A Novel Memristive Chaotic System with Hidden Attractors and a Line of Equilibria

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Abstract: This paper introduces a newly designed four-dimensional memristive chaotic system. The novel oscillator is chaotic regarding the findings that the system's dynamic has one positive Lyapunov exponent (LE). Also, due to the results of the equilibrium points analysis, it is shown that the oscillator has a line of equilibria, so the attractors of this system are hidden. Moreover, the study of energy dissipation of this system, power spectrum, and Poincaré sections are conducted to investigate the system's dynamics. The complex features of this system are investigated with the aid of bifurcation diagrams, LEs spectra, approximate entropy, and basin of attraction.

Keywords: Memristor; Chaos; Hidden Attractor; Line of Equilibria; Dissipative System.

1 Introduction

Chaotic behavior is one of the characteristics of complex and interactive systems [1]. Chaos theory is a branch of mathematics and physics investigating the behavior of dynamical systems whose temporal dynamics, despite not being unstable, are susceptible to initial conditions, making predicting their future impossible [2]. Dynamical systems with these properties are called chaotic systems. Many systems in various fields show this behavior, and in recent decades, much attention has been focused on studying and investigating chaotic systems and signals [3]. Chaos can be considered a nonlinear and deterministic process that is not random but seems random in its time series [4]. The main reason for the output fluctuations, which seem unpredictable in the time series, is related to the internal mechanisms of the nonlinear data generator system and should not be misperceived by external shocks to the system, such as noise and changes in parameters [5].

Memristive elements, known as memory elements, including memristor, memcapacitor, and meminductor, can be easily implemented, and their behavior can be simulated with electronic circuits [6]. A memory resistor or memristor is an element with a function similar to the resistor, which is known as the fourth element of electrical circuits and is made in nano dimensions [7]. The amount of this resistance depends on the amount of voltage and voltage polarity [8]. The hysteretic current-voltage curve in the memristor allows this element to act as a non-volatile resistive memory and remember information until a voltage with a different value and polarity is applied to it, even a year later [9]. In 1971, Leon Chua proved the basis of this technology with mathematical and physical formulas [10]; but it took 37 years for its first technology to be practically registered [11]. Memristors can replace many transistors in some circuits and occupy less space [12]. A memristor has a voltage-dependent resistance, unlike an electrical resistor with a constant value [13]. The memristor material should be a reversible resistor with voltage change [14]. Ideal memristors are symmetrical, but the actual memristor's current-voltage characteristic is usually skewed [15]. Memristive components are nonlinear and can exhibit chaotic behavior in electronic circuits [16]. In recent years, many researchers have worked on designing and implementing chaotic oscillators based on memory elements [17-19]. Considering that the memristor has many industrial and exciting applications, the application of the memristor in a chaotic system is investigated.

From the computational standpoint, it is natural to propose the classification of attractors based on the basin of attraction: the attractor is called a hidden attractor whose basin of attraction does not collide with the small neighborhoods of the equilibrium points; otherwise, it is called a self-excited attractor [20]. The self-excited attractor can be localized numerically by the standard computational methods [21, 22]. On the other hand, the numerical calculation of hidden attractors requires the development of unique analytical-numerical methods [23, 24]. For example, hidden chaotic attractors are attractors in systems with no equilibrium point [25], a stable equilibrium point [26], or a line of equilibrium points [25, 27].

A novel memristive chaotic system with hidden attractors is proposed in this paper. The rest of the article is divided into three more sections. The next part is focused on introducing the newly designed system and examining its properties. The third part is devoted to the outcomes of further numerical simulations of this system to look at it in more depth and discover more exciting behaviors. In the end, the paper is concluded in the last section.

2 The Proposed Memristive System

The mathematical description of the newly designed system is as System (1):

$$\begin{aligned} \dot{x} &= z \\ \dot{y} &= w \\ \dot{z} &= -2.1x - 0.1z - w^2 + 0.11xz + 0.5zw \\ \dot{w} &= z - w + kwY(y) \end{aligned} \tag{1}$$

Where $Y(y) = 1 + 0.24y^2 - 0.0016y^4$ is the memristor function introduced by Pham et al. in 2014 [28], and k is the system parameter. This memristor function's complete analysis and characteristics were conducted before so that this part could be skipped. The rest of the paper is dedicated to studying the properties of the proposed system.

2.1 Equilibrium Points and Stability Analysis

For many years the equilibrium points were used in finding chaotic attractors [29]. Chaotic attractors were found in dynamical systems with unstable equilibrium points (saddle points). However, later, some dynamical systems did not have an equilibrium point at all [30] or had a stable equilibrium point [31] and a chaotic attractor. As the first step in studying a dynamical system, calculating its equilibrium points and investigating their stability is essential. The equilibrium points of dynamical systems can be obtained by setting all velocity variables to zero. In other words, all the right-hand side expressions should be set to zero in the differential equations. By applying this procedure and solving the equations, the equilibrium points of System (1) are as $\{(x, y, z, w) | x = z = w = 0, y \in \mathbb{R}\}$, which indicates that System (1) has a line of equilibria located on the y -axis. Thus, the attractors of this system belong to the group of hidden attractors. It is worth investigating whether this line of equilibria consists of stable or unstable points. The Jacobian matrix of System (1) should be formed to examine this issue. The Eigenvalues of this system for the equilibrium points indicate their stabilities. These Eigenvalues could be obtained by solving $\left|\lambda \boldsymbol{I} - \boldsymbol{J}\,|_{Equilibrium}\right| = 0$, leading to the characteristic equation.

The equilibrium point is stable if the real part of every Eigenvalue is negative. In contrast, the equilibrium point is unstable if just one of them is positive. Zero Eigenvalue without a positive one requires in-depth investigations. Suppose the largest Eigenvalue of the Jacobian matrix is zero or has a zero real part. In that case, the equilibrium point is non-hyperbolic, and its stability cannot be determined from the sign of the Eigenvalues of the Jacobian matrix. Because the Jacobian matrix of this system has a zero column at the equilibrium points, it is singular and thus has a zero Eigenvalue. This procedure is shown in Eq. (2), Eq. (3), Eq. (4), and Eq. (5), step by step.

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.1 + 0.11z & 0 & -0.1 + 0.11x + 0.5w & -2w + 0.5z \\ 0 & kw\left(0.48y - 0.0064y^3\right) & 1 & -1 + k\left(1 + 0.24y^2 - 0.0016y^4\right) \end{bmatrix} \tag{2}$$

$$\boldsymbol{J}\;|_{Equilibrium} = \boldsymbol{J}\;|_{x=z=w=0} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2.1 & 0 & -0.1 & 0 \\ 0 & 0 & 1 & -1+k\left(1+0.24y^2-0.0016y^4\right) \end{bmatrix} \tag{3}$$

$$\begin{gathered} \left|\lambda \boldsymbol{I} - \boldsymbol{J}\;|_{Equilibrium}\right| = 0 \rightarrow \\ \lambda\left(0.0016\lambda^2+0.00016\lambda+0.00336\right)\left(0.00016ky^4-0.024ky^2-0.1k+0.1\lambda+0.1\right)=0 \rightarrow \end{gathered} \tag{4}$$

$$\lambda_1 = 0, \lambda_2 = -0.05+\frac{\sqrt{839}i}{20}, \lambda_3 = -0.05-\frac{\sqrt{839}i}{20}, \lambda_4 = -0.0016y^4+0.24ky^2+k-1 \tag{5}$$

According to the Eigenvalues achieved in Eq. (5), System (1) has a zero, a pair of complex conjugate Eigenvalues with a negative real part, and one more Eigenvalue that its quantity depends on the state variable y and the system parameter k. To analyze λ_4 in detail, the extremums of the fourth-order function are obtained. If the maximum values of λ_4 are negative, then the negativity of λ_4 for all values of y is guaranteed. To find the extremums of λ_4, the derivative of it regarding y should be achieved. Setting the derivative expression equal to zero and substituting the resulting answers in λ_4, giving the extremum points that could be minimum or maximum. The extremums of λ_4 are obtained in Eq. (6) and Eq. (7). λ_4 has three extremums, but the one corresponding to zero is a local minimum, so we are not interested in it.

$$\frac{d\lambda_4}{dy} = -0.0064y^3+0.48ky = y\left(-0.0064y^2+0.48k\right)=0 \rightarrow y_1 = 0, y_{2,3} = \pm 5\sqrt{3k} \tag{6}$$

$$\max \lambda_4 = \lambda_4\;|_{y=\pm 5\sqrt{3k}} = -0.0016\times 5625k^2+0.24\times 75k^2+k-1 = 9k^2+k-1 \tag{7}$$

According to Eq. (7), the maximum value of λ_4 is a quadratic function of k. When the amount of k is between the two roots of Eq. (7), the sign of the resulting value is negative (Since the sign of the coefficient of k^2 is positive). Solving this equation leads to an interval that, if k falls in it, the maximum value of λ_4 is negative. However, because of the zero Eigenvalue, the stability of the equilibria needs more investigations carried out in future parts. The interval of k to have a negative value is found in Eq. (8).

$$9k^2 + k - 1 < 0 \rightarrow \frac{-1-\sqrt{1+36}}{18} < k < \frac{-1+\sqrt{1+36}}{18} \rightarrow -0.393 < k < 0.282 \tag{8}$$

Assume $k = 0.0001$, which is in the interval. To better show this result, the variation of λ_4 concerning y is shown in Fig. 1. It is perceived from Fig. 1 that for all values of y, λ_4 is negative.

2.2 Time Series and State Space

Chaos is a bounded irregular behavior of a system. In chaotic behavior, the attractor is bounded and convergent but not periodic. The time series and the attractor of System (1) with $k = 0.0001$ projected on $X-Y$, $Y-Z$, and $Z-W$ planes after removing the transient parts are depicted in Fig. 2 (a)-(d), respectively. By observing Fig. 2, System (1)'s chaotic behavior in time series and state space can be recognized. The Lyapunov exponents (LEs) of this chaotic attractor are $LE_1 = 0.1428, LE_2 = LE_3 = 0, LE_4 = -3.7978$. Since with $k = 0.0001$, System (1) has one positive LE; it has chaotic behavior.

2.3 Energy Dissipating

A simple analysis of any dynamical system determines how the system treats the energy. This feature can be determined by the sign of the system's divergence, the Jacobian matrix's trace [32]. If the divergence is negative, the system is dissipative; otherwise, the system is classified as conservative if the divergence is zero. If a dynamical system is bounded and has a positive LE, then the system is chaotic. Chaotic behavior can be observed in conservative and dissipative systems [33]. If the sum of the absolute values of the negative exponents is greater than the sum of the positive exponents, or in other words, if the attraction is greater than the repulsion, the chaotic system is dissipative. However, suppose the sum of the absolute values of the negative exponents is equal to the sum of the positive exponents. In that case, the sum of its LEs becomes zero, and the chaotic system is conservative (chaotic sea). Sometimes the divergence computation does not lead to a number, but it depends on the system's state variables. In this case, its average (DC value) determines the divergence. After this substitution, the divergence is transformed into a number that indicates whether the chaotic system is conservative or dissipative. The divergence of System (1) is calculated in Eq. (9).

$$\begin{aligned}
&\nabla = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -0.1 + 0.11x + 0.5w - 1 + k\left(1 + 0.24y^2 - 0.0016y^4\right) = \\
&-1.1 + k + 0.11x + 0.5w + 0.24ky^2 - 0.0016ky^4 \xrightarrow{State\ Dependent} \\
&\nabla = -1.1 + k + 0.11DC(x) + 0.5DC(w) + 0.24kDC\left(y^2\right) - 0.0016kDC\left(y^4\right) \xrightarrow{DC(w)=0} \\
&\nabla = -1.1 + k + 0.11DC(x) + 0.24kDC\left(y^2\right) - 0.0016kDC\left(y^4\right)
\end{aligned} \tag{9}$$

The average value of all velocity variables must be zero to avoid resulting in unbounded behavior. In other words, the DC value of all right-hand side expressions in the differential equations describing the system's bounded dynamics is zero. Thus, regarding System (1), the DC value of the z and w state variables is zero. However, the other expressions in Eq. (9) should be obtained by simulating the system for enough long time and calculating the DC value of x, y^2, and y^4, and replacing them in Eq. (9). Finally, the divergence of the system is achieved numerically. Eq. (9) also depends on the parameter k. Here, this parameter is fixed at 0.0001, and the numerical divergence values of System (1) are acquired over time. The result shown in Fig. 3 means that as time passes, the value of the divergence of System (1) leads to a negative amount, so System (1) is dissipative.

2.4 Power Spectrum

Periodic signals have peaks in a fundamental component and harmonics in frequency space [34]. Also, the peak of quasiperiodic signals is located in linear combinations of two or more related frequencies [34]. Chaotic dynamics have wideband components in the spectrum [35]. This feature is used as a criterion to identify a dynamic as chaos [34, 35]. The power spectrum of System (1) with $k = 0.0001$ is demonstrated in Fig. 4. Since System (1) is four-dimensional, the power spectrum is calculated along four distinct variables and plotted in a different color. When a system is in its chaotic regime, the power spectrum of all system variables is broadband. Usually, the significant frequency interval in which the power spectrum components are prominent and cannot be neglected is similar for all system variables. Although in some unique systems like the well-known Rössler system, the significant power spectrum interval of the system variables is not the same and can be distributed in different ranges [36]. According to Fig. 4, the power spectrum of System (1) with this parameter is wideband and does not exhibit any peaks in specific frequencies. This spectrum is consistent with the results obtained in Fig. 3 and confirms the chaoticity of this newly designed system.

2.5 Poincaré Section

Another well-known tool to investigate dynamical systems is the Poincaré section. As a result of intersecting a plane or hyperplane with the attractor in the N-dimensional state space, the Poincaré section is obtained with a lower dimension. Dimension reduction by this method sometimes enables visualizing the attractors that could not be visualized in the N-dimensional state space. Regarding the fact that System (1) is a four-dimensional system and the most visualizable dimension is three, the Poincaré section of the attractors of System (1) on $X = 0$, $Y = 0$, $Z = 0$, and $W = 0$ hyperplanes with $k = 0.0001$ are presented in Fig. 5 (a)-(d) respectively. The gray diagram depicts the three-dimensional attractors, and the colorful dots demonstrate the system trajectory's intersection with the previously mentioned hyperplanes. In plotting the intersection points, the transition from the negative side of the desired hyperplane to the positive side is considered. The pattern of the dots shows that the attractor of System (1) with the selected parameter is chaotic.

3 Dynamical Analysis of the Proposed System

In this section, a further and more in-depth analysis of System (1) with varying system parameter k and the initial conditions is carried out.

3.1 The Effect of Parameter k on the System Dynamics

In dynamical equations, the solutions of the equations are strongly dependent on the value of the parameters. Studying the equations' behavior is possible by changing the parameters' values. The parameter whose effect of changing is examined is called the control parameter (parameter k in System (1)). The bifurcation diagram of System (1), by changing k using the final solution of the equations in the long term (y_{max}) and with constant initial conditions, is plotted in Fig. 6 (a). Periodic and chaotic behavior and periodic windows in the middle of chaos are observed in System (1). By decreasing parameter k, the period-doubling route to chaos is followed. If the bifurcation parameter changes, the geometrical structure of the phase space also changes, then a critical point occurs. These critical points are called bifurcation points; for instance, in $k = 00210$, $k = 00105$, and $k = 00083$ in Fig. 6 (a).

Moreover, another applicable analytical criterion that can describe chaos is employed. This criterion is called the LE. The LEs describe how quickly a tiny gap between two initially closed states grows. There are various methods of determining LEs. In this work, the Wolf method with a runtime of 10000 seconds is applied [37, 38]. The LEs spectra of System (1) by varying parameter k are shown in Fig. 6 (b). System (1) always has a negative and relatively large LE (LE_4) that assures the system's boundedness. Also, this system always has a zero LE. A system can be considered chaotic in the case of one positive LE.

Previously, a strange attractor in System (1) was confirmed. According to this system's bifurcation diagram, it can also exhibit periodic behaviors. By choosing the system parameter as $k = 0.0025$, $k = 0.002$, $k = 0.0009$, and $k = 0.001$, period one, two, three, and four attractors were achieved, respectively. The projection of these periodic attractors on the $Y - Z$ plane is illustrated in Fig. 7.

One of the time analysis methods for measuring the complexity of signals is to use approximate entropy. Approximate entropy is a statistical measure that quantifies order in a time series. The disorder of the signals in the chaotic dynamic increases, and its amount equals the approximate entropy. Approximate entropy is a parameter used to characterize the irregularity of the signal. This entropy's growth shows the signal's irregularity and complexity during parameter change. To measure this entropy, $N - m + 1$ delayed subsets of the signal are extracted and named $Y_{1:N-m+1}$, where m is the embedding dimension, and N is the number of samples in the signal. In the next step, each Y_k is selected. The number of other Y_i s like it is calculated as N_i in Eq. (10). The similarity criterion is that the difference between attributes of the two subsets must be less than R, the similarity radius, and 1 is the indicator function. Then, as in Eq. (11), Φ_m is calculated based on the N_i s found in the previous step, and in the end, the approximate entropy is obtained by Eq. (12) [39-41].

$$N_i = \sum_{i=1, i \neq k}^{N-m+1} 1(\|Y_i - Y_k\|_\infty < R) \tag{10}$$

$$\Phi_m = \frac{\sum_{i=1}^{N-m+1} \log N_i}{N - m + 1} \tag{11}$$

$$Approximate\ Entropy = \Phi_m - \Phi_{m+1} \tag{12}$$

The approximate entropy of the time series produced by System (1), while varying the parameter k is calculated and shown in Fig. 8. Comparing this result with the bifurcation diagram, in the region of chaotic behavior, the quantity of approximate entropy is larger than the regions with periodic manner. This trend is because the chaotic signal is more irregular and complex than the ordered periodic ones.

3.2 The Effect of Initial Conditions on System Dynamics

As the last part of the dynamical analysis of System (1), the basin of attractions of this system with varying two different initial conditions are depicted in Fig. 9. The results are obtained in $x_0 - y_0$, $y_0 - z_0$, and $z_0 - w_0$ planes with $k = 0.0001$ in parts (a)-(c) of Fig. 9, respectively. A 300×300 grid of initial conditions is used to obtain this figure, and the two other initial conditions are constant. The light orange, red, and white regions correspond to chaotic, fixed points and unbounded solutions. A spectrum of other colors illustrates the periodic behavior; as the number of periods rises, the time series tend to be complex, and the color of these points changes. The black points are the projection of the line of equilibria of System (1). Regarding the equilibrium points stability analysis conducted in part 2.1, the line of equilibria of System (1) with $k = 0.0001$ is stable, and all red zones converge to it. This system has a broad region of initial conditions that leads to chaotic behavior, so it is flexible in choosing the initial conditions.

4 Discussion and Conclusion

This paper studied a chaotic system that took advantage of the characteristics of a formerly introduced memristor. The system was not only able to bring up a strange attractor but also its attractor was hidden. According to the nature of chaotic signals, it is necessary to investigate the state and behavior of these systems with more analytical methods. To verify the possibility of this behavior, numerical simulations were used to check the system's behavior. Chaos can be recognized from the bifurcation diagrams and the largest LE. Some criteria to prove the presence of chaos in this system were investigated: the system's dissipation, time series, a strange attractor, a positive LE, Poincaré mapping, and a broad frequency spectrum. All these criteria showed the existence of chaos in this system, and they have been used to distinguish periodic and chaotic behaviors from each other.

The system had a dissipative chaotic attractor and a line of equilibrium points. Its line of equilibria was located on the y -axis. The system had a zero, two complex conjugates (with negative real parts), and a real Eigenvalue whose value depends on both the parameter k and the y value of the equilibria. Because of the appearance of a zero Eigenvalue, the stability of the equilibria cannot be obtained quickly and needs extensive investigations like basins of attraction. Besides, since the divergence of the system was state-dependent, and the average value of the state-dependent terms could not be algebraically calculated, the average values were computed numerically. After a sufficient run time, the divergence of the system converged to a negative value. It proved that the chaotic trajectories were strange attractors.

The claimed properties of this system were evaluated with the help of powerful dynamical analysis tools like Poincaré sections, power spectrum, bifurcation diagram, LEs spectrum, the basin of attraction, and approximate entropy. The inverse route of period doubling and periodic windows were observed in the bifurcation diagram. Attraction basins verified the coexistence of fixed points, periodic, and chaotic attractors. Moreover, the basin of attractions confirmed that the previously calculated line of equilibria is stable. Due to the outcomes of these examinations and the fact that there has been a promising development in implementing memristor, the proposed system can have potential real-world applications.

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Declaration of Interest Statement

Following ethical obligation as researchers, the authors declare that no conflicts of interest may affect the research reported in this manuscript.

Data Availability Statement

Data generated during the current study will be made available at reasonable request.

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Figure 1. The variation of λ_4 concerning y with $k = 0.0001$. If k is selected from the desired interval, the maximum value of λ_4 is negative.

Figure 2. Time series and projection of attractors of System (1) with $k = 0.0001$ and $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$. **(a)** Time series, **(b)** Projection of attractor on $X - Y$, **(c)** $Y - Z$, **(d)** $Z - W$ planes. System (1) can exhibit chaotic time series and bounded but non-periodic cycles (strange attractors). It should be noted that different colors in the attractors do not have any special meaning, and such a colormap is used for better illustration.

Figure 3. Numerical calculation of System (1)'s divergence over time with $k = 0.0001$ and $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$. The divergence results in a negative value, meaning System (1) is dissipative.

Figure 4. Power spectrum of System (1) with $k = 0.0001$ and $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$. System (1) is a chaotic system due to the wideband spectrum.

Figure 5. Poincaré sections and attractors of System (1) with $k = 0.0001$ and $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$ on **(a)** $X = 0$, **(b)** $Y = 0$, **(c)** $Z = 0$, and **(d)** $W = 0$ hyperplanes. The gray diagram and colored dots correspond to the system trajectory and the Poincaré section.

Figure 6. (a) Bifurcation diagram (y_{max}) and **(b)** Lyapunov exponents (LEs) spectra of System (1) while changing the parameter k with $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$. Periodic solutions approach chaotic solutions along the period-halving route.

Figure 7. The projection of limit cycles of System (1) on the $Y - Z$ plane with $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$ and **(a)** $k = 0.0025$, **(b)** $k = 0.002$, **(c)** $k = 0.0009$, and **(d)** $k = 0.001$. System (1) can exhibit limit cycles with various periods.

Figure 8. Approximate entropy of System (1) while changing the parameter k with $(x_0, y_0, z_0, w_0) = (-28,0,0,0)$. Approximate entropy in chaotic zones is more than that of periodic ones.

Figure 9. Basins of attraction of System (1) with $k = 0.0001$ in a 300×300 grid of initial conditions in **(a)** $(x_0, y_0, z_0, w_0) = (x_0, y_0, 0,0)$, **(b)** $(x_0, y_0, z_0, w_0) = (-28, y_0, z_0, 0)$, and **(c)**

$(x_0, y_0, z_0, w_0) = (-28, 0, z_0, w_0)$. The red, light orange, and white regions correspond to the fixed point, chaotic, and unbounded solutions. The black points are the projection of System (1) equilibrium points in each plane. This system also has many initial conditions that lead to chaos. P1, P2, …, and P15 are periodic oscillations with different period numbers.

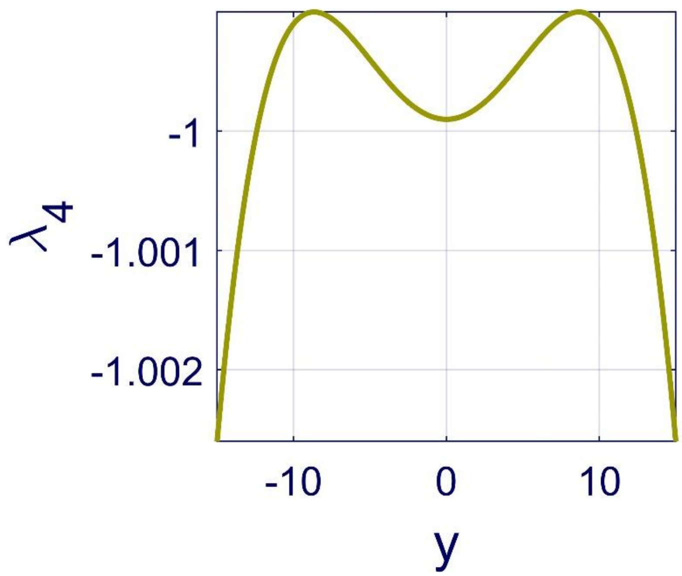


Figure 1.

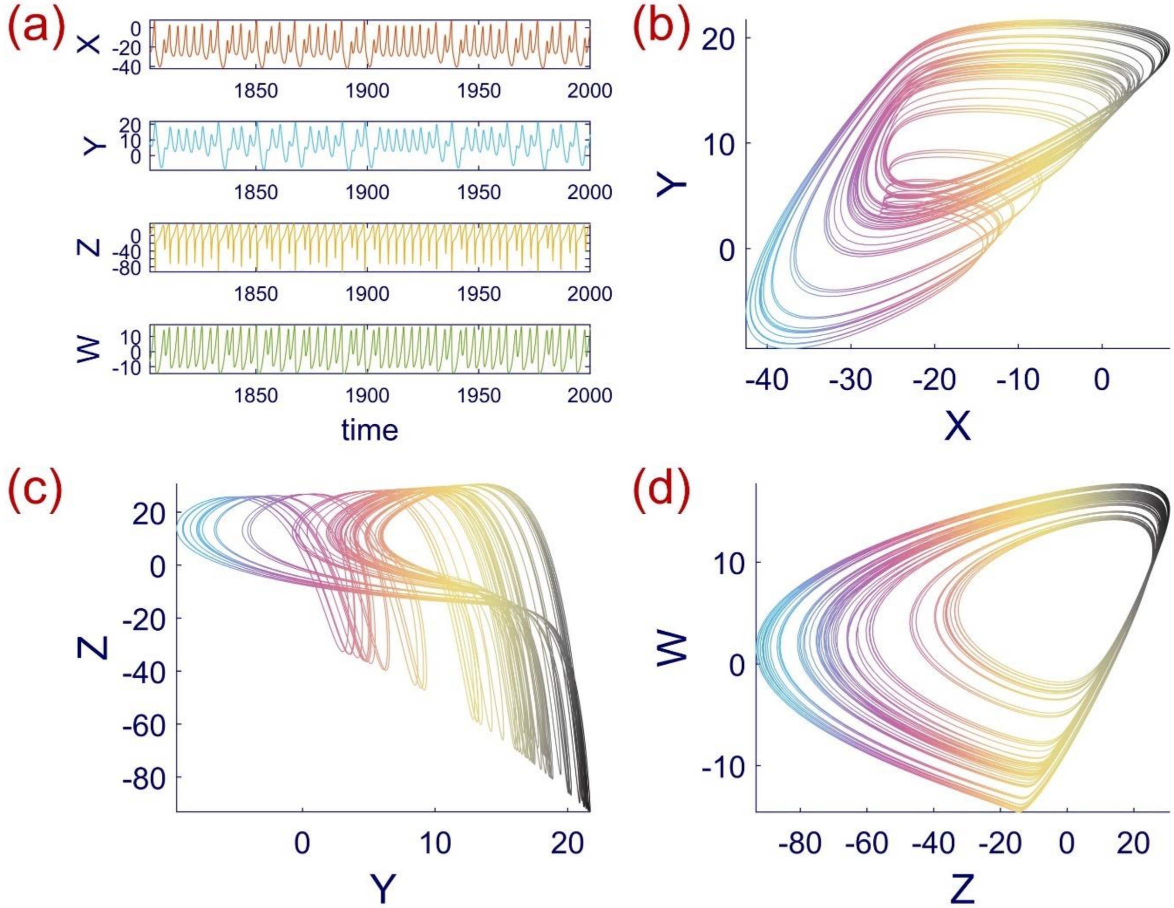

(a)
X
0
-20
-40
1850
1900
1950
2000
Y
20
10
0
Z
0
-40
-80
W
10
0
-10
time
(b)
20
10
0
Y
-40
-30
-20
-10
X
(c)
20
0
-20
-40
-60
-80
Z
0
10
20
Y
(d)
10
0
-10
W
-80
-60
-40
-20
0
20
Z

Figure 2.

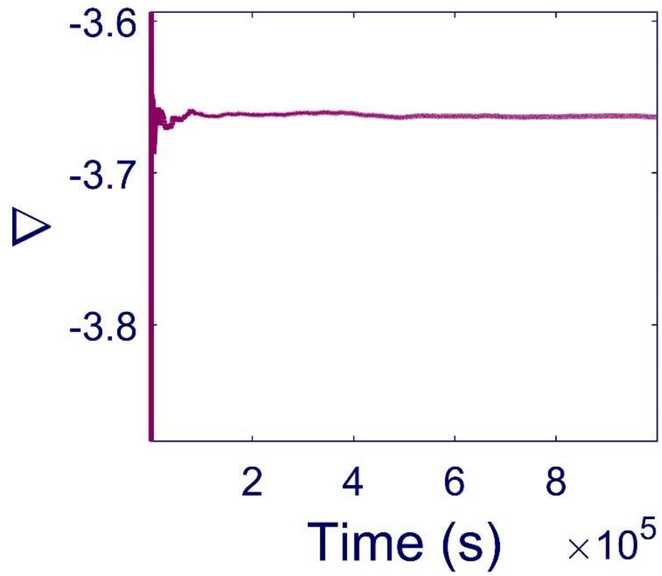


Figure 3.

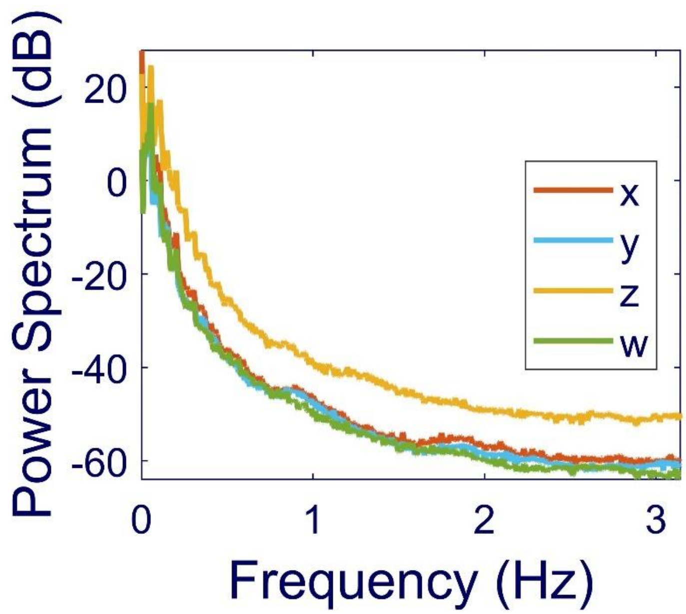


Figure 4.

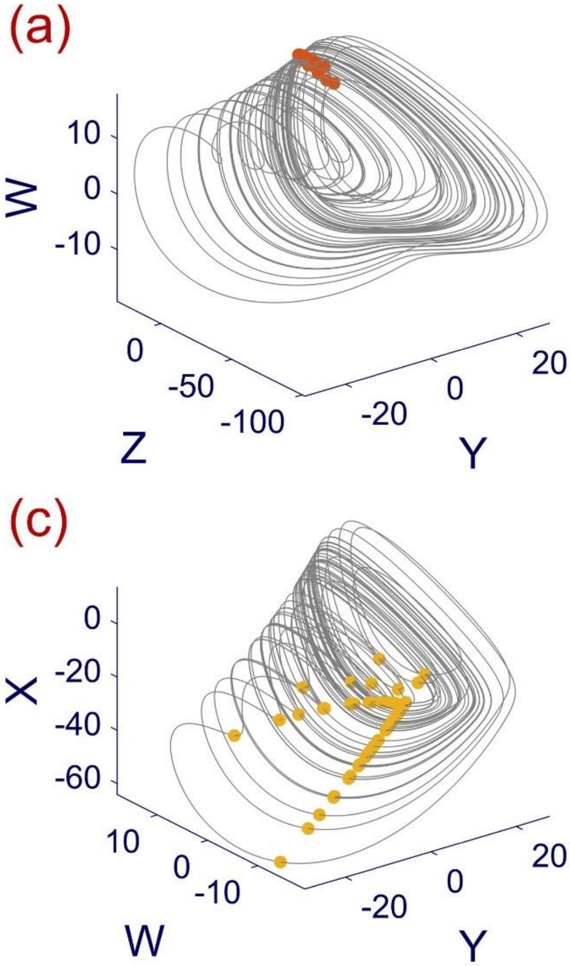


Figure 5.

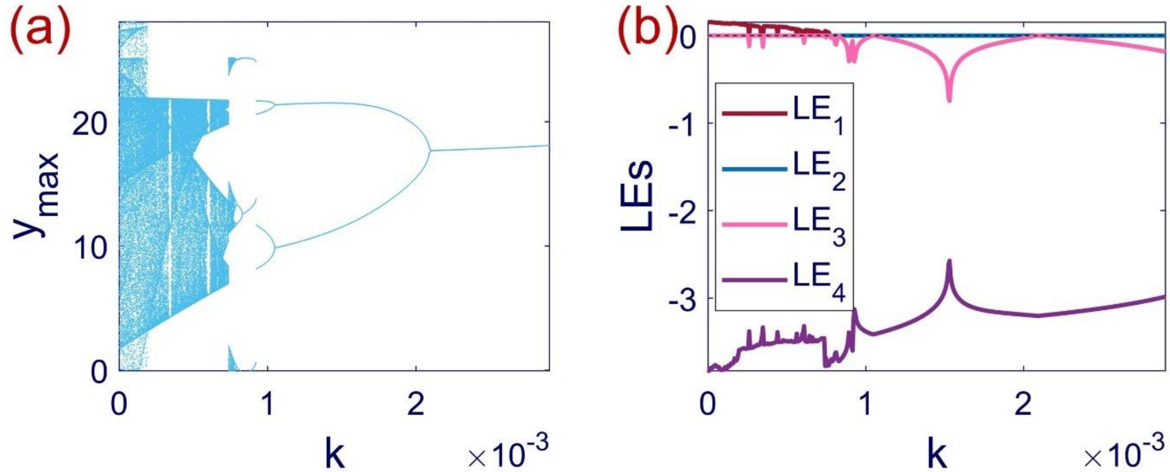
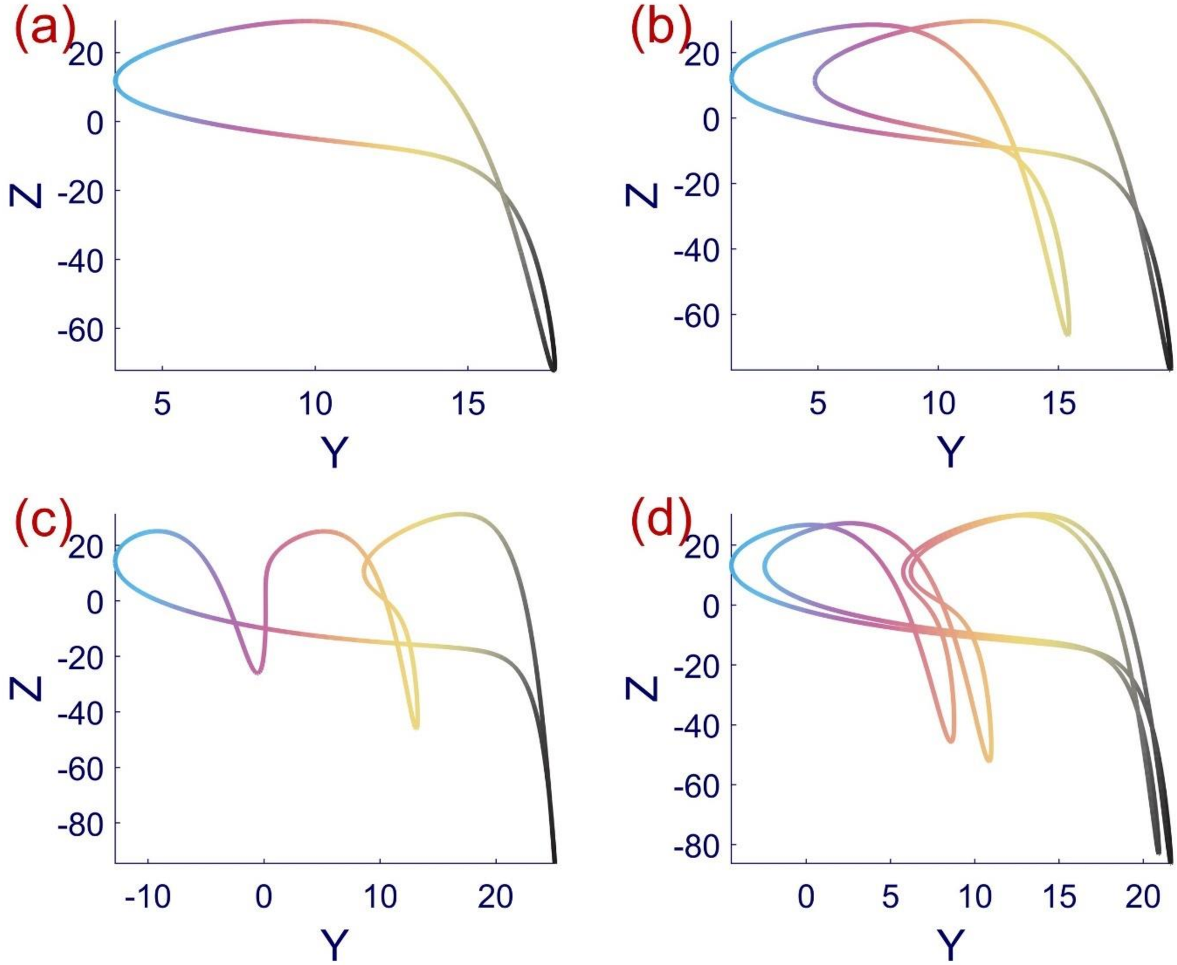


Figure 6.


(a)
20
0
-20
-40
-60
Z
5
10
15
Y
(b)
20
0
-20
-40
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Z
5
10
15
Y
(c)
20
0
-20
-40
-60
-80
Z
-10
0
10
20
Y
(d)
20
0
-20
-40
-60
-80
Z
0
5
10
15
20
Y

Figure 7.

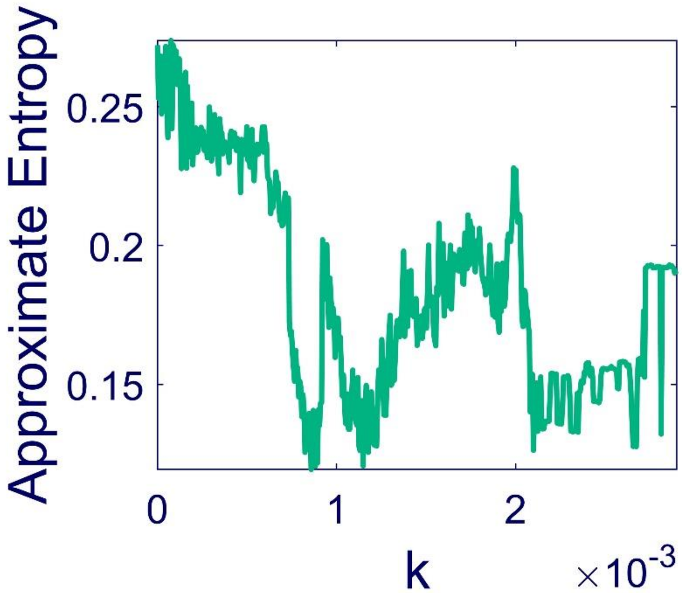

Figure 8.

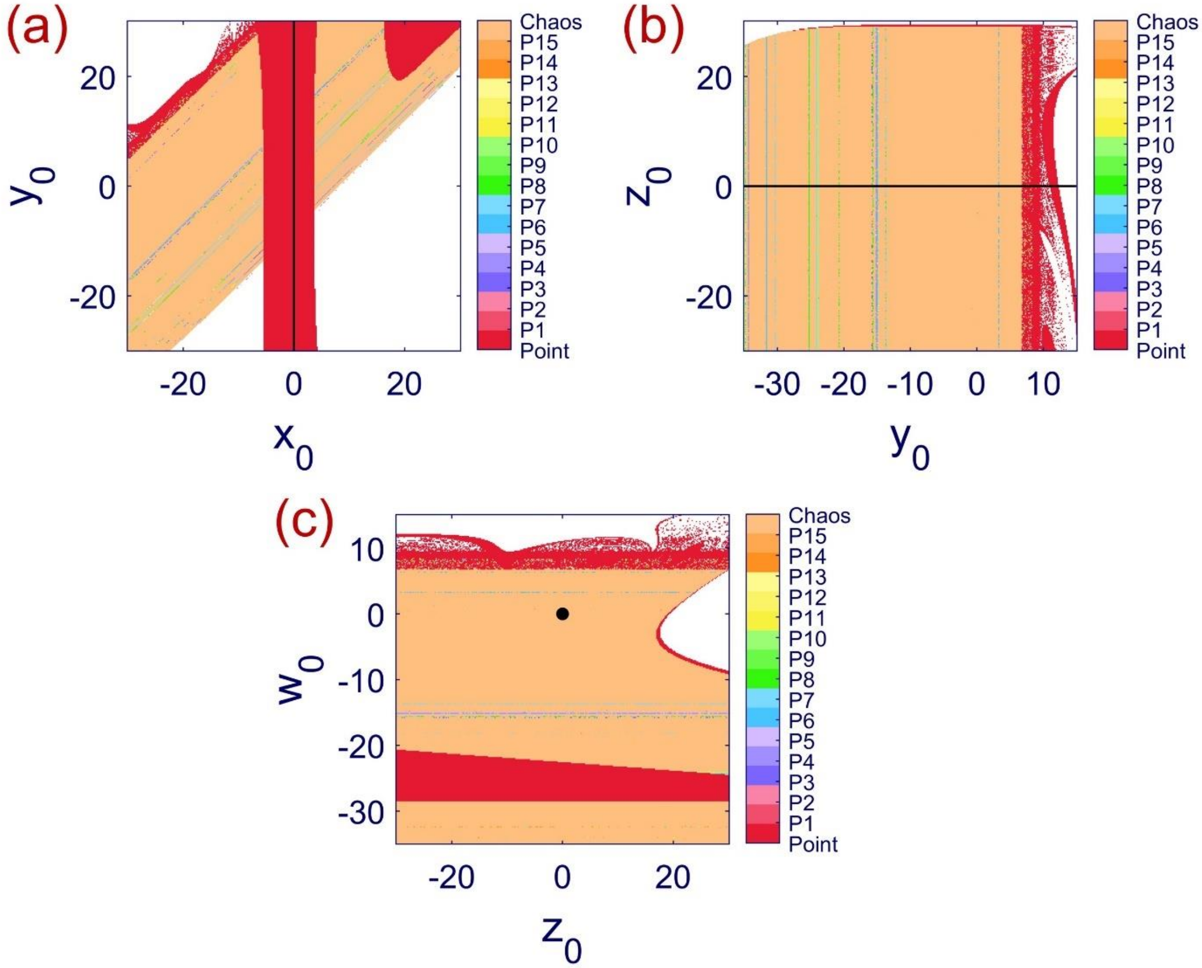
(a)
(b)
(c)
Chaos
P15
P14
P13
P12
P11
P10
P9
P8
P7
P6
P5
P4
P3
P2
P1
Point
x_0
y_0
z_0
w_0

Figure 9.

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