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Dispersion of surface waves in a transversely isotropic half-space underlying a liquid layer

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Anisotropy.

Abstract. Surface waves dispersion is studied in a two-layer half-space consisting of a finite liquid layer overlying a transversely isotropic solid half-space. A couple of complete potential functions are utilized to uncouple the equation of motion of the transversely isotropic solid along with a displacement potential for the liquid. The frequency equation and velocity dispersion curves are developed. Several solid materials are considered for the bed, and both phase and group velocity curves are calculated. Higher modes are also discussed and the respective curves are plotted. Various special cases are considered by letting the liquid layer depth take zero value, to form a solid half-space, or very large values to form a full-space bi-material. Moreover, an isotropic bed material can be obtained as a special case by appropriately setting the respective elastic constants. Reduced frequency equations and numerical results are derived for each case to confirm the results with existing ones.

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1. Introduction

Propagation of elastic surface waves in various media has been of importance and interest in different fields of science such as seismology, earthquake engineering, material engineering, geophysical studies, etc. The media consisting of a liquid layer lying over an isotropic solid half-space has been studied by scientists interested in fields such as ocean engineering, etc. [1-3].

However, in a wide range of cases, there are observations revealing the existence of anisotropy in Earth's crust. It is believed that the fact of following a preferred pattern instead of random orientations by materials depositing in the water and also existence of vertically aligned microcracks result in the transverse

isotropy of the geologic materials [4]. This anisotropy can considerably affect the dispersive properties of surface waves. Studying wave propagation in transversely isotropic solids is the research subject of many investigators due to its importance in different fields of science and technology. Stoneley [5] was the first to point that transverse isotropy significantly influences the wave propagation in a medium in comparison with isotropic one. Later, elastodynamics of transversely isotropic media was subsequently studied by Synge [6], Buchwald [7], and Payton [8].

Rahimian et al. [9] presented an efficient analytical formulation to obtain the response of a three-dimensional transversely isotropic half-space to a time-harmonic surface loading. Khojasteh et al. [10] extended the previous work to present explicit expressions for the three-dimensional dynamic Green's functions in a transversely isotropic half-space and investigated the wave propagation. They continued the work for a two-layer half-space [11] as well. Also, Khojasteh et al. subsequently studied wave propagation in

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transversely isotropic bi-materials, tri-materials, and a multilayered half-space [12–14].

Surface waves dispersion in a model of a liquid layer overlying a transversely isotropic half-space has been investigated in some papers. Abubakar and Hudson [15] studied the dispersive properties of surface waves in such a system in the two-dimensional case. Sharma et al. [16,17] obtained the frequency equation of a three-layer model consisting of a liquid layer lying over poroelastic and anisotropic layers. Sharma [18] also tackled the dispersion of Stoneley waves in an oceanic crust model. In addition to Kumar and Miglani [19], Kumar and Kumar [20] solved the problem for the proposed oceanic crust models. In this paper, the dispersive characteristics of a three-dimensional earth crust model, consisting of a homogeneous transversely isotropic elastic media overlaid by a homogeneous compressible liquid layer, are investigated utilizing the potential method presented by Rahimian et al. [9] and extended by Khojasteh et al. [10]. The velocity dispersion curves of surface waves are plotted. Arrival times of the waves are distinctly dependent on the elastic properties of the bed. Analyzing this phenomenon to realize the earth crust's characteristics by reverse calculation is of significant importance in marine seismology and underground geophysical explorations. The formulation can also be utilized to derive Green's functions of stress and displacement fields for various load types in such a system which is applicable to earthquake engineering.

2. Statement of the problem and governing equations

The geometry of the system and the coordinate axes selected for the problem are shown in Figure 1. For the homogeneous transversely isotropic half-space, the time-harmonic equations of motion are expressed as [9]:

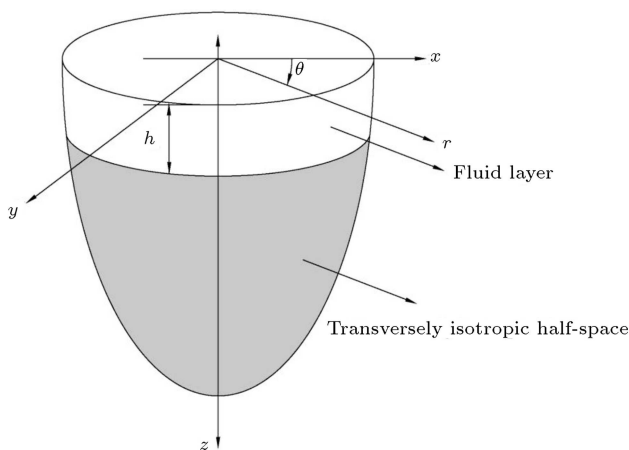


Figure 1. Geometry of the system.

$$\begin{aligned}
 & c_{11} \left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right) + c_{66} \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - 2c_{11} \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \\
 & + c_{44} \frac{\partial^2 u_r}{\partial z^2} + \frac{(c_{11} + c_{12})}{2} \left(\frac{1}{r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \\
 & + (c_{13} + c_{44}) \frac{\partial^2 u_z}{\partial r \partial z} + \rho_s \omega^2 u_r = 0, \\
 & c_{66} \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right) + c_{11} \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + c_{44} \frac{\partial^2 u_\theta}{\partial z^2} \\
 & + 2c_{11} \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} + \frac{(c_{11} + c_{12})}{2} \left(\frac{1}{r} \frac{\partial^2 u_r}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial u_r}{\partial \theta} \right) \\
 & + (c_{13} + c_{44}) \frac{1}{r} \frac{\partial^2 u_z}{\partial \theta \partial z} + \rho_s \omega^2 u_\theta = 0, \\
 & c_{44} \left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} \right) + c_{33} \frac{\partial^2 u_z}{\partial z^2} \\
 & + (c_{13} + c_{44}) \times \left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r} \frac{\partial u_r}{\partial z} + \frac{1}{r} \frac{\partial^2 u_\theta}{\partial \theta \partial z} \right) \\
 & + \rho_s \omega^2 u_z = 0,
 \end{aligned} \tag{1}$$

where the time factor $\exp(i\omega t)$ is suppressed. In order to uncouple these equations, a couple of potential functions introduced by Eskandari-Ghadi [21] are applied. These two potential functions, F and χ , are related to displacement components u_r , u_θ , and u_z as:

$$\begin{aligned}
 u_r(r, \theta, z) &= -\alpha_3 \frac{\partial^2 F(r, \theta, z)}{\partial r \partial z} - \frac{1}{r} \frac{\partial \chi(r, \theta, z)}{\partial \theta}, \\
 u_\theta(r, \theta, z) &= -\alpha_3 \frac{1}{r} \frac{\partial^2 F(r, \theta, z)}{\partial \theta \partial z} + \frac{\partial \chi(r, \theta, z)}{\partial r}, \\
 u_z(r, \theta, z) &= \left[(1 + \alpha_1) \nabla_{r\theta}^2 + \alpha^2 \frac{\partial^2}{\partial z^2} + \frac{\rho_s \omega^2}{c_{66}} \right] F(r, \theta, z),
 \end{aligned} \tag{2}$$

where:

$$\nabla_{r\theta}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \tag{3}$$

and also:

$$\alpha_1 = \frac{c_{12} + c_{66}}{c_{66}}, \quad \alpha_2 = \frac{c_{44}}{c_{66}}, \quad \alpha_3 = \frac{c_{13} + c_{44}}{c_{66}}. \tag{4}$$

Rewriting the equation of motion (Eq. (1)) in terms of the potential functions yields to the two separate partial differential equations as:

$$\begin{aligned}
 & \left(\nabla_1^2 \nabla_2^2 + \delta \omega^2 \frac{\partial^2}{\partial z^2} \right) F = 0, \\
 & \nabla_0^2 \chi = 0,
 \end{aligned} \tag{5}$$

defining:

$$\nabla_i^2 = \nabla_{r\theta}^2 + \frac{1}{s_i^2} \frac{\partial^2}{\partial z^2} + \frac{1}{\mu_i} \frac{\rho_s \omega^2}{c_{66}}, \quad i = 0, 1, 2, \quad (6)$$

and:

$$\begin{aligned} \mu_0 &= 1 \\ \mu_1 &= \alpha_2 = \frac{c_{44}}{c_{66}}, \quad \mu_2 = 1 + \alpha_1 = \frac{c_{11}}{c_{66}}, \\ \frac{\text{delta}}{\rho_s} &= \left[\frac{-1}{c_{44}s_2^2} - \frac{1}{c_{11}s_1^2} + \frac{1}{c_{11}} \left(1 + \frac{c_{33}}{c_{44}} \right) \right], \end{aligned} \quad (7)$$

where:

$$s_0 = \frac{1}{\sqrt{\alpha_2}}. \quad (8)$$

Note that s_1 and s_2 are not zero or pure imaginary numbers with respect to the positive definiteness of the strain energy function [22]:

$$c_{33}c_{44}s^4 + (c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33})s^2 + c_{11}c_{44} = 0. \quad (9)$$

The two real roots of this equation can be easily obtained since no unknowns of odd order exist in Eq. (9). Hence s_1 and s_2 are obtained by Eq. (10) as shown in Box I. With respect to the angular direction, Fourier series expansion is now applied as follows:

$$[F(r, \theta, z), \chi(r, \theta, z)] = \sum_{-\infty}^{\infty} [F_m(r, z), \chi_m(r, z)] e^{im\theta}. \quad (11)$$

Also, with respect to the radial direction, m th order Hankel transform may be expressed as:

$$\begin{aligned} &[\tilde{F}^m(\xi, z), \tilde{\chi}^m(\xi, z)] \\ &= \int_0^\infty [F(r, z), \chi(r, z)] r J_m(r\xi) dr, \\ &[F(r, z), \chi(r, z)] \\ &= \int_0^\infty [\tilde{F}^m(\xi, z), \tilde{\chi}^m(\xi, z)] \xi J_m(r\xi) d\xi. \end{aligned} \quad (12)$$

Applying Hankel transform and Fourier series expansion, one may arrive at the two following ordinary differential equations for transformed components of F and χ as [9]:

$$\begin{aligned} &\left(\tilde{\nabla}_{1m}^2 \tilde{\nabla}_{2m}^2 + \delta \omega^2 \frac{d^2}{dz^2} \right) \tilde{F}_m^m(z) = 0, \\ &\tilde{\nabla}_{0m}^2 \tilde{\chi}_m^m = 0, \end{aligned} \quad (13)$$

in which:

$$\tilde{\nabla}_{im}^2 = \frac{\rho_s \omega^2}{\mu_i c_{66}} - \xi^2 + \frac{1}{s_i^2} \frac{d^2}{dz^2}, \quad i = 0, 1, 2. \quad (14)$$

The general solutions for equations in Eq. (13) are:

$$\begin{aligned} \tilde{F}_m^m(\xi, z) &= A_m(\xi) e^{\lambda_1 z} + B_m(\xi) e^{-\lambda_1 z} + C_m(\xi) e^{\lambda_2 z} \\ &\quad + D_m(\xi) e^{-\lambda_2 z}, \\ \tilde{\chi}_m^m(\xi, z) &= E_m(\xi) e^{\lambda_3 z} + F_m(\xi) e^{-\lambda_3 z}, \end{aligned} \quad (15)$$

introducing:

$$\begin{aligned} \lambda_1 &= \sqrt{a\xi^2 + b + \frac{1}{2}\sqrt{c\xi^4 + d\xi^2 + e}}, \\ \lambda_2 &= \sqrt{a\xi^2 + b - \frac{1}{2}\sqrt{c\xi^4 + d\xi^2 + e}}, \\ \lambda_3 &= s_0 \sqrt{\xi^2 - \frac{\rho_s \omega^2}{c_{66}}}. \end{aligned} \quad (16)$$

In the above terms:

$$\begin{aligned} a &= \frac{1}{2}(s_1^2 + s_2^2), \\ b &= -\frac{1}{2}\rho_s \omega^2 \left(\frac{1}{c_{33}} + \frac{1}{c_{44}} \right), \\ c &= (s_2^2 - s_1^2)^2, \\ d &= -2\rho_s \omega^2 \left[\left(\frac{1}{c_{33}} + \frac{1}{c_{44}} \right) (s_1^2 + s_2^2) \right. \\ &\quad \left. - 2\frac{c_{11}}{c_{33}} \left(\frac{1}{c_{11}} + \frac{1}{c_{44}} \right) \right], \\ e &= \rho_s^2 \omega^4 \left(\frac{1}{c_{33}} - \frac{1}{c_{44}} \right)^2. \end{aligned} \quad (17)$$

The unknown functions A_m , B_m , C_m , D_m , E_m , and F_m will be determined from the boundary conditions.

$$s_{1,2} = \sqrt{\frac{-(c_{13}^2 + c_{13}c_{44} - c_{11}c_{33}) \pm \sqrt{(c_{13}^2 + c_{13}c_{44} - c_{11}c_{33})^2 - 4c_{33}c_{11}c_{44}^2}}{2c_{33}c_{44}}} \quad (10)$$

λ_1 , λ_2 , and λ_3 are multi-valued functions and are made single-valued by specifying the branch cuts emanating from the branch points $\xi_{\lambda_1} = \pm\omega\sqrt{\rho_s/c_{11}}$, $\xi_{\lambda_2} = \pm\omega\sqrt{\rho_s/c_{44}}$, and $\xi_{\lambda_3} = \pm\omega\sqrt{\rho_s/c_{66}}$ on the complex ξ -plane, such that the real parts of λ_1 , λ_2 , and λ_3 remain non-negative for all values of ξ [10]. The positive sign of these branch points denotes the wave numbers of the body waves, P, SV, and SH, respectively. Unlike an isotropic media, transverse isotropy of the medium causes different velocities for SV and SH waves in this kind of solids [5]. The Fourier components of the transformed stress and displacement can be expressed as [10]:

$$\begin{aligned}\tilde{u}_{z_m}^m &= \left[\alpha_2 \frac{d^2}{dz^2} + \frac{\rho_s \omega^2}{c_{66}} - \xi^2 (1 + \alpha_1) \right] \tilde{F}_m^m, \\ \tilde{u}_{r_m}^{m+1} + i \tilde{u}_{\theta_m}^{m+1} &= \alpha_3 \xi \frac{d \tilde{F}_m^m}{dz} - i \xi \tilde{\chi}_m^m, \\ \tilde{u}_{r_m}^{m-1} - i \tilde{u}_{\theta_m}^{m-1} &= -\alpha_3 \xi \frac{d \tilde{F}_m^m}{dz} - i \xi \tilde{\chi}_m^m, \\ \tilde{\sigma}_{z r_m}^{m+1} + i \tilde{\sigma}_{z \theta_m}^{m+1} &= c_{44} \xi \left[(\alpha_3 - \alpha_2) \frac{d^2}{dz^2} + \xi^2 (1 + \alpha_1) \right. \\ &\quad \left. - \frac{\rho_s \omega^2}{c_{66}} \right] \tilde{F}_m^m - c_{44} \xi i \frac{d \tilde{\chi}_m^m}{dz}, \\ \tilde{\sigma}_{z r_m}^{m-1} - i \tilde{\sigma}_{z \theta_m}^{m-1} &= -c_{44} \xi \left[(\alpha_3 - \alpha_2) \frac{d^2}{dz^2} + \xi^2 (1 + \alpha_1) \right. \\ &\quad \left. - \frac{\rho_s \omega^2}{c_{66}} \right] \tilde{F}_m^m - c_{44} \xi i \frac{d \tilde{\chi}_m^m}{dz}.\end{aligned}\quad (18)$$

For the compressible liquid, the governing equation of time-harmonic motion in cylindrical coordinate system is [1]:

$$\frac{K}{\rho_l} \nabla^2 \varphi(r, \theta, z) = -\omega^2 \varphi, \quad (19)$$

where φ , K , and ρ_l are the displacement potential, bulk modulus, and density of the liquid, respectively. The displacement components and pressure can now be expressed in terms of φ as:

$$\begin{aligned}u_r &= \frac{\partial \varphi}{\partial r}, \quad u_z = \frac{\partial \varphi}{\partial z}, \quad u_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta}, \\ P &= -K \nabla^2 \varphi = \rho_l \omega^2 \varphi.\end{aligned}\quad (20)$$

Similar to the solid substratum, the procedure of Fourier series expansion is followed with respect to the angular direction, and applying m th-order Hankel transform is followed with respect to the radial direction:

$$\varphi(r, \theta, z) = \sum_{-\infty}^{\infty} \varphi_m(r, z) e^{im\theta}, \quad (21)$$

$$\tilde{\varphi}^m(\xi, z) = \int_0^\infty \varphi(r, z) r J_m(r\xi) dr, \quad (22)$$

where the inverse transform can be used to return to the real domain as:

$$\varphi(r, z) = \int_0^\infty \tilde{\varphi}^m(\xi, z) \xi J_m(r\xi) d\xi. \quad (23)$$

The same is followed for stress and displacement components. The respective ordinary differential equation of motion in terms of $\tilde{\varphi}_m^m$ can be obtained as:

$$\left(\frac{\rho_l \omega^2}{K} - \xi^2 + \frac{d^2}{dz^2} \right) \tilde{\varphi}_m^m = 0. \quad (24)$$

A general solution to this equation is:

$$\tilde{\varphi}_m^m = S_m(\xi) e^{-\lambda_4 z} + R_m(\xi) e^{\lambda_4 z}, \quad (25)$$

where:

$$\lambda_4 = \sqrt{\xi^2 - \frac{\omega^2}{c_l^2}}. \quad (26)$$

Here, $c_l = \sqrt{K/\rho_l}$ is the velocity of the compressional wave in the liquid, and S_m and R_m will be calculated from the boundary conditions, where R_m is associated with the amplitude of the transmitting wave from the bottom boundary, and S_m is associated with the reflecting wave amplitude from the free surface of the liquid.

Fourier factors of stress and displacement component in the transformed mode are written as:

$$\begin{aligned}\tilde{\sigma}_{zz} &= -\tilde{P} = -\rho_l \omega^2 \tilde{\varphi}_m^m, \\ \tilde{u}_{r_m}^m &= \xi \left[\frac{m+1}{2m} \tilde{\varphi}_m^{m-1} - \frac{m-1}{2m} \tilde{\varphi}_m^{m+1} \right], \\ \tilde{u}_{z_m}^m &= \frac{\partial \tilde{\varphi}_m^m}{\partial z}, \\ \tilde{u}_{\theta_m}^m &= \frac{i\xi}{2} (\tilde{\varphi}_m^{m-1} + \tilde{\varphi}_m^{m+1}).\end{aligned}\quad (27)$$

3. Boundary conditions and frequency equation

The boundary conditions for the problem under consideration are given below. The expressions provided in Eqs. (18) along with Eqs. (27) are utilized. The coefficients A_m , C_m , and E_m are omitted according to the radiation condition.

The free surface condition requires the vanishing of the liquid pressure, P :

$$\sigma_{zz}(z=0)=0. \quad (28)$$

This imposes:

$$S_m(\xi) = -R_m(\xi). \quad (29)$$

At the interface of the solid and liquid, i.e. at $z = h$, the boundary conditions imply continuity of the normal component of the displacements of the two media:

$$u_z(z=h^-) = u_z(z=h^+), \quad (30)$$

which means:

$$\begin{aligned} & (e^{-\lambda_1 h}) \left[\alpha_2 \lambda_1^2 + \left(\frac{\rho_s \omega^2}{C_{66}} - \xi^2(1 + \alpha_1) \right) \right] B_m(\xi) \\ & + (e^{-\lambda_2 h}) \left[\alpha_2 \lambda_2^2 + \left(\frac{\rho_s \omega^2}{C_{66}} - \xi^2(1 + \alpha_1) \right) \right] \\ & D_m(\xi) + \lambda_4 \left(S_m(\xi) e^{-\lambda_4 h} - R_m(\xi) e^{\lambda_4 h} \right) = 0. \end{aligned} \quad (31)$$

Meanwhile, since the liquid does not support shear stresses, the two tangential stress components, $\sigma_{z\theta}$ and σ_{rz} , vanish at the interface:

$$\sigma_{z\theta}(z=h)=0, \quad (32)$$

$$\sigma_{rz}(z=h)=0. \quad (33)$$

These two conditions yield to the following equations, respectively:

$$[2C_{44}\xi i\lambda_3 e^{-\lambda_3 h}] F_m(\xi) = 0, \quad (34)$$

$$\begin{aligned} & \left\{ \left[2C_{44}(\alpha_3 - \alpha_2)\lambda_1^2 + \xi^2(1 + \alpha_1) - \frac{\rho_s \omega^2}{C_{66}} \right] e^{-\lambda_1 h} \right\} B_m(\xi) \\ & + \left\{ \left[2C_{44}(\alpha_3 - \alpha_2)\lambda_2^2 + \xi^2(1 + \alpha_1) \right. \right. \\ & \left. \left. - \frac{\rho_s \omega^2}{C_{66}} \right] e^{-\lambda_2 h} \right\} D_m(\xi) = 0. \end{aligned} \quad (35)$$

The normal stress continuity is preserved at the solid-liquid interface:

$$\sigma_{zz}(z=h^-) = \sigma_{zz}(z=h^+), \quad (36)$$

$$\begin{aligned} & \left[\alpha_3 c_{13} \xi^2 + c_{33} \left(\frac{\rho_s \omega^2}{C_{66}} - \xi^2(1 + \alpha_1) \right) + c_{33} \alpha_2 \lambda_1^2 \right] \\ & (-\lambda_1 e^{-\lambda_1 h}) B_m(\xi) + \left[\alpha_3 c_{13} \xi^2 \right. \\ & \left. + c_{33} \left(\frac{\rho_s \omega^2}{C_{66}} - \xi^2(1 + \alpha_1) \right) + c_{33} \alpha_2 \lambda_2^2 \right] \\ & (-\lambda_2 e^{-\lambda_2 h}) D_m(\xi) - \left[(\rho_s \omega^2)(S_m e^{-\lambda_4 h} \right. \\ & \left. + R_m e^{\lambda_4 h}) \right] S_m(\xi) = 0. \end{aligned} \quad (37)$$

Two separate sets of equations for the five remaining unknown functions, B_m , D_m , F_m , S_m , and R_m , are now provided. These equations constitute the matrix given by Eq. (38) as shown in Box II, where:

$$\begin{aligned} \eta_i &= (\alpha_3 - \alpha_2)\lambda_i^2 + \xi^2(1 + \alpha_1) - \frac{\rho_s \omega^2}{C_{66}}, \\ \vartheta_i &= \alpha_3 \lambda_i^2 - \eta_i, \\ \nu_i &= \left(\eta_i - \alpha_3 \frac{C_{13}}{C_{33}} \xi^2 - \alpha_3 \lambda_i^2 \right) \lambda_i, \quad i = 1, 2. \end{aligned} \quad (39)$$

The nontrivial solution is obtained when the determinant of the matrix vanishes, that is:

$$\det[A] = 0. \quad (40)$$

The expression for χ_m is separated from the other unknowns and reduces to $\lambda_3 = 0$. According to the assumed non-viscous nature of the liquid, no Love wave is expected to appear at the interface which is mathematically observed here, since no roots are obtained in the expression given for χ_m , except one determining the SH wave-number.

Eq. (40) provides the frequency equation with respect to the two-layer half-space; it relates ξ , the wave-number, to ω which is the angular frequency; it presents the dispersive properties of the system. For any value of ω , there is at least one root, $\xi_R > \max(\xi_{\lambda_1}, \xi_{\lambda_2})$, associated with the Rayleigh wave-number which satisfies the frequency equation. Dependent on the liquid layer

$$A = \begin{bmatrix} e^{-\lambda_1 h} \vartheta_1 & e^{-\lambda_2 h} \vartheta_2 & \lambda_4 e^{-\lambda_4 h} & -\lambda_4 e^{-\lambda_4 h} \\ e^{-\lambda_1 h} c_{33} \nu_1 & e^{-\lambda_2 h} c_{33} \nu_2 & -\rho_s \omega^2 e^{-\lambda_4 h} & -\rho_s \omega^2 e^{-\lambda_4 h} \\ 2\xi c_{44} \eta_1 e^{-\lambda_1 h} & 2\xi c_{44} \eta_2 e^{-\lambda_2 h} & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \quad (38)$$

depth, frequency of the excitations, and properties of the liquid and bed material, there might be other roots which are associated with the higher modes. Notwithstanding, only a single root may exceed the value ω/c_l which will be associated with the Stoneley- or Scholte-wave-number. If the compressional wave velocity of the liquid is higher than the SV wave velocity, the only possible root will be related to the Stoneley wave and no higher modes will appear.

4. Special cases

To draw a comparison between the results obtained here with some existing ones, several special cases of the problem are now considered:

- **Case 1.** If ξh tends to infinity or equivalently the liquid layer depth takes very large values compared to the wavelength, the velocity of the surface waves tends to the Stoneley - or Scholte-wave velocity, propagating in a bi-material full space consisting of liquid and transversely isotropic solid half-spaces. This results in the wave-number, ξ_{st} , of Stoneley wave showing exact agreement with the frequency equation presented by Abubakar and Hudson [15]:

$$\rho_l \omega^2 (\eta_2 \vartheta_1 - \eta_1 \vartheta_2) + c_{33} \lambda_4 (\eta_2 \nu_1 - \eta_1 \nu_2) = 0. \quad (41)$$
- **Case 2.** If ξh tends to zero, meaning that the wave-length becomes very large in comparison to the depth of the liquid layer, the frequency equation reduces to that of a Rayleigh wave for a transversely isotropic solid half-space developed by Khojasteh et al. [10], i.e. $\eta_1 \nu_2 - \eta_2 \nu_1 = 0$.
- **Case 3.** If the solid material is considered to be isotropic, the wave-numbers associated with the body waves become $\xi_P = \omega \sqrt{\rho_s / (2\mu + \lambda)}$ and $\xi_{SH} = \xi_{SV} = \omega \sqrt{\rho_s / \mu}$, and the frequency equation reduces to the equation developed by Ewing [3] for an isotropic bed material underlying a liquid layer.
- **Case 4.** If the solid material is reduced to an isotropic medium and ξh tends to zero, the Rayleigh wave velocity for isotropic half-space is obtained. As a more specific case, where $\lambda = \mu$, the well-known function for Rayleigh wave velocity is deduced [10] as: $c_R = \frac{2}{\sqrt{3+\sqrt{3}}} \sqrt{\frac{\mu}{\rho_s}}$. Also, if the solid is simplified to an isotropic case and ξh becomes very large, the interface Stoneley wave is obtained for a full-space bi-materials system consisting of liquid and isotropic solid half-spaces. These special cases are numerically verified in the following section.

5. Numerical results and discussion

The derived frequency equation in the previous section implicitly expresses the wave-number, ξ , in terms of ω ,

Table 1. Material properties.

Material	E_v/E_h	ν_h	\bar{c}_{11}	\bar{c}_{12}	\bar{c}_{13}	\bar{c}_{33}	$\varepsilon(\%)$
1 ^a	1	0.25	3.00	1.00	1.00	3.00	0
2 ^b	2	0.33	3.00	1.12	1.03	5.51	22.78
3 ^b	3	0.17	2.65	0.52	0.79	7.91	33.25
4 ^b	4	0.25	2.72	0.72	0.86	12.93	39.48

^aIsotropic; ^bTransversely isotropic, $\rho_s = 2500 \text{ kg/m}^3$, $\bar{c}_{ij} = c_{ij}/c_{44}$, and $c_{44} = 10 \text{ GPa}$.

which provides the surface waves velocity. By means of a computer program written in Mathematica, the roots of the equation are calculated and the dispersion curves are obtained consequently. The non-dimensional phase and group velocities of the surface wave are plotted versus another dimensionless variable, $\gamma = \xi_n c_n h / 2\pi c_l$, in semi-logarithmic figures. Four solid materials, whose elastic properties are presented in Table 1, are considered as the underlying medium. The elastic constants are normalized with respect to c_{44} . In this table, the materials are sorted with respect to the anisotropy parameter, ε , which is defined as follows [23]:

$$\varepsilon = \frac{c_{11} - c_{33}}{2c_{33}} \times 100. \quad (42)$$

Material 1, shown in Table 1, is an isotropic material; hence, its anisotropy parameter is equal to zero. In all configurations, the liquid is taken to be water with the properties: $\rho_l = 1000 \text{ kg/m}^3$ and $K = 2.2 \text{ GPa}$.

The positive definiteness of the strain energy is considered when defining the material. The elasticity constants should comply with the requirements below [21]:

$$c_{11} > |c_{12}|, \quad (c_{11} + c_{12})c_{33} > 2c_{13}^2, \quad c_{44} > 0. \quad (43)$$

The phase velocity associated with the first mode, and the next three higher modes for systems consisting of half-space of kind Materials 2 and 4 (Table 1) underlying a water layer are provided in Figures 2

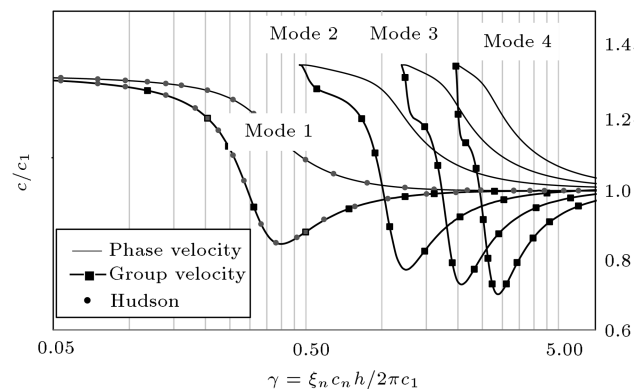


Figure 2. Phase and group velocities for the first four modes of Material 4 underlying a water.

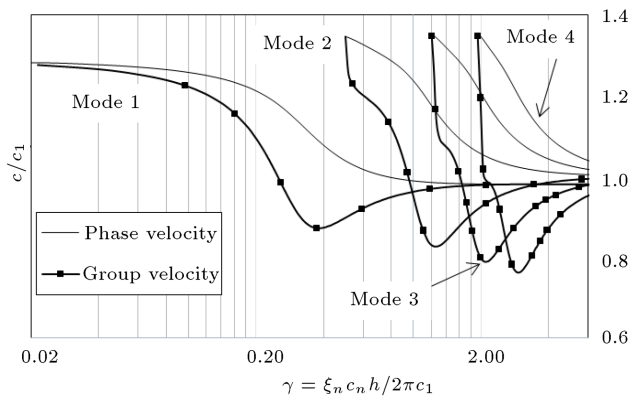


Figure 3. Phase and group velocities for the first four modes of Material 2 underlying a water layer.

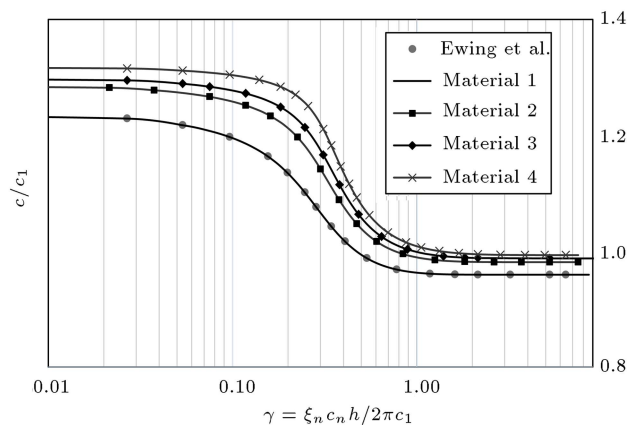


Figure 4. Phase velocity of the first modes for four bed materials as the bed.

and 3. The results show exact agreement with those presented by Abubakar and Hudson [15]. It can be observed that the higher modes take a maximum value of cutoff phase velocity equal to the velocity of SV wave propagation in the solid substratum, i.e. $c_{SV} = \sqrt{c_{44}/\rho_s}$. Nevertheless, for the first mode, it takes the Rayleigh wave velocity for the respective vacuum-solid half-space system.

The dimensionless phase velocities of the first modes of three transversely isotropic materials along with one isotropic material underlying a finite water layer are shown in Figure 4. It can be observed that the curve for the isotropic material is in exact agreement with the results presented by Ewing et al. [3]. An isotropic solid can be degenerated from a transversely isotropic one appropriately setting the elasticity constants as $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, and $c_{44} = c_{66} = \mu$. In an isotropic material, $\lambda_1 = \sqrt{\xi^2 - \xi_P^2}$, and $\lambda_2 = \lambda_3 = \sqrt{\xi^2 - \xi_S^2}$.

As it is apparent in Figures 2, 3, and 5, all values of the phase velocities for higher modes fall above the value c_l , whereas, for the normal mode, there are values of the velocity less than c_1 which tend to the asymptotic value of Stoneley or Scholte wave velocity. Stoneley

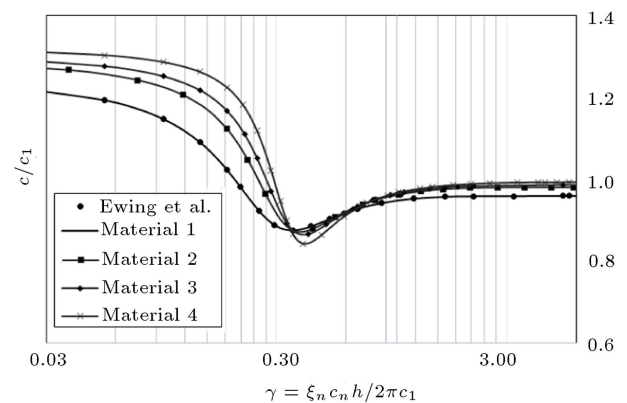


Figure 5. Group velocities of the first modes for four bed materials as the bed underlying a layer of water.

waves can always arise in the systems of contacting liquid and solid media. Nevertheless, it should be noted that in contacting solid layers, the possibility of this type of surface waves depends on the properties of the layers [3]. The Stoneley branch of the curve arises in a higher value of ξh as the elastic constants of the material increase. Normal mode group velocities for the defined materials as the solid half-space are plotted in Figure 5 for comparison. Unlike the phase velocity, the group velocity takes a minimum value (Airy phase) which occurs at a higher ξh value in the higher modes. All the figures indicate that the velocities of SV wave in the solid domain and the P wave in the liquid mainly control the range of the values that the dispersion curves covers.

6. Summary and conclusion

In this paper, dispersion of generalized Rayleigh waves is studied in a model of a homogeneous compressible finite liquid layer lying over a transversely isotropic solid half-space. Hankel transform and Fourier series expansion along with the method of potential functions are used to deal with the equations of motion. The dispersive nature of the surface waves is described through the derived frequency equation. Various special cases are studied to confirm the formulation and numerical results with the existing solutions. In this regard, both the mathematical formulation and numerical results are derived for several simpler half- and full spaces degenerated from the most general case. The results are compared and confirmed with the existing ones. The figures clearly show that the material anisotropy can play an important role in the dispersive characteristics of surface waves. The considerable influence of the liquid layer depth relative to the wave-length, as well as the bed material properties, on the characteristics of the phenomenon as well as the elastic properties of the bed can be observed from the figures presented for different bed materials. The velocity dispersion curves

reveal the fact that the difference between the velocities of SV and SH waves in a VTI solid is important for the lower limit of the velocities of higher modes. As in this case, the lower limit is the SV wave velocity which is different from the SH wave velocity. This indicates the importance of the anisotropy, as the SV and SH wave velocities may significantly differ in the highly anisotropic materials, and hence in the velocity dispersion curves. Consequently, neglecting the effect of the anisotropy may clearly results in considerable inaccuracies.

The obtained results can be efficiently extended to calculate the elastodynamic response of described system subjected to dynamic excitations which is being processed by the authors and can be effectively applied to solve bed-fluid-structure problems.

Nomenclature

c	Dispersive wave velocity;
c_{ij}	Elasticity constants;
c_l	Velocity of compressional wave in the liquid;
c_n	n th mode interface wave velocity;
c_R	Rayleigh wave velocity;
c_{SV}	SV wave velocity;
E_h	Young's moduli in the direction normal to the plane of transverse isotropy;
E_v	Young's moduli in the plane of transverse isotropy;
F	Potential function;
h	Liquid layer depth;
J_m	Bessel function of the first kind and m th order;
K	Liquid bulk modulus;
k_p	Dilatational wave-number;
k_s	Distortional wave-number;
m	Hankel integral transform order;
P	Liquid dynamic pressure;
r	Radial coordinate;
s_1, s_2	Roots of strain energy function;
t	Time variable;
u_r	Displacement component in r -direction;
u_z	Displacement component in z -direction;
u_θ	Displacement component in θ -direction;
ε	Anisotropy parameter;
θ	Angular coordinate;
λ	Lame's constant;
μ	Lame's constant;

ν_h	Poisson ratio characterizing the effects of horizontal strain on complementary horizontal strain;
ξ	Hankel's parameter and wave-number;
ξ_R	Rayleigh wave-number;
ξ_{st}	Stoneley wave-number;
ξ_n	n th Mode wave-number;
ρ_l	Liquid density;
ρ_s	Solid density;
σ_{ij}	Stress components;
χ	Potential function;
ω	Frequency.

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