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Embedded solitons with $\chi^{(2)}$ nonliner susceptibility

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Abstract. This paper recovers optical soliton solutions with $\chi^{(2)}$-nonlinear susceptibility. Bright, dark, singular, bright-dark combo solitons are recovered. A variety of algorithms are implemented. These include the Riccati equation approach, exp-function expansion method, modified simple equation algorithm, sine-Gordon equation scheme, F-expansion approach, trial function method, and functional variable algorithm.

1. Introduction

Optical solitons that exist in the continuous regime of the scattering spectrum are referred to as embedded solitons. These are governed by quadratic nonlinearity that emerges from $\chi^{(2)}$ nonlinear susceptibility. These give way to bright, dark, and singular solitons. These embedded solitons have been studied in the past using a variety of analytical approaches [1–25]. It is now time to revisit the same arena using a wider variety of mathematical approaches. This will yield a fresh set of soliton solutions of a different form, namely bright-dark combo solitons and others, which will be revealed using a wide spectrum of analytical approaches. The results are being reported for the first time in this paper, which will also encompass the previously reported results. These soliton solutions are enumerated with their respective existence criteria that are also presented in Table 1.

1.1. Governing model

The governing model with the quadratic nonlinearity [1–11] reads:

$$iu_t + a_1 u_{xx} + b_1 u_{xt} + c_1 u + \lambda_1 u^* v = i\alpha_1 u_x, \tag{1}$$

$$iv_t + a_2 v_{xx} + b_2 v_{xt} + c_2 v + \lambda_2 u^2 = i\alpha_2 v_x, \tag{2}$$

where x represents the spatial variable, while t denotes the temporal variable. The coefficients a_j, b_j, c_j, λ_j, and α_j $(j = 1, 2)$ are real valued constants. a_js stand for the coefficients of chromatic dispersion, while b_j, s stem from the coefficients of spatio-temporal dispersion. Next, λ_js are the coefficients of the quadratic nonlinearity effect, while α_is depict the coefficients of inter-modal dispersion. The first terms are linear temporal evolution and $i = \sqrt{-1}$. The functions $u = u(x,t)$ and $v = v(x,t)$ are complex-valued functions

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Table 1. Soliton solutions via the integration schemes.

Solitons	Bright	Dark	Singular	Straddled
Riccati equation	N	Y	Y	N
Sine-Gordon equation	N	Y	Y	Y
Functional variable	Y	N	Y	N
F-expansion	Y	Y	Y	Y
Exp-function expansion	N	Y	Y	N
Trial equation	Y	N	Y	N
Modified simple equation	Y	N	Y	N

representing the wave profiles of the forward harmonic and second harmonic waves, respectively. Lastly, $u^* = u^*(x,t)$ is the conjugate of $u = u(x,t)$.

2. Mathematical analysis

To solve Eqs. (1) and (2), the wave transformations are structured as:

$$u(x,t) = U_1(\vartheta)e^{i\varphi_1(x,t)}, \tag{3}$$

$$v(x,t) = U_2(\vartheta)e^{2i\varphi_2(x,t)}. \tag{4}$$

In Eqs. (3) and (4), the amplitude components are $U_j(\vartheta)$ for $j = 1, 2$, and the wave variable is:

$$\vartheta = \eta(x - pt), \tag{5}$$

where the real-valued constants η and p represent the soliton width and the soliton velocity, respectively, while the phase components are:

$$\varphi_j(x,t) = -kx + wt + \zeta, \tag{6}$$

where k, w, and ζ are real-valued constants standing for the soliton frequency, the soliton wave number, and the phase constant, respectively.

Next, the real and imaginary parts are:

$$\eta^2(a_1 - pb_1)U_1'' + \left(c_1 - w - k^2a_1 + kwb_1 - k\alpha_1\right)U_1 + \lambda_1 U_1 U_2 = 0, \tag{7}$$

$$kpb_1 - 2ka_1 + wb_1 - p - \alpha_1 = 0, \tag{8}$$

$$\eta^2(a_2 - pb_2)U_2'' + (c_2 - 2w - 4k^2a_2 + 4kwb_2 - 2k\alpha_2)U_2 + \lambda_2 U_1^2 = 0, \tag{9}$$

$$2kpb_2 - 4ka_2 + 2wb_2 - p - \alpha_2 = 0, \tag{10}$$

respectively, as long as Eqs. (3) and (4) are put in Eqs. (1) and (2). By the use of the balancing rule, Eqs. (7)–(10) reduce to the ordinary differential equation:

$$\eta^2(a - pb)U'' + (2ak^2 - 2bkw + \alpha k + w)U + \lambda U^2 = 0, \tag{11}$$

along with the velocity:

$$p = \frac{4ka - 2wb + \alpha}{2kb - 1}, \tag{12}$$

and the parameter constraints:

$$U_1 = U_2 = U, \quad b_1 = 2b, \quad b_2 = b, \quad a_1 = 2a,$$

$$a_2 = a, \quad \alpha_1 = \alpha_2 = \alpha, \quad \lambda_1 = 2\lambda, \quad \lambda_2 = \lambda,$$

$$c_1 = c_2 = c, \quad c = 6ak^2 - 6bkw + 3\alpha k + 3w. \tag{13}$$

In what follows, we will employ a variety of schemes that will be used to achieve the goals set for this work.

2.1. Riccati equation approach

Assume that the solution structure of Eq. (11) is considered as:

$$U(\vartheta) = \sum_{i=0}^{N} A_i V^i(\vartheta), \tag{14}$$

where A_is are constants to be established later, N is the balancing integer, $A_N \neq 0$ and also the function $V(\vartheta)$ satisfies the Riccati equation:

$$V'(\vartheta) = S_2V^2(\vartheta) + S_1V(\vartheta) + S_0, \quad S_2 \neq 0, \tag{15}$$

where S_2, S_1, and S_0 are constants. The solutions of Eq. (15) are listed as:

$$V(\vartheta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\mu}}{2S_2}\tanh\left(\frac{\sqrt{\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu > 0,$$

$$V(\vartheta) = -\frac{S_1}{2S_2} - \frac{\sqrt{\mu}}{2S_2}\coth\left(\frac{\sqrt{\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu > 0,$$

$$V(\vartheta) = -\frac{S_1}{2S_2} + \frac{\sqrt{-\mu}}{2S_2}\tan\left(\frac{\sqrt{-\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu < 0,$$

$$V(\vartheta) = -\frac{S_1}{2S_2} - \frac{\sqrt{-\mu}}{2S_2}\cot\left(\frac{\sqrt{-\mu}}{2}\vartheta + \vartheta_0\right), \quad \mu < 0,$$

$$V(\vartheta) = -\frac{S_1}{2S_2} - \frac{1}{S_2\vartheta + \vartheta_0}, \quad \mu = 0, \tag{16}$$

where $\mu = S_1^2 - 4S_0S_2$ and ϑ_0 is an arbitrary real constant. Next, Eq. (14) can be rewritten as:

$$U = A_0 + A_1V + A_2V^2, \tag{17}$$

by virtue of the balance principle applied in Eq. (11).

Then, the equations are recovered as:

$$-6\,b\eta^2 pA_2{S_2}^2 + 6\,a\eta^2 A_2{S_2}^2 + \lambda\,{A_2}^2 = 0, \tag{18}$$

$$-2b\eta^2 pA_1{S_2}^2 - 10\,b\eta^2 pA_2S_1S_2 + 2a\eta^2 A_1{S_2}^2 + 10\,a\eta^2 A_2S_1S_2 + 2\lambda\,A_1A_2 = 0, \tag{19}$$

$$-\eta^2 A_1S_1S_0bp - 2\eta^2 A_2{S_0}^2bp + \eta^2 A_1S_1S_0a + 2\eta^2 A_2{S_0}^2a + 2A_0ak^2 - 2A_0bkw + A_0\alpha k + \lambda{A_0}^2 + A_0w = 0, \tag{20}$$

$$-2\,b\eta^2 pA_1S_0S_2 - b\eta^2 pA_1{S_1}^2 - 6\,b\eta^2 pA_2S_0S_1 + 2\,a\eta^2 A_1S_0S_2 + a\eta^2 A_1{S_1}^2 + 6\,a\eta^2 A_2S_0S_1 + 2\,ak^2A_1 - 2bkwA_1 + \alpha\,kA_1 + 2\,\lambda\,A_0A_1 + wA_1 = 0, \tag{21}$$

$$-3b\eta^2 pA_1S_1S_2 - 8b\eta^2 pA_2S_0S_2 - 4\,b\eta^2 pA_2{S_1}^2 + 3a\eta^2 A_1S_1S_2 + 8a\eta^2 A_2S_0S_2 + 4a\eta^2 A_2{S_1}^2 + 2ak^2A_2 - 2bkwA_2 + \alpha kA_2 + 2\lambda A_0A_2 + \lambda{A_1}^2 + wA_2 = 0, \tag{22}$$

by substituting Eqs. (17) and (15) into Eq. (11). So, from Eqs. (18)–(22) we have:

$$S_0 = \pm\sqrt{-\frac{3\left(4\,A_0A_2 - {A_1}^2\right)(a-bp)}{2\,ak^2 - 2\,bkw + \alpha\,k + w}}$$

$$\frac{\begin{pmatrix} 12\,ak^2A_0A_2 - 2\,ak^2{A_1}^2 - 12\,bkwA_0A_2 - \alpha\,k{A_1}^2 \\ +2\,bkw{A_1}^2 + 6\,\alpha\,kA_0A_2 + 6\,wA_0A_2 - w{A_1}^2 \end{pmatrix}}{6\left(4\,A_0A_2 - {A_1}^2\right)\eta\,A_2\,(a-bp)},$$

$$S_1 = \pm\frac{A_1}{3\eta}\sqrt{-\frac{6\,ak^2 - 6\,bkw + 3\,\alpha\,k + 3\,w}{\left(4\,A_0A_2 - {A_1}^2\right)(a-bp)}},$$

$$S_2 = \pm\frac{A_2}{3\eta}\sqrt{-\frac{6\,ak^2 - 6\,bkw + 3\,\alpha\,k + 3\,w}{\left(4\,A_0A_2 - {A_1}^2\right)(a-bp)}},$$

$$\lambda = \frac{2\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)A_2}{4\,A_0A_2 - {A_1}^2},$$

and,

$$\mu = S_1^2 - 4S_0S_2 = \frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a-bp)}. \tag{23}$$

If one employs the solution set Eq. (23) with Eq. (16) in Eq. (17), dark solitons are obtained by Eqs. (24) and (25) as shown in Box I. with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + k\alpha + w\right) > 0.$$

Singular solitons are calculated by Eq. (26) and (27) are shown in Box II, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + k\alpha + w\right) > 0.$$

2.2. Sine-Gordon equation method

The solution of Eq. (11) is structured as follows:

$$U(\vartheta) = \sum_{i=1}^{N}\cos^{i-1}(V(\vartheta))\,[B_i\sin(V(\vartheta)) + A_i\cos(V(\vartheta))] + A_0, \tag{28}$$

where A_is and B_is are real constants to be detected later, $A_i \neq 0$, $B_i \neq 0$, N is the balancing integer, and the function $V(\vartheta)$ satisfies:

$$V'(\vartheta) = \sin(V(\vartheta)). \tag{29}$$

$$u(x,t) = \left\{\begin{array}{l} \dfrac{4\,A_0A_2 - {A_1}^2}{4A_2} - \dfrac{3\left(4\,A_0A_2 - {A_1}^2\right)}{4A_2} \\ \times\tanh^2\left(\sqrt{\dfrac{2\,ak^2 - 2\,bkw + k\alpha + w}{4\,(a-bp)}}\left(x - \dfrac{4ka - 2wb + \alpha}{2kb-1}t\right) + \vartheta_0\right)\end{array}\right\} \times e^{i(-kx+wt+\zeta)}, \tag{24}$$

$$v(x,t) = \left\{\begin{array}{l} \dfrac{4\,A_0A_2 - {A_1}^2}{4A_2} - \dfrac{3\left(4\,A_0A_2 - {A_1}^2\right)}{4A_2} \\ \times\tanh^2\left(\sqrt{\dfrac{2\,ak^2 - 2\,bkw + k\alpha + w}{4\,(a-bp)}}\left(x - \dfrac{4ka - 2wb + \alpha}{2kb-1}t\right) + \vartheta_0\right)\end{array}\right\} \times e^{2i(-kx+wt+\zeta)}. \tag{25}$$

Box I

$$u(x,t)=\left\{\begin{array}{l}\frac{4\,A_0A_2-{A_1}^2}{4A_2}-\frac{3\left(4\,A_0A_2-{A_1}^2\right)}{4A_2}\\ \times\coth^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+k\alpha+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)+\vartheta_0\right)\end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{26}$$

$$v(x,t)=\left\{\begin{array}{l}\frac{4\,A_0A_2-{A_1}^2}{4A_2}-\frac{3\left(4\,A_0A_2-{A_1}^2\right)}{4A_2}\\ \times\coth^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+k\alpha+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)+\vartheta_0\right)\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{27}$$

Box II

Eq. (29) has the following solutions:

$$\sin\left(V\left(\vartheta\right)\right)=\operatorname{sech}\left(\vartheta\right) \quad \text{or} \quad \sin\left(V\left(\vartheta\right)\right)=i\operatorname{csh}\left(\vartheta\right),$$

$$\cos\left(V\left(\vartheta\right)\right)=\tanh\left(\vartheta\right) \quad \text{or} \quad \cos\left(V\left(\vartheta\right)\right)=\coth\left(\vartheta\right). \tag{30}$$

Next, Eq. (28) can be rewritten as:

$$U\left(\vartheta\right)=B_1\sin(V\left(\vartheta\right))+A_1\cos(V\left(\vartheta\right))+\cos(V\left(\vartheta\right))$$
$$[B_2\sin(V\left(\vartheta\right))+A_2\cos(V\left(\vartheta\right))]+A_0, \tag{31}$$

by virtue of the balance principle applied in Eq. (11). Then, the derived equations are:

$$-6\,b\eta^2pB_2+6\,a\eta^2B_2+2\lambda\,A_2B_2=0, \tag{32}$$

$$-6\,b\eta^2pA_2+6\,a\eta^2A_2+\lambda\,{A_2}^2-\lambda\,{B_2}^2=0, \tag{33}$$

$$-2\,b\eta^2pB_1+2\,a\eta^2B_1+2\,\lambda\,A_1B_2$$
$$+2\,\lambda\,A_2B_1=0, \tag{34}$$

$$-2\,b\eta^2pA_1+2\,a\eta^2A_1+2\,\lambda\,A_1A_2$$
$$-2\,\lambda\,B_1B_2=0, \tag{35}$$

$$b\eta^2pB_1-a\eta^2B_1+2\,ak^2B_1-2\,bkwB_1+\alpha\,kB_1$$
$$+2\,\lambda\,A_0B_1+wB_1=0, \tag{36}$$

$$-2\,b\eta^2pA_2+2\,a\eta^2A_2+2\,ak^2A_0-2\,bkwA_0+\alpha\,kA_0$$
$$+\lambda\,{A_0}^2+\lambda\,{B_1}^2+wA_0=0, \tag{37}$$

$$5\,b\eta^2pB_2-5\,a\eta^2B_2+2\,ak^2B_2-2\,bkwB_2$$
$$+\alpha\,kB_2+2\,\lambda\,A_0B_2+2\,\lambda\,A_1B_1$$
$$+wB_2=0, \tag{38}$$

$$2\,b\eta^2pA_1-2\,a\eta^2A_1+2\,ak^2A_1-2\,bkwA_1+\alpha\,kA_1$$
$$+2\,\lambda\,A_0A_1+2\lambda\,B_1B_2+wA_1=0, \tag{39}$$

$$8\,b\eta^2pA_2-8\,a\eta^2A_2+2\,ak^2A_2-2\,bkwA_2+\alpha\,kA_2$$
$$+2\,\lambda\,A_0A_2+\lambda\,{A_1}^2-\lambda\,{B_1}^2$$
$$+\lambda\,{B_2}^2+wA_2=0, \tag{40}$$

by substituting Eqs. (31) and (29) into Eq. (11). So, from Eqs. (32)–(40) we have two results as follows:

– **Result 1:**

$$\eta=\pm\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}},$$

$$A_0=\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda},$$

$$A_2=-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{2\lambda},$$

$$A_1=0, \qquad B_1=0, \qquad B_2=0. \tag{41}$$

If one substitutes Eqs. (41) and (30) with Eq. (31), dark solitons obtained by Eqs. (42) and (43) as shown in Box III, with:

$$(a-bp)\left(2\,ak^2-2\,bkw+k\alpha+w\right)>0.$$

Singular solitons obtained by Eqs. (44) and (45) as shown in Box IV, with:

$$(a-bp)\left(2\,ak^2-2\,bkw+k\alpha+w\right)>0.$$

– **Result 2:**

$$\eta=\pm\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}},$$

$$u(x,t)=\left\{\begin{array}{l}\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{2\lambda}\\ \\ \times\tanh^2\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{42}$$

$$v(x,t)=\left\{\begin{array}{l}\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{2\lambda}\\ \\ \times\tanh^2\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{43}$$

Box III

$$u(x,t)=\left\{\begin{array}{l}\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{2\lambda}\\ \\ \times\coth^2\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{44}$$

$$v(x,t)=\left\{\begin{array}{l}\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{2\lambda}\\ \\ \times\coth^2\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{45}$$

Box IV

$$A_0=\frac{2\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda},\qquad B_1=0,$$

$$A_2=-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda},$$

$$B_2=\pm\frac{3\,i\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda},\quad A_1=0. \tag{46}$$

Combo singular solitons obtained by Egs. (47) and (48) as shown in Box V, with:

$$(a-bp)\left(2\,ak^2-2\,bkw+k\alpha+w\right)>0.$$

2.3. Functional variable methodology

This subsection will apply the functional variable methodology for overcoming Eq. (11). Eq. (11) can be written as follows:

$$\frac{\eta^2\,(a-pb)\left\{V^2\,(U)\right\}'}{2}+(2ak^2-2bkw+\alpha k+w)U+\lambda U^2=0. \tag{49}$$

By employing a functional variable form we can write:

$$U'=V\,(U)\,. \tag{50}$$

Thus, the important result emerging from Eq. (49) is:

$$V\,(U_j)=\pm\sqrt{\frac{2ak^2-2bkw+\alpha k+w}{\eta^2\,(pb-a)}}\,U\sqrt{1+\frac{2\lambda}{3\,(2ak^2-2bkw+\alpha k+w)}U}. \tag{51}$$

$$u(x,t)=\left\{\begin{array}{l}\frac{2\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\+\coth\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\\\times\left(\begin{array}{l}\pm\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\\times\operatorname{csch}\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\\-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\\times\coth\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right)\end{array}\right\}\times e^{i(-kx+wt+\zeta)},\tag{47}$$

$$v(x,t)=\left\{\begin{array}{l}\frac{2\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\+\coth\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\\\times\left(\begin{array}{l}\pm\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\\times\operatorname{csch}\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\\-\frac{3\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{\lambda}\\\times\coth\left[\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{a-bp}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\end{array}\right)\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}.\tag{48}$$

Box V

If we integrate Eq. (51), bright solitons are:

$$u(x,t)=-\frac{3\left(2ak^2-2bkw+\alpha k+w\right)}{2\lambda}\operatorname{sech}^2\left[\sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4\left(pb-a\right)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\times e^{i(-kx+wt+\zeta)},\tag{52}$$

$$v(x,t)=-\frac{3\left(2ak^2-2bkw+\alpha k+w\right)}{2\lambda}\operatorname{sech}^2\left[\sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4\left(pb-a\right)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\times e^{2i(-kx+wt+\zeta)},\tag{53}$$

with:

$$(pb-a)\left(2ak^2-2bkw+\alpha k+w\right)>0.$$

Singular solitons are:

$$u(x,t)=\frac{3\left(2ak^2-2bkw+\alpha k+w\right)}{2\lambda}\operatorname{csch}^2\left[\sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(pb-a)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\times e^{i(-kx+wt+\zeta)}, \quad (54)$$

$$v(x,t)=\frac{3\left(2ak^2-2bkw+\alpha k+w\right)}{2\lambda}\operatorname{csch}^2\left[\sqrt{\frac{2ak^2-2bkw+\alpha k+w}{4(pb-a)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right]\times e^{2i(-kx+wt+\zeta)}, \quad (55)$$

with:

$$(pb-a)\left(2ak^2-2bkw+\alpha k+w\right)>0.$$

2.4. F-Expansion principle

The formal solution of Eq. (11) is given as:

$$U(\vartheta)=\sum_{i=0}^{N}\mu_i F^i(\vartheta), \quad (56)$$

where μ_is are constants that need to be detected, $\mu_N \neq 0$, N is the balancing integer, and also $F(\vartheta)$ ensures:

$$F'(\vartheta)=\sqrt{PF^4(\vartheta)+QF^2(\vartheta)+R}, \quad (57)$$

where P, Q, and R are constants. The solutions of Eq. (57) are presented as below:

$$F(\vartheta)=\operatorname{sn}(\vartheta)=\tanh(\vartheta),\ P=m^2,$$

$$Q=-(1+m^2),\ R=1,\ m\to 1,$$

$$F(\vartheta)=\operatorname{ns}(\vartheta)=\coth(\vartheta),$$

$$P=1,\ Q=-(1+m^2),\ R=m^2,\ m\to 1,$$

$$F(\vartheta)=\operatorname{sc}(\vartheta)=\tan(\vartheta),$$

$$P=1-m^2,\ Q=2-m^2,\ R=1,\ m\to 0,$$

$$F(\vartheta)=\operatorname{cs}(\vartheta)=\cot(\vartheta),$$

$$P=1,\ Q=2-m^2,\ R=1-m^2,\ m\to 0,$$

$$F(\vartheta)=\operatorname{cn}(\vartheta)=\operatorname{sech}(\vartheta),$$

$$P=-m^2,\ Q=2m^2-1,\ R=1-m^2,\ m\to 1,$$

$$F(\vartheta)=\operatorname{ds}(\vartheta)=\operatorname{csch}(\vartheta),$$

$$P=1,\ Q=2m^2-1,\ R=-m^2\left(1-m^2\right),\ m\to 1,$$

$$F(\vartheta)=\operatorname{nc}(\vartheta)=\sec(\vartheta),$$

$$P=1-m^2,\ Q=2m^2-1,\ R=-m^2,\ m\to 0,$$

$$F(\vartheta)=\operatorname{ns}(\vartheta)=\csc(\vartheta),$$

$$P=1,\ Q=-(1+m^2),\ R=m^2,\ m\to 0,$$

$$F(\vartheta)=\operatorname{ns}(\vartheta)\pm\operatorname{ds}(\vartheta)=\coth(\vartheta)\pm\operatorname{csch}(\vartheta),$$

$$P=\frac{1}{4},\ Q=\frac{m^2-2}{2},\ R=\frac{m^2}{4},\ m\to 1,$$

$$F(\vartheta)=\operatorname{sn}(\vartheta)\pm i\operatorname{cn}(\vartheta)=\tanh(\vartheta)\pm i\operatorname{sech}(\vartheta),$$

$$P=\frac{m^2}{4},\quad Q=\frac{m^2-2}{2},\quad R=\frac{m^2}{4},\quad m\to 1,$$

$$F(\vartheta)=\operatorname{ns}(\vartheta)\pm\operatorname{cs}(\vartheta)=\csc(\vartheta)\pm\cot(\vartheta),$$

$$P=\frac{1}{4},\ Q=\frac{1-2m^2}{2},\ R=\frac{1}{4}, m\to 0,$$

$$F(\vartheta)=\operatorname{nc}(\vartheta)\pm\operatorname{sc}(\vartheta)=\sec(\vartheta)\pm\tan(\vartheta),$$

$$P=\frac{1-m^2}{4},\quad Q=\frac{1+m^2}{2},$$

$$R=\frac{1-m^2}{4},\quad m\to 0. \quad (58)$$

Next, Eq. (56) can be rewritten as:

$$U=\mu_0+\mu_1 F+\mu_2 F^2, \quad (59)$$

by virtue of the balance principle applied in Eq. (11). Then, the strategic equations are found as the following:

$$-6\,\eta^2 Pbp\mu_2+6\,\eta^2 Pa\mu_2+\lambda\,{\mu_2}^2=0, \quad (60)$$

$$-2\,\eta^2 Pbp\mu_1+2\,\eta^2 Pa\mu_1+2\,\lambda\,\mu_1\mu_2=0, \quad (61)$$

$$-\eta^2 Qbp\mu_1+\eta^2 Qa\mu_1+2\,ak^2\mu_1-2\,bkw\mu_1+\alpha\,k\mu_1+2\,\lambda\,\mu_0\mu_1+w\mu_1=0, \quad (62)$$

$$-2\,\eta^2 Rbp\mu_2+2\,\eta^2 Ra\mu_2+2\,ak^2\mu_0-2\,bkw\mu_0+\alpha\,k\mu_0+\lambda\,{\mu_0}^2+w\mu_0=0, \quad (63)$$

$$-4\eta^2 Qbp\mu_2 + 4\eta^2 Qa\mu_2 + 2ak^2\mu_2 - 2\,bkw\mu_2 + \alpha\,k\mu_2 + 2\,\lambda\,\mu_0\mu_2 + \lambda\,{\mu_1}^2 + w\mu_2 = 0, \qquad (64)$$

by substituting Eqs. (59) and (57) into Eq. (11). So, from Eqs. (60)–(64) we have:

$$\mu_1 = 0, \quad \eta = \pm\sqrt[4]{-\frac{(2\,ak^2 - 2\,bkw + \alpha\,k + w)^2}{16\,(a-bp)^2\,(3\,PR - Q^2)}},$$

$$\mu_0 = -\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{2\lambda} \pm\sqrt{-\frac{Q^2\,(2\,ak^2 - 2\,bkw + \alpha\,k + w)^2}{4\lambda^2\,(3PR - Q^2)}},$$

$$\mu_2 = \pm\sqrt{-\frac{9P^2\,(2\,ak^2 - 2\,bkw + \alpha\,k + w)^2}{4\lambda^2\,(3PR - Q^2)}}. \qquad (65)$$

If one utilizes the solution set given by Eqs. (65) and (58) into Eq. (59), dark solitons calculated by Eqs. (66) and (67) as shown in Box VI, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) > 0.$$

Singular solitons calculated by Eqs. (68) and (69) as shown in Box VII, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) > 0.$$

Bright solitons calculated by Eqs. (70) and (71) as shown in Box VIII, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) > 0.$$

Singular solitons calculated by Eqs. (72) and (73) as shown in Box IX, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) > 0.$$

Combo singular solitons calculated by Eqs. (74) and (75) as shown in Box X, with:

$$(a-bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) > 0.$$

2.5. Exp-function expansion

The formal solution of Eq. (11) is taken to be:

$$U(\vartheta) = \sum_{i=0}^{N} A_i \left\{exp\left(-V(\vartheta)\right)\right\}^i, \qquad (76)$$

where the coefficients A_i are constants to be designated later, such that $A_N \neq 0$, N is the balancing integer, and also $V(\vartheta)$ satisfies:

$$V'(\vartheta) = exp\left(-V(\vartheta)\right) + S\ exp\left(V(\vartheta)\right) + R, \qquad (77)$$

where S and R are constants. Eq. (77) has the following strategic solutions:

$$V(\vartheta) = \text{In}\left[-\frac{R}{2S} - \frac{\sqrt{\mu}}{2S}\tanh\left(\frac{\sqrt{\mu}}{2}(\vartheta + \vartheta_0)\right)\right],$$

$$S \neq 0, \quad \mu > 0,$$

$$u(x,t) = \left\{ \begin{array}{l} -\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{2\lambda} \pm \sqrt{\frac{\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)^2}{\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)^2}{4\lambda^2}} \\ \times \tanh^2\left(\sqrt{\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a-bp)}}\left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right) \end{array} \right\} \times e^{i(-kx+wt+\zeta)}, \qquad (66)$$

$$v(x,t) = \left\{ \begin{array}{l} -\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{2\lambda} \pm \sqrt{\frac{\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)^2}{\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)^2}{4\lambda^2}} \\ \times \tanh^2\left(\sqrt{\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a-bp)}}\left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right) \end{array} \right\} \times e^{2i(-kx+wt+\zeta)}. \qquad (67)$$

Box VI

$$u(x,t)=\left\{\begin{array}{l} -\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \times\coth^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right) \end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{68}$$

$$v(x,t)=\left\{\begin{array}{l} -\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \times\coth^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right) \end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{69}$$

Box VII

$$u(x,t)=\left\{\begin{array}{l} -\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \times\mathrm{sech}^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right) \end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{70}$$

$$v(x,t)=\left\{\begin{array}{l} -\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}} \\ \times\mathrm{sech}^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right) \end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{71}$$

Box VIII

$$u(x,t)=\left\{\begin{array}{l}-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \times\text{csch}^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{72}$$

$$v(x,t)=\left\{\begin{array}{l}-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \times\text{csch}^2\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{4\,(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{73}$$

Box IX

$$u(x,t)=\left\{\begin{array}{l}-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{\lambda^2}}\\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \times\left\{\begin{array}{l}\coth\left(\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)}{-bp+a}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\\ \pm\text{csch}\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{-bp+a}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\end{array}\right\}^2\end{array}\right\}$$
$$\times e^{i(-kx+wt+\zeta)}, \tag{74}$$

$$v(x,t)=\left\{\begin{array}{l}-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{2\lambda}\pm\sqrt{\frac{\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{\lambda^2}}\\ \pm\sqrt{\frac{9\left(2\,ak^2-2\,bkw+\alpha\,k+w\right)^2}{4\lambda^2}}\\ \times\left\{\begin{array}{l}\coth\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{-bp+a}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\\ \pm\text{csch}\left(\sqrt{\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{-bp+a}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\end{array}\right\}^2\end{array}\right\}$$
$$\times e^{2i(-kx+wt+\zeta)}. \tag{75}$$

Box X

$$V(\vartheta) = \text{In}\left[-\frac{R}{2S} - \frac{\sqrt{\mu}}{2S}\coth\left(\frac{\sqrt{\mu}}{2}(\vartheta+\vartheta_0)\right)\right],$$

$S \neq 0, \quad \mu > 0,$

$$V(\vartheta) = \text{In}\left[-\frac{R}{2S} + \frac{\sqrt{-\mu}}{2S}\tan\left(\frac{\sqrt{-\mu}}{2}(\vartheta+\vartheta_0)\right)\right],$$

$S \neq 0, \quad \mu < 0,$

$$V(\vartheta) = \text{In}\left[-\frac{R}{2S} - \frac{\sqrt{-\mu}}{2S}\cot\left(\frac{\sqrt{-\mu}}{2}(\vartheta+\vartheta_0)\right)\right],$$

$$S \neq 0, \quad \mu < 0, \tag{78}$$

where $\mu = R^2 - 4S$ and ϑ_0 are arbitrary real constants. Next, Eq. (76) can be rewritten as:

$$U_j = A_0 + A_1 exp(-V(\vartheta)) + A_2 exp(-2V(\vartheta)), \tag{79}$$

by virtue of the balance principle applied in Eq. (11). Then, the revealed equations are as follows:

$$-6\,b\eta^2 pA_2 + 6\,a\eta^2 A_2 + \lambda\,{A_2}^2 = 0, \tag{80}$$

$$-10\,Rb\eta^2 pA_2 + 10\,Ra\eta^2 A_2 - 2\,b\eta^2 pA_1 + 2\,a\eta^2 A_1 + 2\,\lambda\,A_1 A_2 = 0, \tag{81}$$

$$-RSb\eta^2 pA_1 - 2\,S^2 b\eta^2 pA_2 + RSa\eta^2 A_1 + 2\,S^2 a\eta^2 A_2 + 2ak^2 A_0 - 2bkwA_0 + \alpha kA_0 + \lambda {A_0}^2 + wA_0 = 0, \tag{82}$$

$$-R^2 b\eta^2 pA_1 - 6\,RSb\eta^2 pA_2 + R^2 a\eta^2 A_1 + 6\,RSa\eta^2 A_2 - 2\,Sb\eta^2 pA_1 + 2\,Sa\eta^2 A_1 + 2\,ak^2 A_1 - 2\,bkwA_1 + \alpha\,kA_1 + 2\,\lambda\,A_0 A_1 + wA_1 = 0, \tag{83}$$

$$-4\,R^2 b\eta^2 pA_2 + 4\,R^2 a\eta^2 A_2 - 3\,Rb\eta^2 pA_1 - 8\,Sb\eta^2 pA_2 + 3\,Ra\eta^2 A_1 + 8\,Sa\eta^2 A_2 + 2\,ak^2 A_2 - 2\,bkwA_2 + \alpha\,kA_2 + 2\,\lambda\,A_0 A_2 + \lambda\,{A_1}^2 + wA_2 = 0, \tag{84}$$

by putting Eq. (79) along with Eq. (77) in Eq. (11). So, from Eqs. (80)–(84) we have two results as follows:

– **Result 1:**

$$R = \frac{A_1}{A_2}, \quad S = \frac{A_0}{A_2}, \quad \eta = \pm\sqrt{-\frac{\lambda\,A_2}{6\,(a-bp)}},$$

$$w = \frac{12\,ak^2 A_2 + 6\,\alpha\,kA_2 + 4\,\lambda\,A_0 A_2 - \lambda\,{A_1}^2}{6A_2\,(2\,bk-1)},$$

and,

$$\mu = R^2 - 4S = -\frac{4\,A_0 A_2 - {A_1}^2}{{A_2}^2}. \tag{85}$$

If one employs Eq. (85) along with Eq. (78) in Eq. (79), singular solitons calculated by Eqs. (86) and (87) as shown in Box XI, with:

$$A_2\,(a-bp)\left(4\,A_0 A_2 - {A_1}^2\right)\lambda > 0.$$

Dark solitons calculated by Eqs. (88) and (89) as shown in Box XII, with:

$$A_2\,(a-bp)\left(4\,A_0 A_2 - {A_1}^2\right)\lambda > 0.$$

– **Result 2:**

$$R = \frac{A_1}{A_2}, \quad S = \frac{6\,A_0 A_2 - {A_1}^2}{2{A_2}^2}, \quad \eta = \pm\sqrt{-\frac{\lambda\,A_2}{6\,(a-bp)}},$$

$$w = \frac{4\,ak^2 A_2 + 2\,\alpha\,kA_2 - 4\,\lambda\,A_0 A_2 + \lambda\,{A_1}^2}{2A_2\,(2\,bk-1)},$$

and,

$$\mu = R^2 - 4S = -\frac{3\left(4\,A_0 A_2 - {A_1}^2\right)}{{A_2}^2}. \tag{90}$$

If one uses Eq. (90) along with Eq. (78) in Eq. (79), singular solitons calculated by Eq. (91) and (92) as shown in Box XIII, with:

$$A_2\,(a-bp)\left(4\,A_0 A_2 - {A_1}^2\right)\lambda > 0.$$

Dark solitons calculated by Eq. (93) and (94) as shown in Box XIV, with:

$$A_2\,(a-bp)\left(4\,A_0 A_2 - {A_1}^2\right)\lambda > 0.$$

2.6. Total equation

The solution of Eq. (11) is introduced as below:

$$(U')^2 = J(U) = \sum_{i=0}^{N} \mu_i U^i, \tag{95}$$

where μ_is are constants to be detected, $\mu_N \neq 0$, and N is the balancing integer. Now, rewriting Eq. (95) in the integral form:

$$\pm(\vartheta - \vartheta_0) = \int \frac{dU}{\sqrt{\sum\limits_{i=0}^{N}\mu_i U^i}}. \tag{96}$$

Next, Eq. (95) can be rewritten as:

$$(U')^2 = \mu_0 + \mu_1 U + \mu_2 U^2 + \mu_3 U^3, \tag{97}$$

by virtue of the balance principle applied in Eq. (11). Then, the equations are exracted as: U^2 coeff.:

$$u(x,t)=\left\{\begin{array}{l} A_0-\dfrac{2A_1A_0}{A_1+\sqrt{A_1{}^2-4\,A_0A_2}\tanh\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)} \\ +\dfrac{4A_2A_0^2}{\left(A_1+\sqrt{A_1{}^2-4\,A_0A_2}\tanh\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2}\end{array}\right\}$$
$$\times e^{i\left(-kx+\frac{12\,ak^2A_2+6\,\alpha\,kA_2+4\,\lambda\,A_0A_2-\lambda\,A_1{}^2}{6A_2(2\,bk-1)}t+\zeta\right)}, \tag{86}$$

$$v(x,t)=\left\{\begin{array}{l} A_0-\dfrac{2A_1A_0}{A_1+\sqrt{A_1{}^2-4\,A_0A_2}\tanh\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)} \\ +\dfrac{4A_2A_0^2}{\left(A_1+\sqrt{A_1{}^2-4\,A_0A_2}\tanh\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2}\end{array}\right\}$$
$$\times e^{2i\left(-kx+\frac{12\,ak^2A_2+6\,\alpha\,kA_2+4\,\lambda\,A_0A_2-\lambda\,A_1{}^2}{6A_2(2\,bk-1)}t+\zeta\right)}. \tag{87}$$

Box XI

$$u(x,t)=\left\{\begin{array}{l} A_0-\dfrac{2A_1A_0}{A_1+\sqrt{A_1{}^2-4\,A_0A_2}\coth\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)} \\ +\dfrac{4A_2A_0^2}{\left(A_1+\sqrt{A_1{}^2-4\,A_0A_2}\coth\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2}\end{array}\right\}$$
$$\times e^{i\left(-kx+\frac{12\,ak^2A_2+6\,\alpha\,kA_2+4\,\lambda\,A_0A_2-\lambda\,A_1{}^2}{6A_2(2\,bk-1)}t+\zeta\right)}, \tag{88}$$

$$v(x,t)=\left\{\begin{array}{l} A_0-\dfrac{2A_1A_0}{A_1+\sqrt{A_1{}^2-4\,A_0A_2}\coth\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)} \\ +\dfrac{4A_2A_0^2}{\left(A_1+\sqrt{A_1{}^2-4\,A_0A_2}\coth\left(\sqrt{\dfrac{\left(4\,A_0A_2-A_1{}^2\right)\lambda}{24A_2(a-bp)}}\left(x-\dfrac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2}\end{array}\right\}$$
$$\times e^{2i\left(-kx+\frac{12\,ak^2A_2+6\,\alpha\,kA_2+4\,\lambda\,A_0A_2-\lambda\,A_1{}^2}{6A_2(2\,bk-1)}t+\zeta\right)}. \tag{89}$$

Box XII

$$u(x,t)=\left\{\begin{array}{l} A_0-\dfrac{A_1\left(6A_0A_2-A_1{}^2\right)}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\tanh\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)} \\ \\ +\dfrac{\left(6A_0A_2-A_1{}^2\right)^2}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\tanh\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2} \end{array}\right\}$$
$$\times e^{i\left(-kx+\frac{4ak^2A_2+2\alpha kA_2-4\lambda A_0A_2+\lambda A_1{}^2}{2A_2(2bk-1)}t+\zeta\right)}, \tag{91}$$

$$v(x,t)=\left\{\begin{array}{l} A_0-\dfrac{A_1\left(6A_0A_2-A_1{}^2\right)}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\tanh\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)} \\ \\ +\dfrac{\left(6A_0A_2-A_1{}^2\right)^2}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\tanh\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2} \end{array}\right\}$$
$$\times e^{2i\left(-kx+\frac{4ak^2A_2+2\alpha kA_2-4\lambda A_0A_2+\lambda A_1{}^2}{2A_2(2bk-1)}t+\zeta\right)}. \tag{92}$$

Box XIII

$$u(x,t)=\left\{\begin{array}{l} A_0-\dfrac{A_1\left(6A_0A_2-A_1{}^2\right)}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\coth\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)} \\ +\dfrac{\left(6A_0A_2-A_1{}^2\right)^2}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\coth\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2} \end{array}\right\}$$
$$\times e^{i\left(-kx+\frac{4ak^2A_2+2\alpha kA_2-4\lambda A_0A_2+\lambda A_1{}^2}{2A_2(2bk-1)}t+\zeta\right)}, \tag{93}$$

$$v(x,t)=\left\{\begin{array}{l} A_0-\dfrac{A_1\left(6A_0A_2-A_1{}^2\right)}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\coth\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)} \\ +\dfrac{\left(6A_0A_2-A_1{}^2\right)^2}{A_2\left(A_1+\sqrt{3(A_1{}^2-4A_0A_2)}\coth\left(\sqrt{\frac{\left(4A_0A_2-A_1{}^2\right)\lambda}{8A_2(a-bp)}}\left(x-\frac{4ka-2wb+\alpha}{2kb-1}t\right)\right)\right)^2} \end{array}\right\}$$
$$\times e^{2i\left(-kx+\frac{4ak^2A_2+2\alpha kA_2-4\lambda A_0A_2+\lambda A_1{}^2}{2A_2(2bk-1)}t+\zeta\right)}. \tag{94}$$

Box XIV

$$3\eta^2 (a-pb)\mu_3 + 2\lambda = 0, \tag{98}$$

U coeff.:

$$\eta^2 (a-pb)\mu_2 + \left(2ak^2 - 2bkw + \alpha k + w\right) = 0, \tag{99}$$

U^0 coeff.:

$$\eta^2 (a-pb)\mu_1 = 0, \tag{100}$$

by substituting Eq. (97) into Eq. (11). So, from Eqs. (98)–(100) we have:

$$\mu_1 = 0, \qquad \mu_2 = -\frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2 (a-pb)},$$

$$\mu_3 = -\frac{2\lambda}{3\eta^2 (a-pb)}. \tag{101}$$

If we use the results Eq. (101) in Eq. (96), we get:

$$\pm(\vartheta - \vartheta_0) = \int \frac{dU}{\sqrt{\mu_0 - \frac{2ak^2 - 2bkw + \alpha k + w}{\eta^2 (a-pb)} U^2 - \frac{2\lambda}{3\eta^2 (a-pb)} U^3}}. \tag{102}$$

Lastly, optical solitons with quadratic nonlinearity are recovered using $\mu_0 = 0$ in Eq. (102). Bright solitons are:

$$u(x,t) = -\frac{3\left(2ak^2 - 2bkw + \alpha k + w\right)}{2\lambda} \operatorname{sech}^2\left[\sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-pb)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb-1} t\right)\right] e^{i(-kx+wt+\zeta)}, \tag{103}$$

$$v(x,t) = -\frac{3\left(2ak^2 - 2bkw + \alpha k + w\right)}{2\lambda} \operatorname{sech}^2\left[\sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-pb)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb-1} t\right)\right] e^{2i(-kx+wt+\zeta)}, \tag{104}$$

with:

$$(a-pb)\left(2ak^2 - 2bkw + \alpha k + w\right) < 0.$$

Singular solitons are:

$$u(x,t) = \frac{3\left(2ak^2 - 2bkw + \alpha k + w\right)}{2\lambda} \operatorname{csch}^2\left[\sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-pb)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb-1} t\right)\right] e^{i(-kx+wt+\zeta)}, \tag{105}$$

$$v(x,t) = \frac{3\left(2ak^2 - 2bkw + \alpha k + w\right)}{2\lambda} \operatorname{csch}^2\left[\sqrt{-\frac{2ak^2 - 2bkw + \alpha k + w}{4(a-pb)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb-1} t\right)\right] e^{2i(-kx+wt+\zeta)}, \tag{106}$$

with:

$$(a-pb)\left(2ak^2 - 2bkw + \alpha k + w\right) < 0.$$

2.7. Modified simple equation

Assume that Eq. (11) has the solution in the form as follows:

$$U(\vartheta) = \sum_{i=0}^{N} \mu_i \left(\frac{Q'(\vartheta)}{Q(\vartheta)}\right)^i, \tag{107}$$

where μ_is are constants to be detected, such that $\mu_N \neq 0$, and $Q(\vartheta)$ is an unknown function to be established later. Next, Eq. (107) can be rewritten as:

$$U(\vartheta) = \mu_0 + \mu_1 \left(\frac{Q'(\vartheta)}{Q(\vartheta)}\right) + \mu_2 \left(\frac{Q'(\vartheta)}{Q(\vartheta)}\right)^2, \tag{108}$$

By virtue of the balance principle applied in Eq. (11). Then, the strategic equations are:

Q^{-4} coeff.:

$$\mu_2 (Q')^4 \left(-6 b\eta^2 p + 6 a\eta^2 + \lambda \mu_2\right) = 0, \tag{109}$$

Q^{-3} coeff.:

$$2(Q')^2\left(\left(5 b\eta^2 p\mu_2 - 5 a\eta^2 \mu_2\right) Q'' + \left(-b\eta^2 p\mu_1 + a\eta^2 \mu_1 + \lambda \mu_1 \mu_2\right) Q'\right) = 0, \tag{110}$$

Q^{-2} coeff.:

$$(-2\,b\eta^2 p\mu_2 + 2\,a\eta^2\mu_2)Q'Q''' + \left(-2\,b\eta^2 p\mu_2 + 2\,a\eta^2\mu_2\right)(Q'')^2 + (3\,b\eta^2 p\mu_1 - 3\,a\eta^2\mu_1)Q'Q'' + (2\,ak^2\mu_2 - 2\,bkw\mu_2 + \alpha\,k\mu_2 + 2\,\lambda\,\mu_0\mu_2 + \lambda\,{\mu_1}^2 + w\mu_2)\,(Q')^2 = 0, \quad (111)$$

Q^{-1} coeff.:

$$\mu_1(\left(-b\eta^2 p + a\eta^2\right)Q'' + (2\,ak^2 - 2\,bkw + \alpha\,k + 2\,\lambda\,\mu_0 + w)Q') = 0, \quad (112)$$

Q^0 coeff.:

$$2\,ak^2\mu_0 - 2\,bkw\mu_0 + \alpha\,k\mu_0 + \lambda\,{\mu_0}^2 + w\mu_0 = 0, \quad (113)$$

by inserting Eq. (108) into Eq. (11). So, from Eqs. (109)–(113) we have:

$$\mu_0 = 0,$$

$$\mu_1 = \pm\sqrt{-\frac{36\eta^2\,(a - bp)\,(2\,ak^2 - 2\,bkw + \alpha\,k + w)}{\lambda^2}},$$

$$\mu_2 = -\frac{6\eta^2\,(a - bp)}{\lambda}, \quad (114)$$

and one gets:

$$Q'' = \pm\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a - bp)}}Q', \quad (115)$$

$$Q''' = -\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a - bp)}Q'. \quad (116)$$

If one employs the Eqs. (115) and (116), one reveals:

$$Q' = \pm\sqrt{-\frac{\eta^2\,(a - bp)}{2\,ak^2 - 2\,bkw + \alpha\,k + w}}\,k_1 e^{\pm\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a - bp)}}\vartheta}, \quad (117)$$

and,

$$Q = -\frac{\eta^2\,(a - bp)}{2\,ak^2 - 2\,bkw + \alpha\,k + w}\,k_1 e^{\pm\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a - bp)}}\vartheta} + k_2, \quad (118)$$

with k_1 and k_2 integration constants. Lastly, if one puts Eqs. (114), (117), (118) into Eq. (108), optical solitons with the quadratic nonlinearity are recovered by Eq. (119) and (120) as shown in Box XV, and also if we set:

$$k_1 = -\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2\,(a - bp)}\, e^{\pm\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{\eta^2(a - bp)}}\vartheta_0}, \quad k_2 = \pm 1. \quad (121)$$

In Eqs. (119) and (120), bright solitons are:

$$u(x,t) = -\frac{3\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)}{2\lambda}\,\mathrm{sech}^2\left[\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a - bp)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right] e^{i(-kx + wt + \zeta)}, \quad (122)$$

$$v(x,t) = -\frac{3\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)}{2\lambda}\,\mathrm{sech}^2\left[\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a - bp)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right] e^{2i(-kx + wt + \zeta)}, \quad (123)$$

with:

$$(a - bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) < 0.$$

Singular solitons are as followes:

$$u(x,t) = \frac{3\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)}{2\lambda}\,\mathrm{csch}^2\left[\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a - bp)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right] e^{i(-kx + wt + \zeta)}, \quad (124)$$

$$v(x,t) = \frac{3\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right)}{2\lambda}\,\mathrm{csch}^2\left[\sqrt{-\frac{2\,ak^2 - 2\,bkw + \alpha\,k + w}{4\,(a - bp)}} \times \left(x - \frac{4ka - 2wb + \alpha}{2kb - 1}t\right)\right] e^{2i(-kx + wt + \zeta)}, \quad (125)$$

with:

$$(a - bp)\left(2\,ak^2 - 2\,bkw + \alpha\,k + w\right) < 0.$$

$$u(x,t)=\left\{\begin{array}{l}\pm\sqrt{-\frac{36\eta^2(a-bp)(2\,ak^2-2\,bkw+\alpha\,k+w)}{\lambda^2}}\\ \times\left(\frac{\pm\sqrt{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}+k_2}\right)\\ -\frac{6\eta^2(a-bp)}{\lambda}\\ \times\left(\frac{\pm\sqrt{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}+k_2}\right)^2\end{array}\right\}\times e^{i(-kx+wt+\zeta)}, \tag{119}$$

$$v(x,t)=\left\{\begin{array}{l}\pm\sqrt{-\frac{36\eta^2(a-bp)(2\,ak^2-2\,bkw+\alpha\,k+w)}{\lambda^2}}\\ \times\left(\frac{\pm\sqrt{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}+k_2}\right)\\ -\frac{6\eta^2(a-bp)}{\lambda}\\ \times\left(\frac{\pm\sqrt{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}}{-\frac{\eta^2(a-bp)}{2\,ak^2-2\,bkw+\alpha\,k+w}k_1e^{\pm\sqrt{-\frac{2\,ak^2-2\,bkw+\alpha\,k+w}{\eta^2(a-bp)}}\eta(x-pt)}+k_2}\right)^2\end{array}\right\}\times e^{2i(-kx+wt+\zeta)}. \tag{120}$$

Box XV

3. Conclusion

This paper revisited embedded solitons that was studied with $\chi^{(2)}$-nonlinear susceptibility. Several integration schemes revealed a wide range of solitons, and especially the combo solitons that are visible in thiswork are being reported for the first time in this paper. The wide spectrum of solitons based on these diverse integration schemes are summarized. This gives a visual perspective to the range of solitons that are available from the schemes.

These results, therefore, pave the way for additional pathways to venture. These would include studying the model with variational principles using these combo solitons or various additional forms of solitons. The conservation laws for the model are yet to be extracted. A wide variety of rich mathematical methodologies are available for implementation [12–25]. Thus, plentiful issues need to be addressed with the dynamics of embedded solitons. Such a lot is floating over the horizon!

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