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On two-stepwise irregular graphs

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Abstract. A graph G is called irregular if the degrees of all its vertices are not the same. A graph is said to be *Stepwise Irregular* (SI) if the difference between the degrees of any two adjacent vertices is always 1. This paper deals with *2-Stepwise Irregular* (2-SI) graphs in which the degrees of every pair of adjacent vertices differ by 2. Here, we discuss some properties of 2-SI graphs and generalize them for k-SI graphs for which the imbalance of every edge is k. Besides, we also compute bounds of irregularity for the Albertson index in any 2-SI graph.

1. Introduction

In this paper, all the graphs considered are simple, undirected and connected. Let a simple graph $G(V, E)$, with vertex set $V(G)$ and edge set $E(G)$, be a structure in which some information can be stored by treating vertices as objects and edges as some relation between two objects. The number of vertices (i.e., the cardinality of $V(G)$) is called the *order* of the graph and is denoted by $|V(G)|$. If u and v are two vertices of G, then we denote uv as an edge with end vertices u and v. A graph G is said to be *bipartite* if we can partition the vertex set $V(G)$ into two sets such that vertices in one partition are adjacent to those of another partition and no two vertices in the same partition are adjacent. A bipartite graph is called a *complete bipartite graph* if each vertex of one partition is adjacent to every vertex of another partition. It is denoted as $K_{m,n}$, where m and n are the sizes of the partitions. A *tree* is a connected graph without any cycle (see the book by Bondy and Murty [1]).

Two vertices $u, v \in V(G)$ of a graph G are said to be *neighbors* of each other if there is an edge between u and v in G. The *degree* of any vertex u in graph G is the number of neighbors of u. It is denoted by $d(u)$ in [1] by Bondy and Murty. A vertex is said to be a *pendant* if its degree is 1. A graph is called *regular* if every vertex of the graph possesses the same degree. The graph in which all the vertices are not of the same degree is said to be an *irregular graph*. In order to define the irregularity and its extent in any graph, much research work has been done and is still in progress by several researchers [2–6]. In articles [7–9], the irregularity of the graph was discussed in more detail. For some basic properties of Stepwise Transmission Irregular (STI) graphs, please refer to [10–12] and their references.

Chemical Graph Theory (CGT) is a branch of mathematical chemistry in which the mathematical aspects of chemical compounds and their behaviors are being studied. In CGT, the vertices of the graph correspond to atoms of some chemical compounds and edges represent the chemical bonds between the atoms. Various researchers [13–16] discussed several chemical applications of graphs and the relation of graph theory with chemistry. Since irregularity is very common in molecular graphs, different approaches have been proposed so far to describe the measure of irregularity in chemical graphs. The *irregularity index* is a numerical value to evaluate the extent of

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irregularity in the whole graph. Several irregularity and topological indices were investigated for different types of graphs by several researchers [3,17–28]. Albertson [29] proposed an irregularity index of a graph G as $Irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)|$. This is the simplest kind of irregularity index. Various applications of the Albertson index are discussed in articles [13,29,30].

1.1. One-Stepwise Irregular (1-SI) graph

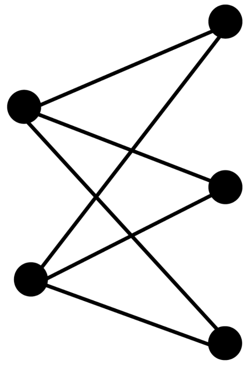
In [5], Ivan Gutman introduced a new class of graphs in each of which the difference in the degrees of any two adjacent pairs of vertices was always 1, i.e., $|d(u) - d(v)| = 1$ held for every adjacent pair of vertices u and v in the graph. Gutman defined these special kinds of graphs as *Stepwise Irregular* (SI) *graphs*. To be more specific, in this paper, we call this class of graphs as *one-Stepwise Irregular* (or *one-SI* or 1-*SI*) graphs, because, in each of these graphs, the degrees of every pair of adjacent vertices differ by 1. Some properties of SI graphs were investigated by Gutman [5]. Inspired by the above class of graphs proposed by Gutman [5], we generalize the idea of fixed degree difference for a value other than 1. As shown in Figure 1(a), the complete bipartite graph $K_{2,3}$ is an empirical example of a 1-SI graph.

1.2. Two-Stepwise Irregular (2-SI) graph and k-SI graph

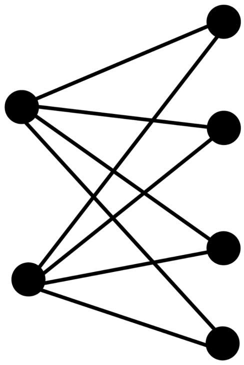
We define a *Two-Stepwise Irregular* (or *Two-SI* or 2-*SI*) graph in which the condition $|d(u) - d(v)| = 2$ holds for every pair of adjacent vertices u and v in the graph. The complete bipartite graph $K_{2,4}$, shown in Figure 1(b), is an intuitive and sensible example of a 2-SI graph. In a similar way, we extend the above definition to define a *k-Stepwise Irregular* (k-SI) graph in which the imbalance (that is, the degree difference) of every edge is k.

1.3. Our contribution

In this paper, in Section 2, we figure out some properties of this extended class of SI graphs. In Theorems 13 and 15, we have assigned methods to increase the number of vertices in a given 2-SI graph to attain another extended 2-SI graph. The existence of these types of graphs is illustrated in Theorem 21. Also, we generalize some of these properties and development ideas for k-SI graphs in Corollary 5, Corollary 7, Theorem 14, and Theorem 16. The last section computes the bounds for the Albertson index for 2-SI graphs.



(a) $K_{2,3}$



(b) $K_{2,4}$

Figure 1. Intuitive or practical examples of (a) a 1-SI graph and (b) a 2-SI graph.

2. Properties of 2-SI graphs and k-SI graphs

Theorem 1. *There does not exist any 2-SI graph with 1, 2, or 3 vertices.*

Proof. For the case having only 1 vertex, we have nothing to prove. For a simple and connected graph having two vertices, the degrees of both vertices are the same. Hence, no irregularity is there. Now, for a graph having 3 vertices, the argument is as follows. Observe that in any 2-SI graph, the least possible case of degree difference can be $3 - 1 = 2$. However, in any simple graph having three vertices, the maximum degree of any vertex can be just $2 < 3$. Hence, the proof. □

Theorem 2. *In a 2-SI graph, either all the vertices are of odd degree or all are of even degree.*

Proof. Without loss of generality, suppose that if possible, there exists a 2-SI graph in which at least one vertex is of odd degree and the rest of the vertices are of even degree. Since the graph is connected, there must be at least one odd degree vertex, say u, which is connected to an even degree vertex, say v, via an edge $e = uv$. However, observe that the imbalance of the edge $e = uv$ is always odd because the difference between an odd number and an even number is always an odd number. It is a contradiction for the case of 2-SI graphs. Hence, in a 2-SI graph, the degrees of all the vertices are either even or odd. □

Theorem 3. *Any 2-SI graph is always bipartite.*

Proof. Consider a 2-SI graph G and let u be one of its vertices. Observe that the degree of any vertex adjacent to u should be $d(u)\pm$ 2. Now, partition the vertex set $V(G)$ in such a way that the vertex u belongs to one partition set and vertices (if exists) of degree $d(u) + 2$ and $d(u) - 2$ both belong to another partition set because vertices of degree $d(u) + 2$ and $d(u) - 2$ cannot be adjacent in any 2-SI graph. Since the choice of the vertex u in G is arbitrary, we can decide the partition for every vertex in the graph. Hence, any 2-SI graph is always bipartite. □

However, the converse of the above theorem is not true, that is, every bipartite graph needs not be a 2-

SI graph. For example, consider a complete bipartite graph $K_{m,m+1},\ m > 0$ which is 1-SI graph, as shown by Das and Mishra [31].

Theorem 4. *Every complete bipartite graph of the form $K_{m,m+2},\ m > 0$ is a 2-SI graph.*

Proof. Since in $K_{m,m+2}$, there are m vertices each of degree $m+2$ and $m+2$ vertices each of degree m. There are only two degrees m and $m+2$ in the graph. Therefore, the vertex of degree m is always adjacent to a vertex of degree $m+2$, and vice versa. Hence, the imbalance of 2 is maintained. Therefore, the graph $K_{m,m+2},\ m > 0$ is a 2-SI graph.

Corollary 1. *Every complete bipartite graph of the form $K_{m,m+k},\ m, k > 0$ is a k-SI graph.*

Proof. This proof is similar to the above and, hence, left for the reader to prove. $\square$

Let us see the definitions of some basic graph products which will be useful for the next few theorems. The *Cartesian product* of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ along with the condition that two vertices $(g_1, h_1) \in V(G) \times V(H)$ and $(g_2, h_2) \in V(G) \times V(H)$ are adjacent if $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $g_1g_2 \in E(G)$ and $h_1 = h_2$. This graph product is denoted by $G\Box H$. See book by Hammack et al. [32]. Thus, we have:

$$V(G\Box H) = \{(g,h)|g \in V(G), h \in V(H)\},$$

$$E(G\Box H) = \{(g_1,h_1)(g_2,h_2)|g_1 = g_2, h_1h_2 \in E(H),$$

$$\text{or } g_1g_2 \in E(G), h_1 = h_2\}.$$

The *Corona product* of two graphs G and H, denoted by $G \circ H$, is obtained by taking one copy of graph G and $|V(G)|$ copy of graph H, and every vertex of the i-th copy of H is joined with the i-th vertex of G , where $1 \leq\ i\ \leq |V(G)|$. Tavakoli et al. [33] studied the corona product of graphs under some graph invariants. The *direct product* of the graph G and H is a graph in which $V(G) \times V(H)$ forms its vertex set and two vertices $(g_1, h_1) \in V(G) \times V(H)$ and $(g_2, h_2) \in V(G) \times V(H)$ are adjacent if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Thus, we have:

$$V(G \times H) = \{(g,h)|g \in V(G), h \in V(H)\},$$

$$E(G \times H) = \{(g_1,h_1)(g_2,h_2)|g_1g_2 \in E(G) \text{ and}$$

$$h_1h_2 \in E(H)\}.$$

Theorem 5. *The Cartesian product of two 2-SI graphs is again a 2-SI graph.*

Proof. Let G and H be two 2-SI graphs. We denote the cartesian product of the graphs G and H by $G\Box H$. We know that the degree of any vertex (a, b) in the product graph $G\Box H$ is given by $d_{G\Box H}(a,b) = d_G(a) + d_H(b)$. See more details in [34] by Maheswari and Maheswari. Of note, in $G\Box H$, any two vertices say (a, b) and (c, d) are adjacent if and only if:

(i) Either $a = c$, and b and d are adjacent in H,

(ii) Or $b = d$, and a and c are adjacent in G.

For case (i), since b and d are adjacent in H, $|d_H(b) - d_H(d)| = 2$. Therefore, $|d_{G\Box H}(a,b) - d_{G\Box H}(c,d)| = 2$. A similar argument holds for case (ii). Hence, the Cartesian product of two 2-SI graphs is also a 2-SI graph.

Corollary 2. *The Cartesian product of two k-SI graphs is also a k-SI graph, where $k > 0$.*

Proof. The proof of the corollary is left as an exercise for the reader.

Remark 1. *The Corona product of two 2-SI graphs needs not be a 2-SI graph. In Corona product of the graphs G and H, we take the graph G and $|V(G)|$ copy of graph H, and we join the i-th vertex of graph G to every vertex of the i-th copy of graph H, where 1$\leq i \leq |V(G)|$. Let $u \in V(G)$ be the i-th vertex of the graph G and v be one of the vertices of the graph H in the i-th copy. Thus, there must be an edge uv in graph $G\circ H$, by definition. Pattabiraman and Kandan [35] stated that the degree of such vertices u and v in the graph $G \circ H$ was given by $d_{G\circ H}(u) = d_G(u)+|V(H)|$ and $d_{G\circ H}(v) = d_H(v) + 1$ [35]. Therefore, in the graph $G \circ H$, the imbalance of the edge $uv = |d_{G\circ H}(u) - d_{G\circ H}(v)|$, which needs not be 2.*

Theorem 6. *The degree of every vertex in a 2-SI tree is always odd.*

Proof. We know that a tree with at least two vertices must have at least two pendant vertices. Also, by Theorem 2, we have the result that if any one vertex in a 2-SI graph possesses an odd degree, then every other vertex must also be of an odd degree. Since the tree has a vertex of degree 1 which is an odd number, every vertex in the tree is of odd degree. $\square$

Theorem 7. *The order of a 2-SI tree is always even.*

Proof. The above theorem states that every vertex in a 2-SI tree is of odd degree. Moreover, we know that the number of odd degree vertices in any graph is always even [1]. Hence, the order of a 2-SI tree is always even.

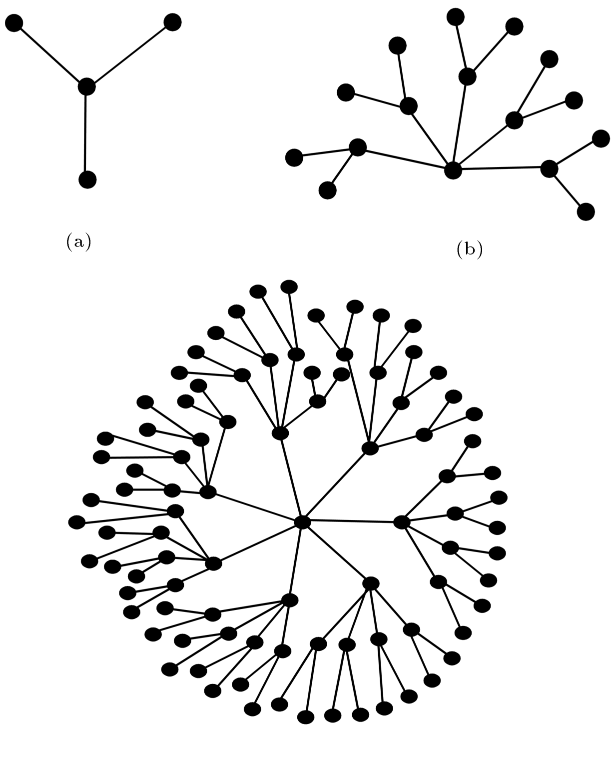


Figure 2. 2-SI trees with (a) $\triangle = 3$, (b) $\triangle = 5$, and (c) $\triangle = 7$.

Remark 2. *The 2-SI tree with maximal degrees 3, 5, and 7 is shown in Figure 2. The 2-SI trees with* $\triangle = 3, 5,$ *and 7 are of order 4, 16, and 92, respectively, where* $\triangle$ *denotes the maximal degree of the tree.*

A circuit/cycle in graph G is said to be an *Eulerian* circuit/cycle if the circuit is simple and contains every edge of G. An Eulerian graph is a graph containing an Eulerian cycle. For more details, please refer to the book [1].

Theorem 8. *For* $m \equiv 0 \pmod 2$, *the graph* $K_{m,m+2}$ *is always an Eulerian graph.*

Proof. Observe that the graph $K_{m,m+2}$ has m vertices each of degree $m+2$ and $m+2$ vertices each of degree m. If m is even, then the degrees of all the vertices in both of the partitions are of even degree. We know that a graph is Eulerian if and only if all the vertices are of even degree [1]. Hence, when m is even, $K_{m,m+2}$ is an Eulerian graph. □

Theorem 9. *If* G_0 *is a 2-SI graph of order* n *having a vertex of degree 1, then we can extend it to have a graph* G_t *of order* $n + 12t$, $t > 0$, *which is also a 2-SI graph.*

Proof. Let G_0 be any 2-SI graph and u be its vertex of degree 1. Therefore, its adjacent vertex, say v, in the graph G_0, must be of degree 3. Therefore, after

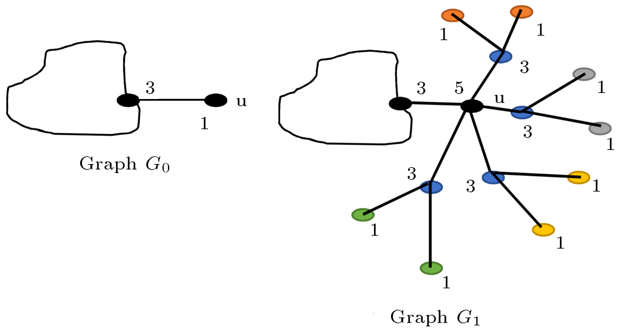


Figure 3. Original 2-SI graph G_0 and its extended 2-SI graph G_1

extension from vertex u, the degree of vertex u should be 5, such that $|d(u) - d(v)| = |5 - 3| = 2$. Thus, we add four vertices adjacent to u. Since the degree of vertex u in the extended graph G_1 is 5, the degree of the 4 newly added vertices should be 3. Therefore, we again add two more vertices corresponding to each of the above 4 newly added vertices, in order to maintain the imbalance of each edge as 2 in the extended graph. Thus, a total of $4 + (4 \times 2) = 12$ vertices are added and 12 edges are also added. Obviously, this new graph G_1 is also 2-SI graph of order $n + 12 \times 1 = n + 12$.

The above idea of construction is also shown in Figure 3. In a similar manner, we can again extend the graph G_i to have the extended 2-SI graph G_{i+1}, $i > 0$.

Theorem 10. *If* G_0 *be a* k*-SI graph of order* n *having a vertex of degree 1, then we can extend it to have a graph* G_t *of order* $n + 2k(k+1)t$ *which is also* k*-SI graph, where* $t > 0$.

Proof. Let G_0 be any k-SI graph and u be its vertex of degree 1. Thus, its adjacent vertex, in the graph G_0, must be of degree $k+1$. Now, after the extension, the vertex u should be of degree $2k+1$; therefore, we add $2k$ new vertices adjacent to u. Since the degree of vertex u in the extended graph G_1 is $2k+1$, the degree of newly added vertices should be $k+1$. Therefore, we add k pendant vertices corresponding to each newly added $2k$ vertices.

Therefore, a total of $2k + (2k \times k) = 2k(k+1)$ vertices are added and $2k(k+1)$ edges are also added to build the extended k-SI graph G_1. In a similar way, we can further extend the graph G_1 to get the k-SI graphs G_t of order $n + 2k(k+1)t$ where $t = 0, 1, 2, 3, \ldots$. □

Theorem 11. *If* G_0 *be a 2-SI graph of order* n *having a vertex of degree 2, then we can extend it to have a graph* G_t *of order* $n + 10t$, *which is also a 2-SI graph.*

Proof. Let G_0 be a 2-SI graph of order n having a vertex u of degree 2. Then, by definition (of 2-SI graph), the vertex u has two adjacent vertices of degree 4. Therefore, the degree of vertex u in the extended graph

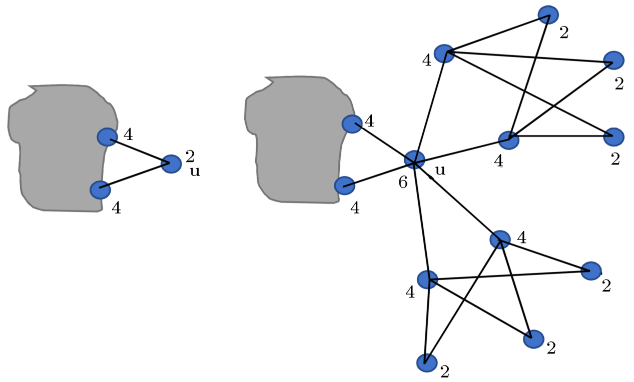


Figure 4. Original graph G_0 with vertex u of degree 2 and its extended graph G_1.

G_1 should be 6 and hence, in order to extend the graph from vertex u, we must add 4 new vertices adjacent to u, whose degree is 4. As shown in Figure 4, it is obvious that 12 new branches must have emerged. We connect these 12 branches with 6 new vertices each of degree 2 because G_1 should be a 2-SI graph.

In this way, we have added a total of 10 new vertices: 4 vertices of degree 4 and 6 vertices of degree 2. In addition, we have updated the degree of vertex u from 2 to 6. Please see Figure 4. Hence, the order of the 2-SI graph G_1 is $n + 10$.

Likewise, we can extend the graph G_1 to construct the 2-SI graph G_t of order $n+10t$ where t is the number of extensions.

Theorem 12. *If G_0 is a k-SI graph of order n having a vertex of degree 2, then we can extend it to have a graph G_t of order $n + k(k + 3)t$, which is also a k-SI graph.*

Proof. Since the proof is similar to Theorem 10 and Theorem 11, we give the hint for proving this theorem as follows:

Let u be the vertex such that $d(u) = 2$ in G_0. Then, the degree of its neighbors should be $k + 2$. As done previously, in order to extend the original graph from the vertex u, $d(u)$ must be $2k+2$ in the extended graph G_1. Then, we add $2k$ new vertices, each of degree $k + 2$, adjacent to u. In addition, to maintain the k-SI property of G_1, we must add $k(k+1)$ vertices, each of degree 2. Therefore, in total, we add $2k + k(k + 1) = k(k + 3)$ vertices. Hence, the extended k-SI graph G_1 is of order $n + k(k + 3)$.

In a similar principle, we can have the k-SI graph G_t of order $n + k(k + 3)t$, where t is essentially the number of extensions.

Lemma 1. *There always exists a 2-SI graph of order n, where n is any even integer except 2.*

Proof. By Theorem 1, it is clear that there does not exist any 2-SI graph of order 2. Moreover, since we know by Theorem 4, every complete bipartite graph of the form $K_{m,m+2}$ is a 2-SI graph.

Also, the order of these types of graphs is $m + (m + 2) = 2(m + 1)$ which is an even number, where $m = 1, 2, 3, 4, \ldots$ Hence, we have a 2-SI graph of order 4, 6, 8, and 10, ...

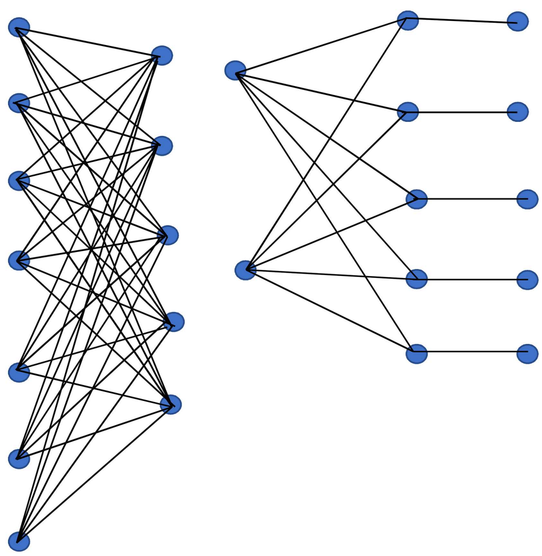


Figure 5. Example of two different 2-SI graphs, each of order 12.

Remark 3. *The uniqueness of even-order 2-SI graphs is maintained only for orders 4, 6, 8, and 10. From Figure 5, it is clear that there exist two different 2-SI graphs of order 12. Note that the first one is a complete bipartite graph $K_{7,5}$.*

Lemma 2. *There always exists a simple 2-SI graph of order n, where n is any odd integer except 1, 3, 5, 7, and 11.*

Proof. 2-SI graphs of orders 9, 13, 15, 17, 21 are shown in Figures 6(a), 6(b), 6(c), 6(d) and 6(e), respectively.

Now, since there exists a 2-SI graph of order 9 and it also has a vertex of degree 2, by Theorem 11, we can draw 2-SI graphs of orders 19, 29, 39, 49, and so on. Also, we have shown a 2-SI graph of order 13 in Figure 6(b); therefore, there exist 2-SI graphs of orders 23, 33, 43, 53, and so on. Similarly, from the 2-SI graphs of order 15, 17, and 21, we can construct 2-SI graphs of order 25, 35, 45, 55, ...; 27, 37, 47, 57, ...; and 31, 41, 51, 61,...., respectively. It is justified that we can have simple 2-SI graphs of every odd order except 3, 5, 7, and 11.□

Remark 4. *Observe that if the graph is not simple, that is, if the graph has multi edges, then we can have 2-SI graphs of order 5, 7 and 11. Please see Figure 7.*

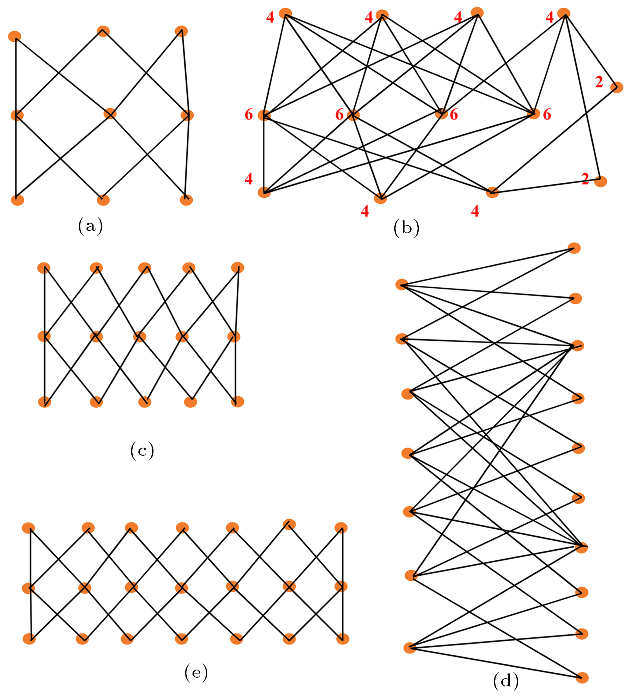


Figure 6. Example of 2-SI graph of order: (a) 9, (b) 13, (c) 15, (d) 17, and (e) 21.

Theorem 13. *There exist simple 2-SI graphs of every order except 1, 2, 3, 5, 7, and 11.*

Proof. The proof directly follows from Lemmas 1 and 2.□

Theorem 14. *For a 2-SI graph, $Irr(G) = 2|E(G)|$.*

Proof. The Albertson Index [29] is defined as follows:

$$Irr(G) = \sum_{uv \in E(G)} |d(u) - d(v)|.$$

We know that in a 2-SI graph, for every pair of adjacent vertices u and v, $|d(u) - d(v)| = 2$ holds. Thus, we have $Irr(G) = 2 \times |E(G)|$.□

Theorem 15. *Among the 2-SI graphs of even order, the graphs of type $K_{m,m+2}$ are the most irregular.*

Proof. It is clear that graphs of the type $K_{m,m+2}$ are of even order. Also, we know by Theorem 3 that any 2-SI graphs are always bipartite. Note that, among the class of bipartite graphs, the complete bipartite graph has the maximum number of edges. Since Theorem 14 states that the irregularity of a 2-SI graph is twice the number of edges. Hence, $K_{m,m+2}$ is the most irregular among 2-SI graphs of order $2m + 2$. □

3. Computational analysis of irregularity in 2-SI graphs

We know that there is a direct relationship between the irregularity of a 2-SI graph and its number of edges (by Theorem 14). Let G_1 be a 2-SI graph of order n and G_2 is its extended 2-SI graph of order $n'(> n)$. Observe that, if G_1 has the minimum number of edges, then the extended graph G_2 has the minimum number of edges among all the 2-SI graphs of order n'. Thus, if the original graph G_1 has minimum irregularity, then its extended graph G_2 has minimum irregularity.

Based on the above observation, we have computed the minimum number of edges possible for different orders of 2-SI graphs. Please refer to Table 1, where the possible order of a 2-SI graph is ranging from 4 to 50 and can be extended further. Moreover, this table gives an idea about the structure of these graphs in terms of the number of vertices along with their degrees.

Remark 5. *From the computations given in Table 1 and by using Theorem 14, we have inferred the following results:*

(i) *If $n \equiv 4(mod\ 12)$ i.e., n is a multiple of 4 and leaves remainder 1 when divided by 3, then Irr(2-SI) $\geq 2(n-1)$;*

(ii) *If $n \equiv 0(mod\ 12)$ i.e., n is a multiple of both 3 and 4, then Irr(2-SI) $\geq 2n$, except $n = 12$;*

(iii) *If $n \equiv 8(mod\ 12)$ i.e., n is a multiple of 4 and leaves remainder 2 when divided by 3, then Irr(2-SI) $\geq 2(n+1)$, except $n = 8$;*

(iv) *If $n \equiv 0(mod\ 3)$ and $n \not\equiv 0(mod\ 12)$ i.e., n is a multiple of 3 but not a multiple of 4, then Irr(2-SI) $\geq \frac{8}{3}n$.*

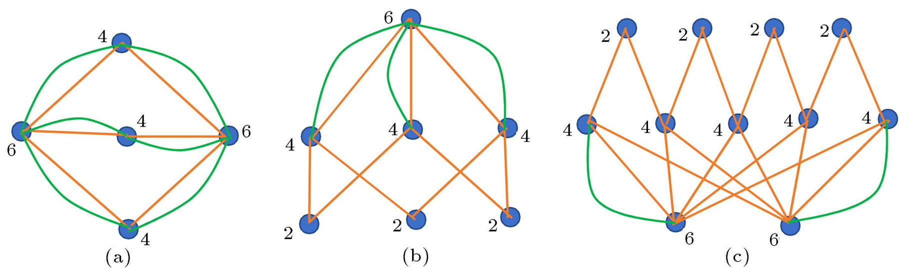


Figure 7. Example of 2-SI multigraph of order (a) 5, (b) 7, and (c) 11.

Table 1. Table for the structure of 2-SI graphs of different order with the minimum number of edges possible

Order of 2-SI graph	Minimum number of edges possible	$Irr(G)$	The structure of the graph
4	3	6	1 vertex of degree 3 and 3 vertices of degree 1
6	8	16	2 vertices of degree 4 and 4 vertices of degree 2
8	15	30	3 vertices of degree 5 and 5 vertices of degree 3
9	12	24	3 vertices of degree 4 and 6 vertices of degree 2
10	24	48	4 vertices of degree 6 and 6 vertices of degree 4
12	15	30	2 vertices of degree 5, 5 vertices of degree 3 and 5 vertices of degree 1
13	28	56	4 vertices of degree 6, 7 vertices of degree 4 and 2 vertices of degree 2
14	24	48	2 vertices of degree 6, 6 vertices of degree 4 and 6 vertices of degree 2
15	20	40	5 vertices of degree 4 and 10 vertices of degree 2
16	15	30	1 vertex of degree 5, 5 vertices of degree 3 and 10 vertices of degree 1
17	28	56	2 vertices of degree 6, 7 vertices of degree 4 and 8 vertices of degree 2
18	24	48	6 vertices of degree 4 and 12 vertices of degree 2
19	28	56	1 vertex of degree 6, 7 vertices of degree 4 and 11 vertices of degree 2
20	21	42	2 vertices of degree 5, 7 vertices of degree 3 and 11 vertices of degree 1
21	28	56	7 vertices of degree 4 and 14 vertices of degree 2
22	32	64	1 vertex of degree 6, 8 vertices of degree 4 and 13 vertices of degree 2
23	36	72	2 vertices of degree 6, 9 vertices of degree 4 and 12 vertices of degree 2
24	24	48	2 vertices of degree 5, 8 vertices of degree 3 and 14 vertices of degree 1
25	36	72	1 vertex of degree 6, 9 vertices of degree 4 and 15 vertices of degree 2
26	38	76	2 vertices of degree 6, 8 vertices of degree 4 and 16 vertices of degree 2
27	36	72	9 vertices of degree 4 and 18 vertices of degree 2
28	27	54	2 vertices of degree 5, 9 vertices of degree 3 and 17 vertices of degree 1
29	44	88	2 vertices of degree 6, 11 vertices of degree 4 and 16 vertices of degree 2
30	40	80	10 vertices of degree 4 and 20 vertices of degree 2
31	44	88	1 vertex of degree 6, 11 vertices of degree 4 and 19 vertices of degree 2
32	33	66	3 vertices of degree 5, 11 vertices of degree 3 and 18 vertices of degree 1
33	44	88	11 vertices of degree 4 and 22 vertices of degree 2
34	48	96	1 vertex of degree 6, 12 vertices of degree 4 and 21 vertices of degree 2
35	52	104	2 vertices of degree 6, 13 vertices of degree 4 and 20 vertices of degree 2
36	36	72	3 vertices of degree 5, 12 vertices of degree 3 and 21 vertices of degree 1
37	52	104	1 vertex of degree 6, 13 vertices of degree 4 and 23 vertices of degree 2
38	52	104	2 vertices of degree 6, 10 vertices of degree 4 and 26 vertices of degree 2
39	52	104	13 vertices of degree 4 and 26 vertices of degree 2
40	39	78	3 vertices of degree 5, 13 vertices of degree 3 and 24 vertices of degree 1
41	60	120	2 vertices of degree 6, 15 vertices of degree 4 and 24 vertices of degree 2
42	56	112	14 vertices of degree 4 and 28 vertices of degree 2
43	60	120	1 vertex of degree 6, 15 vertices of degree 4 and 27 vertices of degree 2
44	45	90	4 vertices of degree 5, 15 vertices of degree 3 and 25 vertices of degree 1
45	60	120	15 vertices of degree 4 and 30 vertices of degree 2
46	64	128	1 vertex of degree 6, 16 vertices of degree 4 and 29 vertices of degree 2
47	68	136	2 vertices of degree 6, 17 vertices of degree 4 and 28 vertices of degree 2
48	48	96	4 vertices of degree 5, 16 vertices of degree 3 and 28 vertices of degree 1
49	68	136	1 vertex of degree 6, 17 vertices of degree 4 and 31 vertices of degree 2
50	66	132	2 vertices of degree 6, 12 vertices of degree 4 and 36 vertices of degree 2

4. Conclusion

This study defined the concept of a 2-Stepwise Irregular (2-SI) graph and extended the idea to delimit a k-SI graph in which the degree difference of every edge is k. Initially, some graph-theoretic properties of the class of 2-SI graphs were introduced. Then, a number of methods were proposed to increase the number of vertices in a given 2-SI graph to obtain another extended 2-SI graph and the existence of these types of graphs was illustrated. Moreover, some of these graph-theoretic attributes and enlargement schemes for k-SI graphs were generalized. In addition, the bounds for the Albertson index were computed for 2-SI graphs.

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