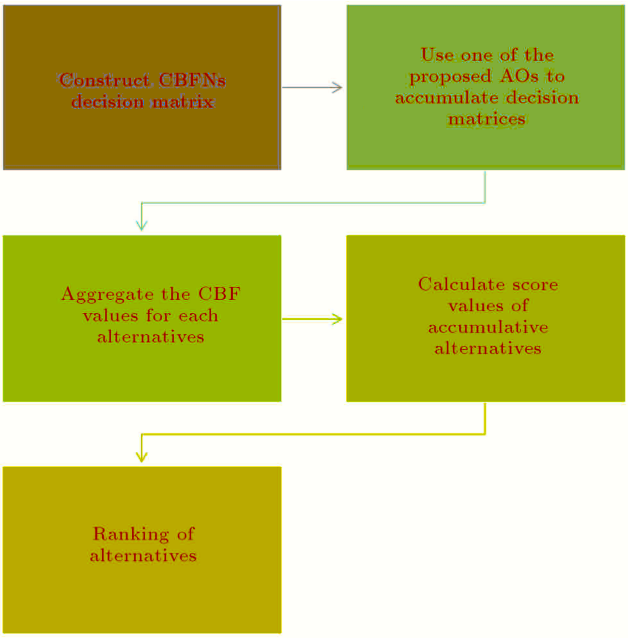

Figure 1. Pictorial structure of the algorithm.

Reverse Logistics (RLs) recycle or reuse goods. Supply chains supply consumers. Supply chain experts measure efficiency with On-Time Delivery (OTD). Supply chain metrics include order-to-delivery time. Service delivery completes the supply chain. Receiving the wrong item, a damaged item, a product that doesn't match the company's logo, or no longer needing the item are all valid reasons for a refund or exchange. The product must be returned, disassembled, inspected, recycled, and repaired. They require frequent supply chain reversals. RLs benefit consumers and industry. Reusing, recycling, and repairing are RLs. Manufacturers can reuse assets. Recycling companies would benefit from RLs. RLs only save materials. E-commerce boosts RLs.

Online retailers expected 414 million in 2018 after replacing shopping carts. Online returns exceeded 30%, compared to 8.89 ± 1% in brick-and-mortar stores. Supply chains struggled with logistics costs as product returns increased. Thus, a reverse logistics system setup requires careful consideration of material reversal metrics. Returns, product types, dollars, and lost profits are included. Return risk metrics can identify issues and grow the business. Reverse logistics pays off. Supply chain turnaround? RLs must be "forward" logistics-efficient (customer support, storage, system integration, etc). Optimizing supply chain returns boosts output, customer satisfaction, and savings.

Key steps of optimization are as follows:

1. RLs reduce shipping costs and resell goods that would have been thrown out if returned. Profit margins will boost if recycled and resold materials generate revenue and the system works well;
2. Your company's return policy can affect customer perceptions. It's possible that an advertised product's defective part caused a bad result. Fixing mistakes is as important as closing deals. Resolve product issues with customers. Customer loyalty can be increased by giving customers multiple return options. You may be able to return an item to a physical shop without the original receipt or packaging and receive a full refund regardless of the reason;
3. Customer satisfaction would increase if you had a well-organized return and replacement system. It speeds up repairing, refurbishing, and reusing products to avoid buying new ones;
4. RLs can help you recycle, resell, or reuse products that would otherwise go to landfills. This raises the brand's social and environmental responsibilities and profits. Remanufacturing or refurbishing extends product life.

Growth adds customers, sites, and manufacturing processes. Some companies lack capital and overhead. Businesses should enhance functionality outside their systems with the aid of 3PRLP. An outside agency provides 3PRLPs for cost savings, productivity, and capability development. 3PRLP services can be intermittent or permanent. Business in need of 3PRLP may find that they outgrow their storage. 3PRLP warehouse management aids storage. Infrastructure, vehicle, and shipping costs may hurt other businesses. 3PRLP's large fleets of specialised trucks and facilities are cheaper. Strong 3PRLPs help US firms enter Canada. 3PRLPs help companies with customer support, delivery times, refunds, order tracking, technical services, stock management, and more. Businesses may increase their 3PRLP associate value. Supply chain specialists aim to increase productivity, speed processes, and lower logistics costs, including transportation. Figure 2 shows logistical cost breakdown.

This shortlisting technique is an MCDM assignment for the 3PRLP. Studies show that 3PRLP selec-

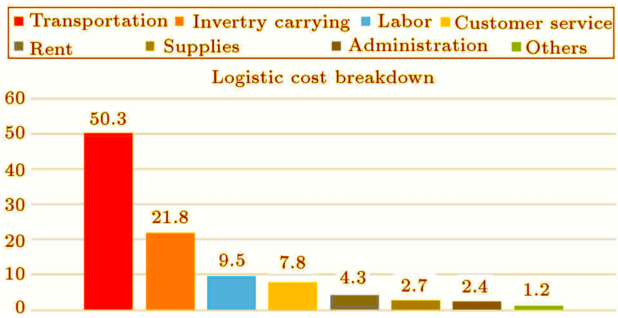

Figure 2. Logistic cost breakdown.

Table 3. Set of criterions [39].

$\mathscr{C}_i$	Criterion	Explanation
$\mathscr{C}_1$	Time deliver	Client-required products or services on time. On-time delivery is the percentage of work completed within the customer's requested or company-committed timeframe. Long delivery times don't help times or by declining difficult business
$\mathscr{C}_2$	Experience	The factory's past service or product achievements will be examined
$\mathscr{C}_3$	Reliability	This ensures that products and services are reliable and improve customer satisfaction
$\mathscr{C}_4$	Knowledge sharing	Traditional information sharing involves sender-receiver data exchanges These exchanges use hundreds of open and proprietary protocols, message formats, and file types. Information sharing is a platform that regulates data and information exchange between clients and providers to ensure privacy, security, and data quality.
$\mathscr{C}_5$	Reputation	Dependent on how others define one's identity. Decentralized, unplanned social control maintains social order through reputation
$\mathscr{C}_6$	Flexibility	System marketability. "Response capability" is a system's ability to quickly and cost-effectively respond to internal or external changes that affect value delivery. Thus, flexibility is how easily a system adapts to uncertainty to maintain or increase value.

Table 4. Set of alternatives.

M_1	M_2	M_3	M_4	M_5	M_6
Company 1	Company 2	Company 3	Company 4	Company 5	Company 6

tion is of scholarly and commercial interest. MCDMs have proliferated in recent years. Models were developed for 3PRLP evaluation. Realistic RLs outsourced assessments are often ambiguous and imprecise due to partial ignorance, imprecise assessment, and partial or unavailable decision-making for further facts. "Outsourcing" first appeared in the American Glossary in 1981 as "outside resourcing". Outsourcing logistics is a major company achievement. A logistics contract provider outsources many companies at once, creating economic balance and lowering costs. Many researchers noted that cost reduction is very seldom the main goal of MCDM outsourcing [9–12].

Numerous academics have described a number of 3PRLP outsourcing modules, including: (1) the advantages and disadvantages of working with a third-party logistics provider; and (2) selecting 3PRLPs for a long-term collaboration.

The second module selects 3PRLPs for DM based on attainability. 3PRLP reduces environmental risks, resource issues, and product life to maximise profits. Sustainable development principles are encouraged and even required in supply chain management 3PRLP configurations in developing nations. Choosing a 3PRLP is well-studied. Researchers have debated the most important 3PRLP selection criteria for 20 years. Researchers surveyed these crucial factors. The best 3PRLP is chosen based on the six criteria listed in Table 3.

6.1. Problem formulation

Company preference determines the 3PRLP selection criterion. The criterion selected diverse sources. Several researchers have spent the last two decades identifying 3PRLP analysis and selection criteria. Researchers identified key factors by surveying. In this article, we use the six criterion for selecting the best 3PRLP, given in Tables 3 and 4.

6.1.1. Parameters

Selection is a difficult problem to solve, criteria and alternatives play a vital role in resolving it. This problem formulation considers the following criteria and alternatives.

6.1.2. Assumption

We have four decision makers $\{D_1, D_2, D_3, D_4\}$ that will assign linguistic values from Table 5 according to their own interest, experience, and knowledge to the above mentioned criteria and alternatives in Table 6.

6.1.3. Calculations

The steps of proposed algorithm are illustrated as follows:

Step 1: The decision matrices are obtained by the decision makers represented in Tables 7–10. The normalized decision matrices are obtained by the decision makers in which each entry represents the viewpoint of decision makers toward the criteria and alternatives shown in Tables 11–14.

Table 5. Linguistic variable and their associated fuzzy values.

No.	Linguistic variable	Signs and code	CBF values
1	Very low	⋆	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
2	Low	⋆⋆	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
3	Satisfactory	⋆⋆⋆	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
4	High	⋆⋆⋆⋆	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
5	Very high	⋆⋆⋆⋆⋆	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$

Table 6. Linguistic terms for alternatives with respect to criterions.

	Strategies	D_1	D_2	D_3	D_4
M_1	$\mathscr{C}_1$	⋆⋆	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆
	$\mathscr{C}_2$	⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_4$	⋆⋆⋆⋆	⋆⋆	⋆	⋆⋆
	$\mathscr{C}_5$	⋆⋆⋆⋆⋆	⋆	⋆⋆	⋆⋆
	$\mathscr{C}_6$	⋆	⋆⋆	⋆⋆⋆	⋆⋆⋆
M_2	$\mathscr{C}_1$	⋆	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_2$	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆	⋆⋆⋆
	$\mathscr{C}_4$	⋆⋆⋆⋆⋆	⋆⋆⋆	⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_5$	⋆	⋆⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_6$	⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆
M_3	$\mathscr{C}_1$	⋆⋆⋆	⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_2$	⋆⋆⋆⋆	⋆	⋆⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆
	$\mathscr{C}_4$	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆
	$\mathscr{C}_5$	⋆⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆	⋆⋆⋆
	$\mathscr{C}_6$	⋆⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆	⋆
M_4	$\mathscr{C}_1$	⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆
	$\mathscr{C}_2$	⋆⋆	⋆⋆⋆⋆⋆	⋆	⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆	⋆⋆⋆⋆	⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_4$	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆	⋆⋆
	$\mathscr{C}_5$	⋆⋆⋆⋆⋆	⋆	⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_6$	⋆	⋆⋆⋆	⋆⋆	⋆⋆⋆
M_5	$\mathscr{C}_1$	⋆	⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_2$	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆
	$\mathscr{C}_4$	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_5$	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆
	$\mathscr{C}_6$	⋆⋆⋆⋆⋆	⋆	⋆⋆	⋆⋆⋆⋆
M_6	$\mathscr{C}_1$	⋆⋆	⋆⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_2$	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆⋆
	$\mathscr{C}_3$	⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆
	$\mathscr{C}_4$	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆	⋆⋆⋆⋆
	$\mathscr{C}_5$	⋆⋆⋆	⋆⋆⋆⋆⋆	⋆⋆⋆⋆	⋆⋆⋆
	$\mathscr{C}_6$	⋆⋆⋆⋆⋆	⋆⋆	⋆⋆	⋆⋆⋆⋆

* The star sign indicated the linguistic terms as declared in Table 5.

Table 7. Decision matrix by D_1

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	[0.7000, 0.9000], [−0.9900, −0.7900], 0.1500, −0.8199	[0.1000, 0.3000], [−0.9900, −0.7900], 0.1500, −0.8199
M_2	[0.0000, 1.0000], [−1.0000, −0.9900], 0.0500, −0.9999	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199
M_3	[0.5000, 0.6999], [−0.7900, −0.5900], 0.4500, −0.6199	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145
M_4	[0.9000, 1.0000], [−1.0000, −0.9900], 0.0500, −0.9999	[0.1000, 0.3000], [−0.9900, −0.7900], 0.1500, −0.8199
M_5	[0.9000, 1.0000], [−1.0000, −0.9900], 0.0500, −0.9999	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199
M_6	[0.7000, 0.9000], [−0.9900, −0.7900], 0.1500, −0.8199	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145
M_2	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150
M_3	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199
M_4	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145
M_5	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150
M_6	[0.5001, 0.8000], [−0.5901, −0.3900], 0.7500, −0.4145	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150	[0.0000, 0.1000], [−1.0000, −0.9900], 0.0500, −0.9999
M_2	[0.0000, 0.1000], [−1.0000, −0.9900], 0.0500, −0.9999	[0.0000, 0.1000], [−1.0000, −0.9900], 0.0500, −0.9999
M_3	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150
M_4	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150	[0.0000, 0.1000], [−1.0000, −0.9900], 0.0500, −0.9999
M_5	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150
M_6	[0.3001, 0.5000], [−0.7900, −0.5900], 0.4500, −0.6199	[0.8001, 1.0000], [−0.3901, −0.0001], 0.9755, −0.1150

Table 8. Decision matrix by D_2.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.5000, 0.6999], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_2	$[0.5000, 0.6999], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.9000, 1.0000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_4	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_5	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_3	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_4	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_5	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_4	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_5	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_6	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$

Table 9. Decision matrix by D_3.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_2	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.0000, 0.1999], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_4	$[0.0000, 0.1999], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_5	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.0000, 0.1999], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_2	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_3	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_4	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_5	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_2	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_4	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$

Table 10. Decision matrix by D_4.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.5000, 0.69999], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_2	$[0.2000, 0.4999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.2000, 0.4999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_4	$[0.4000, 0.9000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_5	$[0.2000, 0.6999], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.5000, 0.6999], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_4	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_5	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_6	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_3	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_4	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_5	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$

Table 11. Normalized decision matrix by D_1.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_3	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_4	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_5	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_6	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_2	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_4	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_2	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_3	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_4	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_5	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$

Table 12. Normalized decision matrix by D_2.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_2	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_4	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_3	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_4	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_5	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_4	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_5	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_6	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$

Table 13. Normalized decision matrix by D_3.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_2	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_4	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_2	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_3	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_4	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_5	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_2	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_4	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_6	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$

Table 14. Normalized decision matrix by D_4.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_2	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_3	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_4	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_5	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_2	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_3	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_4	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$
M_5	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_6	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_2	$[0.8001, 1.0000], [-0.3901, -0.0001], 0.9755, -0.1150$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_3	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.0000, 0.1000], [-1.0000, -0.9900], 0.0500, -0.9999$
M_4	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$
M_5	$[0.1000, 0.3000], [-0.9900, -0.7900], 0.1500, -0.8199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$
M_6	$[0.3001, 0.5000], [-0.7900, -0.5900], 0.4500, -0.6199$	$[0.5001, 0.8000], [-0.5901, -0.3900], 0.7500, -0.4145$

Step 2: To aggregate the decision matrix we will follow these steps:

(a) Calculate the score functions for all the decision matrices and shown in Table 15.

(b) Calculate the $r_i^{d_i} = \frac{\mathfrak{T}_1{}^{d_i}}{\sum_{k=1}^{n}\mathfrak{T}_k{}^{d_k}}$ and $\mathfrak{T}_j = \prod_{k=1}^{n-1}\left(S(\mathscr{C}_k)\right)^{d_k}$.

(c) Accumulate the decision matrices by using Eq. (1) are given matrixes as shown in Box III.

The accumulative matrix is given in Eq. (12).

$$\mathfrak{T}_{ij} = \begin{pmatrix} 1 & 0.36820 & 0.10160 & 0.02290 \\ 1 & 0.32380 & 0.19080 & 0.04940 \\ 1 & 0.66580 & 0.28090 & 0.12320 \\ 1 & 0.34570 & 0.15630 & 0.01010 \\ 1 & 0.52380 & 0.24130 & 0.09050 \\ 1 & 0.54570 & 0.27630 & 0.13880 \end{pmatrix} \quad (12)$$

$$r_i^{(d)} = \begin{pmatrix} 0.6699 & 0.2467 & 0.0681 & 0.0153 \\ 0.6394 & 0.2070 & 0.1220 & 0.0316 \\ 0.4831 & 0.3217 & 0,1357 & 0.0595 \\ 0.6613 & 0,2286 & 0.1034 & 0.0067 \\ 0.5389 & 0.2823 & 0.1300 & 0.0488 \\ 0.5100 & 0.2783 & 0.1409 & 0.0708 \end{pmatrix} \quad (13)$$

Step 3: Perform row-wise accumulation to combine the values of criteria for each alternative shown in Table 16 and Figure 3.

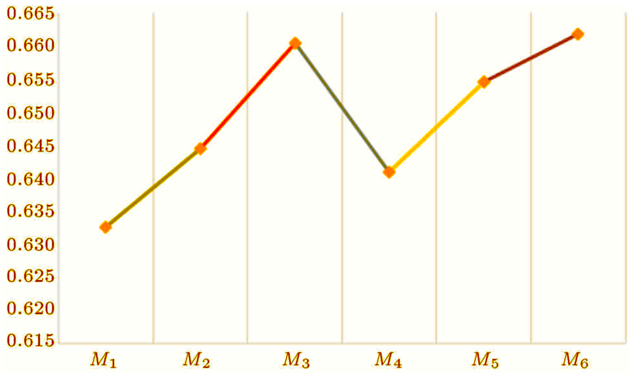

Figure 3. Score functions of alternatives.

Step 4: Calculate the score function for each value of Table 17 and list the values in Table 18 where the priority degree is ordered as :

$$(d_1, d_2, d_3) = (4, 1, 1).$$

Step 5: Rank the alternatives in ascending order and select the best alternative as the optimal solution.

$$M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1.$$

7. Pros and Cons of CBFPDA

Every MCDM technique has advantages and disadvantages; similarly, our proposed method has both stren-

$$\mathfrak{T}_{ij}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \mathfrak{T}_{ij}^{(2)} = \begin{pmatrix} 0.15835 & 0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.02668 \\ 0.02668 & 0.37835 & 0.60925 & 0.87840 & 0.02668 & 0.02668 \\ 0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840 & 0.87840 \\ 0.02668 & 0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.02668 \\ 0.02668 & 0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840 \\ 0.15835 & 0.37835 & 0.60925 & 0.87840 & 0.37835 & 0.87840 \end{pmatrix}$$

$$\mathfrak{T}_{ij}^{(3)} = \begin{pmatrix} 0.05939 & 0.09647 & 0.32943 & 0.09647 & 0.02344 & 0.00422 \\ 0.01001 & 0.22849 & 0.53517 & 0.32943 & 0.02344 & 0.02344 \\ 0.01001 & 0.01625 & 0.32943 & 0.22849 & 0.77159 & 0.32943 \\ 0.00163 & 0.13909 & 0.22849 & 0.53517 & 0.02344 & 0.01001 \\ 0.00163 & 0.32943 & 0.22849 & 0.53517 & 0.32943 & 0.02344 \\ 0.09647 & 0.32943 & 0.22849 & 0.53517 & 0.32943 & 0.13909 \end{pmatrix}$$

$$\mathfrak{T}_{ij}^{(4)} = \begin{pmatrix} 0.03618 & 0.08474 & 0.00879 & 0.00257 & 0.00371 & 0.00158 \\ 0.00610 & 0.20069 & 0.01428 & 0.05217 & 0.00879 & 0.01428 \\ 0.00879 & 0.00609 & 0.20071 & 0.20071 & 0.12218 & 0.20071 \\ 0.00143 & 0.00371 & 0.03618 & 0.01428 & 0.00371 & 0.00159 \\ 0.00099 & 0.05217 & 0.20071 & 0.08474 & 0.20071 & 0.00371 \\ 0.08474 & 0.12355 & 0.20071 & 0.20113 & 0.20071 & 0.02202 \end{pmatrix}$$

Box III

Table 15. Score functions for D_i.

	$\boldsymbol{D_1}$						$\boldsymbol{D_2}$					
	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$
$\boldsymbol{M_1}$	0.15835	0.15835	0.37503	0.60925	0.87840	0.02668	0.37503	0.60925	0.87840	0.15835	0.02668	0.15835
$\boldsymbol{M_2}$	0.02668	0.37503	0.60925	0.87840	0.02668	0.02668	0.37503	0.60925	0.87840	0.37503	0.87840	0.87840
$\boldsymbol{M_3}$	0.37503	0.60925	0.87840	0.37503	0.87840	0.87840	0.02668	0.02668	0.37503	0.60925	0.87840	0.37503
$\boldsymbol{M_4}$	0.02668	0.15835	0.37503	0.60925	0.87840	0.02668	0.60925	0.87840	0.60925	0.87840	0.02668	0.37503
$\boldsymbol{M_5}$	0.02668	0.37503	0.60935	0.87840	0.37503	0.87840	0.60925	0.87840	0.37503	0.60925	0.87840	0.02668
$\boldsymbol{M_6}$	0.15835	0.37503	0.60925	0.87840	0.37503	0.87840	0.60925	0.87840	0.37503	0.60925	0.87840	0.15835
	$\boldsymbol{D_3}$						$\boldsymbol{D_4}$					
	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$	$\mathscr{C}_1$	$\mathscr{C}_2$	$\mathscr{C}_3$	$\mathscr{C}_4$	$\mathscr{C}_5$	$\mathscr{C}_6$
$\boldsymbol{M_1}$	0.60925	0.87840	0.02668	0.02668	0.15835	0.37503	0.37503	0.60925	0.87840	0.15835	0.15835	0.37503
$\boldsymbol{M_2}$	0.60925	0.87840	0.02668	0.15835	0.37503	0.60925	0.60925	0.87840	0.37503	0.60925	0.87840	0.37503
$\boldsymbol{M_3}$	0.87840	0.37503	0.60925	0.87840	0.15835	0.60925	0.60925	0.87840	0.15835	0.15835	0.37503	0.02668
$\boldsymbol{M_4}$	0.87840	0.02668	0.15835	0.02668	0.15835	0.15835	0.15835	0.60925	0.87840	0.15835	0.60925	0.37503
$\boldsymbol{M_5}$	0.60925	0.15835	0.87840	0.15835	0.60925	0.15835	0.60925	0.87840	0.02668	0.60925	0.15835	0.60925
$\boldsymbol{M_6}$	0.87840	0.37503	0.87840	0.37503	0.60925	0.15835	0.60925	0.87840	0.02668	0.60925	0.37503	0.60925

Table 16. Aggregated decision matrix.

	$\mathscr{C}_1$	$\mathscr{C}_2$
M_1	$[0.1905, 0.4115], [-0.9009, -0.6975], 0.2232, -0.7627$	$[0.3036, 1.0000], [-0.8114, -0.3564], 0.2598, -0.7266$
M_2	$[0.1650, 0.6335], [-0.8783, -0.7716], 0.1194, -0.9999$	$[0.4615, 1.0000], [-0.6673, -0.1427], 0.5633, -0.5267$
M_3	$[0.6491, 1.0000], [-0.7611, -0.2093], 0.2541, -0.9692$	$[0.2714, 1.0000], [-0.5990, -0.3404], 0.2979, -0.9653$
M_4	$[0.7220, 1.0000], [-0.8041, -0.3086], 0.1272, -0.9980$	$[0.3575, 1.0000], [-0.7982, -0.1034], 0.2076, -0.8792$
M_5	$[0.2736, 0.5502], [-0.7841, -0.6443], 0.1734, -0.9881$	$[0.5224, 1.0000], [-0.6440, -0.0346], 0.5040, -0.5437$
M_6	$[0.3807, 1.0000], [-0.7496, -0.1987], 0.3180, -0.1795$	$[0.5166, 1.0000], [-0.6382, -0.0498], 0.5740, -0.5072$
	$\mathscr{C}_3$	$\mathscr{C}_4$
M_1	$[0.4838, 1.0000], [-0.6673, -0.0628], 0.4745, -0.7294$	$[0.3886, 0.6923], [-0.7005, -0.5000], 0.4091, -0.7619$
M_2	$[0.5452, 1.0000], [-0.5830, -0.0799], 0.5614, -0.7817$	$[0.6795, 1.0000], [-0.5247, -0.0023], 0.6559, -0.3901$
M_3	$[0.3705, 1.0000], [-0.5473, -0.4647], 0.6565, -0.4201$	$[0.4621, 1.0000], [-0.6624, -0.1618], 0.5518, -0.4686$
M_4	$[0.3404, 1.0000], [-0.7529, -0.5219], 0.4538, -0.6094$	$[0.5627, 1.0000], [-0.5689, -0.0652], 0.5955, -0.7397$
M_5	$[0.4952, 1.0000], [-0.6230, -0.1566], 0.5887, -0.5001$	$[0.7458, 1.0000], [-0.4493, -0.0085], 0.7550, -0.2948$
M_6	$[0.4952, 1.0000], [-0.6230, -0.1566], 0.5887, -0.6419$	$[0.6813, 1.0000], [-0.4904, -0.0048], 0.8086, -0.3085$
	$\mathscr{C}_5$	$\mathscr{C}_6$
M_1	$[0.6629, 1.0000], [-0.5318, -0.0020], 0.4010, -0.9177$	$[0.0542, 0.1946], [-0.9781, -0.8968], 0.0788, -0.9987$
M_2	$[0.3480, 1.0000], [-0.7762, -0.1035], 0.1328, -0.9976$	$[0.3489, 1.0000], [-0.7659, -0.1294], 0.1379, -0.9975$
M_3	$[0.7358, 1.0000], [-0.4616, -0.0006], 0.7226, -0.6781$	$[0.6272, 1.0000], [-0.5476, -0.0087], 0.6150, -0.6287$
M_4	$[0.6605, 1.0000], [-0.5341, -0.0022], 0.4068, -0.9062$	$[0.0905, 0.2364], [0.9451, -0.8563], 0.0939, -0.9985$
M_5	$[0.3218, 0.5488], [-0.7690, -0.5671], 0.4558, -0.6123$	$[0.5995, 1.0000], [-0.5861, -0.0065], 0.1329, -0.9458$
M_6	$[0.5297, 1.0000], [-0.6232, -0.0482], 0.5983, -0.4896$	$[0.6113, 1.0000], [-0.5844, -0.0061], 0.4450, -0.5501$

-gths and weaknesses. A few important ones are listed below:

- The main advantage of the proposed method is that it calculates the weights of criteria automatically, which is more efficient than the weights being given by decision makers;
- Easy to adopt and compute;
- The weight vectors should be non-negative;
- There is usually interaction between the membership and non-membership grades in the AOs, but in the proposed method the membership and non-membership grades are independent.

8. Comparison analysis

We obtained ratings $M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1$ using our proposed method. To validate our optimal alternative, we run the same problem by using the existing operators. The fact that we obtain the same optimal decision shown in Table 19 and illustrated in Figure 4 demonstrates the validity of our suggested AOs.

Table 17. Accumulative decision matrix.

Alternatives	Fuzzy values
M_1	[0.9725,1],[−0.1777,0],0.0004,−1
M_2	[0.9722,1],[−0.1066,0],0.0005,−1
M_3	[0.9915,1],[−0.0418,0],0.0122,−1
M_4	[0.9841,1],[−0.1388,0],0.0003,−1
M_5	[0.9879,1],[−0.0637,0],0.0024,−1
M_6	[0.9912,1],[−0.0532,0],0.0223,−0.9910

Table 18. Score functions of alternatives.

Alternatives	Score value	Ranking
M_1	0.6325	6th
M_2	0.6444	4th
M_3	0.6603	2nd
M_4	0.6409	5th
M_5	0.6544	3rd
M_6	0.6616	1st

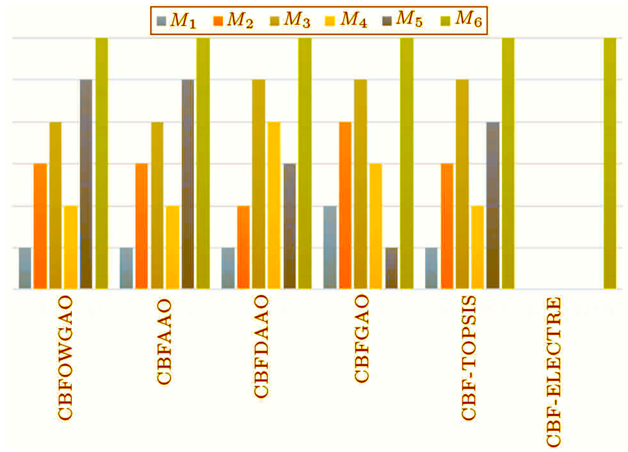

Figure 4. Comparison analysis.

9. Conclusion

This study addresses data ambiguity using positive and negative membership grades and interval values with Cubic Bipolar Fuzzy Numbers (CBFNs). CBF combines CS and BFS models. We defined the cubic bipolar fuzzy prioritised averaging (geometric) operator with priority degrees using strict priority orders. Priority degree theories will help merge massive CBF data. Priority degree hypotheses have been extensively researched and will help integrate multiple CBF data sets. A CBF group Multi-Criteria Decision Making (MCDM) method was developed based on the prioritised Aggregation Operators (AOs). An example illustrates the suggested approach, and the results are compared to other AOs. We also analyse how priorities affect results. Priority levels affect results, making the idea appealing. Decision Making (DM's) freedom to choose the priority degree vector makes this method more resilient and difficult. The CBF framework has a group MCDM strategy based on the prioritised AOs. An analogy illustrates the proposed method, which is compared to many contemporary AOs. Priority degrees affect aggregated results. The DM can choose the priority degree vector based on priorities and problem complexity, strengthening the suggested solution. We applied MCDM to demonstrate the proposed method.

Future work may use fuzzy judgements to imple-

Table 19. Comparison between proposed methods and existing techniques.

Existing techniques	Ranking	Optimal result
CBF ordered weighted geometric AO [40]	$M_6 \geq M_5 \geq M_3 \geq M_2 \geq M_4 \geq M_1$	M_6
CBF averaging AO [21]	$M_6 \geq M_5 \geq M_3 \geq M_2 \geq M_4 \geq M_1$	M_6
CBF Dombi averaging AO [41]	$M_6 \geq M_3 \geq M_4 \geq M_5 \geq M_2 \geq M_1$	M_6
CBF geometric AO [8]	$M_6 \geq M_3 \geq M_2 \geq M_4 \geq M_1 \geq M_5$	M_6
CBF TOPSIS [42]	$M_6 \geq M_3 \geq M_5 \geq M_2 \geq M_4 \geq M_1$	M_6
CBF ELECTRE-I [42]	M_6	M_6

ment the suggested work in practise. AOs and MCDM would improve decision-making, medical diagnosis, pattern recognition, computational intelligence, and artificial intelligence. We will also work on objective priority degree methods soon.

References

1. Zadeh, L.A. "Fuzzy sets", *Information and Control*, **8**(3) pp. 338–353 (1965). http://dx.doi.org/10.1016/S0019-9958(65)90241-X
2. Zadeh, L.A.," Similarity relations and fuzzy ordering", *Information Sciences*, **3**(2), pp. 177–200 (1971). DOI: 10.1016/s0020-0255(71)80005-1
3. Atanassov, K.T. "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, **20**(1), pp. 87–96 (1986). http://dx.doi.org/10.1016/S0165-0114(86)80034-3
4. Atanossov, K.T. "Intuitionistic fuzzy sets", In *Intuitionistic Fuzzy Sets. Studies in Fuzziness and Soft Computing, Physica*, Heidelberg, **35**, pp. 1–137 (1999). https://doi.org/10.1007/978-3-7908-1870-3-1
5. Riaz, M. and Jamil, N. "Topological structures on cubic bipolar fuzzy sets with linear assign- ment model and SIR method for healthcare", *Journal of Intelligent and Fuzzy Systems*, **44**(3), pp. 1191–1212 (2022). DOI: 10.3233/JIFS-222224
6. Xu, Z. "Intuitionistic fuzzy aggregation operators", *IEEE Transactions on Fuzzy Systems*, **15**(6), pp. 1179–1187 (2007). DOI: 10.1109/TFUZZ.2006.890678
7. Li, B. and Xu, Z. "Prioritized aggregation operators based on the priority degrees in multicriteria decision making", *International Journal of Intelligent Systems*, **34**(9), pp. 1985–2018 (2019). DOI: 10.1002/int.22123
8. Riaz, M. and Tehrim, S.T. "Cubic bipolar fuzzy set with application to multi-criteria group decision making using geometric aggregation operators", *Soft Computing*, **24**(21) pp. 16111–16133 (2020). DOI:10.1007/s00500-020-04927-3
9. Koohathongsumrit, N. and Meethom, W. "An integrated approach of fuzzy risk assessment model and data envelopment analysis for route selection in multimodal transportation networks", *Expert Systems with Applications*, **171**, p. 114342 (2021). DOI: 10.1016/j.eswa.2020.114342
10. Büyüközkan, G., Karabulut, Y., and Arsenyan, J. "RFID service provider selection: An integrated fuzzy MCDM approach", *Measurement*, **112**, pp. 88–98 (2017). DOI: 10.1016/j.measurement.2017.08.018
11. Raut, R., Kharat, M., Kamble, S., et al. "Sustainable evaluation and selection of po- tential third-party logistics (3PL) providers: An integrated MCDM approach", *Benchmarking: An International Journal*, **25**(1), pp. 76–97 (2018). DOI: 10.1108/BIJ-05-2016-0065
12. Koohathongsumrit, N. and Meethom, W. "Route selection in multimodal transportation networks: a hybrid multiple criteria decision-making approach", *Journal of Industrial and Production Engineering*, **38**(3), pp. 171–185 (2021). DOI: 10.1080/21681015.2020.1871084
13. Yager, R.R. "Pythagorean fuzzy subsets", In *Joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS)*, Edmonton, AB, Canada, pp. 57–61 (2013). DOI: 10.1109/IFSA-NAFIPS.2013.6608375.
14. Yager, R.R. "Pythagorean membership grades in multicriteria decision making", *IEEE Transactions on Fuzzy Systems*, **22**(4), pp. 958–965 (2014). DOI: 10.1109/TFUZZ.2013.2278989
15. Senapati, T. and Yager, R.R. "Fermatean fuzzy sets", *Journal of Ambient Intelligence and Humanized Computing*, **11**(2), pp. 663–674 (2020). DOI: 10.1007/s12652-019-01377-0
16. Senapati, T. and Yager, R.R. "Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods", *Engineering Applications of Artificial Intelligence*, **85**, pp. 112–121 (2019). DOI: 10.1016/J.ENGAPPAI.2019.05.012
17. Yager, R.R. "Generalized orthopair fuzzy sets", *IEEE Transactions on Fuzzy Systems*, **25**(5), pp. 1220–1230 (2017). DOI: 10.1109/TFUZZ.2016.2604005
18. Zhang, W.R. "Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis", *NAFIPS/IFIS/NASA'94, Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference*, The Industrial Fuzzy Control and Intelligence, San Antonio, TX, USA, pp. 305–309 (1994). DOI: 10.1109/IJCF.1994.375115
19. Zhang, W.R. "(Yin) (Yang) bipolar fuzzy sets", *IEEE International Conference on Fuzzy Systems Proceedings*, IEEE World Congress on Computational Intelligence (Cat. No.98CH36228), pp. 835–840 (1998). DOI: 10.1109/FUZZY.1998.687599
20. Jun, Y.B., Kim, C.S., and Yung, K.O. "Cubic Sets", *Annals of Fuzzy Mathematics and Informatics*, **4**(1), pp. 83–98 (2012). https://doi.org/10.1155/2015/523129
21. Riaz, M. and Tehrim, S.T. "Multi-attribute group decision making based on cubic bipolar fuzzy information using averaging aggregation operators", *Journal of Intelligent and Fuzzy Systems*, **37**(2), pp. 2473–2494 (2019). DOI: 10.3233/JIFS-182751
22. Yager, R.R. "On ordered weighted averaging aggregation operators in multicriteria decision-making", *IEEE Transactions on Systems, Man, and Cybernetics*, **18**(1), pp. 183–190 (1988). https://doi.org/10.1109/21.87068

23. Xu, Z.S. and Yager, R.R. "Some geometric aggregation operators based on intuitionistic fuzzy sets", *International Journal of General Systems*, **35**(4), pp. 417–433 (2006). DOI: 10.1080/03081070600574353

24. Xu, Z.S. "Intuitionistic fuzzy aggreagtion operators", *IEEE Transactions on Fuzzy Systems*, **15**(6), pp. 1179–1187 (2007). DOI: 10.1109/TFUZZ.2006.890678

25. Wang, W. and Liu, X. "Intuitionistic fuzzy information aggregation using Einstein operations", *IEEE Transactions on Fuzzy Systems*, **20**(5), pp. 923–938 (2012). DOI: 10.1109/tfuzz.2012.2189405

26. Wei, G., Alsaadi, F.E., Hayat, T., et al. "Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making", *International Journal of Fuzzy Systems*, **33**(6), pp. 1–12 (2017). DOI: 10.1007/s40815-017-0338-6

27. Kaur, G. and Garg, H. "Multi-attribute decision-making based on Bonferroni mean operators under cubic intuitionistic fuzzy set environment", *Entropy*, **20**(1), p. 65 (2018). DOI: 10.3390/e20010065

28. Shi, L. and Ye, J. "Dombi aggregation operators of neutrosophic cubic sets for multiple attribute decision-making", *Algorithms*, **11**(3), p. 29 (2018). DOI: 10.3390/a11030029

29. Akram, M., Dudek, W.A., and Dar, J.M. "Pythagorean Dombi fuzzy aggregation operators with application in multicriteria decision-making", *International Journal of Intelligent Systems*, **34**(11), pp. 3000–3019 (2019). https://doi.org/10.1002/int.22183

30. Khan, F., Khan, M.S.A., Shahzad, M., et al. "Pythagorean cubic fuzzy aggregation operators and their application to multi-criteria decision making problems", *Journal of Intelligent and Fuzzy Systems*, **36**(1), pp. 595–607 (2019). DOI: 10.3233/JIFS-18943

31. Riaz, M., Fareed, H.M.A., Shakeel, H.M., et al. "Innovative q-rung orthopair fuzzy prioritized aggregation operators based on priority degrees with application to sustainable energy planning: A case study of Gwadar", *AIMS Mathematics*, **6**(11) pp. 12795–12831 (2021). DOI: 10.3934/math.2021739

32. Riaz, M., Pamucar, D., Farid H.M.A., et al. "q-rung orthopair fuzzy prioritized aggregation operators and their application towards green supplier chain management", *Symmetry*, **12**(6), p. 976 (2020). https://doi.org/10.3390/sym12060976

33. Akram, M., Khan, A., Alcantud, J.C.R., et al. "A hybrid decision-making frame- work under complex spherical fuzzy prioritized weighted aggregation operators", *Expert Systems*, **38**(6), e12712 (2021). DOI: 10.1111/exsy.12712

34. Garg, H. and Rani, D. "New prioritized aggregation operators with priority degrees among priority orders for complex intuitionistic fuzzy information", *Journal of Ambient Intelligence and Humanized Computing*, **14**(3), pp. 1373–1399 (2021). DOI: 10.1007/s12652-021-03164-2

35. Riaz, M., Farid, H.M.A., Shakeel, H.M.M., et al. "Innovative q-rung orthopair fuzzy prioritized aggregation operators based on priority degrees with application to sustainable energy planning: A case study of Gwadar", *AIMS Mathematics*, **6**(11), pp. 12795- 12831 (2021). DOI: 10.3934/math.2021739

36. Farid, H.M.A., Riaz, M., Khan, M.J., et al. "Sustainable thermal power equipment supplier selection by Einstein prioritized linear Diophantine fuzzy aggregation operators", *AIMS Mathematics*, **7**, pp. 11201–11242 (2022). DOI: 10.3934/math.2022627

37. Farid, H.M.A., Garg, H., Riaz, M., et al. "Multi-criteria group decision-making algorithm based on single-valued neutrosophic Einstein prioritized aggregation operators and its applications", *Management Decision*, **61**(2), 382–420 (2022). https://doi.org/10.1108/MD-04-2022-0484

38. Farid, H.M.A. and Riaz, M. "Innovative q-rung orthopair fuzzy prioritized interactive aggregation operators to evaluate efficient autonomous vehicles for freight transportation", *Scientia Iranica*, pp. 1–24 (2022). DOI: 10.24200/SCI.2022.59601.6326

39. Akman, G. and Baynal, K. "Logistics service provider selection through an integrated fuzzy multicriteria decision making approach", *Journal of Industrial Engineering*, **2014**, pp. 1–26 (2014). https://doi.org/10.1155/2014/794918

40. Riaz, M. and Tehrim, S.T. "Cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data", *Computational and Applied Mathematics*, **38**(2), pp. 1–25 (2019).

41. Riaz, M., Habib, A., and Aslam, M. "Cubic bipolar fuzzy Dombi averaging aggregation opera- tors with application to multi-criteria decision-making", *Journal of Intelligent and Fuzzy Systems*, **41**(2), pp. 3373–3393 (2021). DOI: 10.3233/JIFS-210667

42. Jamil N. and Riaz, M. "Bipolar disorder diagnosis with cubic bipolar fuzzy informa- tion using TOPSIS and ELECTRE-I", *International Journal of Biomathematics*, (2022). https://doi.org/10.1142/S1793524522500309.

Biographies

Nimra Jamil received her BSc and MSc degrees in Mathematics from the University of the Punjab, Lahore in 2011 and 2013, respectively. She received M.Phil degree in Mathematics from the COM-SATS University Islamabad, Lahore Campus, in 2016. She is currently PhD scholar at Department of Mathematics, University of the Punjab, Lahore, Pakistan. She is the author of 09 SCI research papers and her research interests include bipolar fuzzy sets, cubic sets, cubic bipolar fuzzy sets, cubic m-polar fuzzy sets, multi-criteria decision-making problems, aggregation operators, information measures, information fusion, machine learning, artificial intelligence, and topological data analysis.

Muhammad Riaz has received MSc, M.Phil and PhD degrees in Mathematics from Department of Mathematics, University of the Punjab, Lahore. His research interests include pure mathematics, fuzzy mathematics, topology, algebra, fuzzy sets theory, soft set theory, rough set theory with applications in decision-making problems, medical diagnosis, artificial intelligence, computational intelligence, information measures, image processing, network topology and pattern recognition. He has 25 years regular teaching and research experience. He has published 180 research articles in international peer-reviewed SCIE and ESCI journals with more than 4000 citations. He has supervised 08 PhD students and 27 M.Phil students. Currently, he is supervising 04 M.Phil and 03 PhD students. He has been a reviewer for more than 40 SCI international journals. Professor Riaz is one of top 2% researchers included in the global list released by Stanford University in various disciplines in 2021, 2022, and 2023.