Scientia Iranica (2024) 31(1), 43–54

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions B: Mechanical Engineering
https://scientiairanica.sharif.edu

On numerical solutions of telegraph, viscous, and modified Burgers equations via Bernoulli collocation method

W. Adel[a,b], H. Rezazadeh[c], and M. Inc[d,e,*]

a. *Department of Mathematics and Engineering Physics, Faculty of Engineering, Mansoura University, 35511, Egypt.*
b. *Laboratoire Interdisciplinaire de l'Universite Francaise d'Egypte (UFEID Lab), Universite Francaise d'Egypte, Cairo 11837, Egypt.*
c. *Faculty of Modern Technologies Engineering, Amol University of Special Modern Technologies, Amol, 4615664616, Iran.*
d. *Department of Mathematics, Firat University, 23119, Elazig, Turkey.*
e. *Department of Medical Research, China Medical University, Taichung, Taiwan.*

Received 1 March 2022; received in revised form 30 October 2022; accepted 3 September 2023

KEYWORDS
Nonlinear telegraph;
Burger equations;
Bernoulli collocation;
Finite difference;
Iterative technique.

Abstract. The presented work aims to develop a novel technique for obtaining the solution of linear and nonlinear Partial Differential Equations (PDEs). This technique is based on applying a collocation method with the aid of Bernoulli polynomials and the use of such an algorithm to solve different types of PDEs. The method applies the regular finite difference scheme to the main problem and transforms it into an algebraic system. The obtained system is then solved, the unknown coefficient is acquired, and an approximate solution for the problems is achieved. Some test results of famous equations, including the telegraph, viscous Burger, and modified Burger equations, are tested to ensure that the provided algorithm is effective and robust. In addition, a comparison is provided with other recent techniques from the literature. The current technique proves to have high accuracy concerning the error measure and through graphical representation of the solution.

1. Introduction

Partial Differential Equations (PDEs) are at the heart of many, if not all, the applications and analysis or simulation of multiple physical systems with applications in real-life phenomena, including fluid, electrodynamics, and other related models. These equations have many applications in modeling the vibration of structures such as buildings and beams and are considered the backbone for atomic physics equations. During the last few years, an increasing interest in the development and analysis of the dynamics of PDEs has been noticed due to their particular interest. An unconditionally stable finite difference method has been used to solve the one-dimensional hyperbolic PDE [1]. Applications of PDE in the field of the electromagnetic waveguide are displayed in [2]. Forward and inverse problems in the form of nonlinear PDE are discussed in [3] with application in neural networks. Zang et al. [4] investigated a novel approach for solving high-dimensional PDE, which has a wide range of applications in science. Other models involving PDE can be found in [5,6] with different numerical and analytical methods for solving these types of problems.

In this study, we are concerned with studying a

*. *Corresponding author.*
E-mail addresses: waleedadel85@yahoo.com (W. Adel); h.rezazadeh@ausmt.ac.ir (H. Rezazadeh); minc@firat.edu.tr (M. Inc)

doi: 10.24200/sci.2023.60051.6569

general form of PDE in the following form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + (\alpha - 1)\,\varphi(x,t)\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \delta(x,t)\,\frac{\partial u}{\partial x} + \rho(x,t)\,u\frac{\partial u}{\partial x} + K(x,t)\,u^2 + \omega(x,t)\,u + \psi(x,t)\,, \tag{1}$$

with the boundary conditions:

$$u(a,t) = \eta_1(t)\,, \qquad u(b,t) = \eta_2(t)\,, \tag{2}$$

and with one of the following initial conditions:

$$u(x,0) = g_1(x)\,, \qquad u_t(x,0) = g_2(t)\,, \tag{3}$$

where $\alpha = 1$ or 2, and all of the parameters φ, δ, ρ, K, and ω are finite constants or continuous functions. The general form represented in Eq. (1) possesses a different form of known PDE, including the telegraph, Burger, and modified Burger equations by assigning different values of the parameters in these equations. Setting $\alpha = 2$, $\rho(x,t) = 0$, and $K(x,t) = 0$ into Eq. (1) will lead to a hyperbolic form of PDE known as the telegraph equation. On the other hand, setting $\alpha = 1$ into Eq. (1) will lead to either a form of nonlinear Burger or modified Burger equations having wide application in different domains.

Burger equation was first derived by Bateman [7] back in 1915 and was then used to simulate turbulence in fluid mechanics [8]. Also, this equation is a basic form of the famous Navier Stocks equation used for the modeling of gas dynamics [9] with the presence of the convection and viscosity terms [10]. With a variety of applications of the Burger equation, several mathematicians and researchers try different approaches, both numerical and analytical, to gain more knowledge of the dynamics and behavior of the model and its extended form known as the modified Burger equation. This modified form of the model has been used multiple times for simulating real-life problems such as pollutant transport and shock waves [11]. Due to the above-proposed application, many scientists strived to find different effective approaches for solving Burger and modified Burger equations. For example, Korkmaz and Dag [12] used a Sinc differential quadrature-based method for solving the one-dimensional BE. The quintic B-spline method has been utilized for solving the same problem by Korkmaz et al. in [13] numerically. Also, Korkmaz and Dag [14] applied the polynomial differential quadrature method for solving nonlinear Burger's equation. In addition, a fractional form of the Burger equation has been introduced and solved using several techniques. For example, Wang et al. [15] investigated the solution of a fractal coupled Boussinesq-Burger equation having an important application in simulating shallow-water waves. Also, the multidimensional fractional Burger equation has been studied by Usman et al. [16] using a new finite difference method with a new definition of the differentiation matrix. Hashmi et al. [17] derived some accurate solutions using the B-spline method for simulating the fractional Burger equation. A Homotopy perturbation approach has been proposed by Ahmad et al. in [18] to solve the fractional Burger and KdV Burger equations. A local discontinuous Galerkin technique has been utilized for obtaining solutions for the modified form of the Burger equation by Zhang et al. in [19]. Bratsos and Abdul [20] adapted an exponential time differencing scheme for simulating the Burger and modified the Burger equation. Seydaoğlu [21] employed an algorithm based on the combination of implicit-explicit finite difference schemes for solving the Burger equation. Mohamed [22] investigated the solution of the Burger equation through a fully implicit scheme. Arora and Varun in [23] examined the multi-dimensional Burger equation using the extended B-spline method. Aswin et al. [24] applied the differential quadrature method in obtaining a highly accurate solution to the Buerger equations successfully. A reproducing kernel approach is introduced in [25] by Du et al. to solve Burger's equation while emphasizing its application in diffusive waves in fluid dynamics. Finally, a high-order implicit weighted compact nonlinear scheme is illustrated in [26] to solve the coupled viscous Burger equation.

In this paper, we are concerned with applying a novel and effective collocation method using Bernoulli bases. The use of such a method, in general, using different forms of basis functions has been an efficient numerical tool for solving different forms of application problems. The ability to provide accurate approximate solutions for these types of models was the motivation for scientists to adapt this technique using different types of bases. Chebyshev, Jacobi, Legendre, Bessel, Genouci, and Bernoulli bases are only a few of these types of bases that have been used in combination with the collocation technique for solving models of different behaviors. Bernoulli polynomials, for instance, are one of the bases that have been used extensively in recent years for solving different problems with high accuracy, and this is what motivates us to adapt these polynomials. The main advantage of using these types of bases is that they are simple to represent and easy to use. In addition, the use of a fewer number of Bernoulli bases will guarantee efficient results for complex models. Many researchers applied different techniques for simulating real-life problems in different areas. For example, El-Gamel et al. [27] adopted a new technique for solving the Bratu equation. Also, one-dimensional heat and wave equations were simulated using a similar approach [28]. In addition, Adel and Sabir [29] utilized a collocation technique for solving the general form of the Lane-Emden type equation with delay. Finally, Bazm and Hosseini [30] employed a new approach to solving the two-dimensional integral equation. The operational matrix of Bernoulli polynomials has been studied in [31] by Zeghdane and then used to solve the stochastic integral equations. Also, Adel et al. [32,33] employed the collocation method for solving the Emden Fowler and MHD Jeffery-Hamel flow problems. All of these attempts prove that this method is effective with high accuracy.

The novelty of the proposed technique is indicated through the main steps for solving the presented models. The new method is based on discretizing the time domain using the usual finite difference method, and the use of Bernoulli functions as a basis for the spatial variable.

The new method possesses some lead while simulating such problems in terms of simplicity and robustness compared to other similar techniques. This can be witnessed through the use of fewer bases, which is accompanied by high accuracy and less computational cost. It is worth mentioning that this is the first time that he presented techniques used for solving such models. The paper has multiple novelty sides that can be summarized in the following points:

1. The new method is designed based on the use of Bernoulli polynomials as bases for solving the described model;
2. The method is tested for both linear and nonlinear cases for various models with great importance and impact;
3. The method shows good approximation for the solution of the problems and can be extended to a more complex problem.

The organization of the paper is: in Section 2, we present some preliminaries of the Bernoulli polynomials. Section 3 provides the main steps for the proposed technique based on matrix relations. Section 4 provides the steps for solving Eq. (1) based on the differentiation matrices of Bernoulli polynomials. In Section 5, numerical simulations are dissipated through tables and figures and with several examples of linear and nonlinear type problems. The last stage, Section 6, summarized the conclusion of the study and some extended future work.

2. Basic definitions

We shall provide all the main relations regarding the used Bernoulli basis. We first present the main relations for the Bernoulli matrix and the application of these matrix relations for solving the given problem. Bernoulli polynomials gain increasing interest in solving different types of equations due to their simplicity and the ability to provide accurate solutions.We defined Bernoulli polynomials according to the following relation [27–29]:

$$\frac{ze^{xz}}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n. \tag{4}$$

The derivative of such polynomials can be in the form

$$\frac{d}{dx}B_n(x)=\sum_{n=0}^{\infty}\binom{n}{k}(n-k)B_kx^{n-k-1}$$
$$=nB_{n-1}(x). \tag{5}$$

To explain the use of our novel technique, consider the approximate solution to the main problem in the form of a Bernoulli series:

$$u(x)\cong u_N(x)=\sum_{n=0}^{N}a_nB_n(x), \tag{6}$$

where $n=0,1,2,\cdots,N$ are the unknowns. Next, we shall illustrate the main relations and the applications to Eq. (1).

3. Matrix relation and solution procedure

Here, we will introduce a form for Bernoulli's differentiation matrices and the principle procedure. First, the matrix form of Eq. (6) is in the form:

$$u(x)\cong u_N(x)=B(x)A, \quad A=[a_0,a_1,...,a_N]^T. \tag{7}$$

Also, with the aid of Eq. (5) for $n=0,1,2,\cdots,N$, the kth derivative of $B(x)$ is in the form:

$$B^{(k)}(x)=B(x)(\Theta)^k, \qquad k=0,1,\dots,m, \tag{8}$$

where Θ is the matrix of differentiation using Bernoulli polynomials and takes the form:

$$\Theta=\begin{bmatrix}0&1&0&0&\cdots&0\\0&0&2&0&\cdots&0\\0&0&0&3&\cdots&0\\ & & & &\ddots& \\\cdots&\cdots&\cdots&\cdots& &\cdots\\0&0&0&0&0&N\\0&0&0&0&0&0\end{bmatrix}.$$

Then, we acquire the matrix form for $u^{(k)}(x)$ with the help of Eqs. (7) and (8) as:

$$u^{(k)}(x)\cong u_N^{(k)}(x)=B(x)(\Theta)^kA. \tag{9}$$

Next, we need to obtain the matrix form of the nonlinear part in Eq. (1) through the following theorem:

Theorem 1. The multiple nonlinear terms in Eq. (1) in the form $u(x_j)u^{(1)}(x_j)$,$u^{\nu}(x_j)u^{(1)}(x_j)$,$j=0,1,2,\dots$ can be approximated using the following form:

$$u(x_j)\,u^{(1)}(x_j) = B(x)\,\Theta(\tilde{B}(x)\tilde{A}),$$

$$u^{\nu}(x_j)u^{(1)}(x_j) = B(x)\Theta^{\nu}\left(\tilde{B}(x)\,\tilde{\Theta}\tilde{A}\right), \tag{10}$$

where:

$$\tilde{B} = \begin{pmatrix} B(x_0) & 0 & \cdots & 0 \\ 0 & B(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B(x_N) \end{pmatrix}_{(N+1)\times(N+1)^2}$$

$$\tilde{A} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}_{(N+1)\times(N+1)}$$

$$\tilde{\Theta} = \begin{pmatrix} \Theta & 0 & \cdots & 0 \\ 0 & \Theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Theta \end{pmatrix}_{(N+1)^2\times(N+1)^2}$$

$$B = \begin{pmatrix} B_0(x_0) & B_1(x_0) & \dots & B_N(x_0) \\ B_0(x_1) & B_1(x_1) & \dots & B_N(x_1) \\ B_0(x_2) & B_1(x_2) & \dots & B_N(x_2) \\ \vdots & \vdots & \vdots & \vdots \\ B_0(x_N) & B_1(x_N) & \dots & B_N(x_N) \end{pmatrix}.$$

It should be noted that other nonlinear terms in the form of exponential, for example, can be approximated with the same previous method after expansion in the Taylor series expansion. Next, we will define the basic steps for solving Eq. (1) with the proposed algorithm and with the aid of Eq. (3).

4. Proposed collocation method

In this section, we will illustrate the technique used for finding the solution of Eq. (1). We need first to apply an extension of the regular finite difference scheme where $u_j = u(j\Delta t)$ and we reach:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{(\alpha-1)\,u_{i+1} - \alpha u_i + u_{i-1}}{(-\Delta t)^{\alpha}}. \tag{11}$$

By substituting the finite difference scheme represented in Eq. (11) into the main equation, Eq. (1) may be reduced to the following form:

$$\frac{(\alpha-1)\,u_{i+1} - \alpha u_i + u_{i-1}}{(-\Delta t)^{\alpha}} + (\alpha-1)\,\varphi(x,t_{i+1})\,\frac{u_i - u_{i-1}}{\Delta t} =$$

$$u''_{i+1} + \delta(x,t_{i+1})\,u'_{i+1} + \rho(x,t_{i+1})\,u_{i+1}u'_{i+1}$$

$$+ K(x,t_{i+1})\,u^2_{i+1} + \omega(x,t_{i+1})\,u_{i+1} + \psi(x,t_{i+1}). \tag{12}$$

After some simplifications for Eq. (12), we reach the following form:

$$\frac{d^2 u_{i+1}}{dx^2} + \delta(x,t_{i+1})\,\frac{du_{i+1}}{dx} + \rho(x,t_{i+1})\,u_{i+1}\,\frac{du_{i+1}}{dx}$$

$$+ K(x,t_{i+1})\,u^2_{i+1} + \left[\omega(x,t_{i+1}) - \frac{(\alpha-1)}{(-\Delta t)^{\alpha}}\right] u_{i+1}$$

$$= \zeta(x,t_{i+1}), \tag{13}$$

where:

$$\zeta(x,t_{i+1}) = \psi(x,t_{i+1})$$

$$+ \left[\frac{(\alpha-1)}{\Delta t}\varphi(x,t_{i+1}) - \frac{\alpha}{(-\Delta t)^{\alpha}}\right] u_i$$

$$+ \left[\frac{1}{(-\Delta t)^{\alpha}} - \frac{(\alpha-1)}{\Delta t}\varphi(x,t_{i+1})\right] u_{i-1}. \tag{14}$$

After applying the above simplification, we reach the next theorem.

Theorem 2. If the solution to the model (1) is through the Eqs. (13) and (14), then the discrete Bernoulli series can be in the form:

$$u''(x_k,t_{i+1}) + \delta(x_k,t_{i+1})\,u'(x_k,t_{i+1})$$

$$+ \rho(x_k,t_{i+1})\,u(x_k,t_{i+1})\,u'(x_k,t_{i+1})$$

$$+ K(x_k,t_{i+1})\,u^2(x_k,t_{i+1})$$

$$+ \left[\omega(x_k,t_{i+1}) - \frac{(\alpha-1)}{(-\Delta t)^{\alpha}}\right]$$

$$u(x_k,t_{i+1}) = \zeta(x_k,t_{i+1}), \tag{15}$$

where $x_k, k = 0,1,2,\ldots,N$ is the used equal collocation method with the following form:

$$x_k = \frac{k}{N}. \tag{16}$$

The fundamental matrix representation of Eq. (15) can be in the following form:

$$\Psi A = \zeta, \tag{17}$$

where:

$$\Psi = B\Theta^2 + \delta B\Theta + \rho\left(\tilde{B}\tilde{A}\right)B\Theta + K\left(\tilde{B}\tilde{A}\right)B + JB.$$

Each term in the above equation is in the form by equation is shown in Box I. The matrix form of the boundary condition represented in Eq. (2) can take the form:

$$B(a)\,A = \eta_1(t_{i+1}), \quad B(b)\,A = \eta_2(t_{i+1}), \tag{18}$$

and can be written in the form:

$$\kappa A = \eta_1(t_{i+1}), \qquad \lambda A = \eta_1(t_{i+1}). \tag{19}$$

Replacing the first row of the augmented matrix $[\Psi,\zeta]$

$$\delta = \begin{pmatrix} \delta(x_0, t_{i+1}) & 0 & \cdots & 0 \\ 0 & \delta(x_1, t_{i+1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta(x_N, t_{i+1}) \end{pmatrix}, \qquad \zeta = \begin{pmatrix} \zeta(x_0, t_{i+1}) \\ \zeta(x_1, t_{i+1}) \\ \vdots \\ \zeta(x_N, t_{i+1}) \end{pmatrix},$$

$$\rho = \begin{pmatrix} \rho(x_0, t_{i+1}) & 0 & \cdots & 0 \\ 0 & \rho(x_1, t_{i+1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho(x_N, t_{i+1}) \end{pmatrix},$$

$$K = \begin{pmatrix} K(x_0, t_{i+1}) & 0 & \cdots & 0 \\ 0 & K(x_1, t_{i+1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K(x_N, t_{i+1}) \end{pmatrix},$$

$$J = \begin{pmatrix} \omega(x_0, t_{i+1}) - \frac{(\alpha-1)}{(-\Delta t)^\alpha} & 0 & \cdots & 0 \\ 0 & \omega(x_1, t_{i+1}) - \frac{(\alpha-1)}{(-\Delta t)^\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega(x_N, t_{i+1}) - \frac{(\alpha-1)}{(-\Delta t)^\alpha} \end{pmatrix}.$$

Box I

- Insert the value of integer N.
- Insert (tol).
- İnsert an array ($a_{old} = a_0$) with an initial guess a_0 ($N+1$ dimension).
- $\tilde{\boldsymbol{\Psi}}(a_{old})a_{new} = \tilde{\boldsymbol{\zeta}}$, is solved, and then a_{new} is found.
- If $|a_{old} - a_{new}| < tol$ then $a_{new} = a$, (the algorithm stops).
- Else $a_{old} \leftarrow a_{new}$.

Algorithm 1. Solving the nonlinear system $\left[\tilde{\Psi}; \tilde{\zeta}\right]$.

by $[\kappa, \eta_1(t_i)]$ and the last row with $[\lambda, \eta_2(t_i)]$, a new augmented matrix can be acquired having the following form:

$$\tilde{\Psi} A = \tilde{\zeta}. \tag{20}$$

This new augmented matrix will result in a nonlinear system of an algebraic equation. A novel iterative algorithm that will be used is Algorithm 1.

5. Numerical simulation

Here, the results of the proposed method are presented to prove the effectiveness of the proposed method. We solve different forms of linear and nonlinear PDEs named the telegraph, Burger, and modified Burger equations in different forms. The results are obtained using Matlab 2015. The absolute error is calculated according to the following equation:

$$e_N(x_i) = |u_{Approximate}(x_i) - u_{Exact}(x_i)|,$$

$$i = 0, 1, 2, \ldots, N.$$

Problem 1. In this example, consider the Telegraph equation by assigning $\alpha = 2, \varphi(x,t) = \omega(x,t) = 1$ and $\delta(x,t) = \rho(x,t) = 0$ which gives:

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u - \frac{\partial^2 u}{\partial x^2} = x^2 + t - 1,$$

$$0 < x < 1, \qquad 0 < t < 1. \tag{21}$$

With the initial and boundary conditions:

$$u(x,0) = x^2, \qquad u_t(x,0) = 0,$$

$$u(0,t) = t, \qquad u(1,t) = 1 + t.$$

Whose exact solution is:

$$u(x,t) = x^2 + t.$$

Table 1 provides the error measure of the proposed technique for $t = 1.0$ and $t = 10$ with different step sizes. Figure 1 graphs the approximate solution for the problem for $x \in [0,1]$ and $t \in [0,1]$. As can be seen from the table and figure, the method gives almost the exact solution to the problem with high accuracy.

Table 1. Absolute error for Problem 1 for different values of t.

x_i	$t = 1.0$, $\Delta t = 0.1$	$t = 10$, $\Delta t = 1$
0	2.35E-14	1.85E-13
0.1	1.98E-14	1.49E-13
0.2	4.89E-15	5.86E-14
0.3	1.11E-14	5.68E-14
0.4	1.31E-14	1.49E-13
0.5	2.93E-14	7.11E-14
0.6	2.46E-14	1.49E-13
0.7	1.021E-14	5.68E-14
0.8	2.91E-14	5.68E-14
0.9	3.98E-14	1.49E-13
1.0	2.35E-14	7.28E-14

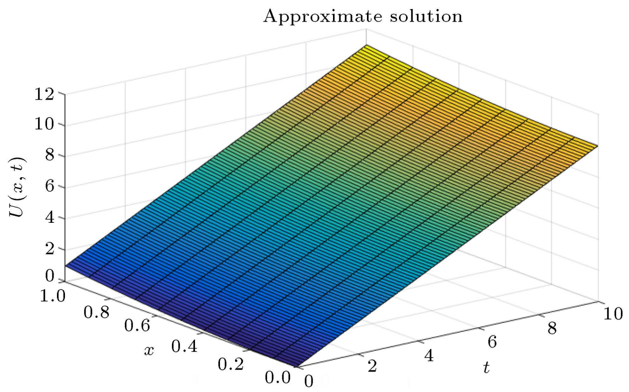

Figure 1. Solution behavior for t and $\nu = 1$ for Problem 1.

Problem 2. Second, we present the first form of the viscous Burger equation while assigning $\alpha = 2$ and $\rho(x,t) = 1$ in the form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\nu}{2}\frac{\partial^2 u}{\partial x^2} = 0, \qquad x \in (0,1). \tag{22}$$

With the initial and boundary conditions:

$$u(x,0) = \sin(x), \qquad u(0,t) = u(1,t) = 0,$$

with the analytic solution in the form:

$$u(x,t) = \frac{2\pi\nu\sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right) n \sin(n\pi x)}{C_0 + \sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2\nu t\right) n \cos(n\pi x)},$$

where:

$$C_0 = \int_0^1 \exp\left(-\frac{1}{2\pi\nu}\left[1-\cos(\pi x)\right]\right) dx,$$

$$C_n \int_0^1 \exp\left(-\frac{1}{2\pi\nu}\left[1-\cos(\pi x)\right]\right) \cos(n\pi x)\, dx.$$

Table 2 presents the solution and error measure with the value $\nu = 1$, $t = 0.1$ among Table 3. which gives the same at $\nu = 10$, $t = 0.01$. A comparison between the approximate solution of our method and the finite difference method in [31] with the absolute error for both is illustrated in Table 4 for $\nu = 10$ with several time and space values. Our method produces more accurate results compared to the method presented in [31]. Figures 2–4 gives the approximate solution for

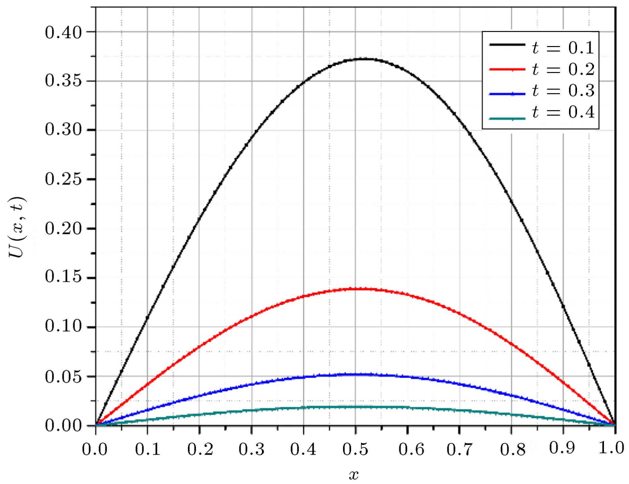

Figure 2. Solution profile at t and $\nu = 1$ for Problem 2.

Table 2. Solutions profile and error for Problem 2 for $\nu = 1$, $t = 0.1$.

x_i	u_{Exact}	$u_{Approximate}$	e_N
0.1	0.109538151270508	0.109590183005071	5.20E-05
0.2	0.209792148910037	0.209893662453422	1.02E-04
0.3	0.291896350825530	0.292038301042883	1.42E-04
0.4	0.347923912365550	0.348093234391569	1.69E-04
0.5	0.371577476146793	0.371758017544956	1.81E-04
0.6	0.359045579984961	0.359219480845147	1.74E-04
0.7	0.309905000631104	0.310054385169234	1.49E-04
0.8	0.227817406627376	0.227926338549254	1.09E-04
0.9	0.120686691089409	0.120743056037676	5.64E-05

Table 3. Solutions profile and error for Problem 2 for $\nu = 10$, $t = 0.01$.

x_i	u_{Exact}	$u_{Approximate}$	e_N
0.1	0.114612984191918	0.115169956659254	5.60E-04
0.2	0.218164263714681	0.219223949428971	1.06E-03
0.3	0.300615471737520	0.302075046467647	1.46E-03
0.4	0.353896911467088	0.355614403091422	1.71E-03
0.5	0.372696489356191	0.374504343571767	1.80E-03
0.6	0.355015826846423	0.356737085848952	1.72E-03
0.7	0.302425928419229	0.303891544645087	1.46E-03
0.8	0.219974737166914	0.221040334833527	1.07E-03
0.9	0.115731926706530	0.116292271087520	5.62E-04

Table 4. Comparison of solutions and error for Problem 2 for $\nu = 0.02$ and different values of t.

t_i	x_i	$u_{Approximate}$	e_N	Finite difference method [31]	Absolute error
$t = 0.4$	0.25	0.3411118	8.03E-04	0.34267	7.60E-04
	0.5	0.6609855	2.75E-04	0.67588	1.52E-02
	0.75	0.9052738	4.99E-03	0.95424	4.40E-02
$t = 0.6$	0.25	0.269281	3.16E-04	0.26908	1.20E-04
	0.5	0.530127	7.09E-04	0.53678	7.36E-03
	0.75	0.765658	1.59E-03	0.79252	2.53E-02
$t = 1.0$	0.25	0.188138	5.60E-05	0.18806	1.30E-04
	0.5	0.37507	6.50E-04	0.37671	2.29E-03
	0.75	0.55596	9.07E-05	0.56535	9.30E-03

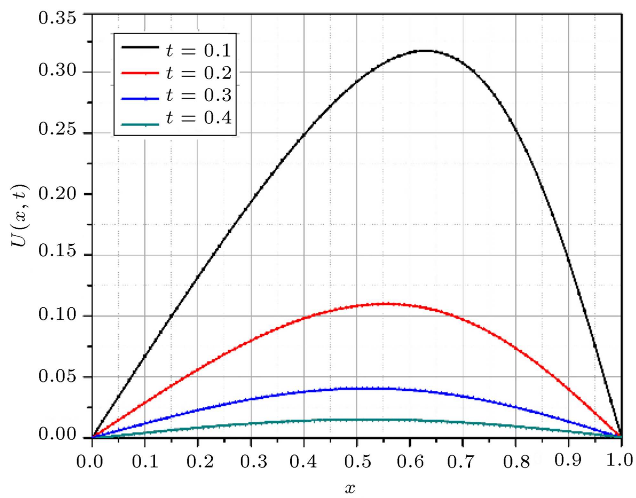

Figure 3. Solution profile at t and $\nu = 0.1$ for Problem 2.

different values of ν. From these figures, the methods provide accurate results, which also prove the physical behavior of the solution.

Problem 3. Next, we present another form of viscous Burger equation represented in Eq. (22) with another form of initial conditions and boundary conditions in the form:

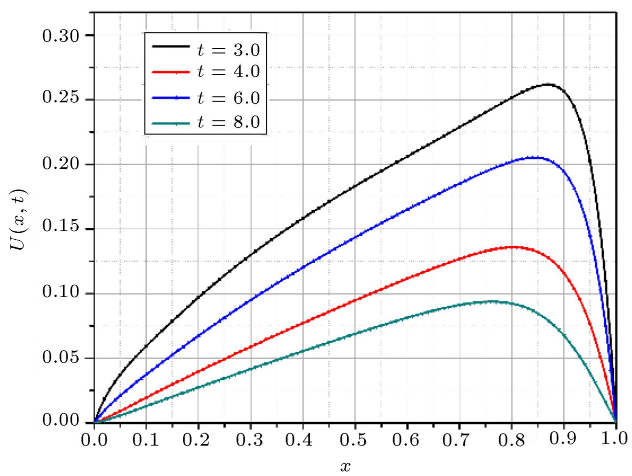

Figure 4. An approximate solution for $t = 3, 4, 6, 8$ and $\nu = 0.01$ for Problem 2.

$$u(x, 0) = 4x(1 - x),$$

with the conditions:

$$u(0, t) = u(1, t) = 0.$$

Results are dissipated in Tables 5 and 6 of the solutions of the model with $\nu = 10$ and a comparison with the finite difference method in [34].

Table 5. Approximate solution and absolute error for $\nu = 10$ and different values of t for Problem 3.

x_i	u_{Exact}	$u_{\text{Approximate}}$	e_N
0.1	0.112892245268291	0.112945548267831	5.55E-05
0.2	0.216252142417222	0.216356779291488	1.07E-04
0.3	0.300965859903397	0.3011128362621s22	1.43E-04
0.4	0.358863061468998	0.359039247340337	1.79E-04
0.5	0.383422416438965	0.383611374946737	1.91E-04
0.6	0.370657835501244	0.370841069190758	1.81E-04
0.7	0.320065690908233	0.320224221869215	1.54E-04
0.8	0.235371149338849	0.235487545002479	1.18E-04
0.9	0.124718046630702	0.124778661126234	5.87E-05

Table 6. Solution and error profiles for Problem 3 for $\nu = 2$, $t = 0.1$.

x_i/t_i	x_i	$u_{Approximate}$	Finite difference method [31]	e_N
	0.25	0.317522880346768	0.31735	6.32E-04
$t = 0.4$	0.5	0.584537259423137	0.58441	2.47E-05
	0.75	0.645615507508048	0.6457	2.70E-03
	0.25	0.246138455741545	0.24603	1.85E-05
$t = 0.6$	0.5	0.457976404556937	0.45786	1.92E-04
	0.75	0.502675751374800	0.50265	3.66E-03
	0.25	0.165598631696975	0.16554	3.06E-05
$t = 1.0$	0.5	0.298343106946419	0.29826	1.29E-05
	0.75	0.295856684503934	0.2958	7.37E-04

Problem 4. Next, we shall illustrate the proposed algorithm for solving the same Eq. (22) with different conditions:

$$u\left(x,0\right)=\frac{2\nu\pi\sin\left(\pi x\right)}{\alpha+\cos\left(\pi x\right)},$$

$$u(0,t)=u(1,t)=0,$$

with the analytic solution in the form:

$$u\left(x,t\right)=\frac{2\nu\pi e^{-\pi^2\nu t}\sin\left(\pi x\right)}{\alpha+e^{-\pi^2\nu t}\cos\left(\pi x\right)}.$$

Table 7 provides a comparison between our method and the differential quadrature method in [35] at $N = 40$, and Table 8 illustrates the compassion of the maximum absolute error for the same problem at $t = 1.0$. Our method proves to acquire effective results and is more accurate compared to the method in [35]. Figures 5 and 6 give the approximate solution for the problem with different t and ν.

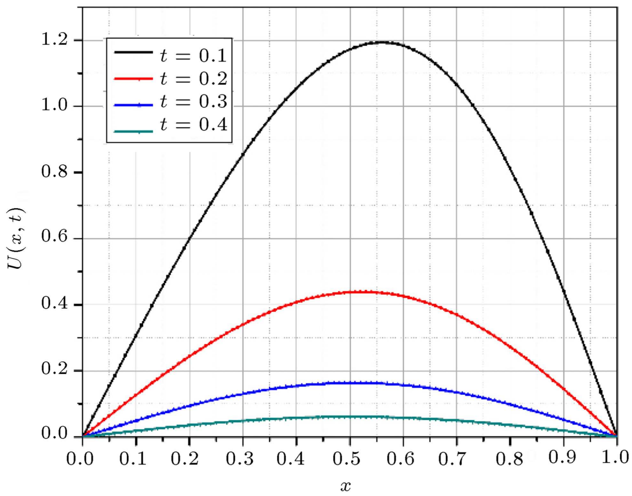

Figure 5. Solution profile for different values of t and $\nu = 1$ for Problem 4.

Problem 5. In our last example, we propose the modified Burger equation in the following form with initial and boundary conditions:

Table 7. Comparison of approximate solution and absolute error for $\nu = 0.01$ and different values of t for Problem 4.

x_i/t_i	x_i	Approximate solution	Differential quadrature [35]	Absolute error
$t = 0.5$	0.25	0.015798965412203	0.01581	1.74E-06
	0.5	0.029903285191579	0.02991	4.16E-08
	0.75	0.031883379375741	0.03197	1.47E-06
$t = 1.0$	0.25	0.015221504528403	0.01523	7.27E-06
	0.5	0.028463574864973	0.02848	1.78E-07
	0.75	0.028890546098028	0.02977	6.03E-06
$t = 2.0$	0.25	0.014109059269918	0.01411	1.52Ee-05
	0.5	0.025789606632468	0.02584	1.26E-06
	0.75	0.025086608119237	0.02590	8.39E-06
$t = 3.0$	0.25	0.013051241736336	0.01305	1.66E-05
	0.5	0.023368093511267	0.02343	3.38E-06
	0.75	0.021913102090887	0.02263	3.15E-06

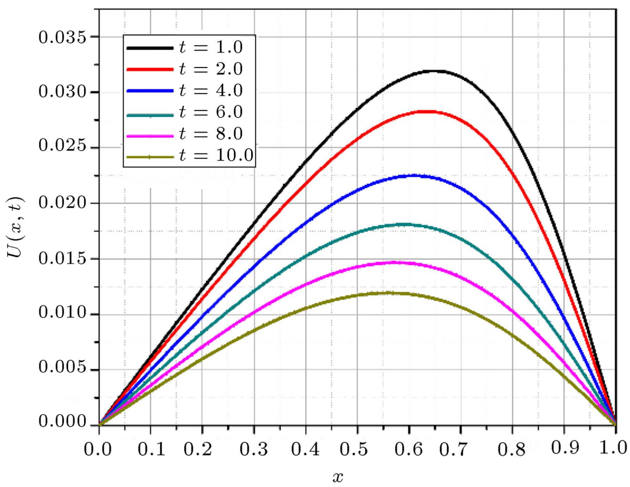

Figure 6. Solution profile for different values of t and $\nu = 0.1$ for Problem 4.

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in (0,1) \times t \geq t_0, \quad (23)$$

with boundary conditions:

$$u(a,t) = \beta_1, \qquad u(b,t) = \beta_2,$$

and exact solution in the form:

$$u(x,t) = \frac{x/t}{1 + \sqrt{t/c_0}\, \exp\left(x^2/4\nu t\right)}.$$

Tables 9 and 10 give the absolute error for the problem with $\nu = 0.01$, 0.001, and at different times. Also, Figures 7 and 8 give the approximate solution for the same values of ν with different t values.

Table 8. Comparison of maximum absolute error for Problem 4.

ν	Maximum absolute error, $t = 1.0$	Bernstein DQM [35]
1.0	8.2977E-06	9.6E-06
0.1	1.4535E-04	1.5E-03
0.01	3.6471E-05	1.6E-04
0.001	7.03406E-07	3.3E-06
0.0001	7.015412e-09	2.7E-08
0.00001	6.981044e-11	5.5E-10

Table 9. Absolute error for Problem 5 at different values of t and $\nu = 0.01$.

x/t	2.0	4.0	6.0	8.0	10
0	1.38E-14	1.47E-14	1.40E-14	1.40E-14	1.30E-14
0.1	4.20E-04	1.57E-04	9.63E-06	5.82E-05	8.03E-05
0.2	3.44E-04	2.76E-06	7.13E-05	1.09E-04	1.42E-04
0.3	8.99E-04	3.74E-04	2.40E-04	2.21E-04	2.48E-04
0.4	6.93E-04	6.38E-04	4.25E-04	3.68E-04	3.97E-04
0.5	2.68E-04	6.26E-04	5.38E-04	5.23E-04	5.84E-04
0.6	6.61E-05	4.44E-04	5.66E-04	6.77E-04	8.10E-04
0.7	3.92E-05	2.70E-04	5.85E-04	8.67E-04	1.09E-03
0.8	7.04E-05	2.21E-04	7.18E-04	1.16E-03	1.44E-03
0.9	2.57E-06	3.51E-04	1.03E-03	1.55E-03	1.83E-03
1	6.59E-07	1.21E-04	5.26E-04	9.63E-04	1.28E-03

Table 10. Absolute error for Problem 5 at different values of t and $\nu = 0.001$.

x/t	2.0	4.0	6.0	8.0	10
0	4.97E-14	4.97E-14	4.97E-14	4.97E-14	4.97E-14
0.1	4.95E-04	4.95E-04	4.95E-04	4.95E-04	4.95E-04
0.2	1.42E-04	1.42E-04	1.42E-04	1.42E-04	1.42E-04
0.3	1.43E-05	1.43E-05	1.43E-05	1.43E-05	1.43E-05
0.4	2.89E-06	2.89E-06	2.89E-06	2.89E-06	2.89E-06
0.5	5.95E-07	5.95E-07	5.95E-07	5.95E-07	5.95E-07
0.6	5.32E-07	5.32E-07	5.32E-07	5.32E-07	5.32E-07
0.7	2.75E-06	2.75E-06	2.75E-06	2.75E-06	2.75E-06
0.8	1.77E-05	1.77E-05	1.77E-05	1.77E-05	1.77E-05
0.9	9.71E-07	9.71E-07	9.71E-07	9.71E-07	9.71E-07
1	5.40E-17	5.40E-17	5.40E-17	5.40E-17	5.40E-17

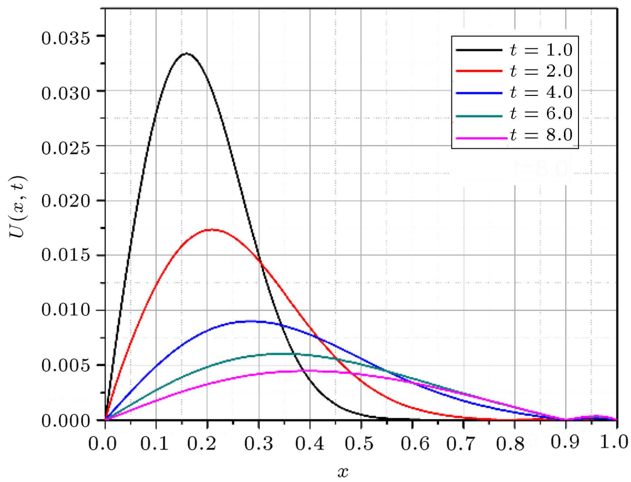

Figure 7. Solution profile for different values of t and $\nu = 0.01$ for Problem 5.

6. Conclusion

In this research work, a novel technique based on the use of Bernoulli bases is introduced for simulating a general form of a partial differential equation. The method utilizes a general form of the finite difference formula to discretize the time domain and the Bernoulli collocation approach for the spatial domain. Thus, the resulting system is worked out for finding the unknown coefficients and, hence, the approximate solution. The use of the Bernoulli collocation approach, along with the novel iterative approach, proved to achieve highly accurate results. Five test problems are solved using this innovative technique of Burger, telegraph, and viscous Burger-type models to test the effectiveness of the new approach. These models have great significance in different domains. The results presented through tables and graphs indicate that the method provides good results and through a comparison made with other methods. In addition, using a small number of bases is one of the advantages of the proposed technique. This may allow us to solve other complex models having potential applications in different areas of science using the proposed technique. Thus, we are interested to see how the method can handle more complex problem geometry.

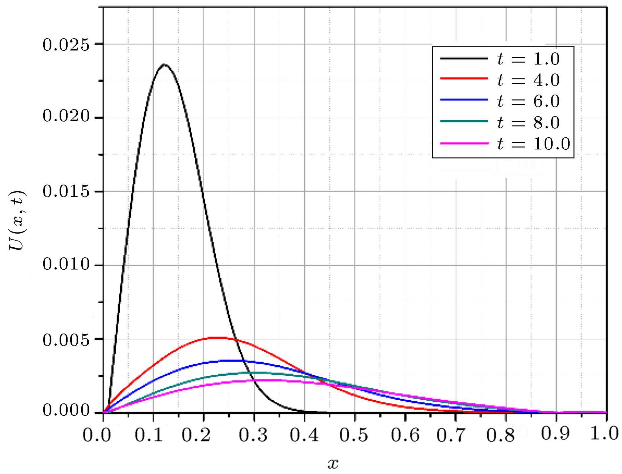

Figure 8. Solution profile for different values of t and $\nu = 0.001$ for Problem 5.

Acknowledgment

The authors would like to thank the anonymous Reviewers and Editor for providing helpful comments and suggestions which further improved this work.

Data availability

The data that support the findings of this study are included in this article.

Declaration of interest

The authors declare that they have no conflict of interest.

Nomenclature

$\psi(x,t)$	Source term
$B_n(x)$	Bernoulli polynomials
a_n	Unknown coefficients
$u_N(x)$	Approximate solution
Θ	Bernoulli differentation matrix
Δt	Temporal step size
x_k	Equal collocation points
$[\Psi, \zeta]$	Augmented matrix form

$\left[\tilde{\Psi};\tilde{\zeta}\right]$ New augmented matrix

$e_N\left(x_i\right)$ Absolute error

References

1. Gao, F. and Chi, C. "Unconditionally stable difference schemes for a one-space-dimensional linear hyperbolic equation", *Appl. Math. Comput.*, **187**, pp. 1272–1276 (2007).
2. Rabczuk, T., Huilong, R., and Xiaoying, Z. "A nonlocal operator method for partial differential equations with application to electromagnetic waveguide problem", *Comput. Mater. Contin.*, **59**, pp. 31–55 (2019).
3. Raissi, M., Paris, P., and George, E.K. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations", *J. Comput. Phys.*, **378**, pp. 686–707 (2019).
4. Zang, Y., Gang, B., Xiaojing, Y., et al. "Weak adversarial networks for high-dimensional partial differential equations", *J. Comput. Phys.*, **411**, 109409 (2020).
5. Alesemi, M., Iqbal, N., and Botmart, T. "Novel analysis of the fractional-order system of non-linear partial differential equations with the exponential-decay kernel", *Mathematics*, **10**(4), p. 615 (2022).
6. Rawani, M.K., Lajja, V., Verma, A.K., et al. "On a weakly L-stable time integration formula coupled with nonstandard finite difference scheme with application to nonlinear parabolic partial differential equations", *Math. Methods Appl. Sci.*, **45**(3), pp. 1276–1298 (2022).
7. Bateman, H. "Some recent researches on the motion of fluids", *Mon. Weather Rev.*, **43**(4), pp. 163–170 (1915).
8. Burgers, J.M. "A mathematical model illustrating the theory of turbulence", *Adv. Appl. Mech.*, **1**, pp. 171–199 (1948).
9. Cole, J.D. "On a quasi-linear parabolic equation occurring in aerodynamics", *Quart. Appl. Math.*, **9**(3), pp. 225–236 (1951).
10. Tari, H., Ganji, D.D., and Babazadeh, H. "The application of He's variational iteration method to nonlinear equations arising in heat transfer", *Phys. Lett. A.*, **363**(7), pp. 213–217 (2007).
11. Bratsos, A.G. "A fourth-order numerical scheme for solving the modified Burgers equation", *Comput. Math. Appl.*, **60**(5), pp. 1393–1400 (2010).
12. Korkmaz, A. and Dag, I. "Shock wave simulations using sinc differential quadrature method", *Eng. Comput.*, **28**(6), pp. 654–674 (2011).
13. Korkmaz, A., Aksoy, M., and Dag, I. "Quartic B-spline differential quadrature method", *Int. J. Nonlinear Sci.*, **11**(4), pp. 403–411 (2011).
14. Korkmaz, A. and Dag, I. "Polynomial based differential quadrature method for numerical solution of nonlinear Burgers' equation", *J. Franklin Inst.*, **348**(10), pp. 2863–2875 (2011).
15. Wang, K., Wang, G.D., and Zhu, H.W. "A new perspective on the study of the fractal coupled Boussinesq-Burger equation in shallow water", *Fractals*, **29**(5), 2150122 (2021).
16. Usman, M., Hamid, M., and Liu, M. "Novel operational matrices-based finite difference/spectral algorithm for a class of time-fractional Burger equation in multidimensions", *Chaos Solitons Fractals*, **144**, 110701 (2021).
17. Hashmi, M.S., Misbah, W., Yao, S.W., et al. "Cubic spline based differential quadrature method: A numerical approach for fractional Burger equation", *Results Phys.*, **26**, 104415 (2021).
18. Ahmad, S., Ullah, A., Akgül, A., et al. "A novel homotopy perturbation method with applications to nonlinear fractional order KdV and Burger equation with exponential-decay kernel", *J. Funct. Spaces*, **2021**, 8770488 (2021).
19. Zhang, R.P., Yu, X.J., and Zhao, G.Z. "Modified Burgers' equation by the local discontinuous Galerkin method", *Chin. Phys. B.*, **22**(3), 030210 (2013).
20. Bratsos, A.G. and Abdul, K. "An exponential time differencing method of lines for the Burgers and the modified Burgers equations", *Numer. Methods Partial Differ. Equ.*, **34**(6), pp. 2024–2039 (2018).
21. Seydaoğlu, M. "An accurate approximation algorithm for Burgers' equation in the presence of small viscosity", *J. Comput. Appl. Math.*, **344**, pp. 473–481 (2018).
22. Mohamed, N. "Fully implicit scheme for solving Burgers' equation based on finite difference method", *Egypt. J. Eng. Sci. Technol.*, **26**(26), pp. 38–44 (2018).
23. Arora, G. and Varun, J. "A computational approach using modified trigonometric cubic B-spline for numerical solution of Burgers' equation in one and two dimensions", *Alex. Eng. J.*, **57**(2), pp. 1087–1098 (2018).
24. Aswin, V.S., Ashish, A., and Mohammad, M.R. "A differential quadrature based numerical method for highly accurate solutions of Burgers' equation", *Numer. Methods Partial Differ. Equ.*, **33**(6), pp. 2023–2042 (2017).
25. Du, M.J., Wang, Y.L., Temuer, C.L., et al. "A modified reproducing kernel method for solving Burgers' equation arising from diffusive waves in fluid dynamics", *Appl. Math. Comput.*, **315**, pp. 500–506 (2017).
26. Zhang, X., Jiang, Y., Hu, Y., et al. "High-order implicit weighted compact nonlinear scheme for nonlinear coupled viscous Burgers' equations", *Math. Comput. Simulation*, **196**, pp. 151–165 (2022).
27. El-Gamel, M., Adel, W., and El-Azab, M.S. "Collocation method based on Bernoulli polynomial and shifted Chebychev for solving the Bratu equation", *J. Appl. Computat. Math.*, **7**, p. 3 (2018).

28. El-Gamel, M. "Two very accurate and efficient methods for solving time-dependent problems", *Appl. Math.*, **9**(11), p. 1270 (2018).

29. Adel, W. and Sabir, Z. "Solving a new design of nonlinear second-order Lane-Emden pantograph delay differential model via Bernoulli collocation method", *Eur. Phys. J. Plus*, **135**(6), p. 427 (2020).

30. Bazm, S. and Hosseini, A. "Bernoulli operational matrix method for the numerical solution of nonlinear two-dimensional Volterra-Fredholm integral equations of Hammerstein type", *Comput. Appl. Math.*, **39**(2), p. 49 (2020).

31. Zeghdane, R. "Numerical solution of stochastic integral equations by using Bernoulli operational matrix", *Math. Comput. Simulation*, **165**, pp. 238–254 (2019).

32. Adel, Waleed. "A numerical technique for solving a class of fourth-order singular singularly perturbed and emden-fowler problems arising in astrophysics", *Int. J. Appl. Comput. Math.*, **8**, pp. 1–18 (2022).

33. Adel, Waleed, Kübra Erdem Biçer, and Mehmet Sezer. "A novel numerical approach for simulating the nonlinear MHD jeffery-hamel flow problem", *Int. J. Appl. Comput. Math.* **7**(3), pp. 1–15 (2021).

34. Pandey, K., Verma, L., and Verma, A.K. "On a finite difference scheme for Burgers' equation", *Appl. Math. Comput.*, **215**(6), pp. 2206–2214 (2009).

35. Mittal, R.C. and Rohila, R. "A study of one-dimensional nonlinear diffusion equations by Bernstein polynomial based differential quadrature method", *J. Math. Chem.*, **55**(2), pp. 673–695 (2017).

Biographies

Waleed Adel received a BSc degree in civil engineering and the MSc and PhD degrees in engineering mathematics from Mansoura University, Egypt, in 2007, 2014, and 2019, respectively. He has worked as an Assistant Professor with the Department of Mathematics and Engineering Physics at the Faculty of Engineering, Mansoura University, since 2019. He published more than 70 papers in high-ranked journals in the field of applied mathematics and was a reviewer for several internationally reputed journals. His current research interests include, but are not limited to, numerical analysis for differential equations, developing novel solutions using spectral methods, computational mathematics, and stability analysis for some biological models.

Hadi Rezazadeh received his PhD degree in Applied Mathematics from Guilan University in 2014. He has worked as an Associate Professor of the Faculty of Engineering Technology at Amol University of special modern technologies. His research interests include applied mathematics, solutions of fractional differential equations, optical soliton solutions, and stability analysis. He raised many master's students. He has published research articles in reputed international journals of mathematical and physical sciences.

Mustafa Inc received his PhD degree in Applied Mathematics from Firat University in 2002. He has worked as a full professor of the Mathematics Department at Firat University. His research interests include applied mathematics, solutions of differential equations, optical soliton, Lie symmetry, and conservation laws. He raised many master's and PhD students. He has published research articles in reputed international journals of mathematical and statistical sciences. He is a referee and editor of mathematical journals.