Robust Portfolio Optimization based on Evidence Theory

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Abstract

During the past few years, there have been some turbulent events on world' economy, which have significantly influenced the performance of companies. Therefore, there is an urgent need to use a robust method to handle the existing uncertainties on the performance of the companies. This paper uses Evidence Theory to present an innovative and practical approach to consider the experts' opinions which are based on the available evidence regarding the factors that influence the stock market. Subsequently, the study proposes a way to determine the changes in these factors from possible scenarios on historical data to find the return range of different investment alters to be used in robust optimization. Moreover, in a case study, the sensitivity of the Iranian stock market to exchange rate fluctuations is examined under a set of scenarios which was due to the ambiguousness of a unified viewpoint of that rate's value midst experts as one of the noticed factors. The preliminary results of a real-life case study reveal that the proposed approach is useful practically and productively.

Keywords

Portfolio Optimization; Evidence Theory; Robust Optimization; Hybrid Uncertainty; Fundamental analysis.

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1. Introduction

Portfolio optimization is one of the most critical subjects in financial issues, where the influential work of Markowitz (1952) provided the fundamental portfolio model that is used as the foundation for modern portfolio theory. However, the valuable function of this framework in the finance administration industry has been discouraging. The fundamental drawback of Markowitz's model is that the idea assumes the appraisals of each stock returns and the variances are accurate but also tends. This issue is analytically reported and empirically tested by many researchers (see[1], [2], [3]and [4]). While the capital market is affected directly or indirectly by numerous factors, making estimation of assets' value is uncertain. Thus, detecting behavioral scenarios and prioritizing influential factors in terms of practical aspects help us overcome some unwanted uncertainties. In addition, in cases where there are historical data, although the data can be used, they may not necessarily end up with an efficient outcome. The stock market is influenced by economic, political, and social elements. In this regard, using the opinion of experts for the process of decision making can be very useful. The present study uses Evidence Theory (ET) in a new manner in portfolio robust optimization problems. The expenses incurred and development time has been reduced considerably compared with other existing models due to repeated expert's interactions as well as the overall complexity of the model which was the chief problem in most of the recent researches. At the same time, we incorporate another level of uncertainty handling mechanism into the portfolio selection model.

One of the most popular methods which can be utilized to deal with uncertainty is the Robust Optimization (RO) methodology [5]. In this regard, a novel robust portfolio selection and optimization approach is provided to deal with the vagueness of the data, boosting the robustness of investment procedure against uncertainty, declining computational complexity, and comprehensive assessments of stocks from various financial aspects and criteria are submitted. For this purpose, the outputs of the ET are been used as the inputs of the robust portfolio optimization model (RPOM) to create more stability in the efficiency of assets and to achieve the nominal portfolio optimization model in real-world conditions even when the empirical distribution of samples is deviated from the normal distribution. This paper is dealing with a hybrid uncertainty in portfolio optimization. After identifying investment options and determining the factors affecting stock returns, we first rank the stocks using the data envelopment analysis approach, filter them based on maximum return and minimum risk, and determine how stock prices are affected by influential factors according to the Sharp multi-factor model. In the portfolio optimization literature, many studies have been conducted to improve the performance of portfolio optimization different nominal models under various circumstances (see[6], [7], [8], [9], [10], [11], [12],[13]).

Furthermore, the current strategy in the investment portfolio might change when an unexpected event or an incident alters an investor's environmental situations. Such changes require a logical and comprehensive evaluation of the portfolio for striking a tradeoff [14]. In this case, various investigations have been conducted based on robust optimization (RO) to face uncertain conditions and parameters related to a portfolio. Ben-tal et al. [15]modeled a robust multi-objective portfolio selection problem using a linear programming approach. According to Moon and Yao [16] a robust absolute value of deviation from the mean model is determined. Kim and Fabozzi [17] discusses the use of robust factor investing in portfolio management. Using a robust approach, Goldfarb and Iyengar [18] introduced a conic programming model for single-period portfolio selection problems by taking advantage of the factor model in portfolio efficiency. Ling and Xu [19] concentrated on giving option contracts in a portfolio using an ellipsoidal uncertainty set with a joint margin. In their method, the risk is controlled by an option contract and existing strategies. Carvalho et al. [20] find the limiting portfolios of RO, formulated with different uncertainty matrices, for the highest and lowest uncertainty levels. Quarant et al. [21] used the Ben-Tal and the Nemirovski's approach to develop a conditional value-at-risk robust model that was a non-linear model.

Bertsimas and Pachamanovab [22] provided a robust optimization for multi-period portfolio selection problems under a number of scenarios. In this case, subsequent portfolio efficiency is considered uncertain coefficients in the optimization problem and an investor's risk-aversion level as the fluctuation level of total efficiency estimation error. Chen and Tan [23] developed a nominal mean variance model using the stochastic optimization method and robust optimization. They assumed that the assets' efficiency was uncertain and used the Bertsimas and Sim approach for RO development. Ghahtarani and Najafi [24] developed a robust multi-objective portfolio selection model for which they used goal programming. Using ellipsoidal uncertainty sets, Pinar and Paç [25] developed their modeling for the single-period and multi-period mean-semi variance model. Ling et al. [26] investigates robust multi-period portfolio selection based on downside risk with asymmetrically distributed uncertainty.

Considering mean vector components and covariance matrix in distance regions, Tütüncü and Koenig [27] solved the portfolio selection problem to achieve a robust answer with non-linear programming. Unlike most researchers in this field, Bertsimas and Sim [28] examined optimization under an uncertain condition with models in which the parameters were not modeled by polygonal sets other than ellipsoidal sets. Afterward, Bertsimas and Thieley [29] provided a model where polygonal uncertainty sets are used instead of ellipsoidal sets. Dimmock et al. [30] provided a different work in uncertainty modeling and created a problem in robust portfolio selection, where certainty explained the probability of asset efficiency distribution. Ji and Lejeune [31] study a class of distributionally robust optimization model with Sharpe, Sortino–Satchel, and Omega ratios. Sharma et al. [32] consider a robust formulation of their respective models under discrete distribution of returns since the true distribution of returns is unknown. In addition, they evaluated the relationships between risk, robustification, and portfolio efficiency. They concluded that as robustness increases, portfolio risk and efficiency declines, diversifying the portfolio. Peykani et al. presented a robust bi-objective model for portfolio selection, which is capable to be used under uncertainty of financial data and also a NSGA-II meta-heuristic algorithm to solve the suggested model of research due to the complexity of the proposed model [33]. Georgantas et al. [34] conducted a comprehensive empirical evaluation of various robust models performance for popular risk criteria in robust portfolio optimization. They also provided a broad comparative analysis of their performance using the US market, where more efficiency was observed in this type of model than nominal models. Ghahtarani et al. [35] provided a comprehensive review of recent advances in robust portfolio selection problems and their extensions, from both operational research and financial perspectives. Incorporating future returns scenarios in the investment decision process and developing a conventional minimax regret criterion formulation, in multi-objective programming problems, Mohammadi et al. [36] verify the validity of the proposed approach through an empirical testing application on the top 75 companies of Tehran Stock Exchange Market. Peykani et al. [37] presented six robust data envelopment analysis (RDEA) models based on the most widely cited and popular classic data envelopment analysis models in the first phase and two robust portfolio optimization models including robust mean-semi variance-liquidity and robust mean-absolute deviation-liquidity in the second phase. Finally, Xidonas et al. [38] provides a categorized bibliographic review which has a broad scope, yet is limited in technical details.

Having utilize the Fuzzy Delphi method in the first stage for crucial factors identification, Thakur et al. [39] hierarchically arranged the stocks by main factors and historical data based on ET. Finally, Ant Colony Simulation for portfolio optimization was used. Skoruchi and Mohammadi [40] examined the high reaction of the Iranian stock market to the dollar value perturbation in the diverse scenarios because of the vagueness of a unified opinion of the amount of that rate among experts and obtained its rate utilizing ET. As seen, the robust model's uncertainty parameters are given without acceptable justification and no admissible mechanism is used to ascertain range. To the best of our knowledge, it is the first time that ET is used to determine this interval according to the available evidence.

The rest of the paper is organized as follows: after a general investigation of the research subject, the fundamental aspects of investment choices based on financial ratios will be analyzed. In section three, ET will be further introduced to receive experts' view under uncertain situations. In section four, we will review the Symbiotic Organisms Search algorithm to answer the problem numerically. Section five will explore how to conduct a portfolio utilizing actual datasets extracted from Tehran capital market. In section six, the obtained results will be provided as well as suggestions for further studies. Fig. 1 illustrates the conceptual flow chart.

(Insert Fig.1 almost here)

2. Fundamental analysis of investment alternatives based on DEA

Fundamental analysis is the process of evaluating a public firm for its investment worthiness by looking at its business at the basic or fundamental financial level, see for example, Thomsett[41]. Moreover, financial ratios are created with the use of numerical values taken from financial statements to gain meaningful information about a company. To begin with, liquidity ratios attempt to measure a company's ability to pay off its short-term debt obligations. This is done by comparing a company's most liquid assets (or those that can be easily converted to cash) and its short-term liabilities. Profitability indicator ratios profitability Indicator Ratios much like the operational performance ratios, give users a good understanding of how well the company utilized its resources in generating profit and shareholder value. Debt ratios are the third series of ratios. A general idea of the company's overall debt load and its mix of equity and debt are given to users by these ratios. Debt ratios can be utilized to define the total level of financial risk an institute. As a result, the greater quantity of debt held by a company the greater risk of bankruptcy.

Operating performance ratios basically, determine how efficiently and effectively a firm is using its assets to generate sales and boost shareholder value. Generally, the better these ratios are, the better it is for shareholders. Cash flow indicator ratios indicators focus on the cash being generated in terms of how much is being generated and the safety net that it provides to the company. These ratios can give users another look at the financial health and performance of a company. Investment Valuation Ratios can be used by investors to estimate the attractiveness of a potential or existing investment and get an idea of its valuation.

3. Analysis of the effective factors based on Evidence Theory (Dempster–Shafer theory)

In the late 1960s, Dempster provided a new viewpoint regarding probability size in his famous paper Dempster [42]. He acknowledged that the classic structure of probability theory does not properly provide the possibility of demonstrating the lack of knowledge. However, in real-world conditions, decision-makers face a type of lack of knowledge in expressing their own subjective probability. In addition, objective probabilities can be aggregated, while empirical investigations do not demonstrate such a property regarding subjective probabilities, and it seems that these probabilities have sequential size. Despite providing a new concept, related researchers neglected this theory for a long time due to its weakness in explaining and establishing its own viewpoints in the mathematics form. Nevertheless, it was provided and redefined in the late 1970s in the form of a proper framework by a famous mathematician named Shafer [43]. One of the important instruments in defining uncertainty is ET, offering the opportunity for a decision-maker (DM) to realize new probabilities. This theory endures with the consultation with regard to existing beliefs of a situation or a system of situations. Beliefs of individuals are dissimilar when dealing with a single type of occurrence, though they can be inspected and combined by a particular method. Indeed, a number of beliefs caused by observation and perception of evidence has shaped Dempster–Shafer theory (DST).

In the following, DST for multi-criteria decision-making analysis with respect to uncertainty conditions will be briefly examine by Mohammadi and Makui [44]. Assume $X=\{x_1,x_2,\ldots,x_m\}$ is a set of options, $W=\{w_1,w_2,\ldots,w_n\}$ a set of weights, $A=\{a_1,a_2,\ldots,a_n\}$ a set of benchmarks, so that $0\le w_j\le 1$, $1\le j\le n$, $\sum_{j=1}^{n} w_j=1$. Let assume P is the evaluation rank of $H_1,H_2,\ldots,H_p$ for multi-criteria evaluation of options. Assume $\beta_{q,j}(x_i)$ indicates a belief degree of the fact that a_j criterion has been assessed for x_i with H_q degree, $0\le\beta_{q,j}(x_i)\le 1$ and

$\sum_{q=1}^{p}\beta_{q,j}(x_i)\le 1$. Assume $S(a_j(x_i))$ indicates criterion evaluation value a_j for x_i option, presented as follows.

$$S(a_j(x_i))=\{H_q,\beta_{q,j}(x_i)\} \tag{1}$$

Where H_q is an evaluation degree, so that $1\le q\le p$, $1\le i\le m$, $1\le j\le n$. Criteria evaluation for options is defined in the form of a decision matrix D, demonstrated as follows.

$$D=(S(a_j(x_i)))_{n\times m} \tag{2}$$

Where $1\le i\le m$ and $1\le j\le n$. Thereby, according to decision matrix D, we can collect criteria evaluation value for x_i option as follows Mohammadi and Makui [44].

Initially, we convert belief degree $\beta_{q,j}(x_i)$ about assessment degree H_q regarding a_j criterion of x_i option to basic probable mass $m_{q,j}(x_i)$. Such that:

$$m_{q,j}(x_i)=w_j\beta_{q,j}(x_i) \tag{3}$$

Where, $1\le q\le p$, $1\le i\le m$, $1\le j\le n$.

Now, assume that $m_{H,j}(x_i)$ indicates the probable residual mass of a_j criteria regarding the assessment of x_i option, which is defined as follows:

$$m_{H,j}(x_i)=\bar{m}_{H,j}(x_i)+\tilde{m}_{H,j}(x_i) \tag{4}$$

$$\bar{m}_{H,j}(x_i)=1-w_j \tag{5}$$

$$\tilde{m}_{H,j}(x_i)=w_j\left(1-\sum_{q=1}^{p}\beta_{q,j}(x_i)\right) \tag{6}$$

Where, $1\le q\le p$, $1\le i\le m$, and $1\le j\le n$. Probable residual mass that is not devoted to each assessment degree consists of two parts: The section pertaining to the violation in the assessment process and the part pertinent to relative weights of criteria [45].

$\bar{m}_{H,j}(x_i)$ is the first part of probable residual mass, which is not yet devoted to assessment degrees. According to the fact that a_j criterion plays a part in the assessment process according to its weight, that is w_j, $\bar{m}_{H,j}(x_i)$ is a descending function of w_j [46]. $\bar{m}_{H,j}(x_i)$ will be equal to 1, if the weight of a_j is equal to zero or $w_j=0$. $\bar{m}_{H,j}(x_i)$ will be zero if a_j dominates the assessment or $w_j=1$. In other words, $\bar{m}_{H,j}(x_i)$ represents the degree to which other criteria could contribution in the assessment process.

The second part of the probable Hesitancy mass is $\tilde{m}_{H,j}(x_i)$, not yet devoted to an assessment degree, and due to violation in the assessment procedure of $S(a_j(x_i))$ will result. $\tilde{m}_{H,j}(x_i)$ will be zero if $S(a_j(x_i))$ is complete or $\sum_{q=1}^{p}\beta_{q,j}(x_i)=1$, otherwise $\tilde{m}_{H,j}(x_i)$ would be a positive value. $\bar{m}_{H,j}(x_i)$ will be proportional to w_j, whose positive quantities will lead to the next constraint's violation.

$G_{I(l)}$, a subset of l number of first criteria, will be proposed as follows:

$$G_{I(l)}=\{a_1,a_2,\ldots,a_l\} \tag{7}$$

Assume that $m_{q,I(l)}(x_i)$ is a probable mass that represents the support degree of a belief that all touchstone existing in $G_{I(l)}$ subset emphasize that x_i option with H_q degree is evaluated. $m_{H,I(l)}(x_i)$ presents the probable hesitancy mass that is not devoted to assessment degrees after all criteria in $G_{I(l)}$ subset are assessed. $m_{q,I(l)}(x_i)$ and $m_{H,I(l)}(x_i)$ can be obtained combining basic probable mass of $m_{q,j}(x_i)$ and $m_{H,j}(x_i)$ for all $q=1,2,\ldots,p$ and $j=1,2,\ldots,l$.

According to the definitions and subjects mentioned above, evidence-based reasoning recursive algorithm can be summarized as follows:

$$\{H_q\}: m_{q,I(l+1)}(x_i)=K_{I(l+1)}\left[m_{q,I(l)}(x_i)m_{q,l+1}(x_i)+m_{H,I(l)}(x_i)m_{q,l+1}(x_i)+m_{q,I(l)}(x_i)m_{H,l+1}(x_i)\right], \tag{8}$$

$$m_{H,I(l+1)}(x_i)=\bar{m}_{H,I(l+1)}(x_i)+\tilde{m}_{H,I(l+1)}(x_i), \tag{9}$$

$$\{H\}: \bar{m}_{H,I(l+1)}(x_i)=K_{I(l+1)}\left[\bar{m}_{H,I(l)}(x_i)\bar{m}_{H,l+1}(x_i)\right], \tag{10}$$

$$\{H\}: \tilde{m}_{H,I(l+1)}(x_i)=K_{I(l+1)}\left[\tilde{m}_{H,I(l)}(x_i)\tilde{m}_{H,l+1}(x_i)+\bar{m}_{H,I(l)}(x_i)\tilde{m}_{H,l+1}(x_i)+\tilde{m}_{H,I(l)}(x_i)\bar{m}_{H,l+1}(x_i)\right], \tag{11}$$

$$K_{I(l+1)}=\left[1-\sum_{u=1}^{p}\sum_{\substack{f=1\\ f\neq u}}^{p}m_{u,I(l)}(x_i)m_{f,l+1}(x_i)\right]^{-1} \tag{12}$$

$K_{I(l+1)}$ is a normalization factor through which $\sum_{q=1}^{p}m_{q,I(l+1)}(x_i)+m_{H,I(l+1)}(x_i)=1$. Note that $m_{q,I(1)}(x_i)=m_{q,1}(x_i)\ (q=1,2,\ldots,p)$ and $m_{H,I(1)}(x_i)=m_{H,1}(x_i)$. Besides, it should be considered that the criteria existing in G are numbered randomly, that is the results of $m_{q,I(l)}(x_i)$, $(q=1,2,\ldots,p)$, and $m_{H,I(l)}(x_i)$ are independent of the sum order of criteria [47].

In the evidence-based reasoning approach, after that all n criteria are aggregated, the combined belief degree β_q is directly calculated from the below equation:

$$\{H_q\}: \beta_q(x_i) = \frac{m_{q,I(n)}(x_i)}{1-\bar{m}_{H,I(n)}(x_i)}, \tag{13}$$

$$\{H\}: \beta_H(x_i) = \frac{\tilde{m}_{H,I(n)}(x_i)}{1-\bar{m}_{H,I(n)}(x_i)}, \tag{14}$$

β_H is a belief degree that is not devoted to any assessment degree after all n criteria are appraised. Indeed, this parameter defines the violation degree existing in the assessment procedure. Therefore, we will have: $\sum_{q=1}^{p}\beta_q(x_i)+\beta_H(x_i)=1$.

Ultimately, utility functions that actually indicate the relative importance of possible exchange rate values based on an aggregation of experts' opinions are indicated in the form of $[u_{min}, u_{max}]$, and are indicated by $u(H_n)$ and calculated as follows:

$$\mathrm{u_{max}}(\mathrm{x}) = \sum_{\mathrm{q}=1}^{\mathrm{P}-1}\mathrm{B_q}.\mathrm{u}(\mathrm{H_q}) + (\mathrm{B_1}+\mathrm{B_H}).\mathrm{u}(\mathrm{H_1}) \tag{15}$$

$$\mathrm{u_{min}}(\mathrm{x}) = (\mathrm{B_N}+\mathrm{B_H}).\mathrm{u}(\mathrm{H_N}) + \sum_{\mathrm{q}=2}^{\mathrm{P}}\mathrm{B_q}\mathrm{u}(\mathrm{H_q}) \tag{16}$$

$$\mathrm{u_{Avg}}(\mathrm{y}) = \frac{\mathrm{u_{max}}(\mathrm{y})+\mathrm{u_{min}}(\mathrm{y})}{2} \tag{17}$$

Where, $1 \le q \le p$ and H_N have the minimum preference degree and H_1 has the maximum preference degree. Besides, $\mathrm{B_N}$ is allocated the maximum belief degree, and $\mathrm{B_1}$ is allocated the minimum belief degree. Notice that if the main assessment of $S(a_j(x_i))$ is complete, we will have $\mathrm{B_H}=0$ [48].

4. Description of portfolio optimization approach

Portfolio optimization is the process of selecting the best portfolio (asset distribution) among the set of all portfolios being considered, according to the specific objective. The risk level and yield of invested assets are two essential parameters in deciding on an investment. Most investors want maximize their efficiency at a specific level of risk or minimize risk for a certain efficiency level. The individuals invest according to their expected utility and in the hope of more gaining in the future, ignore today's consumption. The utility function of each investor is ascertained according to that individual's preferences [49]. Proposing the mean-variance model, Markowitz indicated that selecting a portfolio makes it possible to reduce investment risk at a certain efficiency level. This possibility is attributed to a lack of thorough correlation between the efficiencies of different stocks. Optimal portfolio selection is often carried out by exchange between risk and yield such that the more investors' expected efficiency, the more risk of a portfolio. Accordingly, identifying the efficient frontier pertinent to the portfolio enables investors to obtain their maximum expected efficiency according to the utility function and risk-aversion and risk-taking. Our main model based on the primary framework submitted by Markowitz is developed. On the other hand, as presented by Markowitz, the classical portfolio selection problem completely disregards the uncertainty of the expected asset's returns and the covariance matrix of assets' returns. It is assumed that these parameters are capable of representing the inherent uncertainty associated with the investment returns. Actually, as these parameters are, most of the times, calculated from past data, they are themselves subject to uncertainty. Not acknowledging the uncertainty in the models' parameters substantially degrades the performance of the optimal solution calculated using these Models. The robust formulation of an optimization problem considers the nominal values of the uncertain parameters and the deviations from these nominal values.

4.1. Conventional portfolio model

In 1952, proposing the portfolio optimization problem-solving model (mean-variance theory) for the first time, Markowitz [14] expressed the issue as a Quadratic Programming to minimize portfolio risk when the expected efficiency is a constant value. The main assumption of this model is that all investors are risk-avert. This problem consisted of two applied constraints, according to which the sum of stock weights must be equal to one, and also the weight of each stock in the selected portfolio must be a non-negative number. The standard form of the mean-variance model is as follows:

$$Max\ \mu_p = \sum_{i=1}^{n} x_i \mu_i \qquad (18)$$

Subject to:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j = \sigma_p^2 \qquad (19)$$

$$\sum_{i=1}^{n} x_i = 1, \qquad 20)$$

$$x_i \geq 0, \quad (i = 1, \ldots, n) \qquad 21)$$

$$x_j \geq 0. \quad (j = 1, \ldots, n) \qquad 22)$$

In the Markowitz model, the primary point is that the general risk criterion is variance or standard deviation. This criterion that is generally distributed for stock and exchanged in an efficient market is acceptable. Now, suppose these two features do not exist for a store, for any reason, including ambiguities or severe and unpredictable fluctuations in the influential factors to the stock market. In that case, the variance will not be a suitable index for stock risk. Therefore, another solution must be found.

4.2. Robust optimization

Over the past few years, due to the ever-increasing attention paid to taking account of uncertainty in decision-making processes, many investigations have been conducted in this field, and various approaches have been employed. One of the most efficient methods used in this field is RO [50]. RO is one of the approaches employed to deal with uncertainty in optimization problems and aims to model optimization problems with uncertain data to make the obtained results suitable for all (or most) of the imaginable cases of uncertain parameters, Cornuejols and Tütüncü [51]. A robust answer consists of robustness in terms of optimality and robustness in terms of feasibility. Robustness in terms of feasibility means that the desired response for all (or most) of the imaginable cases remains feasible for uncertain parameters. Robustness in terms of optimality implies that the preferred solution for all (or most) potential cases for uncertain parameters remains close to optimal value or has the minimum standard deviation from the optimal value, Pishvaee et al.[52].

Soyster [53] was the first one investigating the linear optimization problem with uncertain datasets. He generated extremely cautious answers on the basis of the assumption that uncertain datasets are within a certain range. Afterward, Mulvey et al. [54] and Yu and Li [55] proposed scenario-based robust models. El Ghaoui et al. [56] proposed a robust counterpart programming for semidefinite linear and quadratic problems based on the ellipsoidal uncertainty set assumption. Ben-tal and Nemirovsk [57] i indicated that if datasets of a linear optimization problem with uncertainty belong to a semi-ellipsoidal set (conic, quadratic, semidefinite), its robust counterpart or an approximation of it can be solved in polynomial time. Ben-tal and Nemirovski [58] investigated linear programming problems with inaccurate datasets and enumerated the features of an uncertain convex set for the robust counterpart to be exactly equivalent to the initial problem. Ben-tal and Nemirovski [57, 58] indicated that the robust counterpart is conic and quadratic when the uncertainty set is ellipsoidal. Ben-tal and Nemirovski [59] investigated the impact of uncertainty on optimization problems. Considering the semi-ellipsoidal set as the uncertainty set, they proposed a robust counterpart for linear optimization and a relationship for the probability of the problem constraints violation due to uncertainty.

Bertsimas and Sim [60] investigated optimization problems for discrete problems and proposed a robust counterpart model that is computable and capable of being adjusted in terms of robustness, and examined the probability of constraints' violation. Bertsimas and Sim [28] proposed a robust linear counterpart for linear programming problems with an uncertain dataset, which can be adjusted in terms of cautiousness and balanced between acceptability and optimality. They also developed the proposed model for discrete programming problems and examined the probability of constraints' violation. Now, given that the robustification approach employed in this investigation is of Bertsimas and Sim type, this model will be explained in the following. In order to examine this method, the general form of the linear optimization problem can be considered as follows:

$$Max \sum_i c_i x_i \tag{23}$$

$$\sum_i \tilde{a}_{ij} x_i \le b_j , \quad \forall j \tag{24}$$

$$x_i \ge 0, \quad \forall i \tag{25}$$

x_i is the decision variable, j is the number of constraints. c_i and b_j have certain values, while $\tilde{a}_{ij}$ is uncertain. Uncertainty of $\tilde{a}_{ij}$ is defined as follows:

$$\tilde{a}_{ij} = a_{ij} + \xi_{ij}\hat{a}_{ij} \tag{26}$$

a_{ij} is the nominal value of data, $\hat{a}_{ij}$ is a positive constant uncertainty value that indicates disruption value. $\xi_{ij}\hat{a}_{ij}$ is a random value within $[-1,1]$ range and is pertinent to uncertainty, which is indicated as a set.

Therefore, using multi-dimensional uncertainty set and duality theorem, Bertsimas and Sim formulated a robust counterpart of linear optimization problems, which can be adjusted in terms of cautiousness and balanced between feasibility and optimality. This formulation can be employed for discrete optimization problems. Robust counterpart formulation of equations (23) to (25) using this approach are as follows, Bertsimas and Sim [28]:

$$Max \sum_i c_i x_i \tag{27}$$

Subjected to:

$$\sum_i a_{ij} x_i + z_j \Gamma_j \sum_{i \in J_j} p_{ij} \leq b_j, \quad \forall j \tag{28}$$

$$z_j + p_{ij} \geq \hat{a}_{ij} y_i, \quad \forall j, i \in J_j \tag{29}$$

$$-y_i \leq x_i \leq y_i, \quad \forall i \tag{30}$$

$$p_{ij} \geq 0, \quad \forall j, i \in J_j \tag{31}$$

$$y_i \geq 0, \quad \forall i \tag{32}$$

$$z_j \geq 0, \quad \forall j \tag{33}$$

The cautiousness of each constraint can be determined by Γ_j, whose maximum value is equal to the number of $\xi_i \neq 0$ value or $|J_j|$.

After pointing out the necessary subjects, we will address our model and explain it thoroughly in the following section.

5. Case study

In order to analyze any real issue with a mathematical modelling tool, a combination of assumptions with regard to the behavior of system must be considered. In this regard, the return fluctuates within a symmetric range, and all investors have an identical single-period time horizon. Moreover, personal incomes are tax-free, that is investors do not distinction between dividend profit and capital profit, and trade in the market is costless. Furthermore, this problem is not affected by inflation. Moreover, the stock price cannot be solely impacted by any capital according to sell and buy decisions. Ultimately, investors prefer a higher efficiency level at a certain level of risk, and for a certain level of efficiency seek a minimum level of risk.

5-1 symbols of the problem

Parameters

R_i	Expected efficiency
β_i	Measurement unit of systematic risk
R_m	Index efficiency rate
e_i	Random error
$\tilde{r}_i$	The expected value of asset efficiency
$\hat{a}_i$	Maximum possible deviations
Γ	Robustness cost
n	The number of stocks
t	A unique upper bound for all stocks

Variables:

x_i	The weight value of the stock *i* if selected

Thereby, we have:

$$MaxR_p = \sum_{i=1}^{n} \tilde{r}_i x_i \tag{34}$$

Subjected to:

$$\sum_{i=1}^{n} x_i = 1 \tag{35}$$

$$0 \le x_i \le U \tag{36}$$

x_i is the variable of i decision, and $\tilde{r}_i$ is uncertain. Additionally, constraint (1) ascertains that the whole budget must be devoted to different alternatives, it is not allowed to use additional budget. Constraint (2) or sign constraint expresses that using short sell is impossible.

Furthermore, in a linear programming problem in the form mentioned above, the uniform distribution assumption for efficiency can be considered as follows, according to the stated assumption and ignoring the correlation between stocks:

$$\tilde{r}_i \sim U\left(\tilde{r}_i - \hat{a}_i, \tilde{r}_i + \hat{a}_i\right) \tag{37}$$

Ultimately, robust counterpart formulation of equations (34) to (36) using Bertsimas and Sim approach are as follows:

Max U (38)

Subjected to:

$$U - \sum_{i=1}^{n} \hat{r}_i x_i + z\,\Gamma + \sum_{i=1}^{n} P_i \le 0 \tag{39}$$

$$z + P_i \ge \hat{a}_i \; y_i \qquad \forall i = 1,\ldots,n \tag{40}$$

$$-y_i \le x_i \le y_i \qquad \forall i = 1,\ldots,n \tag{41}$$

$$\sum_{i=1}^{n} x_i = 1 \qquad \forall i = 1,\ldots,n \tag{42}$$

$$x_i \le \mathrm{t} \qquad \forall i = 1,\ldots,n \tag{43}$$

$$x_i ,\; y_i ,\; P_i \ge 0 \qquad \forall i = 1,\ldots,n \tag{44}$$

In the proposed formulation, as mentioned earlier, it is assumed that the uncertain coefficient $\hat{a}_i$ is a uniform random value within the range of $[\tilde{r}_i - \hat{a}_i , \tilde{r}_i + \hat{a}_i]$. In this case, if the number of coefficients with uncertainty, that is Γ_i , changes in the raw $i\ th$ in which $1 \le i \le n$, the answer will still be feasible, and if more than number of coefficients change, the constraint will probably be rejected. In addition, the parameter Γ_i might not be integer and can have a value within range of $\left[0, |\Gamma_i|\right]$.

6. Computational results

In this investigation, the collected data belong to 10 firms active in Tehran capital market between 2019, April to 2021, July. The following constraint is applied for the selection of stock sets: A) In the study period, they do not face trading halt for more than half of the year B) Their financial year ends on 20 March C) They have the lowest risk compared to the rest, and their monthly returns are not less than 10%. D) They have clear and complete financial statements and information.

As discussed earlier, the correlation coefficient between dollar rate and each stock's prices plays a pivotal role in the success or failure of the investment, particularly in our country's stock market, and therefore, has captured more interest nowadays than before. Analyzing the correlation coefficient between stock price and stock exchange index, researchers employed the Beta coefficient in order to appraise the rank of stocks. In the present study, due to the impact of dollar rate on the stock market index and the rest of Iran financial markets, the dollar exchange rate is employed as a pivotal factor. As a result, in this section, the quantity of the dollar exchange rate which is an uncertain parameter will be computed by using DST. Afterward, based on the robust Bertsimas and Sim model, the result of solving the model and optimal values of the stock portfolio for different values of Γ will be presented. Accordingly, experts ascertain the belief degree about the dollar exchange rate, depicted in Table 1.

(Insert Table 1 almost here)

It is seen that, the rate of dollar exchange which includes (Scenario A, Scenario B, Scenario C, Scenario D) with values (150000, 200000, 250000, 300000) is predicted using DTS in the [19.607, 25.500] range. Therefore, its percentage of deviation from the mean is in the [-0.130, 0.130] range. Consequently, considering the value of β between stock returns and dollar returns and utilizing this information, the uncertain amount of stock returns is computed intermittently to use the Bertsimas and Sim robust optimization approach to model the stock portfolio represented in Table 2.

(Insert Table 2 almost here)

Finally, due to the linearity of the achieved model, it is solved by GAMS 24.1.3 on a personal computer based on 2.8 GHz at 0.016 seconds with the CPLEX solver. The results are shown in Table 3.

(Insert Table 3 almost here)

Solving the model with the software Gams, we formed an optimal stock portfolio. It is worth mentioning that by increasing the value of Γ due to adding the number of parameters with uncertainty, the return of the optimal portfolio of stocks decreases. If Γ=0, the model is completely certain, and if Γ=10, the relevant parameters can indeed have the maximum fluctuation from central limitation. In other words, when the model is deterministic, due to the higher average returns relative to the other stocks, stock 1, stock 4, stock 6, and stock 8 are selected with the optimal portfolio return of 0.194. Accordingly, the return of the optimal portfolio remarkably declines to 0.104 when uncertainty embraces all of the parameters. The logic of this model is that the probability that all uncertain coefficients have their worst value in nature is less, and usually, some of them face variation. The results are represented in Figure 2.

(Insert Fig.2 almost here)

7. Conclusions

An inherent feature in human mental judgments that should be given special attention to in decisions is Uncertainty. In this study, a novel approach for the portfolio construction problem is proposed in order to deal with data uncertainty, increasing conservatism levels of the investment process, decreasing computational complexity, and assessing comprehensive of stocks. Accordingly, the present study presents a stock portfolio optimization model which is considering the dollar exchange rate, and seeks to consider information deficiencies to improve performance using the logic based on DTS and RO. It is also formulated in a given atmosphere. Finally, in this paper, in order to demonstrate the applicability of the submitted model and exhibit the efficacy and effectiveness of the presented method, a real-life case study from the Tehran stock exchange is implemented. Though this model is implemented here for Iran Stock Market, it can be applied for constructing portfolios in any Stock Exchanges around the world but selection of critical factors can vary in different stock exchanges.

For future studies, to enhance the robustness of the model researchers can think of hybridizing DS evidence theory with other uncertainty handling tools like soft sets and rough sets. Moreover, data-driven robust optimization (DDRO) approach can be employed for proposing data-driven robust portfolio optimization (DDRPO) models (see [61], [62],[63] ,[64]).

1. Best, M.J., and Grauer, R.R. "On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results", *The review of financial studies*, 4(2), pp. 315-342 (1991).
2. Jobson, J.D., and Korkie, B., "Statistical inference in two-parameter portfolio theory with multiple regression software", *Journal of Financial and Quantitative Analysis*, 18(2), pp.189-197 (1983).
3. Yin, C., Perchet, R., and Soupé, F., "A practical guide to robust portfolio optimization", *Quantitative Finance*, *21*(6), pp.911-928 (2021).
4. Chopra, V.K., and Ziemba, W.T., "The Effect of Errors in Means, Variances, and Covariances on Optimal", *Worldwide asset and liability modeling*, pp. 53-61 (1998).
5. Peykani, P., Mohammadi, E., Saen, R.F., Sadjadi, S.J., and Rostamy‐Malkhalifeh, M., "Data envelopment analysis and robust optimization: A review", *Expert Systems*, *37*(4), pp. 12534 (2020).
6. Sharpe, W.F., "A simplified model for portfolio analysis", *Management science*, *9*(2), pp.277-293(1963).
7. Grossman, S.J., and Stiglitz, J.E., "On the impossibility of informationally efficient markets", *The American economic review*, *70*(3), pp. 393-408 (1980).
8. Yunusoglu, M.G., and Selim, H., "A fuzzy rule based expert system for stock evaluation and portfolio construction: An application to Istanbul Stock Exchange", *Expert Systems with Applications*, 40(3), pp. 908-920 (2013).
9. Xidonas, P., Mavrotas, G., and Psarras, J., "A multicriteria methodology for equity selection using financial analysis", *Computers & operations research*, *36*(12), pp. 3187-3203 (2009).
10. Konno, H., and Yamazaki, H., "Mean-absolute deviation portfolio optimization model and its applications to Tokyo stock market". *Management science*, *37*(5), pp.519-531 (1991).
11. Pavlou, A., Doumpos, M., and Zopounidis, C., "The robustness of portfolio efficient frontiers: A comparative analysis of bi-objective and multi-objective approaches", *Management Decision*, pp. 300-313 (2018).
12. Xidonas, P., Mavrotas, G., Zopounidis, C., and Psarras, J., "IPSSIS: An integrated multicriteria decision support system for equity portfolio construction and selection", *European Journal of Operational Research*, *210*(2), pp.398-409 (2011).
13. Feiring, B.R., and Lee, S.W., "A chance-constrained approach to stock selection in hong kong", *International journal of systems science*, *27*(1), pp.33-41 (1996).
14. Markowitz, H.M., "Portfolio selection", *In Portfolio selection*, Yale University Press (1968).
15. Ben-Tal, A., Margalit, T., and Nemirovski, A., "Robust modeling of multi-stage portfolio problems", *High Performance Optimization,* Springer, Boston, MA, pp. 303-328 (2000).
16. Moon, Y., and Yao, T., "A robust mean absolute deviation model for portfolio optimization". *Computers & Operations Research*, *38*(9), pp.1251-1258 (2011).
17. Kim, J.H., Kim, W.C., and Fabozzi, F.J., "Recent advancements in robust optimization for investment management", *Annals of Operations Research*, *266*(1), pp.183-198 (2018).
18. Goldfarb, D., and Iyengar, G., "Robust portfolio selection problems", *Mathematics of operations research*, *28*(1), pp.1-38 (2003).

19. Ling, A.F., and Xu, C.X., “Robust portfolio selection involving options under a “marginal+ joint” ellipsoidal uncertainty set”, *Journal of Computational and Applied Mathematics*, *236*(14), pp.3373-3393 (2012).
20. Heckel, T., de Carvahlo, R., Lu, X., and Perchet, R., “Insights into robust optimization: decomposing into mean–variance and risk-based portfolios”, *The Journal of Investment Strategies*, *6*(1), pp.1-24 (2016).
21. Quaranta, A.G., and Zaffaroni, A., “Robust optimization of conditional value at risk and portfolio selection”, *Journal of Banking & Finance*, *32*(10), pp.2046-2056 (2008).
22. Bertsimas, D., and Pachamanova, D., “Robust multiperiod portfolio management in the presence of transaction costs”, *Computers & Operations Research*, *35*(1), pp.3-17 (2008).
23. Chen, W., and Tan, S., “Robust portfolio selection based on asymmetric measures of variability of stock returns”, *Journal of computational and applied mathematics*, *232*(2), pp.295-304 (2009).
24. Ghahtarani, A., and Najafi, A.A., “Robust goal programming for multi-objective portfolio selection problem”, *Economic Modelling*, *33*, pp.588-592 (2013).
25. Pınar, M.Ç., and Paç, A.B., “Mean semi-deviation from a target and robust portfolio choice under distribution and mean return ambiguity”, *Journal of Computational and Applied Mathematics*, *259*, pp.394-405 (2014).
26. Ling, A., Sun, J., and Wang, M., “Robust multi-period portfolio selection based on downside risk with asymmetrically distributed uncertainty set”, *European Journal of Operational Research*, *285*(1), pp.81-95 (2020).
27. Tütüncü, R.H., and Koenig, M., “Robust asset allocation”, *Annals of Operations Research*, *132*(1), pp.157-187 (2004).
28. Bertsimas, D. and Sim, M., “The price of robustness”, *Operations research*, *52*(1), pp.35-53 (2004).
29. Bertsimas, D., and Thiele, A., “Robust and data-driven optimization: modern decision making under uncertainty”, In *Models, methods, and applications for innovative decision making* pp. 95-122 (2006).
30. Dimmock, S.G., Kouwenberg, R., Mitchell, O.S., and Peijnenburg, K., “Ambiguity aversion and household portfolio choice puzzles: Empirical evidence”, *Journal of Financial Economics*, *119*(3), pp.559-577 (2016).
31. Ji, R., and Lejeune, M.A., “Data-driven optimization of reward-risk ratio measures” *INFORMS Journal on Computing*, *33*(3), pp.1120-1137 (2021).
32. Sharma, A., and Mehra, A., “Extended omega ratio optimization for risk-averse investors”, *International Transactions in Operational Research*, *24*(3), pp.485-506 (2017).
33. PEYKANI, P., BARZINJPOUR, F., and JANDAGHIAN, A., “Portfolio selection under trading constraints and data uncertainty using robust optimization approach and NSGA-II algorithm”, pp. 195-206 (2020).
34. Georgantas, A., Doumpos, M., and Zopounidis, C., “Robust optimization approaches for portfolio selection: a comparative analysis”, *Annals of Operations Research*, pp.1-17 (2021).
35. Ghahtarani, A., Saif, A., and Ghasemi, A., “Robust portfolio selection problems: a comprehensive review”, *Operational Research*, pp.1-62 (2022).
36. Mohammadi, S.E., and Mohammadi, E., “Robust portfolio optimization based on minimax regret approach in Tehran stock exchange market”, *Journal of Industrial and Systems Engineering*, pp.51-62 (2018).
37. Peykani, P., Mohammadi, E., Jabbarzadeh, A., Rostamy-Malkhalifeh, M., and Pishvaee, M.S., “A novel two-phase robust portfolio selection and optimization approach under uncertainty: A case study of Tehran stock exchange”, *Plos one*, *15*(10), p.e0239810 (2020).
38. Xidonas, P., Steuer, R., and Hassapis, C., “Robust portfolio optimization: A categorized bibliographic review”, *Annals of Operations Research*, *292*(1), pp.533-552 (2020).
39. Thakur, G.S.M., Bhattacharyya, R., and Sarkar, S., “Stock portfolio selection using Dempster–Shafer evidence theory”, *Journal of King Saud University-Computer and Information Sciences*, *30*(2), pp.223-235 (2018).
40. Skoruchi, A., and Mohammadi, E., “Uncertain portfolio optimization based on Dempster-Shafer theory”, *Management Science Letters*, *12*(3), pp.207-214 (2022).
41. Thomsett, M.C., *Mastering fundamental analysis*. Dearborn Trade Publishing (1998).

42. Dempster, A.P., “Upper and lower probabilities induced by a multivalued mapping”, In *Classic works of the Dempster-Shafer theory of belief functions*. Springer, Berlin, Heidelberg pp. 57-72 (2008).
43. Shafer, G., “A mathematical theory of evidence”, *A Mathematical Theory of Evidence*. Princeton University Press (2021).
44. Mohammadi, S.E., and Makui, A., “Multi-attribute group decision making approach based on interval-valued intuitionistic fuzzy sets and evidential reasoning methodology”, *Soft Computing*, *21*(17), pp.5061-5080 (2017).
45. Petretto, A., “Optimal social health insurance with supplementary private insurance”, *Journal of Health Economics*, *18*(6), pp.727-745 (1999).
46. Xu, D.L., “An introduction and survey of the evidential reasoning approach for multiple criteria decision analysis”, *Annals of Operations Research*, *195*(1), pp.163-187 (2012).
47. Soraggi, S., Wiuf, C. and Albrechtsen, A., “Powerful inference with the D-statistic on low-coverage whole-genome data”, *G3: Genes, Genomes, Genetics*, *8*(2), pp.551-566 (2018).
48. Huynh, V.N., Nakamori, Y., Ho, T.B., and Murai, T., “Multiple-attribute decision making under uncertainty: the evidential reasoning approach revisited” *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans*, *36*(4), pp.804-822 (2006).
49. Seyedhosseini, S.M., Esfahani, M.J., and Ghaffari, M., “A novel hybrid algorithm based on a harmony search and artificial bee colony for solving a portfolio optimization problem using a mean-semi variance approach”, *Journal of Central South University*, *23*(1), pp.181-188 (2016).
50. Kong, X., Chang, J., Niu, M., Huang, X., Wang, J., and Chang, S.I., “Research on real time feature extraction method for complex manufacturing big data”, *The International Journal of Advanced Manufacturing Technology*, *99*(5), pp.1101-1108 (2018).
51. Cornuejols, G., and Tütüncü, R., *Optimization methods in finance* (Vol. 5). Cambridge University Press (2006).
52. Pishvaee, M.S., Razmi, J., and Torabi, S.A., “Robust possibilistic programming for socially responsible supply chain network design: A new approach”, *Fuzzy sets and systems*, *206*, pp.1-20 (2012).
53. Soyster, A.L., “Convex programming with set-inclusive constraints and applications to inexact linear programming”, *Operations research*, *21*(5), pp.1154-1157 (1973).
54. Mulvey, J.M., Vanderbei, R.J., and Zenios, S.A., “Robust optimization of large-scale systems”, *Operations research*, *43*(2), pp.264-281 (1995).
55. Yu, C.S., and Li, H.L., “A robust optimization model for stochastic logistic problems”, *International journal of production economics*, *64*(1-3), pp.385-397 (2000).
56. El Ghaoui, L., Oustry, F., and Lebret, H., “Robust solutions to uncertain semidefinite programs”, *SIAM Journal on Optimization*, *9*(1), pp.33-52 (1998).
57. Ben-Tal, A., and Nemirovski, A., “Robust convex optimization”, *Mathematics of operations research*, *23*(4), pp.769-805 (1998).
58. Ben-Tal, A., and Nemirovski, A., “Robust solutions of uncertain linear programs”, *Operations research letters*, *25*(1), pp.1-13 (1999).
59. Ben-Tal, A., and Nemirovski, A., “Robust solutions of linear programming problems contaminated with uncertain data”, *Mathematical programming*, *88*(3), pp.411-424 (2000).
60. Bertsimas, D., and Sim, M., “Robust discrete optimization and network flows”, *Mathematical programming*, *98*(1), pp.49-71 (2003).
61. Shang, C., Huang, X., and You, F., “Data-driven robust optimization based on kernel learning”, *Computers & Chemical Engineering*, *106*, pp.464-479 (2017).
62. Bertsimas, D., Gupta, V., and Kallus, N., “Data-driven robust optimization”, *Mathematical Programming*, *167*(2), pp.235-292 (2018).
63. Kang, Z., Li, X., Li, Z., and Zhu, S., “Data-driven robust mean-CVaR portfolio selection under distribution ambiguity”, *Quantitative Finance*, *19*(1), pp.105-121 (2019).

64. Namakshenas, M., and Pishvaee, M.S., “Data-driven Robust Optimization Robust and Constrained Optimization: Methods and Applications”, pp. 1–40 (2019).

Biographies

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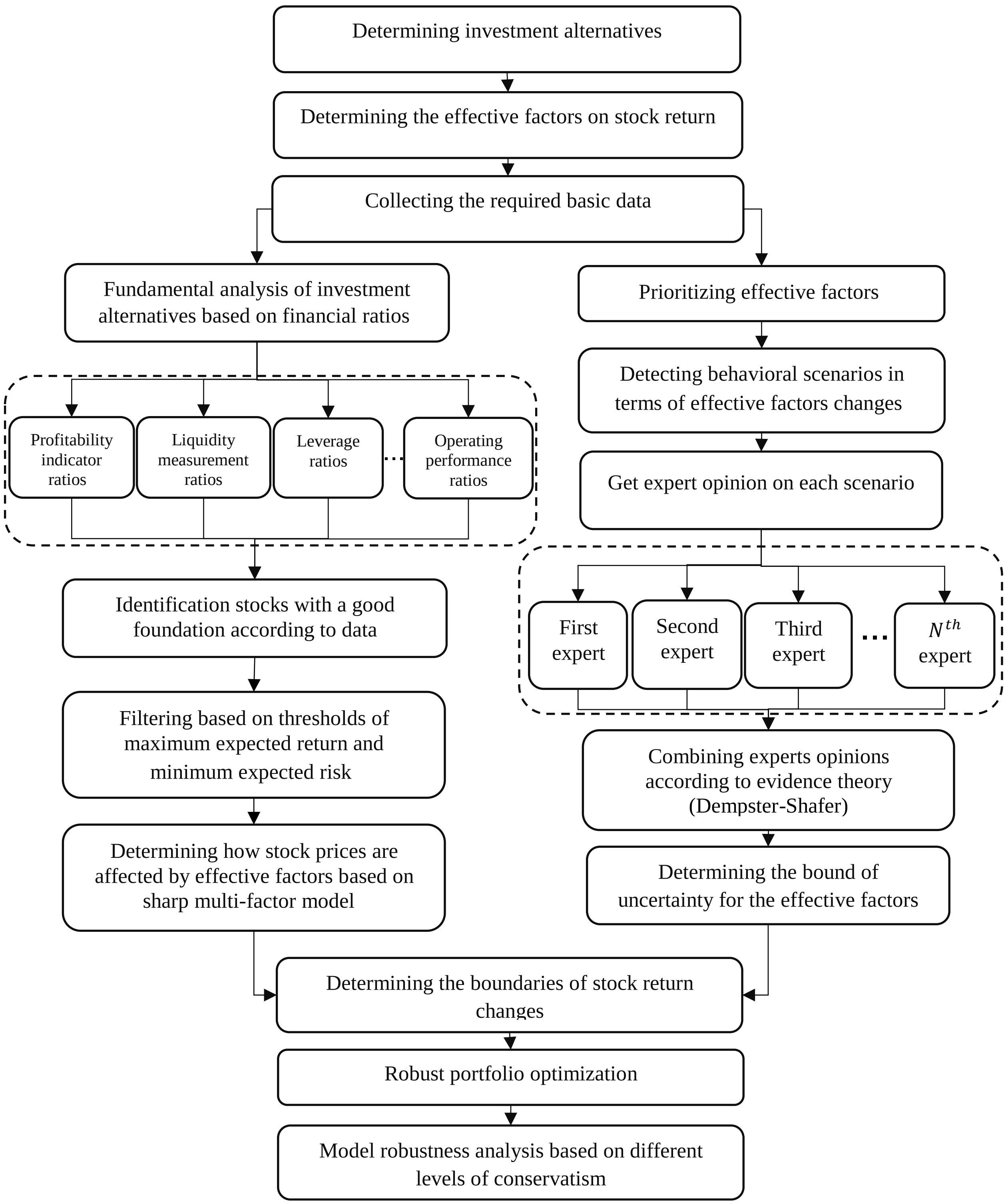
Determining investment alternatives
Determining the effective factors on stock return
Collecting the required basic data
Fundamental analysis of investment alternatives based on financial ratios
Prioritizing effective factors
Profitability indicator ratios
Liquidity measurement ratios
Leverage ratios
...
Operating performance ratios
Detecting behavioral scenarios in terms of effective factors changes
Get expert opinion on each scenario
Identification stocks with a good foundation according to data
First expert
Second expert
Third expert
...
N^{th} expert
Filtering based on thresholds of maximum expected return and minimum expected risk
Combining experts opinions according to evidence theory (Dempster-Shafer)
Determining how stock prices are affected by effective factors based on sharp multi-factor model
Determining the bound of uncertainty for the effective factors
Determining the boundaries of stock return changes
Robust portfolio optimization
Model robustness analysis based on different levels of conservatism

Fig 1.

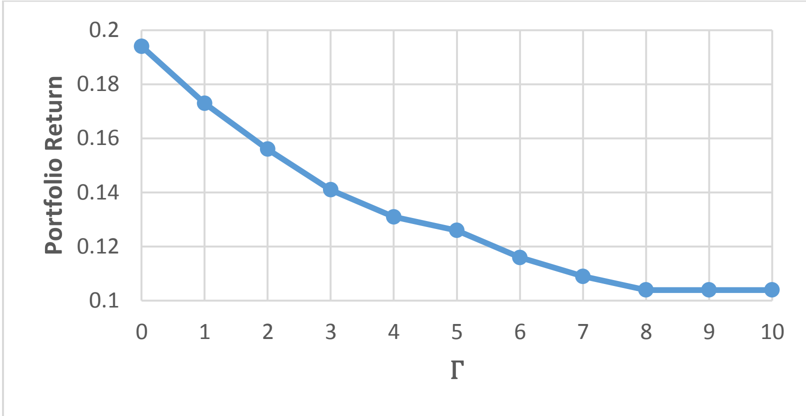

Fig 2.

Table 1.

DM	Weight	Scenario A	Scenario B	Scenario C	Scenario D	B_H
		0.206	0.086	0.107	0.206	0.392
A1	0.100	0.000	0.050	0.300	0.600	0.050
A2	0.100	0.500	0.250	0.100	0.000	0.150
A3	0.450	0.000	0.000	0.100	0.300	0.600
A4	0.350	0.400	0.150	0.050	0.000	0.400

Table 2.

	Stock1	Stock2	Stock3	Stock4	Stock5	Stock6	Stock7	Stock8	Stock9	Stock10
Upper return	0.286	0.241	0.223	0.262	0.189	0.255	0.170	0.224	0.164	0.173
Lower return	0.112	0.016	0.065	0.121	0.063	0.138	0.063	0.120	0.062	0.091
Average return	0.199	0.129	0.144	0.192	0.126	0.197	0.117	0.172	0.113	0.132
Tolerance	0.087	0.082	0.079	0.070	0.062	0.058	0.053	0.051	0.050	0.041

Table3.

Γ	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6	Stock 7	Stock 8	Stock 9	Stock 10	Portfolio returns
0	0.300	0.000	0.000	0.300	0.000	0.300	0.000	0.100	0.000	0.000	0.194
1	0.200	0.000	0.000	0.247	0.000	0.300	0.000	0.253	0.000	0.000	0.173
2	0.187	0.000	0.000	0.231	0.000	0.281	0.000	0.300	0.000	0.000	0.156
3	0.154	0.000	0.169	0.190	0.000	0.230	0.000	0.257	0.000	0.000	0.141
4	0.086	0.090	0.094	0.106	0.000	0.300	0.000	0.144	0.000	0.000	0.131
5	0.012	0.013	0.013	0.300	0.017	0.300	0.020	0.300	0.000	0.025	0.126
6	0.010	0.011	0.011	0.300	0.014	0.300	0.016	0.300	0.017	0.021	0.116
7	0.054	0.057	0.060	0.067	0.075	0.300	0.088	0.091	0.093	0.114	0.109
8	0.000	0.000	0.000	0.100	0.000	0.300	0.000	0.300	0.000	0.300	0.104
9	0.000	0.000	0.000	0.100	0.000	0.300	0.000	0.300	0.000	0.300	0.104
10	0.000	0.000	0.000	0.100	0.000	0.300	0.000	0.300	0.000	0.300	0.104