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Critical investigation of thermally developing nanofluid flow within slippery tubes and channels: An extended Graetz-Nusselt problem with longitudinal conduction and power-law nanofluid

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bvp4c.

Abstract. The Graetz-Nusselt problem is addressed with non-Newtonian power-law nanofluids with slip boundary conditions. In the said fluid model, the power-law coefficient m and flow index n depend on the nanoparticle concentration φ. The Al_2O_3-water nanofluid is considered and results are obtained for typical values of nanoparticle concentration, i.e., $\varphi = 1\%, 2\%, 3\%, 4\%$ and 5%. First of all, we calculate the analytical solution of the fully developed velocity field for power-law nanofluid via the Navier linear slip law. Next, the temperature profile is obtained by utilizing the condition of the specified surface temperature. The longitudinal conduction (which is possible with a low Peclet number) is also considered. The graphical results of mean temperature and local Nusselt number are presented for various values of slip length, nanoparticle concentration, power-law index, and Peclet number. As expected, the concentration of nanoparticles boosts the heat transfer rate, while the slippery boundaries always provide larger flow rates of nanofluid. The analysis reveals that the presence of nanoparticles increases the local Nusselt number and mean temperature. Furthermore, the thermal entry length is considerably enhanced upon raising the nanoparticle concentration and slip length. In addition, decreasing the Péclet number also enhances thermal entrance length.

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1. Introduction

The heat transfer phenomenon by virtue of fluid flow has potential applications in several industrial processes and devices such as MEMS, electronic equipment, energy devices, high-performance gas turbines, chemical processing, and heat exchangers. In modern science and technology, industrial liquids such as distilled water, ink, ethylene glycol, molten plastics, and polymers are widely used. Understanding the heat transfer phenomenon is also important for improving the thermal performance of various complex rheological fluids (low-conductive).

The addition of nanoparticles to base fluid is one of the most famous techniques to increase the thermal performance of many fluids. Nanoparticles exist in the form of carbon, metals, and metal oxides etc. Nanofluids are comparatively more effective in the improvement of thermal conductivity, and this feature makes them popular in engineering applications. Pioneering

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work by Choi [1] revealed the usage of nanofluids to raise the thermal conductivity of fluidic systems. Xuan and Roetzel [2] investigated the heat transport properties of nano fluid in terms of thermal dispersion, which occurs as a result of the random motion of nano particles in base fluid. A few classic studies [3,4] approximated the nano fluid as a Newtonian one with modified physical properties. It is a known fact that most real fluids are non-Newtonian in nature, so Newtonian fluids are not appropriate for predicting the rheological behaviour of many nanofluids. In this context, Cheng et al. [5] elaborated on the rheology of Cu-water nanofluid. He mainly focused on the shear-thinning properties of the nanofluids. Later, carbon-water nanofluid rheology is adopted to increase the shear-thinning nature of the fluid. This can be accomplished by increasing the concentration of nanoparticle volume fraction in the base fluid. Santra et al. [6] investigated heat transfer in a parallel plate duct with Newtonian and non-Newtonian liquids for Cu-water nanofluid flow. Kumar et al. [7] proposed a model in which thermal conductivity is a function of particle diameter and temperature. Prasher et al. [8] explain that a rise in the effective thermal conductivity (k_{eff}) of nanofluid is due to the localized Brownian movement of nanoparticles. Maiga et al. [9] suggested a viscosity correlation formula of γ Al_2O_3-water nanofluid with φ, which shows that the effective viscosity is higher than that of the Brinkman model. Haris et al. [10] conducted experimental studies on Al_2O_3-water and CuO- water nanofluid. They also explained that both of these physical situations predict shear thinning behavior for the typical values of higher shear stress. Furthermore, notable studies related to nanofluid can be found in the [11–13].

The interaction between fluid and walls is another important aspect of fluid mechanical problems. In that context, the integration of slip boundary conditions with the Graetz-Nusselt problem makes them even more versatile. The concept of slip boundary conditions was proposed by Navier [14]. Further, Thompson and Thorian [15] elaborated the slip boundary conditions by using molecular dynamic simulation. Mathews and Hill [16] and Neto et al. [17] conducted two comprehensive studies on the significance of slip boundary conditions. In their analysis, they highlighted the important feature of slip boundary conditions with normal derivative for various flow situations, including pipe flow, slit flow, and annular flow. They also extended the linear slip condition-based analysis of Navier with non-linear slip boundary conditions. To our knowledge, no one has investigated the thermal entry flow in a duct (either pipe or channel) for combined non-Newtonian nanofluid with slip boundary conditions.

The main purpose of this analysis is to elaborate on the influence of the design of so-called plate heat exchangers on their heat transfer performance. Plate heat exchangers are very popular in the food industry and are also used in the plastics industry for a variety of reasons. They are often made out of profiled plates in the shape of many ducts with profiled walls. These ducts have rectangular cross-sections that range from a few millimetres to hundreds of millimetres. Liquids with low to moderate viscosities are heated and cooled from both ends as they pass through these ducts. It is asserted that the profilations have the benefit of not only enhancing the efficiency and the heat exchange area but also improving the heat transfer. This may be valid in the context of laminar flow, in which case the type of profiling may be crucial. In addition, this investigation is also helpful in the fields of biomedical engineering and the chemical industry for the improvement of several types of thermal devices. Motivated by the aforementioned literature, the objective of the present investigation is to elaborate the slip flow analysis along with non-Newtonian nanofluid in both pipe and channel for the specified surface temperature case. Moreover, the axial conduction is also taken into account. A few interesting studies related to the classical Graetz problem can be found in [18–28].

The present article is arranged as follows. The analytical expression of the velocity field with linear slip boundary conditions and the schematic diagram of the flow problem are reported in Section 2. The mathematical formulation and solution of the problem are presented in Sections 3 and 4. The detailed discussions with graphical representation are incorporated in Section 5. Finally, some conclusions are drawn in Section 6.

2. Geometry, constitutive equations, and velocity profile

Let the power-law nano fluid enter either the parallel plate or cylindrical confinement along with fully developed velocity and specified entering temperature (T_i) as depicted in Figure 1. The pipe and channel wall(s) are maintained at a constant wall temperature (T_s). Our goal is to compute the local and mean Nusselt numbers for power-law nanofluids with longitudinal conduction and slip at the wall(s). For this purpose, we require the following relationships:

Continuity equation:

$$\nabla . V = 0, \tag{1}$$

Momentum equation:

$$\rho \, \frac{dV}{dt} = -\nabla P + \nabla . \tau. \tag{2}$$

The constitutive expression for Power-law fluid is [29]:

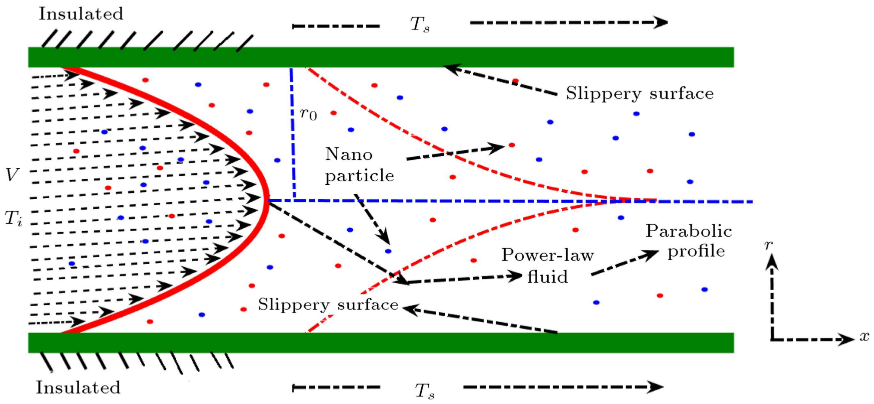

Figure 1. Shemantic sketch of flow problem.

$$\tau = -m\left[\left|\sqrt{\frac{1}{2}\left(\gamma^{\bullet} : \gamma^{\bullet}\right)}\right|^{n-1}\right]\gamma^{\bullet}, \tag{3}$$

and:

$$\gamma^{\bullet} = \nabla V + \nabla V^{t}. \tag{4}$$

2.1. Axisymmetric tube

The present theoretical analysis is based on the following assumptions: (i) spherical nanoparticles, (ii) uniform particle size and shape, (iii) constant physical properties, (iv) thermal equilibrium between liquid and solid particles moving at the same speed, (v) steady state, (vi) fully developed flow, and (vii) laminar flow:

Eq. (2) (under the aforesaid assumptions) takes the following form:

$$0 = -\frac{dP}{dx} + \frac{m}{r}\frac{d}{dr}\left(r\left(\frac{dV}{dr}\right)^{n}\right), \tag{5}$$

where it is assumed that $V = [0, 0, V]$. The non-dimensional power-law velocity expression with Navier (linear) slip law $\left(V = -l\,\frac{dV}{d\bar{r}}\Big|_{\bar{r}=1}\right)$ takes the following form (details in [16]):

$$\bar{V} = \left(1 - (\bar{r})^{1/n+1} + \left(1/n + 1\right)\bar{l}\right), \tag{6}$$

where:

$$\bar{r} = \frac{r}{r_0}, \qquad \bar{l} = \frac{l}{r_0} \qquad \bar{V} = \frac{V}{V_{\max}}. \tag{7}$$

3. Thermal analysis for power-law nanofluid

The energy equation for thermal analysis can be written as:

$$\rho\, c_p \frac{dT}{dt} = k\nabla^2 T + \phi. \tag{8}$$

The expanded energy equation, with longitudinal conduction and negligible viscous dissipation, can be written as [30]:

$$\rho\, c_p\left(V\frac{\partial T(x,r)}{\partial x}\right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T(x,r)}{\partial r}\right) + \frac{\partial^2 T(x,r)}{\partial x^2}\right), \tag{9}$$

where it is assumed that $T = T(x, r)$. In Eq. (9) ρ is the density, c_p the specific heat, and k the effective thermal conductivity of the nanofluid. The density of nanofluid [31] is:

$$\rho = \varphi\rho_p + (1-\varphi)\,\rho_{bf}, \tag{10}$$

where ρ_p and ρ_{bf} are the densities of nano particles and base fluid. The specific heat capacity can be written as [32]:

$$\rho C_p = \varphi\rho_p C_{p,p} + (1-\varphi)\,\rho_{bf} C_{b,bf}. \tag{11}$$

The effective thermal conductivity of nanofluid (based on experimental results [33]) is given as:

$$k = k_{bf}\,(1 + 7.47\varphi). \tag{12}$$

The dimensionless variables are:

$$\bar{r} = \frac{r}{r_0}, \qquad \bar{x} = \frac{x}{R_{eD}\, pr\, r_0},$$

$$R_{eD} = \frac{V_{\max} r_0}{\upsilon}, \qquad \theta = \frac{T - T_s}{\Delta T},$$

$$Pe = R_{eD}\, pr. \tag{13}$$

Using Eq. (13) in Eq. (9), we obtain:

$$\bar{V}(\bar{r})\left(\frac{\partial\theta(\bar{x},\bar{r})}{\partial\bar{x}}\right) = \gamma^{*}\left(\frac{\partial^2\theta(\bar{x},\bar{r})}{\partial\bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial\theta(\bar{x},\bar{r})}{\partial\bar{r}} + \frac{1}{Pe^2}\frac{\partial^2\theta(\bar{x},\bar{r})}{\partial\bar{x}^2}\right), \tag{14}$$

where R_{eD} denotes the Reynolds number and Pe

denotes the Péclet number. The non-dimensional boundary conditions are:

$$\theta\left(0,\,\bar{r}\right)=1,\quad \theta\left(\bar{x},\,1\right)=0,\quad \frac{\partial\theta\left(\bar{x},\,0\right)}{\partial\bar{r}}=0. \tag{15}$$

The classical separation method is employed for the solution of Eq. (14) subject to the boundary conditions (Eq. (14)). The ansatz $\theta\left(\bar{x},\,\bar{r}\right)=N\left(\bar{x}\right)S\left(\bar{r}\right)$ yields the following differential equations and boundary conditions given by Eqs. (14) and (15):

$$N'\left(\bar{x}\right)+\gamma^{*}\lambda^{2}N\left(\bar{x}\right)=0, \tag{16}$$

$$S''\left(\bar{r}\right)+\frac{1}{\bar{r}}S'\left(\bar{r}\right)+\lambda^{2}\left(\frac{\lambda^{2}}{Pe^{2}}\pm\left(1-\left(\bar{r}\right)^{1/n+1}+\left(1/n+1\right)\bar{l}\right)\right)S\left(\bar{r}\right)=0, \tag{17}$$

$$S'\left(0\right)=0,\quad S\left(1\right)=0. \tag{18}$$

Eq. (16) is integrated as:

$$N\left(\bar{x}\right)=C\exp\left(-\left|\gamma^{*}\right|\lambda^{2}\bar{x}\right). \tag{19}$$

Eq. (17) is not a regular SL-Problem and its eigenfunctions are not mutually orthogonal with respect to its weight function. As a result, the modified Gram-Schmidt procedure (see [34,35]) is used in conjunction with the MATLAB routine bvp4c to compute the eigenfunctions $S(\bar{r})$ and constant in Eq. (19).

Now the solution to Eq. (14) can be written as:

$$\theta\left(\bar{r},\;\bar{x}\right)=\sum_{0}^{i}C_{i}S_{i}\left(\bar{r}\right)\exp\left(-\left|\gamma^{*}\right|\lambda_{i}^{2}\bar{x}\right). \tag{20}$$

We will use the following relationship for the mean temperature:

$$\theta_{m}\left(\bar{x}\right)=\frac{\int_{0}^{1}\bar{V}\theta\left(\bar{r},\,\bar{x}\right)\;\bar{r}\,d\bar{r}}{\int_{0}^{1}\bar{V}\;\bar{r}\,d\bar{r}}. \tag{21}$$

The following formulas can be used to calculate the local and mean Nusselt numbers:

$$Nu\left(\bar{x}\right)=\frac{(-2)}{\theta_{m}\left(\bar{x}\right)}\frac{\partial\theta\left(\bar{x},\,1\right)}{\partial\bar{r}}, \tag{22}$$

$$Nu_{m}\left(\bar{x}\right)=\frac{1}{\bar{x}}\int_{0}^{\bar{x}}Nu\left(x^{*}\right)dx^{*}. \tag{23}$$

4. Flat channel

The momentum equation for channel flow analysis is:

$$0=-\frac{dP}{d\bar{x}}+m\frac{d}{d\bar{r}}\left(\left(\frac{dV}{d\bar{r}}\right)^{n}\right). \tag{24}$$

Following the same solution procedure as given in [16], the only valid expression of power-law fluid velocity with linear Navier-Slip law is as follows:

$$V=\left(1-\left(\bar{r}\right)^{1/n+1}+\left(1/n+1\right)\bar{l}\right), \tag{25}$$

where:

$$\bar{r}=\frac{r}{r_{0}},\qquad \bar{l}=\frac{l}{r_{0}},\qquad \bar{V}=\frac{V}{V_{\max}}. \tag{26}$$

For thermal analysis, the non-dimensional heat equation with associated boundary conditions is:

$$\bar{V}\left(\bar{r}\right)\left(\frac{\partial\theta\left(\bar{x}\,,\bar{r}\right)}{\partial\bar{x}}\right)=\gamma^{*}\left(\frac{\partial^{2}\theta\left(\bar{x}\,,\bar{r}\right)}{\partial\bar{r}^{2}}+\frac{1}{Pe^{2}}\frac{\partial^{2}\theta\left(\bar{x}\,,\bar{r}\right)}{\partial\bar{x}^{2}}\right), \tag{27}$$

$$\theta\left(0,\,\bar{r}\right)=1,\quad \theta\left(\bar{x},\,1\right)=0,\quad \frac{\partial\theta\left(\bar{x},\,0\right)}{\partial\bar{r}}=0. \tag{28}$$

The solution procedure is the same as described in the previous section. The mean temperature, local and mean Nusselt number are achieved through the following formulae:

$$\theta_{m}\left(\bar{x}\right)=\frac{\int_{0}^{1}\bar{V}\theta\left(\bar{r},\,\bar{x}\right)\;d\bar{r}}{\int_{0}^{1}\bar{V}\,d\bar{r}}, \tag{29}$$

$$Nu\left(\bar{x}\right)=\frac{(-4)}{\theta_{m}\left(\bar{x}\right)}\frac{\partial\theta\left(\bar{x},\,1\right)}{\partial\bar{r}}, \tag{30}$$

$$Nu_{m}\left(\bar{x}\right)=\frac{1}{\bar{x}}\int_{0}^{\bar{x}}Nu\left(x^{*}\right)dx^{*}. \tag{31}$$

5. Results and discussion

The computed results obtained via the aforesaid technique are discussed in this section. The problem investigated in the present study depends on several parameters, which are power-law index (n), nanoparticle concentration φ, slip length (l) and Peclet number Pe. Solutions for both channel and tube cases are obtained for several suitable sets of emerging parameters. Moreover, to make sure of the accuracy of the current methodology, we compare our obtained results with those that have already been published in the literature. First, we validate our results for Newtonian liquid ($n=1$ and $l=0$) with no axial conduction effects ($Pe \rightarrow \infty$) with the corresponding results presented by Johnson [36]. The variations in local Nusselt number against axial distance in a steady state are compared in Table 1. According to the table, our computed findings are a perfect match for Johnson's results [36]. Furthermore, the fully developed Nusselt number for the uniform surface temperature case is fairly close to the expected analytical value of 3.66.

In this analysis, the classical Graetz problem is elaborated for non-Newtonian power-law rheology with Al_2O_3-water nanofluid and Navier slip boundary condition. The empirical constants appearing in the power-law model are taken to be dependent on the nanofluid solid volume fraction. Putra et al. [37] (experimentally) presented the relationship between the shear strain and shear stress for Al_2O_3-water nano fluid. Using this result, they calculated the values of the empirical constants m and n for various solid volume fraction values. These values are appropriately interpolated and extrapolated, keeping in mind that the shear stress decreases with a gradual increase in φ for a particular shear rate in the mixture. The numerical values of empirical constants appearing in the power-law model and thermophysical properties are displayed in Tables 2 and 3, respectively. It is also noted that for a shear thinning fluid, the value of flow index n is always less than unity. The present analysis is carried out for Al_2O_3-water nanofluid (based on the experimental data provided by Putra et al. [37]). Next, we will present the heat transfer analysis through engineering quantities, i.e., mean temperature, local and mean Nusselt numbers in the presence of nanoparticle concentration φ, slip length l and longitudinal conduction (Pe).

The profiles of mean temperature for various values of nano particle concentration φ, slip length (l), and Peclet number (Pe) are presented in Figures 2 and 3 for flat channel configuration. It is observed that mean temperature increases in the presence of nanoparticle concentration. The effect of axil conduction is more prominent in the entrance region as compared to the fully developed region. Moreover, both slip length and Peclet number tend to increase the mean temperature. A similar trend is observed

Table 1. Comparison of current approach with existing data for $n = 1$ and $Pe = 1000$.

P.R. Johnson [36]	Present approach	
$\bar{x}$	$Nu(\bar{x})$	$Nu(\bar{x})$
0.01	7.47	7.47
0.02	6.00	6.00
0.04	4.91	4.91
0.06	4.44	4.44
0.1	4.00	4.00
0.2	3.71	3.71
0.3	3.66	3.66

Table 2. Numerical values for (m, n) [37].

Solid fraction, φ (%)	m (Nsec^n m^{-2})	n
0.5	0.00187	0.880
1	0.00230	0.830
1.5	0.00283	0.780
2	0.00347	0.730
2.5	0.00426	0.680
3	0.0053	0.625
3.5	0.00641	0.580
4	0.0075	0.540
4.5	0.00876	0.500
5	0.01020	0.460

Table 3. Thermophysical properties at 20°C [37].

Property	Liquid	Solid
c_p (J/kgK)	4181.80	383.1
ρ (kg/m^3)	1000.52	8954.0
K (W/mK)	0.597	386.0

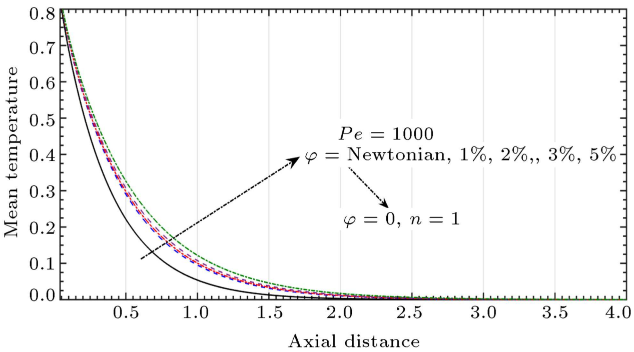

Figure 2. Mean channel temperature in the presence of a nanofluid volume fraction and a fixed slip length ($l = 0.10$).

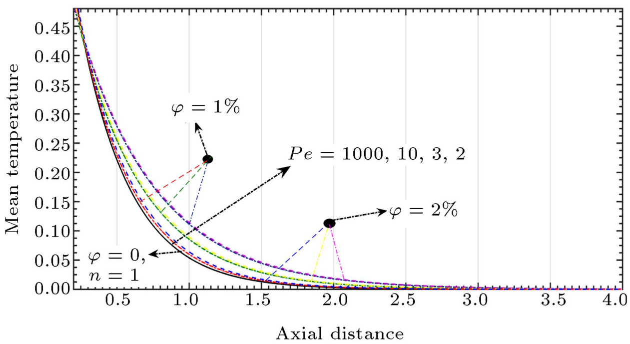

Figure 3. Mean channel temperature in the presence of nanofluid volume fraction and Peclet number at ($l = 0$).

for the tube case from Figures 4 and 5. The local Nusselt number for the channel case is presented in Figures 6 and 7 for different values of nanoparticle concentration, slip length, and Peclet number. The presence of nanoparticle concentration increases the local Nusselt number at a fixed slip length. Further, the axial conduction and slip parameters both amplify the local Nusselt number. Similar observations are obtained for tube geometry. This fact is confirmed by Figures 8 and 9. Further, the higher Nusselt numbers are observed in the channel than in the tube. Figure 10 depicts the magnitude of both the local and mean

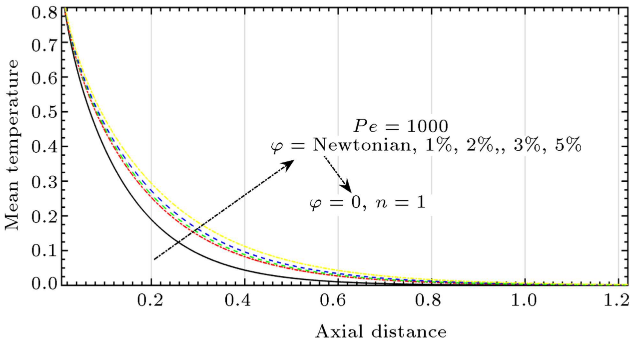

Figure 4. Tube mean temperature in the presence of a nanofluid volume fraction and a fixed slip length ($l = 0.10$).

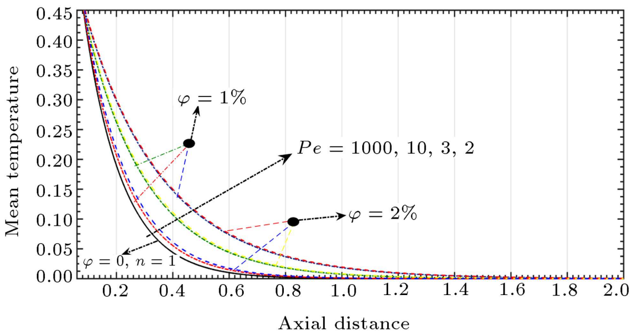

Figure 5. Mean temperature in the presence of nanofluid volume fraction and Peclet number at slip length for the tube ($l = 0$).

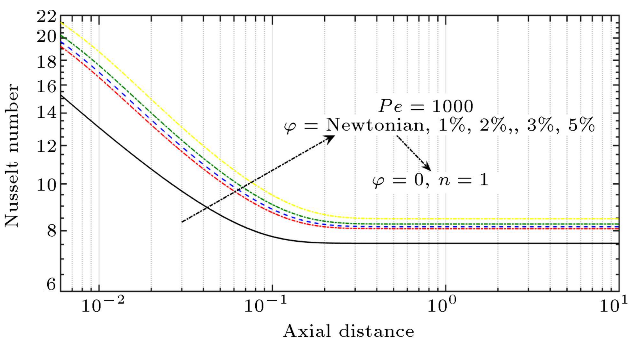

Figure 6. The local Nusselt number for a channel with a nanofluid volume fraction and a fixed slip length ($l = 0.10$).

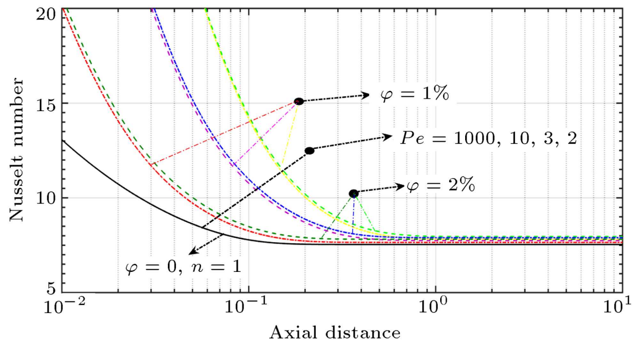

Figure 7. Local Nusselt number and Peclet number for a channel as a function of nanofluid volume fraction at slip length ($l = 0$).

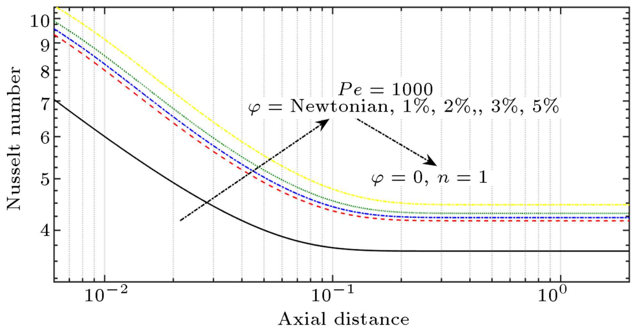

Figure 8. The local Nusselt number for tubes with a nanofluid volume fraction and a fixed slip length ($l = 0.10$).

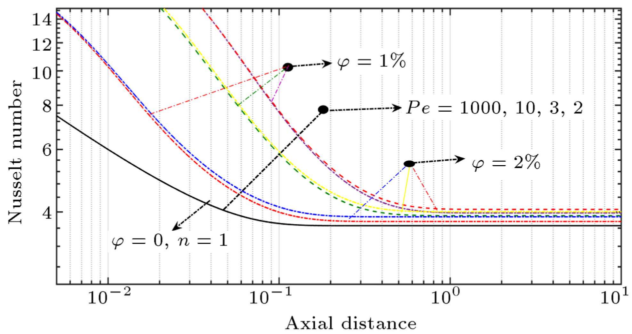

Figure 9. Local Nusselt number and Peclet number at slip length for tubes with nanofluid volume fraction ($l = 0$).

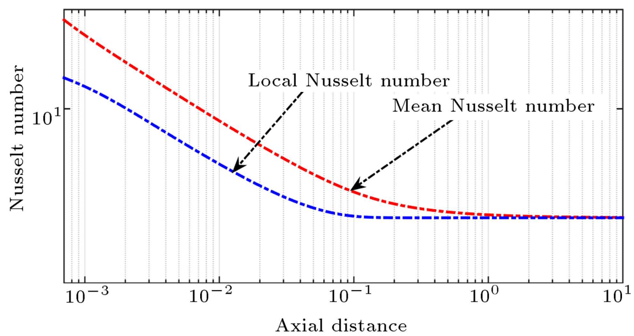

Figure 10. Local and mean Nusselt numbers for ($l = 0, \varphi = 0, Pe = 1000$).

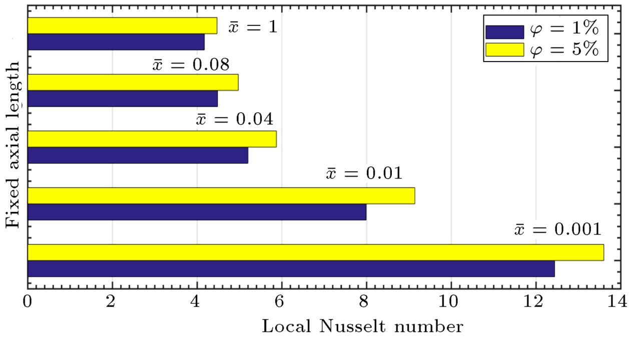

Figure 11. Nusselt number for parallel plate in the presence of nanofluid volume fraction and fixed slip length ($l = 0.10$).

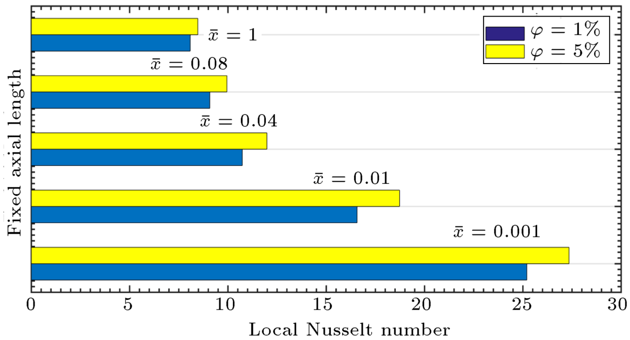

Figure 12. Nusselt number for pipe in the presence of nanofluid volume fraction and fixed slip length ($l = 0.10$).

Nusselt numbers. Clearly, the mean Nusselt number is higher than the local Nusselt number. This trend is typical for all values of nanoparticle concentration, slip parameter, and Peclet number. A bar graph depicts the magnitude of the local Nusselt number for both geometries. Figure 11 is for the channel configuration, while Figure 12 is for the tube case. It is clearly seen that the magnitude of local Nusselt number is higher for the fully developed regime in the presence of nanoparticle concentration and slip length. Moreover, channel geometry achieves the highest local Nusselt number in comparison with tube geometry.

6. Concluding remarks

The classical Graetz-Nusselt problem is extended to non-Newtonian power-law nanofluid in both circular tubes and channels with slip boundary conditions. The analytical expression of the velocity profile is obtained for the Navier-slip boundary condition for both confinements. The temperature fields of the power-law nanofluid for the uniform surface temperature case for both tube and channel geometries are obtained in terms of an infinite series solution. Finally, the mean temperature and local Nusselt number showing the temperature distribution of the power law nanofluid are calculated. The obtained results revealed the effects of nanoparticle volume fraction, Peclet number, and slip length on nanofluid heat transfer. Here are a few highlights:

- The mean temperature for pipe and channel confinement rises as the volume fraction of nanoparticles increases;
- Increasing the nanoparticle volume fraction increases the local Nusselt number for a fixed Peclet number and slip length;
- The heat transfer rate at the wall(s) increases as the slip length increases;
- The local Nusselt number is always less than the mean Nusselt number;
- Heat transfer significantly depends on the shape of the geometry. Channel geometry is more appropriate in that context;
- Lowering the Peclet number and slip length, as well as tuning the non-Newtonian nanofluid characteristics, allows the fully developed condition to be reached sooner.

Nomenclature

ρ	Density (kg/m^3)
d/dt	Material derivative
r	Radial coordinate (m)
V_m	Mean velocity (m/s)
c_p	Specific heat (J/kg. K)
V	Velocity profile (m/s)
θ	Non-dimensional temperature
pr	Prandtl number
θ_m	Bulk temperature
Nu	Local Nusselt number
∇	Gradient operator
γ^*	Is equal to $\frac{A^*}{B^*}$
A^*	Is equal to $[1 + 1.47\varphi]$
B^*	Is equal to $[1 + \varphi(\frac{\rho_p C_{p,p} - \rho_{bf} C_{b,bf}}{\rho_{bf} C_{b,bf}})]$
T	Temperature (K)
φ	Nanoparticles concentration
x	Axial coordinate (m)
m	Model parameter (Nsex^n m^{-2})
n	Flow index
R_{eD}	Reynolds number
r_0	Half width (m)
λ	Eigenvalue
Pe	Peclet number
Nu_m	Average Nusselt number

S	Eigenfunction
i	1, 2, 3,...
τ	Shear stress (Pa)
$-$	Dimensional parameter
ΔT	Temperature difference (K)
T_s	Surface temperature (K)
T_{in}	Inlet temperature (K)
k	Thermal conductivity (W/m.K)
k_{bf}	Thermal conductivity of base fluid (W/m.K)
ρ_p	Density of nanoparticle (kg/m^3)
ρ_{bf}	Density of base fluid (kg/m^3)
$C_{p,p}$	Specific heat of the nanoparticle (J/kg.K)
$C_{b,bf}$	Specific heat of the base fluid (J/kg.K)
$\bar{l}$	Slip parameter

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