An exact iterative algorithm to solve a linear fractional programming problem

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Abstract

The Linear Fractional Programming (LFP) problem that optimizes the ratio of two linear objective functions under linear constraints has a wide range of application areas. Based on the traditional definition of continuity, we developed an exact iterative algorithm that does not depend on big-M coefficients. Removing the nonlinearity in the fractional objective function by converting the objective function into a linear form, an equivalent linear-iterative problem is obtained and a computationally efficient algorithm is proposed. We also analyze the unbounded and asymptotic solution case of the LFP. To demonstrate the efficiency of the proposed method, illustrative numerical examples are provided for all solution cases. Also, we analyze the validity of our algorithm and compare our results with the existing algorithm from the literature by generating random large-scale test problems.

Keywords: Linear Fractional Programming, Iterative Optimization, Asymptotic Solution, Unboundedness, Large Scale Optimization.

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1. Introduction

Linear fractional programming (LFP) problems are of great interest because of their extensive application areas such as resource allocation, transportation, production, finance, location theory, stochastic processes, Markov renewal programs, information theory, applied linear algebra, large scale programming, game theory, etc. In many practical applications like cutting stock problems, ore blending problems, shipping schedules problems, the optimal policy for a Markovian chains, the sensitivity of linear programming problems, optimization of ratios of criteria gives more insight into the situation than the optimization of each criterion. To study relative efficiency in different fields such as education, hospital administration, court systems, air force maintenance units, Bank branches, etc., fractional programming solves more efficiently the above type of problems. In real world decision situations, decision makers, sometimes, may face up with the decision to optimize inventory/sales, actual cost/standard cost, output/employee, etc. with respect to some constraints [[1]]. Reviews of various fractional programming applications are given by Schaible ([2],[3],[4]), and Craven [5]. Further references are listed in Schaible's bibliography of this field [2], which covers more than 550 articles. There are also comprehensive books related to LFP problems, such as that of Bajalinov [6].

A usual linear fractional programming problem is a special case of a non-linear programming problem. Isbell and Marlow [7] presented an algorithm for the LFP problem of the maximization of a linear fractional function under linear constraints. An LFP problem can be transformed into a linear programming (LP) problem by using the variable transformation method of Charnes and Cooper [8], or it can be solved by adopting the updated objective function method of Bitran and Novaes [9]. Bitran and Magnanti [10] considered algorithms, duality, and sensitivity analysis for optimization problems that they, called fractional, whose objective function is the ratio of two real-valued functions. Assuming that the denominator of the objective function of the LFP problem over the feasible region is positive, Swarup [11] extended the wellknown simplex method to solve LFP problems, with the object of giving an algorithm for the solution of programming problems with linear fractional functions without reducing them to LP problems. Singh [12] extended the saddle-point and stationary point theory of optimality in nonlinear programming to nonlinear fractional programming problems. Bajalinov [13] considered a special problem in the context of linear and LFP: given an objective function on a feasible bounded set S, the optimal vertex, and a neighboring vertex $\bar{x}$, adjust the objective function to make $\bar{x}$ the new optimum. Considering the presented literature up to [13], it can be concluded that almost all of the methods developed for LFP are analytical methods.

As an iterative method, Tantawy [14] focused on LFP problems with inequality constraints and presented a method based on the conjugate gradient projection that is applied to solve nonlinear programming problems with linear constraints. Tantawy [15] proposed an iterative method for solving LFP problems and showed that this method can be used for sensitivity analysis when a scalar parameter is introduced into the objective function coefficients. Effati and Pakdaman [16] introduced an interval-valued LFP problem and proved that an interval-valued LFP problem can be converted to an optimization problem with an interval-valued objective function whose bounds are linear fractional functions. Tantawy [17] presented a concept of duality for the LFP problem in which the objective function is a linear fractional function and the constraint functions are in the form of linear inequalities. Using the concept of duality, Simi and Talukder [18] presented a new approach for solving LFP problem. Biswas et al. [19] discussed the theory of LFP and presented some proof for the optimality and convexity of LFP. Fu et al. [20] proposed simulation-based linear fractional programming model, which integrates a runoff simulation model into a LFP framework, is developed for optimal water resource planning. Ozkok [21] presented an efficient iterative algorithm to solve LFP problem. In their paper they noticed that in order to improve their algorithm they should determine the good big-M values. Based on this idea, we improved their algorithm by removing the big-M parameter in their approach. Moreover, we demonstrated the effectiveness of our new algorithm, which we developed by eliminating big-M from Ozkok's algorithm, on large-scale examples that we have produced, by comparing the results.

As another type of LFP, Das et al. [22] focused on the LFP problem with absolute value functions. They applied the traditional unrestricted variable transformation and convert the LFP into an LP problem. As a real-life application of LFP, there are some current studies. Considering the challenge of solving large-scale LFP problems, Mohammed and Lomte [23] proposed a secure and verifiable scheme to offload the computations on the cloud side. Mohammed et al. [24] also focused on outsourcing of scientific computations and shown how to use certificate validation to obtain correctness guarantees for privacy-preserving outsourcing of LFP problem. Adding some parameters obtained by solving a linear fractional programming problem, [25] presented an extended version of the Tikhonov regularization method. In [26], an LFP is converted to an LP problem by using the duality concept, and a traditional data envelopment analysis model is provided to demonstrate the applicability.

Considering the objective function is continuous at every point of the feasible region of LFP problem, a convergence condition is obtained by the traditional definition of continuity. We combine our convergence condition and the objective function of the LFP problem to create an iterative constraint. Using this constraint, we construct a new iterative LP problem to obtain the optimal solution of the LFP problem. This iterative LP problem can solve all LFP problems that have a bounded feasible region.

This paper is organized as follows: In Section 2, we present the definition of the LFP problem and some preliminaries. In the next section, we present our methodology and give the steps of our new algorithm and its flow chart. Finally, Section 4 and Section 5 consist of our numerical examples and conclusions, respectively.

2. Problem definition and preliminaries

The mathematical model of an LFP problem can be stated as follows:

$$\max Z(\mathbf{x}) = \frac{\mathbf{N}(\mathbf{x})}{\mathbf{D}(\mathbf{x})} = \frac{\mathbf{c^T x} + \alpha}{\mathbf{d^T x} + \beta} \qquad \text{subject to } \mathbf{x} \in S \tag{1}$$

where $S = \left\{ \mathbf{x} \in \mathbf{R}^n \left| \mathbf{Ax} \leq \mathbf{b}, \quad \mathbf{x} \geq 0, \quad \mathbf{b} \in \mathbf{R}^m, \quad \left(\mathbf{d}^T \mathbf{x} + \beta\right) > 0 \right. \right\}$ is a convex and compact nonempty set and the fractional objective function is defined on S , that is $Z : S \subset \mathbf{R}^n \to \mathbf{R}(S) \subset \mathbf{R}$. We note here that assuming that is the domain of the objective function is not a restriction, since the problem (1) aims to find an optimal solution on S.

Remark 1: Resulting from the basic assumption $\left(\mathbf{d}^T \mathbf{x} + \beta\right) > 0$, the fractional function $Z(\mathbf{x})$ is continuous on $\mathbf{R}$, and also on its domain S, that is, $Z(\mathbf{x})$ is continuous on $\forall \mathbf{x}_i \in S$ and the corresponding function value is $Z(\mathbf{x}_i) = Z_i$.

Definition 1: A *neighborhood* of a point $\mathbf{x}_i \in \mathbf{R}^n$ is the set

$$B(\mathbf{x}_i, r) = \{\mathbf{x} \in \mathbf{R}^n \,| |\mathbf{x} - \mathbf{x}_i| < r\}$$

where $r \in \mathbf{R}^+$ is some positive number. The neighborhood is also called a ball with radius r and centre $\mathbf{x}_i$.

Definition 2: $Z(\mathbf{x})$ is continuous on $S \subset \mathbf{R}^n$ provided that for every point $\mathbf{x}_i \in S$ and for $\varepsilon > 0$, there exists a number $\delta > 0$ such that $Z(\mathbf{x})$ satisfies $|Z(\mathbf{x}) - Z(\mathbf{x}_i)| < \varepsilon$ whenever $\mathbf{x} \in S$ and the distance between $\mathbf{x}$ and $\mathbf{x}_i$ satisfies $|\mathbf{x} - \mathbf{x}_i| < \delta$.

Using Definition 1, Definition 2 can be re-expressed in terms of neighborhoods as follows: $\forall \varepsilon \in \mathbf{R}^+$, if there exists $\delta > 0$ such that $\forall \mathbf{x} \in B(\mathbf{x}_i, \delta)$ and $Z \in B(Z_i, \varepsilon)$ where $Z_i = Z(\mathbf{x}_i)$, then $Z(\mathbf{x})$ is continuous at the given $\mathbf{x}_i \in \mathbf{R^n}$.

3. Our methodology

Considering the algorithm in Ozkok [21], the convergence condition for the LFP (1),

$$Z\mathbf{x} = Z_i \mathbf{x} + \mathbf{x}_i Z - Z_i \mathbf{x}_i \tag{2}$$

is obtained.

It is obvious that the fractional objective function $Z = \frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta}$ can be written as follows:

$$\mathbf{d}^T\mathbf{x}Z + \beta Z = \mathbf{c}^T\mathbf{x} + \alpha. \tag{3}$$

Combining (2) and (3), we have the following linear equation:

$$\left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\bar{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha. \tag{4}$$

Obviously, the structures of these two functions are different from each other. Therefore, from now on, the objective function will be expressed with Z for the LFP problem and $\bar{Z}$ for the LP problem.

Thus, the problem (1) can be converted to the following iterative LP problem:

$$\max \bar{Z} \tag{5a}$$

$$\text{s.t.} \quad \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\bar{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha \tag{5b}$$

$$\mathbf{x} \in S \tag{5c}$$

Here, the subscript $i \in \{0, 1, 2, \ldots\}$ denotes the iteration counter and (5b) represents our iterative constraint. Starting with an initial solution $(\mathbf{x}_0, Z_0)$, that is $i = 0$, the the sub-problem (5) generates a second feasible solution $(\mathbf{x}_1^*, Z_1^*)$ by using the initial solution, then moves to a third feasible solution $(\mathbf{x}_2^*, Z_2^*)$ by using the previous feasible solution $(\mathbf{x}_1^*, Z_1^*)$, and so on. In general, if $(\mathbf{x}_i^*, Z_i^*)$ is a feasible solution obtained at iteration i, then our algorithm finds a new feasible solution $\left(\mathbf{x}_{i+1}^*, Z_{i+1}^*\right)$ at iteration $i + 1$.

Proposition 1: The gradient vectors of the fractional objective function Z and the linear objective function $\bar{Z}$ are equal at every point $\mathbf{x}_i \in S$.

Proof: We omit the proof, the interested reader can see it in Ozkok [21].

Result 1: Since the gradient direction determines the direction of the increase of a function, an increase in the linear objective function $\bar{Z}\left(\mathbf{x}_i\right) \leq \bar{Z}(\mathbf{x}^*)$ implies an increase in the fractional objective function $Z\left(\mathbf{x}_i\right) \leq Z(\mathbf{x}^*)$.

Proposition 2: Starting with an initial point $\mathbf{x}_0 \in S$, let the successive optimal solutions of (5) be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots$, $\forall i \in N$. Then, the fractional objective function values generate an increasing sequence for the successive optimal solutions $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_i, \mathbf{x}_{i+1}, \ldots$, that is $Z_0 \leq Z_{_1} \leq \ldots \leq Z_{_i} \leq Z_{_{i+1}} \leq \ldots$ until reaching the optimal solution of (1) where $\forall i \in N$.

Proof: We omit the proof, the interested reader can see it in Ozkok [21].

Theorem 1: If an increasing sequence $\{Z_i\}_{i\in N}$ is bounded above, then it converges.

Proof: Straightforward.

Result 2: If the feasible region S is bounded, then $\{Z_i\}_{i\in N}$ always converges since it is bounded above. This means that successive optimization of the problem (5) guarantees obtaining the optimal solution of problem (1).

Result 3: If the feasible region S is unbounded, then $\{Z_i\}_{i\in N}$ is either bounded above (convergent) or unbounded (divergent).

Result 4: The case of unbounded solution of problem (5) $\left(\lim\limits_{\mathbf{x}\to\infty} \bar{Z}(\mathbf{x}) \to \infty\right)$ implies $\lim\limits_{\mathbf{x}\to\infty} Z(\mathbf{x}) = \lim\limits_{\mathbf{x}\to\infty} \frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta}$ which is the asymptotic solution of problem (1). This solution can be obtained by an additional analysis which we give in the next subsection.

3.1. Unbounded feasible region case

Assume that the feasible region

$$S = \left\{\mathbf{x} \in \mathbf{R}^n \,\middle|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0, \mathbf{b} \in \mathbf{R}^m, \left(\mathbf{d}^T\mathbf{x} + \beta\right) > 0\right\}$$

is unbounded and the problem (5) has an unbounded solution, that is $\lim\limits_{\mathbf{x}\to\infty} \bar{Z}(\mathbf{x}) \to \infty$. By adding the constraint $x_1 + x_2 + \cdots + x_n \leq M$ to the feasible region S , we get

$$\overline{S} = \left\{\mathbf{x} \in \mathbf{R}^n \,\middle|\, \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq 0, \quad \mathbf{b} \in \mathbf{R}^m, \quad \left(\mathbf{d}^T\mathbf{x} + \beta\right) > 0\,, \ \mathbf{1}\cdot\mathbf{x} \leq M \ \right\}$$

where M is a very big number. As seen, $\overline{S}$ is the restricted form of with the hyperplane $x_1 + x_2 + \cdots + x_n = M$. The optimal solution in $\overline{S}$ occurs at an extreme point of the intersection of the original constraints $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and the hyperplane $x_1 + x_2 + \cdots + x_n = M$. Let reduce the feasible region $\overline{S}$ by the following transformation for each decision variable:

$$\frac{x_j}{M} = v_j \Rightarrow x_j = M\, v_j \Rightarrow \mathbf{x} = M \cdot \mathbf{V}$$

By this transformation, the constraint $\mathbf{1} \cdot \mathbf{x} \leq M$ can be converted to $v_1 + v_2 + \ldots + v_n \leq 1$ or $\mathbf{1} \cdot \mathbf{V} \leq 1$. The original constraints $\mathbf{Ax} \leq \mathbf{b}$ can be converted to $\mathbf{AV} \leq \mathbf{0}$ since the limit of $\frac{\mathbf{b}}{M}$ is zero as M approaches to infinity. The nonnegativity constraints $\mathbf{x} \geq 0$ and the assumption $\left(\mathbf{d}^T\mathbf{x} + \beta\right) > 0$ can be written as $\mathbf{V} \geq 0$ and $\mathbf{d^T V} > 0$ respectively.

Thus, the reduced feasible region $\widehat{S}$

$$\widehat{S} = \left\{ \mathbf{V} \in \mathbf{R^n} \,\middle|\, \mathbf{AV} \leq \mathbf{0},\ \mathbf{V} \geq \mathbf{0},\ \mathbf{d^T V} > \mathbf{0},\ \mathbf{1} \cdot \mathbf{V} \leq \mathbf{1} \right\},$$

and the objective function of (1) is:

$$Z\left(\mathbf{x}\right) = \frac{\mathbf{c}^T\mathbf{x} + \alpha}{\mathbf{d}^T\mathbf{x} + \beta} = \frac{M\mathbf{c}^T\mathbf{V} + \alpha}{M\mathbf{d}^T\mathbf{V} + \beta} = \frac{M\left(\mathbf{c}^T\mathbf{V} + \frac{\alpha}{M}\right)}{M\left(\mathbf{d}^T\mathbf{V} + \frac{\beta}{M}\right)}.$$

The limit of the objective function as M approaches to infinity can be evaluated as:

$$\lim_{M\to\infty} Z(\mathbf{x}) = \frac{\mathbf{c}^T\mathbf{V}}{\mathbf{d}^T\mathbf{V}} = \bar{Z}(\mathbf{V}),$$

thus the problem to find the maximum of asymptotic values of the fractional objective function Z($\mathbf{x}$) can be written as follows:

$$\max\left(\lim_{M\to\infty} Z(\mathbf{x})\right) = \max\bar{Z}(\mathbf{V}) = \frac{\mathbf{c^T V}}{\mathbf{d^T V}} \tag{6a}$$

$$\text{s.t.} \quad \mathbf{AV} \leq \mathbf{0} \tag{6b}$$

$$\mathbf{1} \cdot \mathbf{V} = \mathbf{1} \tag{6c}$$

$$\mathbf{V} \geq \mathbf{0}. \tag{6d}$$

Note that $\mathbf{1} \cdot \mathbf{V} \leq 1$ is handled in equality form as (6c), since the optimal solution is on the hyperplane $x_1 + x_2 + \cdots + x_n = M$.

Assume that $\bar{Z}$ is continuous at $\mathbf{V_i} \in \widehat{S}$ and $\bar{Z}_i = \frac{\mathbf{c}^T\mathbf{V_i}}{\mathbf{d}^T\mathbf{V_i}}$.

By Definition 2, $\forall \varepsilon \in R^{+}$ correspond to a $\delta > 0$ such that $\left|\bar{Z} - \bar{Z}_i\right| < \varepsilon$ for each $\mathbf{V}$ satisfies $|\mathbf{V} - \mathbf{V_i}| < \delta$. Then, $\left|\bar{Z} - \bar{Z}_i\right| |\mathbf{V} - \mathbf{V_i}| < \varepsilon.\delta$ and $\left(\bar{Z} - \bar{Z}_i\right)(\mathbf{V} - \mathbf{V_i}) = 0$ imply the following:

$$\bar{Z}\mathbf{V} = \bar{Z}_i\mathbf{V} + \mathbf{V}_i\bar{Z} - \bar{Z}_i\mathbf{V_i}. \tag{7}$$

Combining (7) and $\bar{Z} = \frac{\mathbf{c}^T\mathbf{V}}{\mathbf{d}^T\mathbf{V}}$, the following linear approximation equation can be written as:

$$\left(\bar{Z}_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{V} + \bar{Z}\mathbf{d}^T\mathbf{V}_i = \bar{Z}_i\mathbf{d}^T\mathbf{V_i}. \tag{8}$$

Obviously, the objective function of (6) is nonlinear whereas (8) is linear. Therefore, from now on the objective function will expressed with $\bar{Z}$ for the LFP problem while $\widehat{Z}$ for LP problem.

Theorem: The fractional objective function $\bar{Z}$ and the linear objective function $\widehat{Z}$ take the same value at the point $\mathbf{V_i}$ and also their gradient vectors are the same at that point.

Proof: Straightforward.

Then, the transformed problem can be written as:

$$\max\ \widehat{Z} \tag{9a}$$

$$\text{s.t.}\quad \mathbf{AV} \leq \mathbf{0} \tag{9b}$$

$$\mathbf{1} \cdot \mathbf{V} = \mathbf{1} \tag{9c}$$

$$\left(\bar{Z}_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{V} + \bar{Z}\mathbf{d}^T\mathbf{V}_i = \bar{Z}_i\mathbf{d}^T\mathbf{V_i} \tag{9d}$$

$$\mathbf{V} \geq \mathbf{0}. \tag{9e}$$

Let the optimal solution of (9) and the corresponding optimal objective function value are denoted by $\mathbf{V}^* = \mathbf{V_{i+1}}$ and $\bar{Z}\left(\mathbf{V}^*\right) = \bar{Z}\left(\mathbf{V_{i+1}}\right) = \bar{Z}_{i+1}$, respectively.

Remark 2: If $\bar{Z}_{i+1} \to \infty$, then the unbounded solution occurs.

Remark 3: If $\bar{Z}_{i+1} = \bar{Z}_i$, then the maximum of asymptotic values $\max\left(\lim\limits_{M\to\infty} Z(\mathbf{x})\right) = \max \bar{Z}(\mathbf{V}) = \bar{Z}_i$ is found by the optimal solution of (9).

Remark 4: If $\bar{Z}_{i+1} > \bar{Z}_i$, then the iterative process is continued by setting $i = i + 1$.

Remark 5: Assume that $\bar{Z}_i = \max\left(\lim_{M\to\infty} Z(\mathbf{x})\right)$.
Case 1: If $Z\left(\mathbf{x_{i+1}}\right) < \bar{Z}_i$ for all $\mathbf{x_{i+1}} \in \mathbf{S}$, $\bar{Z}_i$ is the supremum of the objective function at the feasible region S, that is, $\sup_S Z(\mathbf{x}) = \bar{Z}_i$.
Case 2: If $Z\left(\mathbf{x_{i+1}}\right) > \bar{Z}_i$ for $\exists\, \mathbf{x_{i+1}} \in S$, by setting $Z_i = \bar{Z}_i$, the problem

$$
\begin{aligned}
&\max \widehat{Z} && \text{(10a)}\\
&\text{s.t.} \quad \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x} + \left(\mathbf{d}^T\mathbf{x}_i + \beta\right)\widehat{Z} = Z_i\mathbf{d}^T\mathbf{x}_i + \alpha && \text{(10b)}\\
&\qquad \mathbf{x} \in S && \text{(10c)}
\end{aligned}
$$

is solved iteratively until either $Z_{i+1} \to \infty$ (unbounded solution) or $Z_{i+1} = Z_i$ (optimal solution) is obtained.

Result 5: The problem (10) and the problem

$$
\begin{aligned}
&\max \quad \left(\mathbf{c^T} - Z_i\mathbf{d^T}\right)\mathbf{x} && \text{(11a)}\\
&\text{s.t.} \qquad \mathbf{x} \in S && \text{(11b)}
\end{aligned}
$$

have the same optimal solution.
Proof: From (10b), $\widehat{Z}$ can be written as $\widehat{Z} = \frac{Z_i\mathbf{d}^T\mathbf{x}_i + \alpha - \left(Z_i\mathbf{d}^T - \mathbf{c}^T\right)\mathbf{x}}{\left(\mathbf{d}^T\mathbf{x}_i + \beta\right)}$.

Since $Z_i\mathbf{d}^T\mathbf{x}_i + \alpha$ and $\mathbf{d}^T\mathbf{x}_i + \beta$ are constants, and $\mathbf{d}^T\mathbf{x}_i + \beta > 0$ by the assumption, handling the object function $\widehat{Z}$ as $\left(c^T - Z_i d^T\right)\mathbf{x}$ does not change the optimal solution.

Similar conclusion can be made for the problem (9) and the following problem:

$$
\begin{aligned}
&\max \quad \left(\mathbf{c^T} - Z_i\mathbf{d^T}\right)\mathbf{V} && \text{(12a)}\\
&\text{s.t.} \quad \mathbf{AV} \leq \mathbf{0} && \text{(12b)}\\
&\qquad \mathbf{1}\cdot\mathbf{V} = \mathbf{1} && \text{(12c)}\\
&\qquad \mathbf{V} \geq \mathbf{0}. && \text{(12d)}
\end{aligned}
$$

3.2. Finding an initial objective function value

An initial objective function value Z_i can be found by solving one of the following problems over the feasible region S or selecting an arbitrary value.

$$\max_{\mathbf{x}\in S}\ 0 \tag{13a}$$

$$\max_{\mathbf{x}\in S}\ \mathbf{c^T x} \tag{13b}$$

$$\max_{\mathbf{x}\in S}\ -\mathbf{d^T x} \tag{13c}$$

$$\max_{\mathbf{x}\in S}\ \left(\mathbf{c^T} - \mathbf{d^T}\right)\mathbf{x} \tag{13d}$$

We will choose the initial value by maximizing $\left(\mathbf{c^T} - \mathbf{d^T}\right)\mathbf{x}$ over S.

Remark 2: If an unbounded solution is encountered in problems (13b), (13c) or (13d), problem (13a) can be used to determine an initial objective function value.

Remark 3: The problems (13a)-(13d) have no notable advantage over each other in terms of the number of iterations. However, it can be interpreted that the problem (13d) may require the least number of iterations since it considers both the numerator and the denominator of the fractional objective function, while problem (13a) may require the greatest number of iterations.

3.3. A stopping criterion

As stated in the Proposition 2, the problem (5) generates an increasing sequence for the fractional objective function. If the problem (1) has an optimal solution $(\mathbf{x}^*, Z^*)$, then Z^* will be a finite value, so the corresponding sequence will be bounded above. In each iteration, the solution becomes closer to $(\mathbf{x}^*, Z^*)$. Thus, the stopping criterion can be defined as to obtain the same objective function value in two successive iterations, $Z^*_{i+1} = Z^*_i$.

3.4. Statement of our algorithm

Based on the previous sections, we are now ready to give our algorithm:

Step 0: Load the LFP problem (1).

Step 1: Select an initial objective function value Z_0 and set $i = 0$.

Step 2: Solve the problem (11) and find the optimal solution $\mathbf{x}^*$.

Step 2a: If the problem (11) is bounded, set $\mathbf{x_{i+1}} = \mathbf{x}^*$ and $Z_{i+1} = Z(\mathbf{x_{i+1}})$, and go to Step 3.

Step 2b: If the problem (11) is unbounded, set $\bar{Z}_i = Z_i$ and go to Step 4.

Step 3: Consider the objective function value Z_{i+1}.

Step 3a: If $Z_{i+1} \to \infty$, then the LFP problem (1) is unbounded. STOP.

Step 3b: If $Z_{i+1} < Z_i$, then $\sup\limits_S Z(\mathbf{x}) = Z_i$. STOP.

Step 3c: If $Z_{i+1} = Z_i$, then the optimal solution of (1) is found, that is $\mathbf{x}^* = \mathbf{x_i}$ and $Z^* = Z_i$. STOP.

Step 3d: Otherwise, set $i = i + 1$ and go to Step 2.

Step 4: Solve the problem (12), find the optimal solution $\mathbf{V}^*$ and evaluate $\bar{Z}(\mathbf{V_{i+1}}) = \bar{Z}_{i+1}$.

Step 4a: If $\bar{Z}_{i+1} \to \infty$, then the LFP problem (1) is unbounded. STOP.

Step 4b: If $\bar{Z}_{i+1} > \bar{Z}_i$, set $i = i + 1$ and go to Step 4.

Step 4c: Otherwise, set $Z_i = \bar{Z}_i$ and go to Step 5.

Step 5: Solve the problem (11) and find the optimal solution $\mathbf{x}^*$.

Step 5a: If the problem (11) is bounded, set $\mathbf{x_{i+1}} = \mathbf{x}^*$ and $Z_{i+1} = Z(\mathbf{x_{i+1}})$, and go to Step 6.

Step 5b: If the problem (11) is unbounded, then the LFP problem (1) is unbounded. STOP.

Step 6: Consider the objective function value Z_{i+1}.

Step 6a: If $Z_{i+1} \to \infty$, then the LFP problem (1) is unbounded. STOP.

Step 6b: If $Z_{i+1} < Z_i$, then $\sup\limits_S Z(\mathbf{x}) = Z_i$. STOP.

Step 6c: If $Z_{i+1} = Z_i$, then the optimal solution of (1) is found, that is $\mathbf{x}^* = \mathbf{x_i}$ and $Z^* = Z_i$. STOP.

Step 6d: Otherwise, set $i = i + 1$ and go to Step 5.

The flow chart of our algorithm is given in Figure-1.

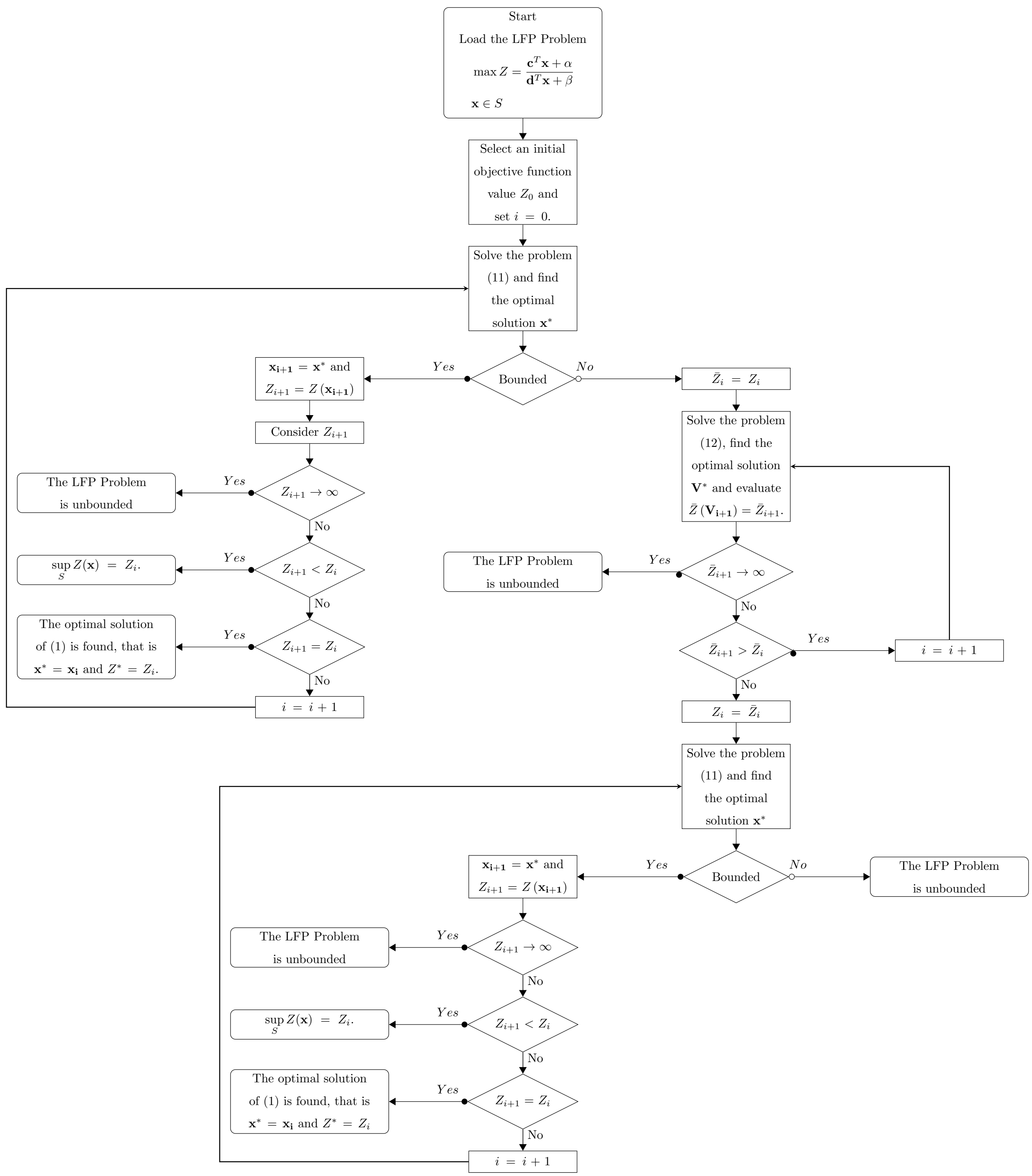

Figure 1: Flow chart of our algorithm

4. Numerical experiments

In this section, after presenting some examples for each solution case of the LFP problem, we analyze the validity of our algorithm by generating random large-scale test problems.

4.1. Illustrative examples

Example 1: (Optimal solution case) Consider the following example:

$$\begin{aligned} &\max Z(\mathbf{x}) = \frac{2x_1 + 3x_2 - x_3 + 5}{x_1 + 2x_2 + 3x_3 + 2} \\ &\text{s.t.} - 2x_1 + x_2 + 3x_3 \le 2, \\ &- x_1 + x_2 + 5x_3 \ge 1, \\ &x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{aligned} \tag{14}$$

Step 1: The initial point $\mathbf{x}_0 = (0, 0, 0.2)$ and the objective function value $Z_0 = 1.8462$ are determined by solving the following LP problem:

$$\begin{aligned} &\max\ 0 \\ &\text{s.t.} - 2x_1 + x_2 + 3x_3 \le 2, \\ &- x_1 + x_2 + 5x_3 \ge 1, \\ &x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{aligned} \tag{15}$$

Step 2: With these initial values, the LP problem corresponding to (11) is formed as:

$$\begin{aligned} &\max\ 0.1538x_1 - 0.6924x_2 - 6.5386x_3 \\ &\text{s.t.} - 2x_1 + x_2 + 3x_3 \le 2, \\ &- x_1 + x_2 + 5x_3 \ge 1, \\ &x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{aligned} \tag{16}$$

The optimal solution of (16) is $\mathbf{x}_1 = (0, 1, 0)$ and the objective function value is $Z_1 = 2$.

Step 2a: Since the problem (16) is bounded, then $\mathbf{x}_2 = \mathbf{x}_1{}^* = (0, 1, 0)$ and $Z_2 = Z(\mathbf{x_2}) = 2$, and go to Step 3.

Step 3d: Since $Z_1 \neq Z_0$, then set $i = 1$, and go to Step 2.

Step 2: With these values, the LP problem corresponding to (11)is formed as:

$$\begin{aligned}
&\max \ -x_2 - 7x_3 \\
&\text{s.t.} -2x_1 + x_2 + 3x_3 \le 2, \\
&-x_1 + x_2 + 5x_3 \ge 1, \\
&x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.
\end{aligned} \tag{17}$$

The optimal solution of (17) is $\mathbf{x}_2 = (0, 1, 0)$ and the objective function value is $Z_2 = 2$.

Step 3c: Since $Z_2 = Z_1$, then the optimal solution of Example 1 and the optimal objective value are $\mathbf{x}^* = \mathbf{x}_2^* = (0, 1, 0)$ and $Z^* = Z_2^* = 2$, respectively.

Example 2: (an asymptotic solution)

To demonstrate an asymptotic solution for the LFP problem, let us consider the following example:

$$\begin{aligned}
&\max Z(\mathbf{x}) = \frac{2x_1 + 3x_2 - x_3}{x_1 + 2x_2 + 3x_3} \\
&\text{s.t.} -2x_1 + x_2 + 3x_3 \le 2, \\
&-x_1 + x_2 + 5x_3 \ge 1, \\
&x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.
\end{aligned} \tag{18}$$

Step 1: The initial point $\mathbf{x}_0 = (0, 0, 0.2)$ and the objective function value $Z_0 = -0.3333$ are determined by solving the following LP problem:

$$\begin{aligned}
&\max \ 0 \\
&\text{s.t.} -2x_1 + x_2 + 3x_3 \le 2, \\
&-x_1 + x_2 + 5x_3 \ge 1, \\
&x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.
\end{aligned} \tag{19}$$

Step 2: With these initial values, the LP problem corresponding to (11) is formed as:

$$\begin{aligned} &\max\ 2.3333x_1 + 3.6667x_2 \\ &\text{s.t.} - 2x_1 + x_2 + 3x_3 \leq 2, \\ &-x_1 + x_2 + 5x_3 \geq 1, \\ &x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned} \tag{20}$$

The Problem (20) is unbounded, then go to Step 2b.

Step 2b: Since the problem (20) is unbounded, then set $\bar{Z}_0 = Z_0 = -0,3333$, and go to Step 4.

Step 4: Consider the following problem

$$\begin{aligned} &\max\ 2.3333V_1 + 3.6667V_2 \\ &\text{s.t.} - 2V_1 + V_2 + 3V_3 \leq 0, \\ &-V_1 + V_2 + 5V_3 \geq 0, \\ &V_1 + V_2 + V_3 = 1, \\ &V_1 \geq 0,\ V_2 \geq 0,\ V_3 \geq 0. \end{aligned} \tag{21}$$

The optimal solution of (21) is $\mathbf{V}_1 = (0.3333, 0.6667, 0)$ and the objective function value is $\bar{Z}\left(\mathbf{V_1}\right) = \bar{Z}_1 = 1.6$.

Step 4b: Since $\bar{Z}_1 > \bar{Z}_0$, set $i = 1$ and go to Step 4.

Step 4: Consider the following problem

$$\begin{aligned} &\max\ 0.4V_1 - 0.2V_2 - 5.8V_3 \\ &\text{s.t.} - 2V_1 + V_2 + 3V_3 \leq 0, \\ &-V_1 + V_2 + 5V_3 \geq 0, \\ &V_1 + V_2 + V_3 = 1, \\ &V_1 \geq 0,\ V_2 \geq 0,\ V_3 \geq 0. \end{aligned} \tag{22}$$

The optimal solution of (22) is $\mathbf{V}_2 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_2 = 1.6667$.

Step 4b: Since $\bar{Z}_2 > \bar{Z}_1$, set $i = 2$ and go to Step 4.

Step 4: Consider the following problem

$$\begin{aligned}
&\max\ 0.3333V_1 - 0.3334V_2 - 6.0001V_3 \\
&\text{s.t.} - 2V_1 + V_2 + 3V_3 \leq 0, \\
&- V_1 + V_2 + 5V_3 \geq 0, \\
&V_1 + V_2 + V_3 = 1, \\
&V_1 \geq 0,\ V_2 \geq 0,\ V_3 \geq 0.
\end{aligned} \tag{23}$$

The optimal solution of (23) is $\mathbf{V}_3 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_3 = 1.6667$.

Step 4c: Since $\bar{Z}_2 = \bar{Z}_3$, set $Z_3 = \bar{Z}_3$, and go to Step 5.

Step 5: Consider the following problem

$$\begin{aligned}
&\max\ 0.3333x_1 - 0.3334x_2 - 6.0001x_3 \\
&\text{s.t.} - 2x_1 + x_2 + 3x_3 \leq 2, \\
&- x_1 + x_2 + 5x_3 \geq 1, \\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.
\end{aligned} \tag{24}$$

The optimal solution of (24) is $\mathbf{x}_4 = (0, 1, 0)$ and the objective function value is $Z_4 = 1.5$.

Step 5a: The problem (24) is bounded. Then, set $\mathbf{x}_4 = \mathbf{x}^*$ and $Z_4 = Z(\mathbf{x_4})$ go to Step 6b.

Step 6b: Since $Z_3 > Z_4$, then $\sup\limits_S Z(\mathbf{x}) = Z_3 = 1.6667$.

Example 3: (an unbounded solution)

To demonstrate an unbounded solution for the LFP problem, let us consider the following example:

$$\begin{aligned}
\max Z\left(\mathbf{x}\right) = &\frac{2x_1 + 3x_2 - x_3}{x_1 + 2x_2 + 3x_3 - 1} \\
&\text{s.t.} - 2x_1 + x_2 + 3x_3 \leq 2, \\
&- x_1 + x_2 + 5x_3 \geq 1, \\
&x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.
\end{aligned} \tag{25}$$

Step 1: The initial point $\mathbf{x}_0 = (0, 0, 0.2)$ and the objective function value $Z_0 = 0.5$ are determined by solving the following LP problem:

$$\begin{aligned} & \max\ 0 \\ & \text{s.t.} - 2x_1 + x_2 + 3x_3 \leq 2, \\ & - x_1 + x_2 + 5x_3 \geq 1, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned} \tag{26}$$

Step 2: With these initial values, the LP problem corresponding to (11) formed as:

$$\begin{aligned} & \max\ 1.5x_1 + 2x_2 - 2.5x_3 \\ & \text{s.t.} - 2x_1 + x_2 + 3x_3 \leq 2, \\ & - x_1 + x_2 + 5x_3 \geq 1, \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned} \tag{27}$$

The Problem (27) is unbounded, then go to Step 2b.

Step 2b: Since the problem (27) is unbounded. Then, set $\bar{Z}_0 = Z_0 = 0.5$, and go to Step 4.

Step 4: Consider the following problem

$$\begin{aligned} & \max\ 0.3333V_1 + 0.6667V_2 \\ & \text{s.t.} - 2V_1 + V_2 + 3V_3 \leq 0, \\ & - V_1 + V_2 + 5V_3 \geq 0, \\ & V_1 + V_2 + V_3 = 1, \\ & V_1 \geq 0,\ V_2 \geq 0,\ V_3 \geq 0. \end{aligned} \tag{28}$$

The optimal solution of (28) is $\mathbf{V}_1 = (0.3333, 0.6667, 0)$ and the objective function value is $\bar{Z}(\mathbf{V_1}) = \bar{Z}_1 = 4$.

Step 4b: Since $\bar{Z}_1 > \bar{Z}_0$, set $i = 1$ and go to Step 4.

Step 4: Consider the following problem

$$
\begin{aligned}
&\max \ -2V_1 - 5V_2 - 13V_3 \\
&\text{s.t.} -2V_1 + V_2 + 3V_3 \leq 0, \\
&-V_1 + V_2 + 5V_3 \geq 0, \\
&V_1 + V_2 + V_3 = 1, \\
&V_1 \geq 0, \ V_2 \geq 0, \ V_3 \geq 0.
\end{aligned}
\tag{29}
$$

The optimal solution of (29) is $\mathbf{V}_2 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_2 = 5$.

Step 4b: Since $\bar{Z}_2 > \bar{Z}_1$, set $i = 2$ and go to Step 4.

Step 4: Consider the following problem

$$
\begin{aligned}
&\max \ -3V_1 - 7V_2 - 16V_3 \\
&\text{s.t.} -2V_1 + V_2 + 3V_3 \leq 0, \\
&-V_1 + V_2 + 5V_3 \geq 0, \\
&V_1 + V_2 + V_3 = 1, \\
&V_1 \geq 0, \ V_2 \geq 0, \ V_3 \geq 0.
\end{aligned}
\tag{30}
$$

The optimal solution of (30) is $\mathbf{V}_3 = (0.5, 0.5, 0)$ and the objective function value is $\bar{Z}_3 = 5$.

Step 4c: Since $\bar{Z}_2 = \bar{Z}_3$, set $Z_3 = \bar{Z}_3$, and go to Step 5.

Step 5: Consider the following problem

$$
\begin{aligned}
&\max \ -3x_1 - 7x_2 - 16x_3 \\
&\text{s.t.} -2x_1 + x_2 + 3x_3 \leq 2, \\
&-x_1 + x_2 + 5x_3 \geq 1, \\
&x_1 \geq 0, \ x_2 \geq 0, \ x_3 \geq 0.
\end{aligned}
\tag{31}
$$

Step 5b: The problem (31) is unbounded, then the LFP problem is unbounded.

4.2. Generation of test problems

We randomly generate a number of test instances to validate the proposed solution method. The size of an instance is given by the Number of Variables

(NoV) and Number of Constraints (NoC). For each problem instance, we generate the coefficient matrix $\mathbf{A}$, the right-hand side vector $\mathbf{b}$, and the objective coefficients $\mathbf{c}$, $\mathbf{d}$ and constants α and β that are unrestricted in sign for a specified (NoV, NoC) to meet the requirement $\left(\mathbf{d}^T\mathbf{x} + \beta\right) > 0$. All these parameters are generated as random integer numbers chosen from the uniform distribution.

By the generation of test problems, we aim to assess the effect of NoV and NoC on the total execution time and the iteration number of our algorithm. The sizes of the generated test problems, the maximum and minimum of the number of iterations and the CPU times (in seconds) of the algorithm are given in Table 1. Here, we note that one iteration refers to a new point generated by our algorithm. Also, the results in Table 1 do not contain unbounded, infeasible or asymptotic solution cases. For each class, ten problem instances are generated and solved using the software GAMS 24.1.3. All the computational experiments were carried out on a computer with a 2.7 GHz processor and 8GB RAM running Windows 10 Pro.

Table 1 also presents the comparison results of our algorithm with Ozkok [21]. The same starting point is used in the executions to make a reasonable comparison. In reporting CPU times, the time to obtain the starting point has been excluded.

Table 1 shows that our algorithm can reach the optimal solution for different sizes of the problem within a reasonable number of iterations, even with large problem instances. As the size of the problem increases, the CPU time is increasing as well for both algorithm. We can clearly say that our algorithm outperforms Ozkok [21] when both iteration numbers and CPU times are taken into account.

Additionally, Figure 2 and Figure 3 represent the average number of iterations and average CPU time (in seconds) of our algorithm for the specified (NoV, NoC), respectively.

Table 1: Comparison results of the proposed algorithm and [21].

Class		Number of iterations				CPU Time (in seconds)			
	(NoV, NoC)	Min [Our Method]	Min [21]	Max [Our Method]	Max [21]	Min [Our Method]	Min [21]	Max [Our Method]	Max [21]
1	$(5,5)$	1	2	4	4	0.171	0.195	0.689	0.874
2	$(10,10)$	2	2	4	10	0.176	0.256	0.779	0.895
3	$(20,20)$	2	5	6	10	0.365	0.369	0.826	0.917
4	$(30,30)$	3	6	5	10	0.604	0.71	1.078	1.264
5	$(40,40)$	3	3	6	15	0.545	0.556	1.148	1.209
6	$(50,50)$	3	7	10	11	0.42	0.499	0.733	0.767
7	$(60,60)$	5	6	8	14	0.457	0.494	0.895	0.967
8	$(70,70)$	4	6	8	14	0.566	0.581	0.765	0.797
9	$(80,80)$	5	6	8	11	0.483	0.493	0.912	0.987
10	$(90,90)$	5	5	9	18	0.497	0.499	0.958	1.071
11	$(100,100)$	4	6	9	12	0.697	0.882	1.271	1.382
12	$(200,200)$	6	6	9	8	1.031	1.106	1.476	1.645
13	$(350,350)$	6	6	8	7	1.445	1.514	2.226	2.666
14	$(500,500)$	6	6	7	9	3.604	3.72	6.176	6.876
15	$(750,750)$	6	7	8	11	11.965	12.217	17.825	18.684
16	$(1000,1000)$	6	7	7	10	32.642	34.01	39.135	39.372

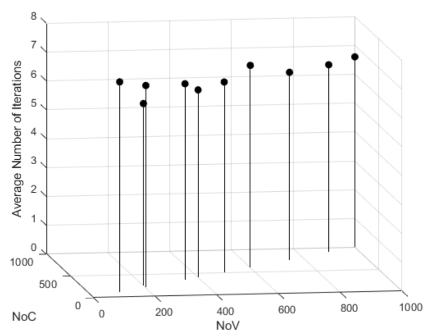

Figure 2: (NoV, NoC) vs average number of iterations.

5. Conclusion

In this paper, an exact iterative algorithm by providing an extension and improvement to Ozkok [21] based on the traditional continuity definition is developed for the LFP problem. With this new extended algorithm, we aim at removing the model dependency on the big-M coefficients. Our proposed algorithm converts the LFP problem to a linear programming problem and constructs an iterative procedure. Whereas most of the existing methods in the

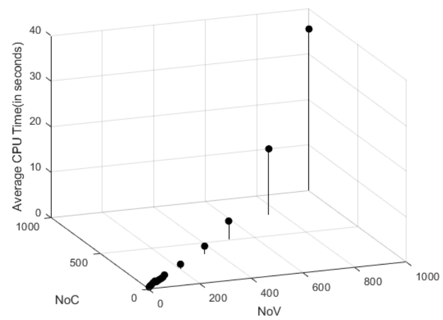

Figure 3: (NoV, NoC) vs average CPU time (in seconds)

literature find the optimal solution over a bounded convex region, our algorithm is able to solve all LFP problems that have a bounded or unbounded feasible region. Moreover, our algorithm is easily applicable and is capable of generating all types of solutions of LFP such as optimal, unbounded or asymptotic. As a result, considering the performance of our algorithm on large-scale examples, it can be said that it is more effective than Ozkok's algorithm. A limitation of our algorithm is that it cannot solve LFP problems involving frequently encountered uncertainty situations in real life. In future studies, our algorithm will be modified to solve fuzzy LFP problems. In addition, we aim to apply our algorithm to real life problems by investigating specific application areas such as resource allocation, transportation, production, finance, location theory, etc.

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Captions of figures

Figure 1. Flow chart of our algorithm

Figure 2. (NoV, NoC) vs average number of iterations

Figure 3. (NoV, NoC) vs average CPU time (in seconds)

Captions of tables

Table 1. Comparison Results of the proposed algorithm and [21]

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