Vibration Behavior of Misaligned Rotor with the Asymmetric Shaft Using Timoshenko Beam Theory

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Abstract

In the present study, an accurate model is used for analyzing an asymmetric shaft with misalignment and under unbalance forces in rotating coordinates verified by developing an experimental test. The asymmetric rotor is modeled by the Timoshenko beam theory. The differential equations of motion are discretized by Rayleigh-Ritz method and derived from Hamilton's principle. Thereafter, According to the results, a frequency range is detected in which, resonance and instability will occur. The dynamic behavior of the rotor is investigated near the boundary of the instability region. Various parameters, including unbalanced and misalignment induced forces, and the dimensions of the shaft are taken into account. Finally, vibration responses of the system and their Fast Fourier Transform (FFT) are presented graphically to determine the frequencies of the harmonic responses. It is concluded that the asymmetry of the shaft and misalignment fault severely affect the dynamic response of the rotor. Moreover, the accuracy of the results is increased by applying the Timoshenko beam theory. By developing an experimental test for the rotor system with misalignment fault, verifying this model has been carefully done. The experimental results obviously obtained the unstable operating area as acquired by the differential equations of motion.

Key words: Asymmetric Rotor, Timoshenko Beam Theory, Misalignment, mass unbalance, Vibration
1. Introduction

With the growing trend of deriving a proper analytical approach for rotor systems, which are very important in the design, identification, and control of such systems, the need for presenting more reliable mathematical models in rotor dynamics is increasing [1]. Also with the use of a high-speed rotor with a thick shaft, it is important to understand the dynamic behaviors of the system under misalignment and mass unbalance excitations [2]. There are several factors that cause misalignment in rotor system that some of them are differential thermal increase of systems, asymmetry in applied loads, unequal base settlement [3]. The problem of modeling the rotors under the misalignment forces was addressed by a number of studies. However, an analytical model that satisfactorily addresses the dynamic behavior of an asymmetric rotor with the Timoshenko beam model, under the influence of unbalance and misalignment, is still highly demanded. Gibbons [4] presented a model for deriving the reaction forces and moments generated of misaligned couplings as shown in Fig A1. An experimental study on the
misalignment influence on cylindrical and three-lobe journal bearings was carried out by Prabhu [5]. S. Ganesan and C. Padmanabhan [6] considered two shafts with flexible coupling under parallel misalignment forces. The response of the system under the influence of parametric excitations has been found and their effects are obvious at $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ of the principal parametric resonance. Zhao Wan et al. [7] analyzed the dynamic behavior of a multi-disk rotor under the misalignment force and the nonlinear oil film force in bearings theoretically and experimentally. In the frequency response, the frequencies of 2X, 3X, 4X were seen due to the misalignment coupling. Redmond and Al-Hussain [8] and [9] studied the lateral and torsional behavior of two shafts with the parallel misalignment coupling. Hujare and Karnik [10] analyzed the Aluminum shaft rotor system with parallel misalignment coupling using FEA. Verma et al. [11] investigated Misalignment Effects on Rotor Shaft Vibration and Stator Current Signature experimentally. Ma et al. [12] analyzed the instability of an overhung rotor under parallel and angular misalignment forces in the run-up and run-down period. Jalan and Mohanty [13] identified the effect of misalignment and unbalance in response of a rotor system experimentally using model-based method. They showed 2X vibration response as the misalignment result. Sudhakar and Sekhar [14] summarized the different studies on coupling misalignment modeling. As shown in previous studies, the characteristics of a rotor with a misalignment defect can be detected in vibration magnitude at a speed of 2X in FFT analysis. Wang and Jiang [15] investigated the vibration features of a dual-rotor with unbalanced and misaligned coupling numerically and experimentally. Gao et al. [16] investigated the vibration response of a rotor-bearing system under misalignment forces and integral squeeze film damper (ISFD). Wang and Gong [17] analyzed the dynamic behavior of coupling misalignment between two rotors. In the case of rotors with asymmetric shafts, Tondl [18] investigated the dynamic behavior of these shafts with
different stiffness and presented the boundaries of instability. Also, Badlani et al. [19] analyzed the stability of this asymmetric rotor by modeling it. Srinath [20] investigated instability of an asymmetric rotor. He utilized complex model that consist of higher degrees of freedom. Hili et al. [21] analyzed the stability and dynamic behavior of an unbalanced and asymmetric rotor to indicate the effect of dynamic features of Active Magnetic Bearings (AMBs). Raffa and Vatta[22] derived the differential equations of motion for an asymmetric shaft using Timoshenko beam theory. Jamshidi and Jafari [23] analyzed the dynamic response of a rotor with an asymmetric shaft subject to unbalance and misalignment forces. They extracted differential equations of motion using the Euler-Bernoulli theory, considering the effect of high order large deformations. Khadem and Shahgholi [24] studied the stability of steady-state response of the asymmetric shaft, considering the effects of dissimilar mass moment of inertia. They used the multiple scales method.

Feng et al. [25] investigated the vibration behavior of an asymmetric rotor under misalignment force. They observed that the maximum amplitude occurs near half of the natural frequency $\frac{\omega_n}{2}$, as a result of asymmetry in the shaft, while the dominant frequency can be seen near $\omega_n$ when subject to parallel misalignment. Liet al. [26] studied the nonlinear vibration behavior of the asymmetric with parallel misalignment between two rotors. The responses showed that the super-synchronous component in the frequency of 2X is important to identify and diagnose misalignment of the rotor. Rahi [27] used the improved couple stress theory to see the size effect on dynamic response in a micro drill that is under the effect of a mounted mass at its free end.

Soheili and Farshidianfar [28] analyzed the influence of shaft rotation on its natural frequency. After deriving the equation of motion, it is solved by using the new analytical method. Wang et al. [29] introduced a new model for analyzing a rotor bearing system using rolling element
bearings. Matteo and Erasmo [30] used high-order finite beam elements to investigate the stability and transient response of un-symmetric rotors. They analyzed the influence of angular acceleration on the response of a realistic gas turbine under anisotropic bearings. H.Jamshidi, P. Jamshidi and Jafari [31] and [32] studied accurate models using asymmetric shaft with misalignment and under unbalance forces in rotating coordinates verified by developing an experimental test. They also utilized Timoshenko theory to improve the accuracy of the results. Bab, Khadem, Abbasi and Shahgholi. [33] investigated dynamic stability and nonlinear vibration analysis of a rotor system with flexible/rigid blades. They improved the method using numerical method. Jiaqiao Lu [34] modeled coupling of rotors as cantilever dual-rotor system for the first time. The proposed method in this study can provide guidance for analyzing coupling of rotors. In some studies, vibration behavior of a rotor under misalignment forces is investigated by modeling the asymmetric shaft as Euler-Bernoulli beam model. Some other researchers have considered the derivation of differential equations of motion of a rotor with an asymmetric Timoshenko shaft in the rotating coordinate, without modeling misalignment forces and considering their effects on dynamic behavior of the rotor.

The present study intends to introduce a model for a rotor with Timoshenko asymmetric shaft, under mass unbalance and misalignment induced forces. Effects of parameters such as the dimensions of the cross-section of the shaft, mass moment of inertia, unbalanced forces and misalignment forces are considered. For deriving the differential equations of motion for this system, Hamilton’s principle is used and the coupled equations are solved by numerical method, thereafter. Finally, Campbell curves are plotted for two shafts with different cross-sections. According to the obtained equations and the results, one can predict the influences of the
asymmetry in shafts and the mass unbalance induced force on the dynamic response of the rotor system.

2. The Dynamic Model and Differential Equations of Motion

Fig. 1 represents a model of a flexible rotor connected to an electric motor through a coupling. This rotor consists of a disk mounted at the distance \( L_2 \) from bearing 2. The cross-section of the shaft is rectangular with length \( L \). The mass of the disk is denoted by \( M_D \), and is assumed to be rigid with radius \( R_D \).

The displacements of the rotating shaft are denoted by \( u(z,t), v(z,t), \) and \( w(z,t) \), where these variables refer to the deflections of the shaft in an arbitrary location \( z \) in the \( x, y, \) and \( z \) directions, respectively. Misalignment is shown in A.1.

The transformation between the inertial coordinate \( x_0,y_0,z_0 \) and the rotating coordinate \( x,y,z \) is described by Euler angles \( \phi(z,t), \theta(z,t), \) and \( \psi(t) \) as shown in Fig. 2.

The angular velocities of the rotating shaft can be obtained by the Euler angles. The angular velocities can be derived as [3]:

\[
\dot{\omega} = \psi e_{x_0} + \dot{\theta} e_{y_0} + \dot{\phi} e_{z_0} = R_x R_y \psi (\tilde{k}) + R_y \dot{\psi} (\tilde{j}) + \dot{\phi} (\tilde{i}) = \\
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} R_y \dot{\psi} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} \dot{\phi} + \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Where \( R_x, R_y \) are rotational matrices about \( x \) and \( y \), respectively. Since torsional deformations are neglected and the rotational speed of the rotor is constant, the magnitude of \( \psi \) is equal to the rotational rotor speed \( \Omega \).

By solving Eq. 1, angular velocities \( \omega_x, \omega_y, \) and \( \omega_z \) are determined as follows:

\[
(2)
\]
For this asymmetric rotating shaft, which is modeled by the Timoshenko beam, kinetic and potential energies are derived in the rotating coordinate system in order to have a constant mass moment of inertia about x and y axes during the rotation.

The kinetic energy of the rotor consists of rotational and translational kinetic energies for the shaft and the disk:

\[
T_{\text{Rotor}} = T_{\text{Ro-S}} + T_{\text{Ro-D}} + T_{\text{Tr-S}} + T_{\text{Tr-D}}
\]  
(3)

The rotational kinetic energy of the rotor can be determined as follows:

\[
T_{\text{Ro-S}} = \frac{1}{2} \int_0^L \left(J_{\alpha x} \dot{\omega}_x^2 + J_{\alpha y} \dot{\omega}_y^2 + J_{\rho} \dot{\omega}_z^2\right) dz
\]  
(4)

Also, the rotational kinetic energy of the shaft and disk of this rotor, using Eq.2, is equal to:

\[
T_{\text{Ro-D}} = \frac{\rho L}{2} \left(\int_0^L \left(\dot{\omega}_x^2 - \Omega^2 \omega_x^2 - 2 \Omega^2 \Omega \dot{\phi} \dot{\phi}\right) dz + \right)
\]

\[
\frac{\rho A L}{2} \left(\int_0^L \left(\dot{\omega}_y^2 - \Omega^2 \omega_y^2 - 2 \Omega^2 \Omega \dot{\phi} \dot{\phi}\right) dz\right)
\]

\[
T_{\text{Ro,D}} = \frac{1}{2} J_{\alpha d} \left(\dot{\phi}^2 + \dot{\theta}^2\right) - \frac{1}{2} J_{\alpha d} \Omega^2 (\dot{\phi} \dot{\phi} + \dot{\theta} \dot{\theta}) - \frac{1}{2} J_{\alpha d} \Omega^2 (\dot{\phi}^2 + \dot{\theta}^2)
\]  
(5)

Where \(I_x = \frac{I_x + I_y}{2}\), \(\Delta I = \frac{I_y - I_x}{2}\), \(I_x\) and \(I_y\) are the moments of inertia about x and y axes respectively.

The translational kinetic energy of the rotor can be determined as follows:

\[
T_{\text{Tr}} = \frac{1}{2} \int_0^L \rho A \left(V_x^2 + V_y^2 + V_z^2\right) dz
\]  
(7)
Deformation vectors can be determined as follows [3]:

\[
\begin{bmatrix}
    u_p(z,t) \\
v_p(z,t) \\
w_p(z,t)
\end{bmatrix} = \begin{bmatrix}
u(z,t) \\
w(z,t)
\end{bmatrix} + \begin{bmatrix}
cos \theta & 0 & sin \theta \\
0 & 1 & 0 \\
-sin \theta & 0 & cos \theta
\end{bmatrix} \begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix}
\]

Where \( \begin{bmatrix} u(z,t) + e_x \\
v(z,t) + e_y \\
w(z,t) \end{bmatrix} \) is the position of the mass center of the disk, considering mass eccentricity distribution with respect to x, y, and z respectively, and \( \begin{bmatrix} u_p(z,t) \\
v_p(z,t) \\
w_p(z,t) \end{bmatrix} \) is the position of the mass center of the disk in \( x_1, y_1, \) and \( z_0 \) coordinate system, as shown in Fig2, and is called the rotating coordinate system. Translational velocities in the rotational coordinates can be determined as follows:

\[
\begin{bmatrix}
    V_1 \\
    V_2 \\
    V_3
\end{bmatrix} = \begin{bmatrix}
    \dot{u}_p \\
    \dot{v}_p \\
    \dot{w}_p
\end{bmatrix} + \Omega(\vec{k}) \begin{bmatrix}
u_p(\vec{i}) \\
v_p(\vec{j}) \\
w_p(\vec{k})
\end{bmatrix}
\]

Therefore, translational kinetic energy for this rotor is equal to:

\[
T_{Tr,s} = \frac{1}{2} M_s [\dot{u}^2 + \dot{v}^2 + \Omega^2 (u^2 + v^2) + 2\Omega (u\dot{v} - \dot{u}v)] dz
\]

\[
T_{Tr,B} = \frac{1}{2} M_p [\dot{u}^2 + \dot{v}^2 + \Omega^2 (u^2 + v^2) + 2\Omega (u\dot{v} - \dot{u}v)]_{z=L_4} +
M_p [\Omega^2 (v \epsilon_x + u \epsilon_y) - \Omega (\dot{u} \epsilon_x - \dot{v} \epsilon_y)]
\]

Where \( \mu_s = \rho A \).

Considering Timoshenko model, the strain energy for the shaft of this rotor can be obtained:

\[
U_s = \frac{1}{2} \int_0^{L_s} \left[ KGA \left( \frac{\partial u}{\partial z} - \theta \right)^2 + KGA \left( \frac{\partial v}{\partial z} + \phi \right)^2 + E(I - \Delta I) \left( \frac{\partial \phi}{\partial z} \right)^2 + E(I + \Delta I) \left( \frac{\partial \theta}{\partial z} \right)^2 \right] dz
\]
Where E, G, and K are Young’s shear modulus and Timoshenko shear coefficient, respectively. It should be noted that the Timoshenko shear coefficient is considered 0.8 in this study.

In order to model the misalignment induced forces, the reaction forces proposed by Gibbons [4] are used in this study, and the dynamic model of misalignment induced forces according to the model introduced by Jafari [23] for the inertial coordinate system is as follows:

\[
F_{mis,x} = f_{mis1}(\sin(\Omega t)+\sin(2\Omega t)) \tag{13}
\]
\[
F_{mis,y} = f_{mis2}(\cos(\Omega t)+\cos(2\Omega t))
\]

Where \( f_{mis1} \) and \( f_{mis2} \) are obtained according to Gibbons’ equations (Appendix A1).

By applying the transfer matrix to this force, the dynamic model of the misalignment forces applied in the rotating coordinate system will be obtained as follows:

\[
F_{mis,r} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{mis,x} \\ F_{mis,y} \\ 0 \end{bmatrix}
\]

\( \text{Using} \ (14) \)

That the work done by this force is equal to:

\[
W_{mis,x} = [\cos(\Omega t) f_{mis1}(\sin(\Omega t)+\sin(2\Omega t)) + \sin(\Omega t) f_{mis2}(\cos(\Omega t)+\cos(2\Omega t))]u(t_2) \tag{15}
\]
\[
W_{mis,y} = [-\sin(\Omega t) f_{mis1}(\sin(\Omega t)+\sin(2\Omega t)) + \cos(\Omega t) f_{mis2}(\cos(\Omega t)+\cos(2\Omega t))]v(t_2)
\]

It is worthy to note that the differences between the results in this paper and the work done by Jafari [23] are due to the coordinate systems used. The rotating coordinates is used in this study, while Jafari utilized global coordinates.

**Application of the Rayleigh-Ritz Method and Hamilton’s Principle**

In order to approximate the functions of \( u(z,t) \) and \( v(z,t) \), displacements in x and y directions, and \( \theta(z,t) \) and \( \phi(z,t) \), rotational angles about y and x axes, the Rayleigh-Ritz method can be used. In this method, the functions \( u, v, \theta, \) and \( \phi \) can be expressed as follows:
\( u(z, t) = a(t)\bar{u}(z), \quad v(z, t) = b(t)\bar{v}(z) \quad \text{(16)} \)
\( \theta(z, t) = c(t)\bar{\theta}(z), \quad \phi(z, t) = d(t)\bar{\phi}(z) \)

Where \( a(t), \; b(t), \; c(t), \; \text{and} \; d(t) \) are independent time functions. The chosen displacement is the exact first bending mode shape of a beam, simply supported at both ends; therefore, the shape modes can be assumed as [3]:

\[
\bar{u}(z) = \bar{v}(z) = \sin \frac{\pi z}{L}
\]
\[
\bar{\theta}(z) = \bar{\phi}(z) = \cos \frac{\pi z}{L}
\]

By applying the Hamilton’s principle to the kinetic and strain energies, we have:

\[
\delta \int_0^L (T_{r_{r-D}} + T_{r_{r-s}} + T_{R_{r-D}} + T_{R_{r-s}} - U_s + W_{m_s}) dt = 0
\]

(18)

Now, the differential equations of motion can be derived as:

\[
\begin{bmatrix}
M_{11} & M_{22} & M_{33} & M_{44} \\
K_{11} & K_{13} & K_{31} & K_{42} \\
K_{12} & K_{22} & K_{24} & K_{43} \\
K_{31} & K_{32} & K_{33} & K_{44}
\end{bmatrix}
\begin{bmatrix}
\ddot{r} \\
\dot{\omega}
\end{bmatrix}
\begin{bmatrix}
C_{11} & C_{22} \\
-\Omega^2 & N_{33}
\end{bmatrix}
\begin{bmatrix}
\ddot{r} \\
\dot{\omega}
\end{bmatrix}
\begin{bmatrix}
-G \\
0
\end{bmatrix}
\]

(19)

where \([r] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\) and the components of the matrices in Eq.(19) are as follows:
\begin{align}
M_{11} &= \int_0^{L_2} \mu_s \bar{u}_c^2 \, dz + M_B \bar{u}_{(L_1)}^2, \\
M_{22} &= \int_0^{L_2} \mu_s \bar{v}_c^2 \, dz + M_B \bar{v}_{(L_1)}^2, \\
M_{33} &= \int_0^{L_2} \rho(I - \Delta I) \bar{\theta}_c^2 \, dz + J_{D_y} \bar{\theta}_{(L_1)}^2, \\
M_{44} &= \int_0^{L_2} \rho(I + \Delta I) \bar{\phi}_c^2 \, dz + J_{D_y} \bar{\phi}_{(L_1)}^2, \\
C_{11} &= C_{22} = c_i, \\
G &= \int_0^{L_2} \mu_s \bar{V}_c^2 \, dz + M_B \bar{V}_{(L_1)}^2 \\
K_{11} &= -\int_0^{L_2} KG_{\bar{u}_c} \bar{u}_c \, dz, \quad K_{12} = \int_0^{L_2} KG_{\bar{v}_c} \bar{v}_c \, dz, \\
K_{22} &= -\int_0^{L_2} KG_{\bar{V}_c} \bar{V}_c \, dz, \quad K_{23} = \int_0^{L_2} (KG_{\bar{\theta}_c} \bar{\theta}_c - E(I + \Delta I) \bar{\theta}_{(L_1)} \bar{\theta}_{(L_1)}) \, dz, \\
K_{33} &= -\int_0^{L_2} KG_{\bar{\phi}_c} \bar{\phi}_c \, dz, \quad K_{34} = \int_0^{L_2} (KG_{\bar{\phi}_c} \bar{\phi}_c - E(I - \Delta I) \bar{\phi}_{(L_1)} \bar{\phi}_{(L_1)}) \, dz, \\
K_{42} &= \int_0^{L_2} KG_{\bar{\phi}_c} \bar{\phi}_c \, dz, \quad K_{44} = \int_0^{L_2} (KG_{\bar{\phi}_c} \bar{\phi}_c - E(I - \Delta I) \bar{\phi}_{(L_1)} \bar{\phi}_{(L_1)}) \, dz, \\
N_{11} &= \int_0^{L_2} \mu_s \bar{u}_c^2 \, dz + M_B \bar{u}_{(L_1)}^2, \quad N_{22} = \int_0^{L_2} \mu_s \bar{v}_c^2 \, dz + M_B \bar{v}_{(L_1)}^2, \\
N_{33} &= \int_0^{L_2} \rho(I - \Delta I) \bar{\theta}_c^2 \, dz + J_{D_y} \bar{\theta}_{(L_1)}^2, \quad N_{44} = \int_0^{L_2} \rho(I + \Delta I) \bar{\phi}_c^2 \, dz + J_{D_y} \bar{\phi}_{(L_1)}^2, \\
F_x &= [\cos(\Omega t) f_{\text{rot}}(\sin(\Omega t) + \sin(2\Omega t)) + \sin(\Omega t) f_{\text{rot}}(\cos(\Omega t) + \cos(2\Omega t))] \bar{u}_{(L_1)} + M_B \Omega^2 \epsilon_x \bar{u}_{(L_1)}, \\
F_y &= [-\sin(\Omega t) f_{\text{rot}}(\sin(\Omega t) + \sin(2\Omega t)) + \cos(\Omega t) f_{\text{rot}}(\cos(\Omega t) + \cos(2\Omega t))] \bar{v}_{(L_1)} + M_B \Omega^2 \epsilon_y \bar{v}_{(L_1)}.
\end{align}

**Numerical Study**

Two rotors with geometrical and material properties as introduced in Table 1 are considered. By substituting the term $\bar{r} e^{s t}$ in matrices in Equation (19) with $r$ and ignoring the excitation, the Campbell diagram is derived and shown in Fig. 3. It should be noted that $s$ is a complex number and has two parts. The real part and the imaginary part versus rotational speed are plotted separately and presented in Fig. 3 and Fig. 4 for two rotors with different cross-sections. The rotors 1 and 2 are used to compare the Timoshenko and Euler-Bernoulli beam theories for varying dimension of the shaft’s cross-section, as shown in Fig. 3 and 4.

In Fig. 3 and 4 (i), (B), four frequencies can be seen versus each rotational speed. This is due to the four variables in $[r]$ in the Timoshenko beam theory, where two of which usually will not
appear in the response due to the small amplitude of the high-frequency response [2], while In Fig 3 and 4 (D), there are only two frequencies versus rotational speed.

Due to the asymmetry in shaft, rotor system will be unstable in a certain frequency range [23]. In Fig. 3 (i), it is observed that in the rotational speed range between 1760 and 2320 (rad/s), the real part of the response consists of some positive values, which means that the rotor is unstable in this range. On the other hand, one of the diagrams corresponding to the imaginary responses has a zero magnitude in this frequency range, which means that the resonance is occurring in the system. In Fig 3 (ii), this frequency range for Euler-Bernoulli beam theory is between 1800 and 2360 (rad/sec).

In order to investigate the effects of shaft dimensions on this frequency range, the cross-sectional area is changed in case 2 as depicted in Fig. 4. In both Figs 4 (i) and (ii), it is observed that in the rotational speed range between 125 and 183 (rad/s), the real part of the response consists of some positive values, and also one of the diagrams of the imaginary responses has a zero magnitude. Furthermore, according to the reference [23], it can be seen that the instability of the rotor, which has a thin shaft of case 2, starts from the rotational speed of 1477 (RPM) or 155 (radian per second). The difference between the results is less than 3.5%.

In Fig. 5, high frequencies of Fig. 3 (B) and Fig. 4 (B) are neglected, in order to find out the differences between the results of Timoshenko and Euler Bernoulli’s theories more clearly. From the results, it is clear that the Euler-Bernoulli theory responses for the rotors with larger ratios of cross-section to length are not as accurate as those of the Timoshenko theory.
**Experimental model:**

The Experimental setup used in this research is a model of the Machinery Fault Simulator, shown in Fig.6. Test Rig is a rotor with the ability to install one or more discs connected via a flexible coupling to a variable drive motor. To simulate the effect of asymmetry in rotating equipment, a rectangular shaft has been designed and replaced the shaft with a circular cross-section to determine the effect of the rotor asymmetry in the designed experiments. The disk also has the ability to install a bolt to simulate the effect of mass unbalance in different radii to determine its effect on vibrations. To investigate the parallel misalignment, the shims will be placed underneath the Bearings No. 1 and 2 as shown in Fig6. Instrument Leonova Diamond will be used for data acquisition. MATLAB software is used for further analysis on the data acquired. The general characteristics of the device are given in Table 2. An experimental test is performed for further evaluation of rotor 2 as shown in Fig.7. In this test, mass unbalance and misalignment forces in rotor 2 are considered for finding the frequency range in which the rotor will become unstable. As shown in Fig. 6., it is clearly observed that the instability zone is between 1160 and 1690 (rpm), or between 121 and 176 (rad/s), where the difference between this observed frequency range and the values obtained from Eq. (19) is less than 5%.

Also, based on the results obtained from the theoretical method, it was noted that the instability frequency range for rotor 1 starts from 1760 (rad/s) or 16815 (rpm), which can only be achieved by a powerful motor, not usually available in academic facilities. Thereafter, by solving Eq. (19), the misalignment and mass unbalance response of the rotor will be derived. The Runge-Kutta numerical method is used to solve this equation. The differential equations of motion of an aligned rotor with unbalance mass are solved to investigate the effects of the unbalance mass, so that the effects of the misalignment can be studied. As introduced in Eq. (16), a and b are
independent time functions of the $u(z,t)$ and $v(z,t)$ respectively. The responses of the unbalanced rotor consist of three frequencies, as shown in Figs. 8 and 9, where the first one is equal to zero due to application of the constant unbalance force. The other two frequencies are due to the rotor’s critical speed in the rotating coordinate at $\Omega=3000$ (rad/s), as can be found in the imaginary plots of Figs. 3.(B) and (D) and the small effects of the high-frequency responses are neglected for the Timoshenko beam model. As shown in Fig. 3, although the responses of the rotor in both Timoshenko and Euler-Bernoulli beam are close together, but there are much less differences due to the accuracy of the Timoshenko model. Despite the fact that there is not much difference in the instability frequency range between the two theories, this small difference, as shown in Figs. 8 and 9, leads to a large difference in the frequency response of the system.

Solving Eq. (19) for rotor_1 and considering misalignment induced forces using the Timoshenko model, these effects can be seen in Fig.10. The parallel misalignment $\Delta X=0.1\text{mm}$ and $\Delta Y=0\text{mm}$ is considered for rotor_1 at $\Omega=3000$ (rad/s)[23].

In Fig.10, the responses include some frequencies due to the parallel misalignment. In part (B) of this figure, there is no zero frequency response, unlike the part (D), and this is because of applying the misalignment induced forces in Eq.(15). As it can be observed, the response of the parallel misalignment induced forces consists of the frequencies of 1X (478 Hz), 2X (956 Hz), 3X (1434 Hz) in the x-direction, and the frequencies of 0X, 1X, 2X, 3X in the y-direction. Also, there are two critical speed frequencies of 125(Hz), 680(Hz) in both directions. The rotor orbits are plotted for the three frequencies. The first one is less than the resonance frequency, the second one is within the resonance frequency range, and the third one is higher than the range, as shown in Figs.11 and12.
As it is found, the vibration amplitude is acceptable when the operating speed is out of the resonance frequency range and the rotor is unstable within this range.

Figs. 11(B) and 12(B) show a straight line, which means that the rotor has a straight motion from the observer's point of view in the rotating coordinate system. In other words, at any given moment, the observer sees that the rotor gets farther and farther away from him, and as shown in the Campbell diagrams, the observer notices zero frequency in the resonance frequency range.

3. Conclusion

Although numerous studies have been performed on the asymmetric rotors, Timoshenko shaft model has not been considered in some of the existing researches, and some others have not considered the effect of misalignment forces on dynamic behavior of the rotor. While in the present study, dynamic behavior of a rotor with a rectangular shaft modeled by Timoshenko beam theory is studied under influence of mass unbalance and misalignment induced forces in the rotating coordinates. Gibbons’ equations are applied for calculating the forces due to coupling misalignment.

By plotting Campbell diagrams, the natural frequencies of the system are investigated and it is found that there is a frequency range that leads to the system instability. Also, the effects of the cross-section on this frequency range are studied considering two rotors with different cross-sections. By comparing Timoshenko and Euler-Bernoulli beam theories, it is concluded that for thin shafts, both theories have the same results, while for thick shafts, the Euler-Bernoulli beam theory does not lead to accurate results. In this study, it is proved that using the Timoshenko’s beam theory can increase the accuracy of the results, especially in cases where the ratio of the cross-section to the length of the shaft is larger. Thereafter, vibration waveforms of the rotors and their Fast Fourier Transforms are plotted so that the frequencies of the vibrations are found.
to specify the behavior of the rotor and identify defects in the system. It is shown that response of the misaligned rotor consists of the frequencies of 0X, 1X, 2X, 3X, and the response of the unbalanced rotor consists of the frequency of 0X. Furthermore, orbit plots are figured for the rotors in three frequencies. As expected, it is shown that the vibration amplitude is acceptable when the operating speed is out of the resonance frequency range and the rotor is unstable within this range. Therefore, the results obtained in this study clearly showed the difference between the theory of Timoshenko and Euler-Bernoulli, and also from the comparison with previous studies conducted in the inertial coordinate system, we can see the difference between the results of this study and them.

To verify the theoretical results for the model, an experimental setup was constructed to test the asymmetric rotor connected to an electric motor with misalignment. The measurements and experimental results show that the resonance frequency range is close to what is obtained from the Campbell diagram.

4. Acknowledgements

We would like to express our deep gratitude to Dr. P. Jamshidi, Head of Turbomachinery Maintenance of the South Pars Gas Complex (SPGC) and Dr. F. Samsami, Faculty Member of West Tehran Branch, Islamic Azad University for their advices and assistances in keeping our progress on schedule and organizing the experimental tests.

We would also like to extend our thanks to the technicians of the laboratory of the South Pars Gas Complex (SPGC) Refinery Plant Phase 2&3 CBM Department for their help in offering us the resources in running the program and using their Machinery Fault Simulator to verify our theoretical results.
References


Appendices

A.1 Misalignment induced reaction forces and moments

For parallel misalignment:

\[ MX_1 = Tq \sin \theta_1 + Kz \phi_1, \quad MX_2 = Tq \sin \theta_2 - Kz \phi_2 \]
\[ MY_1 = Tq \sin \phi_1 - Kz \theta_1, \quad MY_2 = Tq \sin \phi_2 + Kz \theta_2 \]
\[ FX_1 = (-MY_1 - MY_2) / Z3, \quad FY_1 = (MX_1 + MX_2) / Z3 \]

Where,

\[ \theta_1 = \text{Arcsin}(\Delta X_1 / Z3), \quad \theta_2 = \text{Arcsin}(\Delta X_2 / Z3) \]
\[ \phi_1 = \text{Arcsin}(\Delta Y_1 / Z3), \quad \phi_2 = \text{Arcsin}(\Delta Y_2 / Z3) \]
Fig. 1. Flexible rotor system with rectangular cross-section shaft

Fig. 2. Transformation between inertial and rotating coordinate using Euler angles [3]

Table 1. Geometrical and material properties of rotor

<table>
<thead>
<tr>
<th>Shaft Properties</th>
<th>Cross-section’s Dimension</th>
<th>Case 1</th>
<th>Cross-section’s Dimension</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho = 7800kgm^{-3}$, $E = 200 \times 10^9 Nm^{-2}$, $c = 0.001$, $L_1 = 92mm$, $L = 240mm$, $L_2 = 320mm$</td>
<td>$sx = 20mm$, $sy = 15mm$</td>
<td>$sx = 4mm$, $sy = 6mm$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shaft Thin Ratio</td>
<td></td>
<td>Shaft Thin Ratio</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{20}{240} = \frac{1}{12}$</td>
<td></td>
<td>$\frac{6}{240} = \frac{1}{40}$</td>
</tr>
<tr>
<td>Disk Properties</td>
<td>$R_2 = 150mm$, $R_1 = 30mm$, $h = 10mm$, $J_{Dh} = J_{Dv} = M_p(3R_1^2 + 3R_2^2 + h^2)/12$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Fig. 3. Campbell diagram of the rotor-1 in the rotating coordinate (i) Timoshenko (ii) Euler-Bernoulli beam theory: (A) and (C) real part of responses, (B) and (D) imaginary part of responses.

Fig. 4. Campbell diagram of the rotor-2 in the rotating coordinate (i) Timoshenko (ii) Euler-Bernoulli beam theory: (A) and (C) real part of responses, (B) and (D) imaginary part of responses.
Fig. 5. Comparing Timoshenko and Euler-Bernoulli theories by plotting imaginary part of the responses considering (A) rotor-1, (B) rotor-2

Fig. 6. Machinery Fault Simulator

Table 2. Specifications of the rotor used in the test

<table>
<thead>
<tr>
<th>Rotor-Bearing-Pedestal Feature</th>
<th>Value (unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Rotor shaft</strong></td>
<td></td>
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<tr>
<td>Shaft material</td>
<td>steel</td>
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<tr>
<td>Shaft length between bearing</td>
<td>240 mm</td>
</tr>
<tr>
<td>Total Shaft length</td>
<td>362 mm</td>
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<tr>
<td>Shaft No. 2 cross section</td>
<td>circle</td>
</tr>
<tr>
<td>Diameter</td>
<td>19.05 mm</td>
</tr>
<tr>
<td>Shaft No. 2 cross section</td>
<td>rectangular</td>
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<tr>
<td>Length and width shaft cross section</td>
<td>4<em>6 mm</em>mm</td>
</tr>
<tr>
<td>Density of shaft material</td>
<td>7800 kg/ m³</td>
</tr>
<tr>
<td>Young’s modulus (E)</td>
<td>200 GPa</td>
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<tr>
<td><strong>Disk</strong></td>
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<tr>
<td>Diameter</td>
<td>151 mm</td>
</tr>
<tr>
<td>Mass</td>
<td>0.571 kg</td>
</tr>
<tr>
<td>Thickness</td>
<td>10 mm</td>
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<tr>
<td><strong>Motor</strong></td>
<td></td>
</tr>
<tr>
<td>Speed range</td>
<td>0–4000 rpm</td>
</tr>
<tr>
<td>Motor power rating</td>
<td>1.0 HP AC</td>
</tr>
<tr>
<td>Bearing No.</td>
<td>ER-10K</td>
</tr>
</tbody>
</table>
Fig. 7. Experimental run up test of rotor 2 considering misalignment and unbalance forces.

Fig. 8. Unbalance vibration response of the aligned rotor-1 considering Euler-Bernoulli beam model at $\Omega = 3000$ (rad/s) or $\Omega = 478$(Hz): (A) and (C) vibration waveforms, (B) and (D) FFT.
Fig. 9. Unbalance vibration response of the aligned rotor-1 considering Timoshenko beam model at $\Omega = 3000$ (rad/s) or $\Omega = 478$ (Hz): (A) and (C) vibration waveforms, (B) and (D) FFT.

Fig. 10. Vibration response of the balanced rotor-1 with the parallel misalignment considering Timoshenko beam model at $\Omega = 3000$ (rad/s) or $\Omega = 478$ (Hz): (A) and (C) vibration waveforms, (B) and (D) FFT.
Fig.11. Orbit plot of the rotor-1 with the parallel misalignment and unbalance forces A) $\Omega=1500$ (rad/s) B) $\Omega=1800$ (rad/s) C) $\Omega=3000$ (rad/s)

Fig.12. Orbit plots of the rotor-2 with the parallel misalignment and unbalance forces A) $\Omega=100$ (rad/s) B) $\Omega=200$ (rad/s) C) $\Omega=300$ (rad/s)

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