SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica

A method for Sub-optimal control of the delayed fractional order linear time varying systems with computation reduction approach

Mehdi Yousefi Tabari [a,*], Zahra Rahmani[a], Ali Vahidian Kamyad [b], Seyed Jalil Sadati Rostami[a]

a. Department of Electrical and Computer Engineering, Babol Noshirvani University of Technology, Babol, Iran.
b. Ferdowsi University of Mashhad, Mashhad, Iran.

KEYWORDS
Fractional optimal control;
Delay system;
Linear programming;
Caputo derivative;
Grünwald–Letnikov approximation.

Abstract. A method for designing suboptimal control for a class of delayed fractional systems is proposed in this paper. Despite theoretical advances in fractional mathematics and computational techniques for solving fractional optimal control (FOC) problems, as well as a lack of comprehensive analytical methods, numerical methods have been developed. For this purpose, in this study, the necessary optimal conditions for the time-delay fractional optimal control (TDFOC) problem are presented first; Then an algorithm for the numerical solution to this problem is suggested. This algorithm is based on a fractional derivative approximation and linear interpolation for delayed arguments. According to this method, the TDFOC problem is transformed into a system of algebraic equations that can be solved numerically. The proposed method's efficiency is assessed by solving several numerical examples.

1. Introduction

The generalization of classical optimal control theory dates back to 1950 with its applications in the aerospace industry during the Cold War [1, 2]. The dynamics considered for the system in the case of classical optimal control have been in the form of integer order differential equations, whereas research has shown that more accurate dynamics of many systems are expressed based on differential equations including fractional derivatives [3-7]. Time-delay fractional systems is an important category of systems that control and optimization of them have been attracted by most researchers. But, the time delay makes the analysis and control of these systems more complex than ever. Therefore, the suggestion of new analytical and numerical methods for finding a solution for delay fractional optimal control (DFOC) problem is one of the challenging issues that are under study. However, little research has been conducted in this area. The presence of delay in fractional variation problems has been discussed in [8-11], but no specific solution method for the obtained equations has been presented in these references. Bernstein polynomials (BP) are used as basic functions in [12], to solve the FOC problem with constant delays in the system's states. The authors of [13], used BPs to solve the FOC problem with constant delay in both control and system state. Furthermore, a linear dynamic system is considered in this paper, and the initial state is also a constant function. New operational matrices for the orthogonal polynomial of Legendre Shift are proposed in [14], which are used as basic functions to solve the DFOC problem. The DFOC problem is transformed into a linear programming (LP) problem in [15], for sub-optimal control of delayed fractional-order systems whose dynamics include Riemann-Liouville fractional derivatives (RFDs) using an algorithm based on an approximation for fractional derivatives and a discretization approximation for delay and

*. Corresponding author. Tel.: 09352582091
E-mail address: Mehdi.yt61@gmail.com

advance terms. In [16], using the Pade approximation to solve the DFOC problem, the problem of delay is transformed into a problem without delay. Then the FOC problem is reduced to a nonlinear programming problem using the Fractional Derivative functional matrix of the Montes polynomial and the spectral method. [17], presents a method for solving DFOC problems based on fractional order Lagrangian polynomials (FLP) and a correlation method. The DFOC problems were solved using a combination of the fractional power series method and neural networks in [18]. In [19], an efficient direct numerical method for obtaining an approximate solution to the DFOC problems is proposed using orthogonal Chelishkov wavelets and operational matrices. Based on an embedding process and the use of fractional derivative, [20], proposes a new definition of fractional integral and derivative. Because of recent advances, many studies have focused on DFOC problems, numerical solutions, and applications. Active damping of flexible structures, generalized viscoelastic models, biomedical and bioengineering, optics and image processing are some applications [21-25].

Given that the field of fractional optimal control research is a new area for systems with time delay and usually has complex problems, generally there is no analytical technique to solve them. For this reason, searching strong and accurate numerical methods to solve their various types has become an active research field [15, 20].

The aim of this paper is to present a method of designing sub-optimal control for a class of delay fractional systems whose system dynamics are expressed by Caputo Fractional Derivative (CFD). To that end, this study first presents the necessary optimal conditions for the time-DFOC problem, followed by a new technique for numerically solving this problem. This algorithm is based on a fractional derivative approximation and linear interpolation for delayed terms. According to this method, the TDFOC problems are transformed into a system of algebraic equations that can be solved numerically using this method. The proposed method's efficiency is assessed by solving several numerical examples.

So the innovation of this article is due to:

- Proposing an approximation for Caputo Fractional Derivative (CFD).
- Linear interpolation is proposed to discretize delay terms in delay systems.
- Using Grünwald–Letnikov approximation (GLA) for fractional derivatives.
- The research problem is converted into a linear programming (LP) problem.
- The method enjoys less complexity, computational load, and processing time.

The following sections of this paper are organized as follows. Part two explains the problem's definition. Section three explains the proposed method. Section 4 shows the simulation results and comparison results. Section 5 contains the conclusions.

2. Definitions and Statement of the Problem

2.1 Definitions

In this section, common definitions are given for fractional derivatives:

Suppose $p(z):[a,b]\rightarrow\mathbb{R}$ is a time-dependent function, $\beta>0$ is a fractional derivative order, and $n=[\beta]+1$.

So, RFD and CFD are defined as [3-5]:

The left RFD

$$ {}_{a}D_{z}^{\beta}p(z)=\frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dz}\right)^{n}\int_{a}^{z}\frac{p(\nu)d\nu}{(z-\nu)^{\beta-n+1}}\,, \tag{1}$$

The right RFD

$$ {}_{z}D_{b}^{\beta}\,p(z)=\frac{(-1)^{n}}{\Gamma(n-\beta)}\left(\frac{d}{dz}\right)^{n}\int_{z}^{b}\frac{p(\nu)d\nu}{(\nu-z)^{\beta-n+1}}\,, \tag{2}$$

The left CFD

$$ {}_{a}^{C}D_{z}^{\beta} p(z) = \frac{1}{\Gamma(n-\beta)} \int_{a}^{z} \frac{d^n p(\nu) / d\nu^n}{(z-\nu)^{\beta-n+1}} d\nu, \tag{3} $$

The right CFD

$$ {}_{z}^{C}D_{b}^{\beta} p(z) = \frac{(-1)^n}{\Gamma(n-\beta)} \int_{z}^{b} \frac{d^n p(\nu) / d\nu^n}{(\nu-z)^{\beta-n+1}} d\nu. \tag{4} $$

Assume that $0<\beta<1$ and $0<\beta\le 1$ are the generalizations of binomial coefficients, which means:

$$ \binom{\beta}{k} = \frac{\Gamma(\beta+1)}{\Gamma(k+1)\Gamma(\beta-k+1)}, \tag{5} $$

β can be any integer or real, then left and right Grünwald–Letnikov fractional derivatives (GLFDs) to-be defined as follows:

The left GLFD

$$ {}_{a}^{GL}D_{z}^{\beta} p(z) = \lim_{\ell\to 0^+} \frac{1}{\ell^{\beta}} \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{\mathrm{k}} p(z-k\ell), \tag{6} $$

The right GLFD

$$ {}_{z}^{GL}D_{b}^{\beta} p(z) = \lim_{\ell\to 0^+} \frac{1}{\ell^{\beta}} \sum_{k=0}^{\infty} (-1)^k \binom{\beta}{k} p(z+k\ell). \tag{7} $$

Also, if β is an integer, it can be written

$$ {}_{a}D_{z}^{n} p(z) = {}_{a}^{c}D_{z}^{n} p(z) = {}_{a}^{GL}D_{z}^{n} p(z) = \frac{d^n p(z)}{dz^n}, \tag{8} $$

$$ {}_{z}D_{b}^{n} p(z) = {}_{z}^{c}D_{b}^{n} p(z) = {}_{z}^{GL}D_{b}^{n} p(z) = (-1)^n \frac{d^n p(z)}{dz^n}\ . \tag{9} $$

The following relationships are also useful relationships between Riemann–Liouville and Caputo derivatives [26]:

$$ {}_{a}^{C}D_{z}^{\beta}\ p(z) = {}_{a}D_{z}^{\beta} p(z) - \sum_{k=0}^{n-1} \frac{p^k(a)}{\Gamma(k-\beta+1)} (z-a)^{k-\beta}, \tag{10} $$

$$ {}_{a}^{C}D_{b}^{\beta}\ p(z) = {}_{z}D_{b}^{\beta} p(z) - \sum_{k=0}^{n-1} \frac{(-1)^k p^k(b)}{\Gamma(k-\beta+1)} (b-z)^{k-\beta}\ . \tag{11} $$

2.2 Statement of the Problem

The performance index (PI) or dynamics of the system, or both, contained at least one fractional derivative as well as the time delay in a DFOC problem. The goal of this paper is to solve the delay fractional optimal control problem in order to obtain optimal control for systems whose dynamics include fractional derivatives of Caputo as well as the state variable with constant time delay, which is defined as (12) and (13). The goal of the delay optimal control problem is to find the optimal control $u^*(z)$ for a delay fractional dynamic system so that the following PI is minimized.

$$ J = \frac{1}{2} \int_{z_0}^{z_f} \left(p^T(z) Q(z) p(z) + u^T(z) R(z) u(z) \right) dz, \tag{12} $$

In this case, the system dynamics and initial state are considered as follows:

$$\xi(z) \tag{13}$$

Where $p \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and control vectors, respectively. $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ positive semi-definite and positive matrices, The parameter ${}_{z_0}^{C}D_z^{\beta} p(z)$ shows the left CFD of $p(z)$, $a(z)$ and $c(z)$ are continuous matrices with the appropriate dimensions, $\eta > 0$ denotes delay which is supposed constant, and b is constant matrix, and $\gamma(z)$ is the specified initial state function. In addition, it isassumed to be $0 < \beta \le 1$.

3. The Proposed Method

3.1 Formulation of the DFOC PROBLEM

This research is done in two main parts. The first section describes how to obtain the Euler-Lagrange equations in systems with delay fractional dynamics to establish the necessary optimality conditions. The necessary optimality conditions for a boundary value problem, including the delay arguments, are obtained in this section by introducing the Lagrangian multiplier $\xi(z)$ and using variational methods. Then the optimal control law $u^*(z)$ is computed by solving this boundary value problem and using equation (23). Assume that the initial state $\gamma(z)$ and initial time z_0 are definite to demonstrate the necessary optimality conditions. If we define Hamilton's function $\mathcal{H}$ as follows:

$$\begin{aligned}&\left(p(z), p(z-\eta), u(z), \xi(z), z\right)\\ &\quad \triangleq p^T(z) Q(z) p(z)\\ &\quad + u^T(z) R(z) u(z)\\ &\quad + \xi^T(z)\left[a(z) p(z) + b u(z) + c(z) p(z-\eta)\right]\end{aligned} \tag{14}$$

Then the following equations are necessary conditions which must be established in the whole-time range $[z_0, z_f]$, [15]:

$${}_{z_0}^{C}D_z^{\beta} p(z) = \frac{\partial(\mathcal{H})}{\partial \xi(z)},\ z \epsilon [z_0, z_f], \tag{15}$$

$${}_{z}^{C}D_z^{\beta} \xi(z) = \frac{\partial(\mathcal{H})}{\partial p(z)} + \delta(z+\eta),\ z \epsilon [z_0, z_f - \eta], \tag{16}$$

$${}_{z}^{C}D_{z_f}^{\beta} \xi(z) = \frac{\partial(\mathcal{H})}{\partial p(z)},\ z \epsilon [z_f - \eta, z_f], \tag{17}$$

$$\frac{\partial(\mathcal{H})}{\partial u(z)} = 0 . \tag{18}$$

And $\delta(z+\eta) = \dfrac{\partial \mathcal{H}}{\partial p(z-\eta)}$.

The equations above represent the Euler-Lagrange equations for DFOC problems. These equations take into account both the right and left fractional derivatives. Furthermore, these equations have both time delay and advance, making solving them difficult and complex. The following equations can also be obtained by minimizing the cost function (12), [15]:

$$\begin{aligned}{}_{z}^{C}D_{z_f}^{\beta} \xi(z) &= Q(z) p(z) + a^T(z) \xi(z)\\ &\quad + g(z) \xi(z+\eta), \quad z \epsilon [z_0, z_f]\end{aligned} \tag{19}$$

$$R(z) u(z) + b^T \xi(z) = 0, \quad z \epsilon [z_0, z_f] \tag{20}$$

$$g(z)=X_{[z_0,z_f-\eta]}(z)c^T(z+\eta), \tag{21}$$

Based on (21) to (19) which are substituted into (13), the following equations are achieved:

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta}p(z)=a(z)p(z)+c(z)p(z-\eta) \\ \qquad +V(z)\xi(z), \qquad z\epsilon[z_0,z_f], \\ {}_{z}^{C}D_{z_f}^{\beta}\xi(z)=Q(z)p(z)+a^T(z)\xi(z) \\ \qquad +g(z)\xi(z+\eta), \quad z\epsilon[z_0,z_f], \\ p(z)=\gamma(z), \qquad z\epsilon[z_0-\eta,z_0], \\ p(z)=\gamma(z), \qquad z\epsilon[z_0-\eta,z_0], \end{cases} \tag{22}$$

and $V(z)=-bR^{-1}(z)b^T$. Therefore, the state and co-state variables $p(z)$ and $\xi(z)$ are considered by the boundary value problem. Then, using equation (20), the optimal control law $u^*(z)$ becomes as follows [15]:

$$u^*(z)=-R^{-1}(z)b^T\xi(z). \quad z\epsilon[z_0,z_f] \tag{23}$$

As it turns out, (22) is a boundary value problem with delay and advance terms. This problem is not insurmountable, but it is extremely difficult to solve. To address this issue, a novel numerical method for determining the state and co-state variables will be developed in the next part.

3.2. Discretization and Approximation

In the second part of this research, in order to find a solution for the resulting equations numerically, the whole time domain $[z_0, z_f]$ is separated into N equal sub-domains first, each scale of which is represented by $\ell=1/N$ and the time at node j is $z_j=j\ell$. The nodes are labelled *0, 1, ..., N*. The GLA for the RFDs is as follows [27] and [5]:

$$_{z_0}D_z^{\beta}p_i=\frac{1}{\ell^{\beta}}\sum_{j=0}^{i}\omega_j^{\beta}.p_{i-j}, \qquad i=1 to N \tag{24}$$

$$_{z}D_{z_f}^{\beta}\xi_i=\frac{1}{\ell^{\beta}}\sum_{j=i}^{N}\omega_{j-i}^{\beta}.\xi_j, \quad i=0 to N-1 \tag{25}$$

Where ξ_i and p_i are numerical approximations for $\xi(z)$ and $p(z)$ in node i and we have $\omega_j^{\beta}=(-1)^j\binom{\beta}{j}$.

From equations (10), (11), (24) and (25), the GLA for Caputo fractional derivatives can be obtained, which is a new method for approximating CFDs in solving DFOC problems.

$$_{z_0}^{C}D_z^{\beta}p_i= \frac{1}{\ell^{\beta}}\sum_{j=0}^{i}\omega_j^{\beta}.p_{i-j}-\frac{1}{\Gamma(1-\beta)}\frac{p_0}{(z_i-z_0)^{\beta}}, \; i=1 to N \tag{26}$$

$$_{z}^{C}D_{z_f}^{\beta}\xi_i= \frac{1}{\ell^{\beta}}\sum_{j=0}^{N-i}\omega_{j+i}^{\beta}.\xi_j-\frac{1}{\Gamma(1-\beta)}\frac{\xi_N}{(z_N-z_i)^{\beta}}, i=0 to N-1 \tag{27}$$

By using the GLA and equations (26) and (27), equation (22) is rewritten for numerical solution:

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta}\, p_i = a_i p_i + c_i p\left(z_i - \eta\right) + V_i \xi_i \,, \qquad i = 1 to N \\ {}_{z}^{C}D_{z_f}^{\beta}\, \xi_i = Q_i p_i + a_i^T \xi_i + g_i \xi\left(z_i + \eta\right) \,, \; i = 0 to N-1 \end{cases} \tag{28}$$

Moreover, linear interpolation [28] is applied to approximate delay terms. The linear function T between two points $\left(a, f\,(a)\right)$ and $\left(b, f\,(b)\right)$, for the function *f* in the domain $\left(a, b\right)$ is as follows:

$$\begin{aligned} T\left(s\right) = &\left(\left(s-b\right).f\left(a\right)\right)/\left(a-b\right) \\ &+\left(s-a\right).f\left(b\right)/\left(b-a\right), \end{aligned} \tag{29}$$

In equation (28), the purpose is to do linear interpolation for the function $p(z_i - \eta)$ and $\xi\left(z_i + \eta\right)$ in the range $z_i - \eta \epsilon \left[z_{q-1}, z_q\right]$, and $\eta \epsilon R$ and $q \epsilon Z$ for delay arguments, the interpolation function is as follows:

$$\begin{aligned} p\left(z_i - \eta\right) = &(z_i - \eta - z_q).p\left(z_{q-1}\right).\left(z_{q-1} - z_q\right)^{-1} \\ &+\left(z_i - \eta - z_{q-1}\right).p\left(z_q\right).(z_q - z_{q-1})^{-1}, \\ &z_{q-1} \le z_i - \eta \le z_q \end{aligned} \tag{30}$$

$$\begin{aligned} \xi\left(z_i + \eta\right) = &\left(z_i + \eta - z_q\right).p\left(z_{q-1}\right).\left(z_{q-1} - z_q\right)^{-1} \\ &+\left(z_i + \eta - z_{q-1}\right).p\left(z_q\right).(z_q - z_{q-1})^{-1}, \\ &z_{q-1} \le z_i + \eta \le z_q \end{aligned} \tag{31}$$

To solve equations (28) numerically and convert them to linear equations, equations (30) and (31) can be used to approximate the delay and advance terms with linear interpolation:

$$\begin{aligned} m_1 &= \frac{z_i - \eta - z_q}{z_{q-1} - z_q} \\ m_2 &= \frac{z_i - \eta - z_{q-1}}{z_q - z_{q-1}}, \qquad z_{q-1} \le z_i - \eta \le z_q \end{aligned} \tag{32}$$

$$\begin{aligned} m_3 &= \frac{z_i + \eta - z_q}{z_{q-1} - z_q} \\ m_4 &= \frac{z_i + \eta - z_{q-1}}{z_q - z_{q-1}}, \qquad z_{q-1} \le z_i + \eta \le z_q \end{aligned} \tag{33}$$

$$\begin{aligned} p\left(z_i - \eta\right) = m_1 p\left(z_{q-1}\right) + m_2 p\left(z_q\right), \\ z_{q-1} \le z_i - \eta \le z_q \end{aligned} \tag{34}$$

$$\begin{aligned} \xi\left(z_i + \eta\right) = m_3 \xi\left(z_{q-1}\right) + m_4 \xi\left(z_q\right), \\ z_{q-1} \le z_i + \eta \le z_q \end{aligned} \tag{35}$$

Now, by replacing (34) and (35) in (28), the linear form of system for solving with i equation is obtained.

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta}\, p_i = a_i p_i + m_1 c_i\, p\left(z_{q-1}\right) \\ \qquad + m_2 c_i p\left(z_q\right) + V_i \xi_i \,, \qquad i = 1 to N, \\ {}_{z}^{C}D_{z_f}^{\beta}\, \xi_i = Q_i p_i + a_i^T \xi_i + m_3 g_i\, \xi\left(z_{q-1}\right) \\ \qquad + m_4 g_i \xi\left(z_q\right). \qquad i = 0 to N-1, \end{cases} \tag{36}$$

The proposed method converts the DFOC problem (28) into a linear system of algebraic equation (36), with linear interpolation approximation for the delay arguments, which can be easily solved.

3.3. Proposed Method Algorithm

Block diagram of the proposed algorithm has been revealed in Figure 1.

Therefore, the special highlight of the proposed algorithm can be considered as follows:

- The system's dynamics are based on the CFD. The delay is in the state variable.
- Using the indirect method of the Pontryagin principle in solving the optimal control problem.
- Using linear interpolation and GLA for delay arguments and CFDs.
- Converting the DFOC problem into a LP problem.
- Capability of solving DFOC problems, with non-integer delay.
- Efficiency, simplicity and less processing time than other methods.

4. Numerical Results

The effectiveness and validity of the proposed method are investigated in this section using numerical examples. For the examples presented, the numerical results obtained by the proposed algorithm are compared to similar existing results in the literature.

Example 1. The following DFOC problem is as follows:

$$\min J = \frac{1}{2}\int_{z_0}^{z_f}\left(p^2(z)+u^2(z)\right)dz, \tag{37}$$

subject to

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta} p(z) = p(z-1)+u(z), & z\epsilon[0,2], \\ p(z)=1, & z\epsilon[-1,0]. \end{cases} \tag{38}$$

$a(z)=0$, $b=1$, $\eta=1, c(z)=1$, $Q=R=1,$ are considered in example 1. So, the optimality conditions provide in (28) are converted as follows:

$$\begin{cases} {}_{0}^{C}D_z^{\beta} p(z) = p(z-1)-\xi(z), & z\epsilon[0,2], \\ {}_{z}^{C}D_2^{\beta} \xi(z) = p(z)+g(z)\xi(z+1), & z\epsilon[0,2], \\ p(z)=1, & z\epsilon[-1,0]. \\ \xi(2)=0. \end{cases} \tag{39}$$

Using the equations in section 3.2, we first discretize the problem, then, then, to numerically solve (39) and convert them to linear equations, use (30) and (31), we approximate the delay and advance terms with linear interpolation:

$$m_1 = \frac{z_i-1-z_q}{z_{q-1}-z_q} \qquad m_2 = \frac{z_i-1-z_{q-1}}{z_q-z_{q-1}}, \quad z_{q-1}\le z_i-1\le z_q \tag{40}$$

$$m_3 = \frac{z_i+1-z_q}{z_{q-1}-z_q} \qquad m_4 = \frac{z_i+1-z_{q-1}}{z_q-z_{q-1}}, \quad z_{q-1}\le z_i+1\le z_q \tag{41}$$

$$p(z_i - 1) = m_1\, p(z_{q-1}) + m_2 p(z_q), \quad z_{q-1} \le z_i - 1 \le z_q \tag{42}$$

$$\xi(z_i + 1) = m_3\, \xi(z_{q-1}) + m_4 \xi(z_q), \quad z_{q-1} \le z_i + 1 \le z_q \tag{43}$$

From equations (42), (43) and (39), the linear form is obtained for solving arguments including (39), using equation (36) with i equation.

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta}\, p_i = m_1 c_i\, p(z_{q-1}) + m_2 c_i p(z_q) - \xi_i\,, & i = 1 to N \\ {}_{z}^{C}D_{z_f}^{\beta}\, \xi_i = p_i + m_3 g_i\, \xi(z_{q-1}) + m_4 g_i \xi(z_q)\,. & i = 0 to N-1 \end{cases} \tag{44}$$

The proposed scheme in Section 3 is used to approximate solving the DFOC problem (39), for various values of 'β'. Table 1 compares the PI value results of the proposed algorithm to those reported in other references for $\beta = 1$. Based on the results in Table 1, it is clear that the GLA method produces results that are in good agreement with the true solution given by [29], [30], [15] and [19]. Furthermore, it is clear that the values reported in [12] and [14] are less than those obtained in the other references; however, those values do not agree with the true value stated by [29]. For non-integer values of 'β', the method proposed to solve this problem is used. Figures 2 and 3 show the control and state functions for various values of ℓ and 'β'. As ℓ decreases, these figures confirm the convergence of the state and control variables. The graphs and responses get closer to the integer-order solutions as the fractional derivative of 'β' approaches one. The results in Table 2 show that the numerical results have good convergence as the number of interpolation intervals increases, and the accuracy increases as the time step size ℓ is decreased. Furthermore, Table 2 lists the elapsed CPU time (in seconds) and the PI values J for various values of β. The results demonstrated that the LP method, when combined with the GLA and linear interpolation, is simple to implement and efficient in terms of computational effort reduction.

Remark 1. Example 1 is solved by using the methods presented in the following papers. [29], employed Y^N subspaces made up of functions that are piecewise constant on the delay interval [-h,0]. This is the well-known averaging approximation scheme. The authors of [30] used Walsh functions to solve the DFOC problem. The authors of [12], used the BPs to solve the FOC problem with constant delays in both control and state in system. New operational matrices for the orthogonal polynomial of Legendre Shift that are used as the basic functions to solve the DFOC problem are proposed in [14]. In [15], the Euler-Lagrange equations are investigated and explained for systems with delay fractional dynamics to establish the necessary optimality conditions for systems with RFDs in their dynamics. The DFOC problem was then converted into an LP problem using an algorithm based on a fractional derivative approximation and a delay term discretization approximation. In [19], using orthogonal Chelishkov wavelets and operational matrices, an effective numerical method is proposed to solve DFOCs approximately. [31], introduces discrete Chebyshev polynomials and thoroughly investigates their properties. Then a comparison has been made between the required CPU time and accuracy of the discrete and continuous Chebyshev polynomials methods. In [32] make use of these discrete orthonormal polynomials and their operational matrix, an efficient direct numerical method is proposed to approximate the solution of the TDFOC problem

Example 2. Let the DFOC problem is as follows:

$$\min J = \frac{1}{2}\int_{z_0}^{z_f} \left(p^2(z) + u^2(z)\right) dz, \tag{45}$$

subject to

$$\begin{cases} {}_{z_0}^{C}D_z^{\beta}\, p(z) = z p(z) + p(z-1) + u(z), & z \epsilon [0,2], \\ p(z) = 1, & z \epsilon [-1,0]. \end{cases} \tag{46}$$

$a(z) = z,\ b = 1,\ \eta = 1,\ c(z) = 1,\ Q = R = 1$ are considered in example 1. So, the optimality conditions provide in (28) are converted as follows:

$$\begin{cases} {}_0^C D_z^\beta p(z) = zp(z) + p(z-1) - \xi(z), & z\epsilon[0,2], \\ {}_z^C D_2^\beta \xi(z) = p(z) + z\xi(z) + g(z)\xi(z+1), & z\epsilon[0,2], \\ p(z) = 1, & z\epsilon[-1,0]. \\ \xi(2) = 0. \end{cases} \tag{47}$$

Using the equations in section 3.2, we first discretize the problem, then, then, to numerically solve (47) and convert them to linear equations, use (30) and (31), we approximate the delay and advance terms with linear interpolation:

$$m_1 = \frac{z_i - 1 - z_q}{z_{q-1} - z_q} \qquad m_2 = \frac{z_i - 1 - z_{q-1}}{z_q - z_{q-1}}, \quad z_{q-1} \le z_i - 1 \le z_q \tag{48}$$

$$m_3 = \frac{z_i + 1 - z_q}{z_{q-1} - z_q} \qquad m_4 = \frac{z_i + 1 - z_{q-1}}{z_q - z_{q-1}}, \quad z_{q-1} \le z_i + 1 \le z_q \tag{49}$$

$$p(z_i - 1) = m_1\, p(z_{q-1}) + m_2 p(z_q), \quad z_{q-1} \le z_i - 1 \le z_q \tag{50}$$

$$\xi(z_i + 1) = m_3\, \xi(z_{q-1}) + m_4 \xi(z_q), \quad z_{q-1} \le z_i + 1 \le z_q \tag{51}$$

By inserting equations (50), (51) and (47), the linear form is obtained to solve the system equation (47) using (36) with i equation, which can be easily solved.

$$\begin{cases} {}_{z_0}^C D_z^\beta\, p_i = z_i p_i + m_1 c_i\, p(z_{q-1}) + m_2 c_i p(z_q) - \xi_i\,, & i = 1 to N \\ {}_z^C D_{z_f}^\beta\, \xi_i = p_i + z_i \xi_i + m_3 g_i\, \xi(z_{q-1}) + m_4 g_i \xi(z_q)\,. & i = 0 to N-1 \end{cases} \tag{52}$$

The proposed described scheme that is presented in the section 3 is used to solve DFOC problem, (52), approximately for different amount of 'β'. For β = 1, Table 3 compares the PI value results of the proposed algorithm to those of other algorithms in other references. According to Table 3, there is good agreement between the obtained results and those reported in [33], [34], and [19]. Furthermore, it is clear that the values reported in [35] and [36], are not good in compare with the other results.

The method suggested to solve this problem is employed for non-integer values of 'β'. To investigate the convergence issue of the suggested algorithm, the obtained optimal values of PI with various ℓ and β are presented in Table 4. In addition, in Figures 4 and 5, the state and control functions for various values of ℓ and 'β' are illustrated. The results indicate that the LP method combined with the GLA and linear interpolation is efficient, accurate, and simple. As the fractional derivative of 'β' approaches 1, the graphs and responses get closer to the integer-order solutions. The results displayed in Table 5 reveal that the numerical results have a good convergence when the number of interpolation intervals rises. Moreover, in Table 4, the elapsed CPU time and the PI values for different values of β are listed. It is observed that for the time step size $\ell = 0.004$, the value of the cost functional is obtained PI within 0.069477 s of CPU time while in [35] is achieved within 62 s of CPU time and the value of PI for the [33] with step size $\ell = 0.004$ is obtained within 0.354545 s of CPU time (Table 5). Therefore, the proposed algorithm is efficient, straightforward and reduces the computational labor, effectively; and less processing time than other methods.

Remark 2: [36] solves an example using the Walsh functions presented in [30]. [35], investigates the optimal control of time-delay systems using iterative dynamic programming with systematic region contraction and accessible grid points. [19], proposes an efficient direct numerical method to solve the problem using orthogonal Chebyshev wavelets and operational matrices. [33], uses a second-order finite variation formula and a Hermite interpolation polynomial to modify DFOC problems with first order derivatives and a delay system in the form of a linear LP. For the proposed algorithm in [34], a hybrid of block-pulse functions and orthonormal Taylor polynomials is used. The proposed hybrid functions show convergence. The Laplace transform technique is used to build a hybrid function associated with a fractional integration operational matrix. [31], introduces discrete

Chebyshev polynomials and thoroughly investigates their properties. Then a comparison has been made between the required CPU time and accuracy of the discrete and continuous Chebyshev polynomials methods. In [32] make use of these discrete orthonormal polynomials and their operational matrix, an efficient direct numerical method is proposed to approximate the solution of the TDFOC problem. In [23] the method based on a hybrid of block-pulse and fractional-order Legendre functions is presented.

5. Conclusion

In this paper, a novel method for solving the DFOC problem using approximation and discretization is proposed. First, the necessary optimality conditions for the DFOC problem are introduced, and then a novel algorithm for numerical solution to this problem is proposed. This algorithm is based on the GLA for fractional derivatives and linear interpolation for delay arguments.

The DFOC problem is transformed into a system of algebraic equations using this method. The research problem is converted into a LP problem, which is much easier to solve than the original problem, using this method. The results show that the LP method is efficient, effective, accurate, and simple because it is based on the GLA and linear interpolation discretization. It has less complexity and computational load than other methods, so it requires less time to process. The more fractional derivative of 'β' approaches one, the closer graphs and responses get to an integer-order problem. Another key feature of this method is that it can solve the DFOC problem with a non-integer delay, which is impossible in a few methods. Finally, two examples demonstrate the efficiency and effectiveness of the proposed method.

In future work, we will attempt to solve applied problems using the new method presented in this paper. Also, we intend to extend this method for solving DFOC problems with delay in control, multi-delay, and, nonlinear systems.

Acknowledgments

The author is very grateful to the Reviewers and the Associate Editor for their useful comments and valuable suggestions that improved the paper.

Reference

1. Sussmann, H J. and Willems, J C. "300 years of optimal control: from the brachystochrone to the maximum principle", *IEEE Control Systems Magazine*, 17(3), pp. 32-44, (1997).
2. Bryson, A E. "Optimal control-1950 to 1985", *IEEE Control Systems Magazine*, 16(3), pp. 26-33, (1996).
3. Bagley, R L. and Calico, R. "Fractional order state equations for the control of viscoelasticallydamped structures", *Guidance, Control, and Dynamics*, 14(2), pp. 304-311, (1991).
4. Gorenflo, R. and Mainardi, F. "Fractals and fractional calculus in continuum mechanics", *Fractional calculus: Integral and differential equations of fractional order. Wien, New York: Springer-Verlag*, pp. 223-76, (1997).
5. Podlubny, I, "Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications", *New York: Academic Press.*, (1998).
6. Hilfer, R, "Applications of fractional calculus in physics", *World scientific*, (2000).
7. Machado, J T. and Guest, J. "Special issue on fractional calculus and applications", *Nonlinear Dynamics*, 29(1-4), pp. 1-385, (2002).
8. Baleanu, D. Maaraba, T. and Jarad, F. "Fractional variational principles with delay", *Physics A: Mathematical and Theoretical*, 41(31), pp. 315403, (2008).
9. Jarad, F. and Baleanu, D. "Fractional variational principles with delay within Caputo derivatives", *Reports on Mathematical Physics*, 65(1), pp. 17-28, (2010).
10. Jarad, F. Abdeljawad, T. and Baleanu, D. "Fractional variational optimal control problems with delayed arguments", *Nonlinear Dynamics*, 62(3), pp. 609-614, (2010).
11. Jarad, F. Abdeljawad, T. and Baleanu, D. "Higher order fractional variational optimal control problems with delayed arguments", *Applied Mathematics and Computation*, 218(18), pp. 9234-9240, (2012).
12. Safaie, E. Farahi, M H. and Ardehaie, M F. "An approximate method for numerically solving multi-dimensional delay fractional optimal control problems by Bernstein polynomials", *Computational and applied mathematics*, 34(3), pp.

831-846, (2015).
13. Safaie, E. and Farahi, M H. “An approximation method for numerical solution of multi-dimensional feedback delay fractional optimal control problems by Bernstein polynomials”, *IRANIAN JOURNAL OF NUMERICAL ANALYSIS AND OPTIMIZATION*, 4(1), pp. 77-94, (2014).
14. Bhrawy, A. and Ezz-Eldien, S. “A new Legendre operational technique for delay fractional optimal control problems”, *Calcolo*, 53(4), pp. 521-543, (2016).
15. Jajarmi, A. and Baleanu, D. “Suboptimal control of fractional-order dynamic systems with delay argument”, *Journal of Vibration and Control*, 24(12), pp. 2430-2446, (2018).
16. Hosseinpour, S. Nazemi, A. and Tohidi, E. “Müntz–Legendre spectral collocation method for solving delay fractional optimal control problems”, *Computational and Applied Mathematics*, 351, pp. 344-363, (2019).
17. Sabermahani, S. Ordokhani, Y. and Yousefi, S-A. “Fractional-order Lagrange polynomials: an application for solving delay fractional optimal control problems”, *Transactions of the Institute of Measurement and Control*, 41(11), pp. 2997-3009, (2019).
18. Kheyrinataj, F. and Nazemi, A. “Fractional power series neural network for solving delay fractional optimal control problems”, *Connection Science*, 32(1), pp. 53-80, (2020).
19. Moradi, L. Mohammadi, F. and Baleanu, D. “A direct numerical solution of time-delay fractional optimal control problems by using Chelyshkov wavelets”, *Vibration and Control*, 25(2), pp. 310-324, (2019).
20. Ziaei, E. and Farahi, M H. “The approximate solution of non-linear time-delay fractional optimal control problems by embedding process”, *IMA Journal of Mathematical Control and Information*, 36(3), pp. 713-727, (2018).
21. Erturk, V., Godwe, E., Baleanu, D., et al. “Novel fractional-order Lagrangian to describe motion of beam on nanowire”, *Acta Physica Polonica A*, 140(3), pp. 265-272, (2021).
22. Jajarmi, A., Baleanu, D., Vahid, K Z., et al. “A new and general fractional Lagrangian approach: a capacitor microphone case study”, *Results in Physics*, 31, 104950, (2021).
23. Marzban, H R. “A new fractional orthogonal basis and its application in nonlinear delay fractional optimal control problems”, *ISA transactions*, 114, pp. 106-119, (2021).
24. Baleanu, D., Ghassabzade, F A., Nieto, J J., et al. “On a new and generalized fractional model for a real cholera outbreak”, *Alexandria Engineering Journal*, 61(11), pp. 9175-9186, (2022).
25. Jajarmi, A., Baleanu, D., Sajjadi, S S., et al. “Analysis and some applications of a regularized Ψ–Hilfer fractional derivative”, *Journal of Computational and Applied Mathematics*, 415, 114476, (2022).
26. Kilbas, A A. Srivastava, H M. and Trujillo, J J, “Theory and applications of fractional differential equations”, 204, *elsevier*, (2006).
27. Agrawal, O P. and Baleanu, D. “A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems”, *Vibration and Control*, 13(9-10), pp. 1269-1281, (2007).
28. Burden, R L. Faires, J D. and Reynolds, A. “Numerical Analysis Prindle”, *Weber & Schmidt*, pp. 28-34, (1985).
29. Banks, H. and Burns, J. “Hereditary control problems: numerical methods based on averaging approximations”, *SIAM Journal on Control Optimization*, 16(2), pp. 169-208, (1978).
30. Palanisamy, K. and Prasada, R G. “Optimal control of linear systems with delays in state and control via Walsh functions”, *IEE Proceedings D (Control Theory and Applications)*, 130(6), pp. 300-312, (1983).
31. Moradi, L. and Mohammadi, F. “A Comparative Approach for Time-Delay Fractional Optimal Control Problems: Discrete Versus Continuous Chebyshev Polynomials”, *Asian Journal of Control*, 22(1), pp. 204-216, (2020).
32. Mohammadi, F. “A discrete orthogonal polynomials approach for fractional optimal control problems with time delay”, *Iranian Journal of Numerical Analysis Optimization*, 9(2), pp. 49-75, (2019).
33. Jajarmi, A. and Hajipour, M. “An efficient finite difference method for the time-delay optimal control problems with time-varying delay”, *Asian Journal of Control*, 19(2), pp. 554-563, (2017).
34. Marzban, H R. and Malakoutikhah, F. “Solution of delay fractional optimal control problems using a hybrid of block-pulse functions and orthonormal Taylor polynomials”, *Journal of the Franklin Institute*, 356(15), pp. 8182-8215, (2019).
35. Dadebo, S. and Luus, R. “Optimal control of time-delay systems by dynamic programming”, *Optimal Control Applications and Methods*, 13(1), pp. 29-41, (1992).
36. Palanisamy, K. Balachandran, K. and Ramasamy, R. “Optimal control of linear time-varying delay systems via single-term Walsh series”, *IEE Proceedings D-Control Theory and Applications*, 135(4), pp. 332, (1988).

Mehdi Yousefi Tabari was born in Bojnourd, Iran, in 1982. He received the M.Sc. degree from Islamic Azad University of Gonabad in 2012. From 2017, he is a Ph.D Candidate of Control Engineering in the Department of Electrical and Computer Engineering, Babol Noshirvani University of Technology. His research interests are optimal control, fractional-order control, time delay systems.

Zahra Rahmani was born in Tehran, Iran, in 1975. She received B.Sc. degree in Control engineering from Sharif University of Technology, Tehran, Iran in 1998, M.Sc. degree and Ph.D. degree in Control engineering from Iran University of Science and Technology, Tehran, Iran in 2000 and 2007, respectively. She works in the department of electrical and computer engineering of Babol University of Technology as an assistant professor from 2008. Her research interests are Nonlinear, Complex, and Hybrid systems, Intelligent methods.

Ali Vahidian Kamyad received the B.Sc. degree from Ferdowsi University of Mashhad, Mashhad, Iran, in 1970, the M.Sc. degree from the Institute of Mathematics Tehran, Tehran, Iran, in 1973, and the Ph.D. degree from Leeds University, Leeds, U.K., in 1988, under supervision of J. E. Rubio.,Since 1972, he has been at Ferdowsi University of Mashhad, where he is currently a Full Professor in the College of Mathematics. His research interests include optimal control of distributed parameter systems and applications of fuzzy theory

Seyed Jalil Sadati was born in Behshahr, Iran, in 1979. He received the M.Sc. degree from Ferdowsi University of Mashhad in 2005, and the Ph.D. degree in control engineering from the University of Mazandaran, Iran, in 2011. From 2011, he is an Assistant Professor of Control Engineering with the Department of Electrical and Computer Engineering, Babol Noshirvani University of Technology. His research interests are fractional-order control, time delay systems, nonlinear control, adaptive systems, and model predictive control.

Figure 1. Block diagram of the proposed method.

Table 1. The estimated value of PI for with $\beta=1$ in compare with the other references (Example 1).

Table 2. Simulation results of PI and CPU time given by the proposed algorithm for various values of β, N and ℓ (Example 1).

Figure 2. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.001 and various amount of β (Example 1).

Figure 3. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.1 and various amount of β (Example 1).

Table 3. The estimated value of PI for with $\beta=1$ in compare with the other references (Example 2).

Table 4. Simulation results of PI and CPU time given by the proposed algorithm for various values of β, N and ℓ (Example 2).

Figure 4. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.001 and various amount of β (Example 2).

Table 5. Comparison of the estimated value of CPU time with the other methods for ℓ=0.004 and β=1 in Example 2.

Figure 5. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.1 and various amount of β (Example 2)

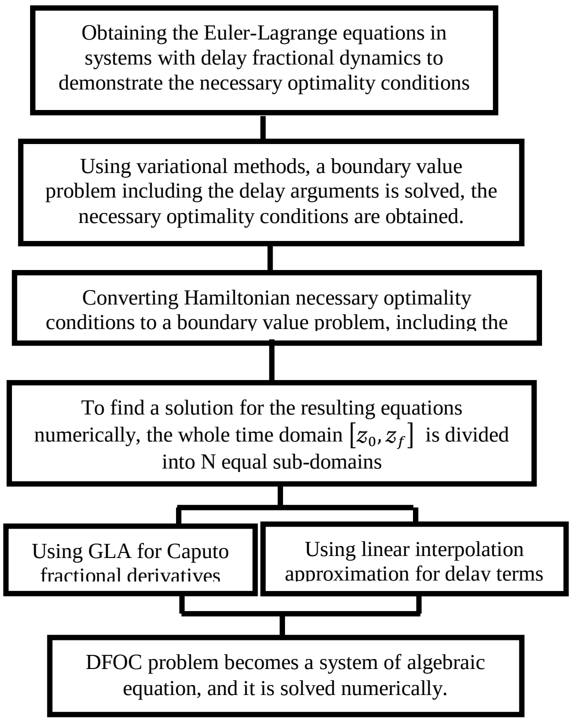

Figure 1. Block diagram of the proposed method

Table 1. The estimated value of PI for with $\beta = 1$ in compare with the other references (Example 1)

Method	PI value J
[12]	0.6381
[14]	0.4727464
[29]	1.6419
[30]	1.6497
[15]	1.6488
[32]	1.6473
[19]	1.6478
[31]	1.6448
Present method	1.6497

Table 2. Simulation results of PI and CPU time given by the proposed algorithm for various values of β, N and ℓ (Example 1)

β	N	ℓ	J	Cpu time
0.7	20	0.1	2.00184	0.001264
	40	0.05	1.85569	0.001716
	200	0.01	1.73703	0.011111
	400	0.005	1.71900	0.048459
	2000	0.001	1.69903	2.308480
	2500	0.0008	1.69744	3.409173
0.8	20	0.1	1.96718	0.001831
	40	0.05	1.82710	0.003027
	200	0.01	1.71817	0.010413
	400	0.005	1.70300	0.049863
	2000	0.001	1.68739	2.355534
	2500	0.0008	1.68620	3.418871
0.9	20	0.1	1.93109	0.002626
	40	0.05	1.79488	0.022449
	200	0.01	1.69297	0.008884
	400	0.005	1.68026	0.050579
	2000	0.001	1.66924	2.095672
	2500	0.0008	1.66858	3.498754
1.0	20	0.1	1.90721	0.031153
	40	0.05	1.77198	0.035863
	200	0.01	1.67185	0.057104
	400	0.005	1.65981	0.105847
	2000	0.001	1.65025	2.311920
	2500	0.0008	1.64977	3.450183

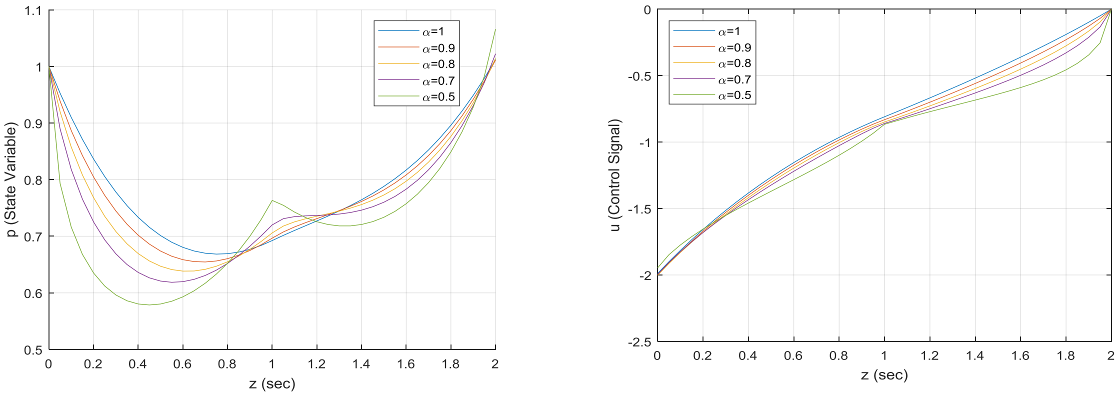

Figure 2. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.001 and various amount of β (Example 1)

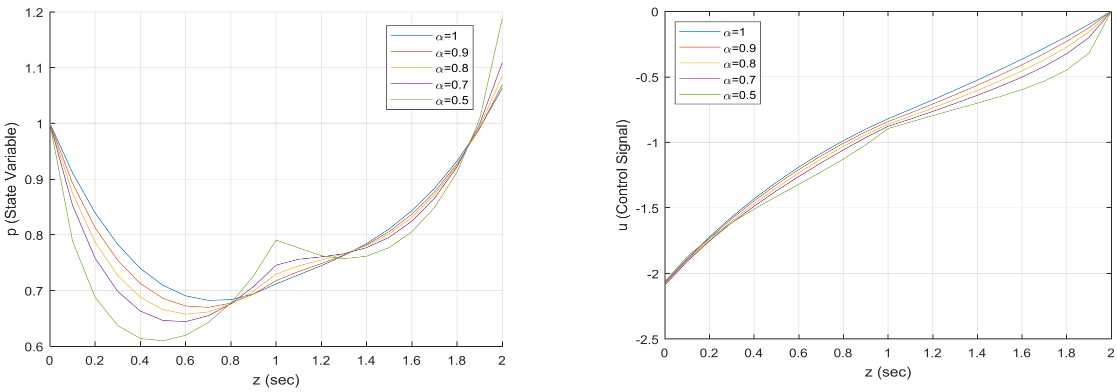

Figure 3. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.1 and various amount of β (Example 1)

Table 3. The estimated value of PI for with $\beta = 1$ in compare with the other references (Example 2)

Method	PI value J
[35]	6.2677
[36]	6.0079
[33]	4.7967
[34]	4.7985
[32]	4:7921
[19]	4.7967
[31]	4.7832
[23]	4.7967
Present method ($\ell = 0.0002$)	4.7978

Table 4. Simulation results of PI and CPU time given by the proposed algorithm for various values of β, N and ℓ (Example 2)

β	N	ℓ	J	Cpu time
0.7	20	0.1	5.96514	0.001310
	40	0.05	5.42788	0.001707
	200	0.01	5.00867	0.033933
	400	0.005	4.94809	0.052347

	2000	0.001	4.88428	2.267731
	2500	0.0008	4.87943	3.660245
0.8	20	0.1	5.92840	0.001206
	40	0.05	5.39150	0.002673
	200	0.01	4.98512	0.017695
	400	0.005	4.93063	0.048707
	2000	0.001	4.87734	2.205480
	2500	0.0008	4.87353	3.694311
0.9	20	0.1	5.88059	0.001017
	40	0.05	5.33676	0.005412
	200	0.01	4.93665	0.012265
	400	0.005	4.88748	0.212383
	2000	0.001	4.84585	2.386488
	2500	0.0008	4.84344	3.389471
1.0	20	0.1	5.86231	0.010807
	40	0.05	5.30239	0.043361
	200	0.01	4.89390	0.058210
	400	0.005	4.84511	0.070769
	2000	0.001	4.80642	2.619932
	2500	0.0008	4.80449	3.329568

Table 5. Comparison of the estimated value of CPU time with the other methods for ℓ=0.004 and β=1 in Example 2.

Methods	[35]	[33]	Present method
Cpu time (s)	62	0.354545	0.069477

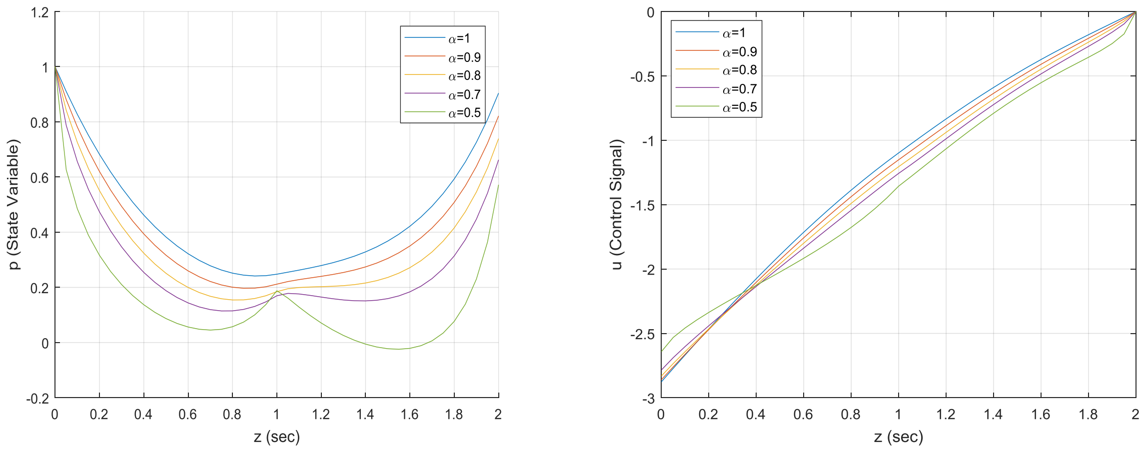

Figure 4. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.001 and various amount of β (Example 2)

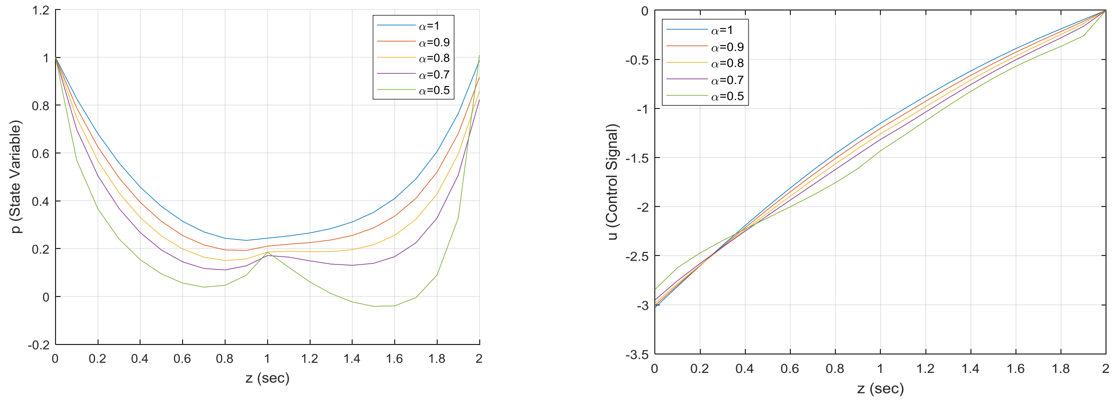

Figure 5. Simulation curves of $p(z)$ and $u(z)$ done by the proposed algorithm for ℓ=0.1 and various amount of β (Example 2)