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Fuzzy confidence interval construction and its application in recovery time for COVID-19 patients

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Abstract. An approach is proposed to construct fuzzy confidence intervals for unknown parameters in statistical models. In this approach, a family of confidence intervals of the unknown crisp parameters is considered. Such confidence intervals are used to obtain a fuzzy confidence interval for the parameter of interest. The proposed approach enjoys a wide range of confidence intervals to obtain a trapezoidal shaped fuzzy set of the parameter space as the fuzzy confidence interval for the parameter of interest. By using the resolution identity, it is shown that the constructed fuzzy confidence intervals are really fuzzy sets of the parameter space. Some numerical examples are provided to explain the functionality of the approach at one-sided and two-sided fuzzy confidence intervals. Moreover, the application of this proposed approach in health sciences is provided for the case of the recovery time of olfactory and gustatory dysfunctions for COVID-19 patients.

1. Introduction

Statistical methods in fuzzy environments have been studied in both theory and practice over the last decades. This issue comes with interesting applications in many fields [1]. Confidence interval is an important subject in statistics and is a fundamental element in statistical inference. In this respect, fuzzy confidence interval is the basic subject of statistical analysis in fuzzy environments. Some authors have investigated the issue through certain approaches. Corral and Gil [2] formulated the problem of interval estimation for the cases in which the experimental outcomes were fuzzy rather than crisp. Parchami et al. [3] and Ramezani et al. [4] introduced two classes of fuzzy confidence intervals for the fuzzy process capability indices based on ranking functions. The application of fuzzy set theory to statistical confidence intervals for unknown fuzzy parameters based on fuzzy random variables was studied by Wu [5]. Škrjanc [6] presented an approach to confidence interval for fuzzy models by combining a fuzzy identification methodology with some methods from applied statistics. The main idea is to determine the confidence interval defined by the lower and upper fuzzy bounds which construct the band that contains all the output measurements. Couso and Sánchez [7] extended the concept of confidence interval to fuzzy information by introducing a pair of fuzzy inner and outer confidence intervals. Using extension principle, Viertl [1] investigated the problem of confidence interval based on fuzzy data. Chachi and Taheri [8] introduced a fuzzy confidence interval for mean of Gaussian distribution on the basis of fuzzy random variables. The developed approaches to constructing fuzzy confidence intervals during recent decades were

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reviewed and analyzed by Kahraman et al. [9]. Moreover, they presented two interval-valued intuitionistic and hesitant confidence intervals for intuitionistic and hesitant observations, respectively [10]. Berkachy and Donze [11] proposed an approach functioning based on the likelihood ratio method to estimate the fuzzy confidence interval in case fuzziness occurred. Chukhrova and Johannssen [12] presented an extended sign test based on fuzzy categories and fuzzy hypotheses to improve the generality, versatility, and practicability of the common sign test. Testing hypotheses for multivariate normal distribution were investigated by Hesamian and Akbari on the basis of fuzzy random variables [13].

Chachi et al. [14] proposed fuzzy testing hypotheses about a fuzzy unknown parameter when the available information was fuzzy. Their proposed method functions based on the relationship between the acceptance region of statistical tests at level γ and confidence intervals for the parameter of interest at confidence level $1-\gamma$. After constructing a fuzzy confidence interval for the fuzzy parameter of interest, they have constructed a fuzzy test function using such a fuzzy confidence interval. Moreover, fuzzy testing of multi-alternative hypotheses was studied by Harikrishnan et al. [15] through the relation between fuzzy confidence interval and region of acceptance.

An extensive and comprehensive systematic review was guided by Chukhrova and Johannssen [16] with several key research questions, containing absolute essence on the topic of testing fuzzy hypothesis. To review some new articles about testing fuzzy hypotheses, we only refer the interested readers to [17–23] for the sake of brevity.

Besides the above studies, Buckley [24,25] developed a method to estimate an unknown parameter in a statistical model. He used a set of confidence intervals producing a triangular fuzzy number for estimation of the interested parameter. Falsafain et al. [26] investigated a procedure to derive an explicit and unique membership function of the above triangular fuzzy number. Also, Falsafain and Taheri [27] demonstrated that Buckley's method was subject to some drawbacks and therefore, presented an improved method. Another possibilistic testing crisp hypothesis was developed by Mylonas and Papadopoulos based on Buckley's method in which fuzzy estimators were employed to construct the test statistic and the possibility of rejection/acceptance of the null hypothesis [28,29].

The present paper extends Buckley's estimation method to construct $100(1-\gamma)\%$ fuzzy confidence interval by a trapezoidal shaped fuzzy subset in the parameter space. This paper is organized as follows. Preliminaries and notations are presented in Section 2. Buckley's fuzzy point estimation method is reviewed in Section 3. Then, the proposed method to construct a fuzzy confidence interval is developed in Section 4. Several illustrative examples are presented in Section 5. An application in health sciences related to Covid-19 is provided in Section 6. The final section includes conclusions and future works.

2. Preliminaries and notations

Let X be a universal set and $F(X)=\{\tilde{A}|\tilde{A}: X \rightarrow [0,1]\}$. Any $\tilde{A} \in F(X)$ is called a fuzzy subset on X for which $\tilde{A}(x)$ denotes the degree of membership of x in the fuzzy set $\tilde{A}$. We denote a fuzzy set using an overline on symbols. An α-cut of $\tilde{A}$ is defined as $\tilde{A}[\alpha]=\{x \in X|\tilde{A}(x) \geq \alpha\}$ for $0 \leq \alpha \leq 1$.

In particular, let $\mathbb{R}$ be the set of real numbers. A *triangular shaped fuzzy number* $\tilde{N} \in F(\mathbb{R})$ is a fuzzy subset of $\mathbb{R}$ satisfying *(i)* $\tilde{N}(x)=1$ for exactly one $x \in \mathbb{R}$; *(ii)* for $\alpha \in (0,1]$, the α-cut of $\tilde{N}$ is a bounded and closed interval denoted by $\tilde{N}[\alpha]=[n_1(\alpha), n_2(\alpha)]$ where $n_1(.)$ and $n_2(.)$ are the increasing and decreasing continuous functions, respectively [24,25]. A *trapezoidal shaped fuzzy number* $\tilde{M} \in F(\mathbb{R})$ is a fuzzy subset of $\mathbb{R}$ satisfying: *(i)* $\tilde{M}(x)=1$ for any $x \in [x_1, x_2]$ such that $x_1, x_2 \in \mathbb{R}$ and $x_1 \leq x_2$; *(ii)* for $\alpha \in (0,1]$, the α-cut of $\tilde{M}$ is a bounded and closed interval, denoted by $\tilde{M}[\alpha]=[m_1(\alpha), m_2(\alpha)]$ where $m_1(.)$ and $m_2(.)$ are the increasing and decreasing continuous functions, respectively. It is obvious that a triangular shaped fuzzy number is a special case of the trapezoidal shaped fuzzy number. The symbols and notations used in the article are summarized in Table 1.

3. Fuzzy point estimation: A brief review of Buckley's method

This section briefly reviews Buckley's method [24,25] for fuzzy estimation of an unknown parameter in statistical models, with some new modifications. Let $X_1, \cdots, X_n$ be a random sample with observed values $x_1, \cdots, x_n$ from a distribution with probability density/mass function $f(x;\theta)$, where θ is a single unknown crisp parameter of interest. Based on these observations, we construct $100(1-\alpha)\%$ confidence intervals for the crisp parameter θ, $0 \leq \alpha \leq 1$. We denote such confidence intervals by $[\theta_1(\alpha), \theta_2(\alpha)]$. In two special cases, $[\theta_1(1), \theta_2(1)]$ is a point estimation and the whole parameter space Θ is a 100% confidence interval for $\alpha=0$. Usually, Θ is one of the forms$(-\infty,+\infty)$, $(-\infty, b]$, $[a,+\infty)$, or $[a,b]$; $a<b$, $a,b \in \mathbb{R}$. When α increases, the length of $[\theta_1(\alpha), \theta_2(\alpha)]$ decreases, and vice versa. Therefore, these confidence intervals are nested. Thus, we have a family of $100(1-\alpha)\%$ confidence intervals for θ, where $0 \leq \alpha \leq 1$. Now, we place the confidence intervals one on top of the other from $\alpha=0$ to $\alpha=1$ to construct a triangular shaped fuzzy number $\tilde{\theta}$, whose α-cuts are as follows:

Table 1. Symbols and notations used in the article.

Notation	Meaning
$\mathbb{R}$	The set of real numbers
$F(X)$	The set of all fuzzy sets on X
$F(\mathbb{R})$	The set of all fuzzy sets on $\mathbb{R}$
θ	Unknown parameter (which is crisp)
Θ	Parameter space
$\tilde{N}$, $\tilde{M}$, $\tilde{A}$, $\tilde{B}$, $\tilde{\theta}$ and $\tilde{\mu}$	Fuzzy sets
$\tilde{A}[\alpha]=[a_1(\alpha),a_2(\alpha)]$	α-cut of the fuzzy set $\tilde{A}$
$\tilde{A}\subseteq\tilde{B}$	$\tilde{A}$ is a fuzzy subset of $\tilde{B}$
$\tilde{C}_\theta^{100(1-\gamma)\%}$	$100(1-\gamma)\%$ fuzzy confidence interval for θ (which is a fuzzy subset of Θ)
$\tilde{C}_\theta^{100(1-\gamma)\%}[\alpha]$	α-cut of $100(1-\gamma)\%$ fuzzy confidence interval for θ
$[\theta_1(\alpha),\theta_2(\alpha)]$	$100(1-\alpha)\%$ confidence interval for θ (which is a crisp subset of Θ)
z_α	α-quantile for the standard normal distribution
$t_{n,\alpha}$	α-quantile for the student's t-distribution with n degrees of freedom
χ^2_n	Chi-square distribution with n degrees of freedom
$\chi^2_{n,\alpha}$	α-quantile for the chi-square distribution with n degrees of freedom

$$\tilde{\theta}[0]=\Theta,$$

$$\tilde{\theta}[\alpha]=[\theta_1(\alpha),\theta_2(\alpha)],\qquad \text{for}\quad 0<\alpha<1,$$

$$\tilde{\theta}[1]=\theta_1(1)=\theta_2(1).$$

Using the above method, we have more information about the unknown parameter θ rather than merely having a classical single interval estimate [24,25]. In fact, now, we are employing much more information from the random sample, similar to all confidence intervals between 0% and 100%.

Note 1. It should be noted that the parameter of interest θ is crisp. However, we try to estimate such a crisp parameter by constructing a fuzzy confidence interval.

Example 1. Assume that $X_1,X_2,\cdots,X_{64}$ is a random sample with observed values $x_1,x_2,\cdots,x_{64}$ from the normal distribution $N(\mu,\sigma^2=100)$. Using Buckley's method, we are going to estimate the parameter μ. Suppose that the sample mean is obtained as $\bar{x}=28.6$. Then, the usual $100(1-\alpha)\%$ confidence interval for μ is $[28.6-1.25z_{1-\frac{\alpha}{2}},28.6+1.25z_{1-\frac{\alpha}{2}}]$, where $z_{1-\frac{\alpha}{2}}$

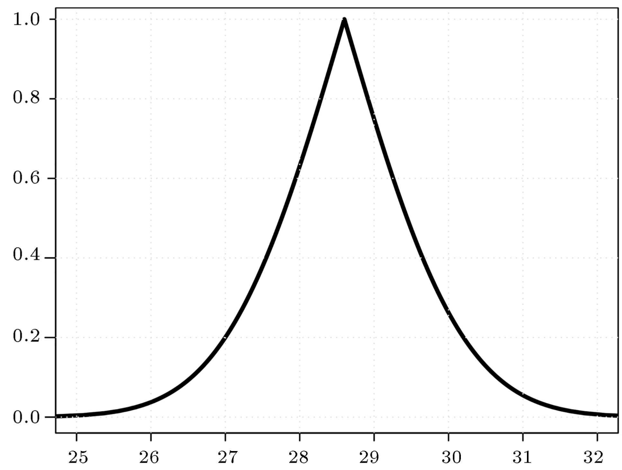

Figure 1. The membership function of $\tilde{\mu}$ in Example 1 based on Buckley's method.

denotes the $(1-\frac{\alpha}{2})$-quantile for the standard normal distribution. Therefore, the α-cuts of triangular shaped fuzzy number $\tilde{\mu}$ are obtained as follows:

$$\tilde{\mu}[0]=\mathbb{R},$$

$$\tilde{\mu}[\alpha]=\left[28.6-1.25z_{1-\frac{\alpha}{2}},28.6+1.25z_{1-\frac{\alpha}{2}}\right],$$

$$\text{for}\quad 0<\alpha<1,$$

$$\tilde{\mu}[1]=\bar{x}=28.6.$$

By placing the α-cuts of $\tilde{\mu}$ one on top of the other from $\alpha=0$ to $\alpha=1$, it is possible to produce a triangular shaped fuzzy point estimate for μ, where its membership function is depicted in Figure 1.

4. Fuzzy confidence interval: the proposed method

In this section, a procedure is proposed to construct $100(1-\gamma)\%$ fuzzy confidence intervals.

Let $X_1,\cdots,X_n$ be a random sample with observed values $x_1,\cdots,x_n$ from a distribution with probability density/mass function $f(x;\theta)$. Assume that θ is an unknown crisp parameter and a fuzzy confidence interval is constructed for θ. Inspired by Buckley's method presented in Section 2, statistical confidence intervals of θ are employed to construct the fuzzy confidence interval.

Definition 1. A $100(1-\gamma)\%$ fuzzy confidence interval for θ, denoted by $\tilde{C}_\theta^{100(1-\gamma)\%}$, (as a fuzzy subset of Θ) is constructed by the following intervals as its α-cuts:

$$\tilde{C}_\theta^{100(1-\gamma)\%}[0]=\Theta,$$

$$\tilde{C}_\theta^{100(1-\gamma)\%}[\alpha]=[\theta_1(\alpha\gamma),\theta_2(\alpha\gamma)],\quad \text{for}\quad 0<\alpha<1,$$

$$\tilde{C}_\theta^{100(1-\gamma)\%}[1]=[\theta_1(\gamma),\theta_2(\gamma)],$$

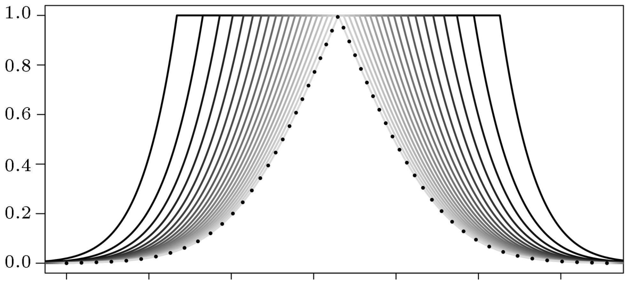

Figure 2. Buckley's fuzzy point estimation (dotted line) and the nested fuzzy confidence intervals based on the proposed method (solid lines) in the parameter space in Theorem 2.

where $[\theta_1(\alpha), \theta_2(\alpha)]$ is the usual/precise $100(1-\alpha)\%$ confidence interval for θ. Note that $\tilde{C}_\theta^{100(1-\gamma)\%}[1]$, as the extreme situation, is the ordinary/crisp $100(1-\gamma)\%$ confidence interval for θ.

Remark 1. It should be noted that the notation α is used for α-cut of the fuzzy confidence interval (which is a crisp subset of Θ for each $\alpha \in [0,1]$), while the notation γ is used for the confidence level of the fuzzy confidence interval.

Remark 2. It is obvious that Buckley's fuzzy point estimate is a particular case of the proposed fuzzy interval estimate when $\gamma = 1$ is selected. In fact, for such a reduced case, Buckley's fuzzy point estimation is a fuzzy subset of the $100(1-\gamma)\%$ fuzzy confidence interval (see Figure 2). This assertion is analogous to the classical condition where the point estimation of any parameter belongs to the related confidence interval.

Theorem 1. *The intervals* $\tilde{C}_\theta^{100(1-\gamma)\%}[\alpha]$, *for* $0 \le \alpha \le 1$, *in Definition 1 are* α*-cuts corresponding to a fuzzy set.*

Proof. For any $0 \le \alpha_1 \le \alpha_2 \le 1$, the usual/crisp confidence intervals of unknown parameter θ are nested, i.e., $[\theta_1(\alpha_2), \theta_2(\alpha_2)] \subseteq [\theta_1(\alpha_1), \theta_2(\alpha_1)]$. That is because $\theta_1(.)$ and $\theta_2(.)$ are increasing and decreasing functions, respectively. Moreover, $\tilde{C}_\theta^{100(1-\gamma)\%}[0] = [\theta_1(0), \theta_2(0)] = \Theta$ is defined in Definition 1. Therefore, by resolution identity [30], the introduced $100(1-\gamma)\%$ fuzzy confidence interval is a fuzzy set in the parameter space Θ.

Proposition 1. *Any two-sided fuzzy confidence interval is a fuzzy number and any one-sided fuzzy confidence interval is a fuzzy limit in the parameter space* Θ*. Refer to [31] to see details of lower and upper fuzzy limits.*

Theorem 2. *The* $100(1-\gamma)\%$ *fuzzy confidence intervals are nested fuzzy subsets of the parameter space* Θ *(see Figure 2). In other words:*

$$\tilde{C}_\theta^{100(1-\gamma_2)\%} \subseteq \tilde{C}_\theta^{100(1-\gamma_1)\%}, \qquad 0 \le \gamma_1 \le \gamma_2 \le 1.$$

Proof. It suffices to show that the α-cut of $100(1-\gamma_2)\%$ fuzzy confidence interval is a fuzzy subset of the α-cut of $100(1-\gamma_1)\%$ fuzzy confidence interval for every $\alpha \in [0,1]$. Since $\theta_1(.)$ is an increasing function, $\theta_1(\alpha\gamma_1) \le \theta_1(\alpha\gamma_2)$ for any $\alpha \in [0,1]$ and $0 \le \gamma_1 \le \gamma_2 \le 1$. Similarly, $\theta_2(.)$ is a decreasing function and so for any $\alpha \in [0,1]$ and $0 \le \gamma_1 \le \gamma_2 \le 1$, we have $\theta_2(\alpha\gamma_2) \le \theta_2(\alpha\gamma_1)$. Hence, considering the fact that $\theta_1(\alpha\gamma_2) \le \theta_2(\alpha\gamma_2)$, it is obvious that:

$$[\theta_1(\alpha\gamma_2), \theta_2(\alpha\gamma_2)] \subseteq [\theta_1(\alpha\gamma_1), \theta_2(\alpha\gamma_1)],$$

for any $0 \le \gamma_1 \le \gamma_2 \le 1$ and $\theta \in \Theta$. Therefore, by Definition 1, $\tilde{C}_\theta^{100(1-\gamma_2)\%}[\alpha] \subseteq \tilde{C}_\theta^{100(1-\gamma_1)\%}[\alpha]$ for any $0 \le \gamma_1 \le \gamma_2 \le 1$. Hence, we have finally $\tilde{C}_\theta^{100(1-\gamma_2)\%} \subseteq \tilde{C}_\theta^{100(1-\gamma_1)\%}$, $0 \le \gamma_1 \le \gamma_2 \le 1$, $\forall \theta \in \Theta$.

Remark 3 (Interpretation). Note that the interpretation of a fuzzy set is based on the possibility. Moreover, the occurrence possibility of an event is defined as the supremum of the possibility of single elements of the set, i.e., [32]:

$$\text{Poss}(A) = \sup\{\text{Poss}(\{x\}); \quad x \in A\}.$$

This interpretation is used in numerical examples.

5. Illustrative examples

In this section, four numerical examples are provided to elaborate on the proposed method. All calculations have been done using the software *R* version 3.5.2 [33].

5.1. Two-sided fuzzy confidence interval

Example 2 (Mean of normal distribution). Assume that $X_1, X_2, \cdots, X_{36}$ is a random sample with observed values $x_1, x_2, \cdots, x_{36}$ from the normal distribution $N(\mu, \sigma^2 = 144)$. Suppose that the sample mean turns out to be $\bar{x} = 28.6$ and we seek to construct a 90% fuzzy confidence interval for the unknown parameter μ. Considering $\gamma = 0.10$ and using the standard normal distribution for the statistic $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$, the $100(1-0.10\alpha)\%$ confidence interval based on the observed random sample of size n can be obtained as follows:

$$\left[\bar{x} - z_{1-\frac{0.1\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{x} + z_{1-\frac{0.1\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right], \qquad \alpha \in (0,1).$$

Hence, the α-cut of 90% fuzzy confidence interval of μ is obtained as follows:

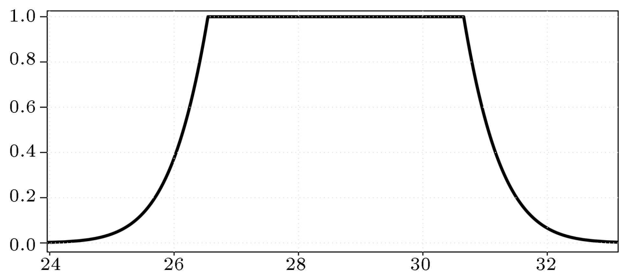

Figure 3. The membership function of 90% fuzzy confidence interval $\tilde{C}_{\mu}^{90\%}$ for μ in Example 2.

$$\tilde{C}_{\mu}^{90\%}[0]=(-\infty,+\infty),$$

$$\tilde{C}_{\mu}^{90\%}[\alpha]=\left[28.6-2z_{1-\frac{0.1\alpha}{2}},28.6+2z_{1-\frac{0.1\alpha}{2}}\right],$$

$$\text{for}\quad 0<\alpha<1,$$

$$\tilde{C}_{\mu}^{90\%}[1]=\left[28.6-2z_{1-\frac{0.1}{2}},28.6+2z_{1-\frac{0.1}{2}}\right]$$

$$=[25.31,31.89].$$

Note that $\tilde{C}_{\mu}^{90\%}[1]$, as an extreme situation, is an ordinary 90% confidence interval for μ. By placing α-cuts of $\tilde{C}_{\mu}^{90\%}$ one on top of the other from $\alpha=0$ to $\alpha=1$, one can produce a trapezoidal shaped fuzzy confidence interval for μ at a confidence level of 0.90. The membership function of such a fuzzy confidence interval is shown in Figure 3. The obtained fuzzy confidence interval includes the 90% classical confidence interval with a membership degree of one, i.e., the interval $[25.31,31.89]$. It includes other possible values of the unknown parameter μ with a degree of membership between 0 and 1. These results are clearly more useful than using only a single exact confidence interval to make an inference about the unknown population mean.

Example 3 (Mean of normal distribution-σ unknown). Assume that $X_1,\cdots,X_9$ is a random sample from normal distribution $N(\mu,\sigma^2)$ where both μ and σ^2 are unknowns. Suppose that the observed values of mean and standard deviation are $\bar{x}=3$ and $s=4$. A fuzzy confidence interval for μ with confidence coefficient 0.90 is constructed, i.e., $\gamma=0.10$. By using the t-distribution with $n-1$ degrees of freedom for the statistic $\frac{\sqrt{n}(\bar{X}-\mu)}{S}$, the related $100(1-0.10\alpha)\%$ confidence interval can be constructed based on the observed random sample of size n as:

$$\left[\bar{x}-t_{n-1,1-\frac{0.1\alpha}{2}}\frac{s}{\sqrt{n}},\bar{x}+t_{n-1,1-\frac{0.1\alpha}{2}}\frac{s}{\sqrt{n}}\right],$$

$$\text{for any}\quad \alpha\in(0,1),$$

where $t_{n,\alpha}$ is the α-quantile for the student's t-distribution with n degrees of freedom. Hence, the

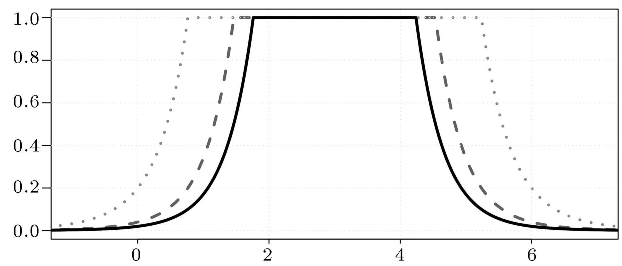

Figure 4. Three nested membership functions of fuzzy confidence intervals $\tilde{C}_{\mu}^{90\%}$ (solid line), $\tilde{C}_{\mu}^{95\%}$ (dashed line), and $\tilde{C}_{\mu}^{99\%}$ (dotted line) in Example 3.

α-cuts of 90% fuzzy confidence interval for μ are obtained as follows:

$$\tilde{C}_{\mu}^{90\%}[0]=(-\infty,+\infty),$$

$$\tilde{C}_{\mu}^{90\%}[\alpha]=\left[3-\frac{2}{3}t_{8,1-\frac{0.1\alpha}{2}},3+\frac{2}{3}t_{8,1-\frac{0.1\alpha}{2}}\right],$$

$$\text{for}\quad 0<\alpha<1,$$

$$\tilde{C}_{\mu}^{90\%}[1]=\left[3-\frac{2}{3}t_{8,1-\frac{0.1}{2}},3+\frac{2}{3}t_{8,1-\frac{0.1}{2}}\right]=[1.76,4.24].$$

Note that $\tilde{C}_{\mu}^{90\%}[1]$, as an extreme situation, is a classical 90% confidence interval for μ. By placing the α-cuts of $\tilde{C}_{\mu}^{90\%}$ one on top of the other, one can construct the membership function of the trapezoidal shaped fuzzy confidence interval for μ at level 0.90, which is shown in Figure 4 by solid line.

To make a comparison between different fuzzy confidence intervals and studying the effect of value γ, we provide fuzzy confidence intervals for $\gamma=0.10,0.05$, and 0.01. The results are three nested fuzzy confidence intervals. The related membership functions are shown in Figure 4.

5.2. One-sided fuzzy confidence interval

Using the proposed approach made it possible to obtain a one-sided fuzzy confidence interval, too. In the following, two numerical examples for such a case are presented.

Example 4. Suppose that a 90% left one-sided fuzzy confidence interval for μ is constructed at level of 0.90 in Example 3. It is easy to obtain $100(1-0.1\alpha)\%$ left one-sided confidence interval for μ based on the observed random sample of size n as follows:

$$\left[\bar{x}-t_{n-1,1-0.1\alpha}\frac{s}{\sqrt{n}},+\infty\right),\qquad \alpha\in(0,1).$$

Therefore, the α-cuts of 90% fuzzy left one-sided confidence interval for μ are obtained as follows:

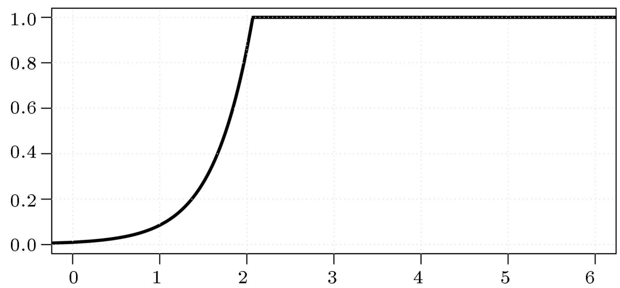

Figure 5. The membership function of 90% one-sided fuzzy confidence interval for μ in Example 4.

$$\tilde{C}_{\mu}^{90\%}[0] = (-\infty, +\infty),$$

$$\tilde{C}_{\mu}^{90\%}[\alpha] = \left[3 - t_{8,1-0.1\alpha} \times \frac{2}{3}, +\infty\right),$$

for $0 < \alpha < 1$,

$$\tilde{C}_{\mu}^{90\%}[1] = \left[3 - t_{8,1-0.1} \times \frac{2}{3}, +\infty\right) = [2.069, +\infty).$$

Note that $\tilde{C}_{\mu}^{90\%}[1]$ is an ordinary 90% left one-sided confidence interval for μ. By placing the α-cuts of $\tilde{C}_{\mu}^{90\%}$ one on top of the other, one can construct a fuzzy confidence interval for μ at level 0.90 (see Figure 5).

Example 5 (Beta distribution). Assume that $X_1, X_2, \cdots, X_n$ is a random sample from beta distribution $Beta(\theta, 1)$, i.e., $f_\theta(x) = \theta x^{\theta-1}$, $\theta > 0$. Considering pivotal quantity $-2\theta \sum_{i=1}^{n} \ln X_i \sim \chi^2_{2n}$, one can obtain $100(1-\alpha)\%$ right one-sided confidence interval $\left(0, \frac{\chi^2_{2n,1-\alpha}}{-2\sum_{i=1}^{n} \ln x_i}\right]$ based on the observed random sample of size n. Now, let $n = 23$ and the observations are as follows:

0.454,	0.755,	0.975,	0.796,	0.374,	0.835,
0.885,	0.585,	0.971,	0.811,	0.456,	0.939,
0.682,	0.691,	0.917,	0.899,	0.858,	0.913,
0.892,	0.722,	0.863,	0.678,	0.642.	

Since $\sum_{i=1}^{23} \ln x_i = -6.84$, the α-cuts of 95% fuzzy right one-sided confidence interval for θ are obtained as follows:

$$\tilde{C}_{\theta}^{95\%}[0] = (0, +\infty),$$

$$\tilde{C}_{\theta}^{95\%}[\alpha] = \left(0, \frac{\chi^2_{46,1-0.05\alpha}}{(-2)(-6.84)}\right],$$

for $0 < \alpha < 1$,

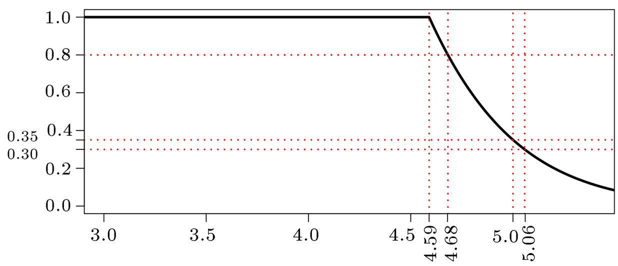

Figure 6. The membership function of 95% one-sided fuzzy confidence interval for θ in Example 5.

$$\tilde{C}_{\theta}^{95\%}[1] = \left(0, \frac{\chi^2_{46,1-0.05}}{(-2)(-6.84)}\right] = (0, 4.590].$$

Note that $\tilde{C}_{\theta}^{95\%}[1]$, as the extreme situation, is an ordinary 95% right one-sided confidence interval for θ. Similar to the previous examples, one can construct a fuzzy confidence interval for θ at level 0.95, where its membership function is depicted in Figure 6.

Regarding Remark 3, the obtained fuzzy interval has a probabilistic-possibilistic interpretation. For instance, at a confidence level of 0.95:

- It is completely possible that the unknown parameter θ be less than 4.59;
- With possibility 0.3, θ is equal to 5.06;
- With possibility 0.8, θ is greater than 4.68;
- With possibility 0.35, θ is between 5 and 6.

6. Application in health sciences: Recovery time for COVID-19 patients

Corona virus exhibits different clinical manifestations in paucity-symptomatic patients; Olfactory Dysfunction (OD) and Gustatory Dysfunction (GD) may represent the first or only symptoms. This important subject is currently arousing great interests, and a growing number of studies are being published. One hundred and twenty Iranian patients with the Covid-19 were considered in a research study (68 were males and 52 females) [34]. The mean duration of OD and GD for male patients was 16.87 and 12.80 days, respectively. Moreover, the mean duration of OD and GD for female patients was 25.80 and 27.13 days, respectively. It appears that the mean recovery time from OD or GD was longer for females than males. Therefore, we are going to construct four 90% fuzzy confidence intervals in this section for:

1. The mean of recovery time of GD in male patients;
2. The mean of recovery time of GD in female patients;
3. The mean of recovery time of OD in male patients;
4. The mean of recovery time of OD in female patients.

On the basis of Kolmogorov-Smirnov non-parametric test [35], the assumption of exponential distribution can be accepted for the recorded data set with p-values 0.97, 0.31, 0.54, and 0.83, respectively. Therefore, it is assumed that each data set has an exponential distribution with unknown parameter $\lambda > 0$, i.e., $f_\lambda(x) = \lambda e^{-\lambda x}$, $x > 0$. Thus, by considering pivotal quantity $2\lambda \sum_{i=1}^{n} X_i \sim \chi^2_{2n}$, the classical $100(1-\alpha)\%$ confidence interval for the mean of recovery time, i.e., $\mu = \frac{1}{\lambda}$, based on the observed values is:

$$\left[\frac{2\sum_{i=1}^{n} x_i}{\chi^2_{2n,1-\frac{\alpha}{2}}}, \frac{2\sum_{i=1}^{n} x_i}{\chi^2_{2n,\frac{\alpha}{2}}}\right],$$

for any $\alpha \in (0,1)$.

One of the limitations of the study was that patients did not have specific medical examinations and detailed medical tests to assess olfactory and gustatory dysfunctions. Hence, patients reported the process of improving their olfactory and gustatory dysfunctions based on mental perception and this fact can be one reason to use the fuzzy confidence interval for the mean of recovery time. Therefore, the α-cuts of 90% fuzzy confidence interval for the recovery mean time of OD in male patients are obtained as follows:

$$\tilde{C}^{90\%}_{\mu}[0] = (0, +\infty),$$

$$\tilde{C}^{90\%}_{\mu}[\alpha] = \left[\frac{2\times 1147.17}{\chi^2_{2\times 68,1-\frac{0.10\alpha}{2}}}, \frac{2\times 1147.17}{\chi^2_{2\times 68,\frac{0.10\alpha}{2}}}\right],$$

For $0 < \alpha < 1$,

$$\tilde{C}^{90\%}_{\mu}[1] = \left[\frac{2\times 1147.17}{\chi^2_{2\times 68,1-\frac{0.10}{2}}}, \frac{2\times 1147.17}{\chi^2_{2\times 68,\frac{0.10}{2}}}\right]$$

$$= [10.60, 15.81].$$

By using the proposed method, we can construct the fuzzy confidence interval for μ at level 0.90. The membership function of 90% fuzzy confidence interval for the mean of recovery time of OD in male patients is shown by solid line in Figure 7. Also, the 90% fuzzy confidence interval for the recovery mean time of OD is shown by dashed line in Figure 7 for female patients. Similarly, Figure 8 shows 90% fuzzy confidence intervals for the recovery mean time of GD for male and female patients.

7. Discussion and interpretation

Results for recovery time of gustatory dysfunction are as follows: Both of the constructed 90% fuzzy intervals in Figure 7 have probabilistic-possibilistic interpretations. For instance, at confidence level 0.90:

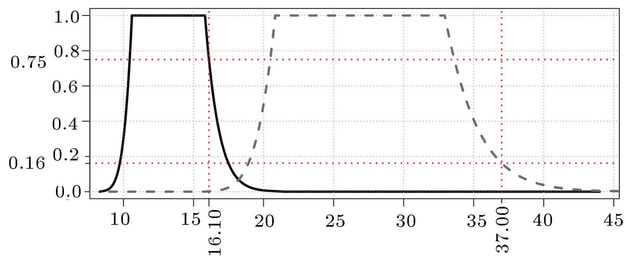

Figure 7. The membership functions of 90% fuzzy confidence intervals for the recovery mean time of gustatory dysfunction for males (solid line) and for females (dashed line) COVID-19.

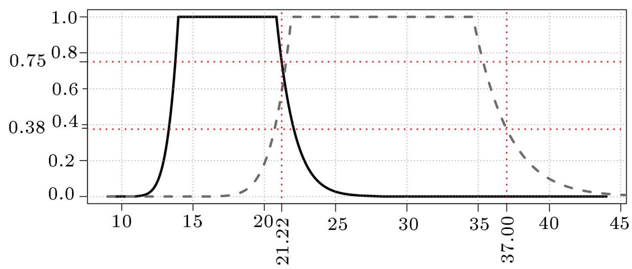

Figure 8. The membership functions of 90% fuzzy confidence intervals for the recovery mean time of olfactory dysfunction for males (solid line) and for females (dashed line) COVID-19.

- It is completely possible that the mean recovery time duration of GD for male patients be between 10.60 and 15.82 days;
- With possibility 0.75, the mean recovery time duration of GD for male patients is more than 16.10 days;
- It is completely possible that the mean recovery time duration of GD for female patients is between 20.83 and 32.94 days;
- With possibility 0.16, the mean recovery time duration of GD for female patients is more than 37 days.

Also, results for recovery time of OD are as follows: Considering two 90% fuzzy intervals in Figure 8, some probabilistic-possibilistic interpretations of results are as follows. For instance, at confidence level 0.90:

- It is completely possible that the mean recovery time duration for OD be between 13.97 and 20.85 days for male patients;
- With possibility 0.75, the mean recovery time duration of OD for male patients is more than 21.22 days;
- It is completely possible that the mean recovery time duration for OD is between 21.90 and 34.63 days for female patients;

– With possibility 0.38, the mean recovery time duration of OD for female patients is more than 37 days.

For a statistician, such results can be more useful than using only a single classical confidence interval to make an inference about the unknown mean of recovery time in health sciences. Note that such fuzzy confidence intervals benefit from both probabilistic and possibilistic sources of information.

8. Conclusions

An approach was presented to construct a fuzzy confidence interval for an unknown crisp statistical parameter. The proposed fuzzy confidence interval was a trapezoidal shaped fuzzy subset in the parameter space. Some numerical and applied examples were presented to explain the proposed approach. The benefits of the approach are as follows:

1. The proposed approach enjoys greater generality than Buckley's fuzzy point estimation method;
2. There is no limitation to construct fuzzy confidence interval for any unknown parameter;
3. The interpretation of the proposed fuzzy confidence interval is based on both probability and possibility;
4. The method is applicable in various contexts, as the applicability of the proposed method was explained based on a real-world problem about COVID-19.

9. Future works

Regarding the proposed approach in this paper, the following topics can be considered for future works:

1. Considering the relationship between statistical tests and confidence intervals, the study of testing fuzzy hypotheses [36,37] based on fuzzy/precise observations presents two potential subjects for further research;
2. As another related subject, one can investigate the fuzzy confidence interval from the Bayesian perspective;
3. Moreover, the developments of various types of confidence interval are some potential directions for future research. For instance, the shortest fuzzy confidence interval, the unbiased fuzzy confidence interval, the equal tails fuzzy confidence interval, and the asymptotic fuzzy confidence interval may be considered in future research based on the proposed approach.

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