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Separating bichromatic polylines by fixed-angle minimal triangles

M. Hosseinzadeh Moghaddam[a] **and A. Bagheri**[b,*]

a. *Department of Computer Engineering, North Tehran Branch, Islamic Azad University, Tehran, Iran.*
b. *Department of Computer Engineering, Amirkabir University of Technology, Tehran, Iran.*

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Abstract. Separation of the desired from undesired objects is one of the most important challenges in computational geometry. In this respect, this study tends to cover the desired objects in one or a couple of geometric shapes such that all of the desired objects are included in the covering shapes while excluding the undesired objects. To this end, this study investigated the separation of polylines by minimal triangles with a given fixed angle and using present $O(NlogN)$-time algorithm, where N is the number of all the desired and undesired polylines. A minimal triangle represents a triangle whose edges are tangential to the convex hull of the desired polylines. The motivation behind analyzing this separation problem lies in the need for separating the bichromatic objects that are modeled by polylines rather than points in real-life scenarios.

1. Introduction

Separation of points is one of the essential and practical problems in computational geometry. Two sets P and Q of blue and red points and a specific geometric shape in the plane were taken into consideration in this study. The blue points of P are to be covered by a specific geometric shape such that none of the red points of Q are covered. Different versions of the separation of objects have been defined with the main focus on the optimization of parameters such as perimeter, area, and number of edges of the separating shape. The polygon with the minimum number of edges is triangle which is considered as a separating shape in this paper. Separation of the points has different applications in facility location [1], VLSI layout design [2], image processing [3], data mining [4], computer graphics [5], and statistics and classification [6].

This study took into account a new version of the problem and presented an $O(NlogN)$-time algorithm to separate two bichromatic object sets called P and Q by fixed-angle minimal triangles which contain all objects of P and exclude all objects of Q. As mentioned earlier, these triangles are minimal; in other words, their three sides are tangential to the convex hull of P. The objects considered here are polylines. The motivation for investigating this separation problem lies in the need for separating the bichromatic objects which are modeled by polylines rather than the points in real-life scenarios. For this purpose, first, the algorithm needed for the lines was proposed and then, it was extended to polylines given in Section 3.6.

The four main steps of the algorithm are elaborated in Sections 3.2–3.5. Each section outlines one step, proves the involved lemmas, and gives its time complexity. In Section 3.2, Seara's [1] idea is considered in making a star-shaped structure to compute the separating triangles. In Section 3.3, the trajectory of $\theta - vertex$ of $\theta - triangles$ is taken into account.

*. *Corresponding author. Tel.: +98 21 64542742; Fax: +98 21 66495521; Orcid: 0000-0002-3542-7763*
E-mail addresses: mh.moghaddam@yahoo.com (M. Hosseinzadeh Moghaddam); ar_bagheri@aut.ac.ir (A. Bagheri)

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In Section 3.4, by using the third main step of the algorithm, the event points are computed. Finally, in Section 3.5, event points as well as the pseudo-code of the proposed algorithm are considered as a whole.

2. Related works

A fair number of studies have been conducted on different types of separators so far. For instance, in 1983, a linear-time algorithm was proposed for separation of the colored points by a line [7]. In 1986, O'Rourke et al. solved the problem of separating the points by a circle [8].

Edelsbrunner and Preparata [9] solved the above-mentioned problem using a convex polygon with the minimum number of edges. In addition, Fekete [10] remarked that separation of points by a general polygon with the minimum number of edges was an $NP-$ *complete* problem and an approximation algorithm was proposed based on polynomial time in [11]. In a study on the separation of points, separators in the form of strips and wedges were also explored [12]. Sarkar and Stojmenovic [13] considered the polygon separation problem and presented a parallel algorithm to construct a separating convex $k-gon$ with the smallest k. In 2002, Eckstein et al. [14] worked on the problem of finding an axis-aligned rectangle B such that $P \cap B = \phi$ and the cardinality of $Q \cap B$ was maximized. Liu and Nediak conducted a study on finding an axis-aligned rectangle B that could maximize the difference between points of Q and P inside B, i.e., $|B \cap Q| - |B \cap P|$ [15].

Demaine et al. [16] studied the separability of the two-point sets inside a polygon using chords or geodesic lines and provided necessary and sufficient conditions for the existence of a chord or geodesic path to separate the two sets. Sheikhi et al. studied the separability of the two-point sets in the case of an L-shaped separator with the orientation of θ [17]. Moslehi and Bagheri solved the problem of whether or not the two-point sets could be separated by two disjoint isothetic rectangles [18]. Sheikhi et al. investigated the separability of imprecise bichromatic points [19] and proposed several algorithms to obtain certain separators (which could separate the bichromatic points with a probability of 1), possible separators (which could separate the bichromatic points with non-zero probability), most likely separators (which could separate such points with maximum probability), and maximal separators (which could maximize the expected number of correctly classified points). Bandyapadhyay and Banik [20] considered a collection of geometric problems containing the points in two colors, referred to as bichromatic problems, and presented low polynomial time exact algorithms for those problems. Xue et al. [21] studied the linear separability problem for stochastic geometric objects under the well-known uni-point and multi-point uncertainty models. They assumed that $S = S_R \cup S_B$ was a given set of stochastic bichromatic points and $n = \min\{|S_R|, |S_B|\}$ and $N = \max\{|S_R|, |S_B|\}$ were defined. They also demonstrated that the separable-probability of S was computed within $O(nN^{d-1})$ time for $d \geq 3$ and $O(\min\{nN \log N, N^2\})$ time for $d = 2$, while the expected separation margin of S was computed in $O(nN^d)$ time for $d \geq 2$.

In 2018, Har-Peled and Jones [22] studied the separability of n points in the plane using the minimum number of lines to separate all its pairs from each other. They found that the minimum number of lines that could separate n points, which were randomly and uniformly picked, in the unit square, was calculated as $\theta(n^{2/3})$.

Bonnet and Lampis [23] worked on the separation of two bichromatic point sets using a minimum-size set of lines and separated them from each other. According to their findings, parameterized by the number of lines k in the solution, the problem was unlikely to be solved significantly faster than the brute-force $n^{O(k)}$-time algorithm, where n represents the total number of points. In addition, the problem of separation with a minimum-size set of axis-parallel lines can be solved in $O(9^{\beta})$ time (assuming that β is the smallest set).

Abrahamsen et al. [24] remarked that separation of groups of objects in the plane by the shortest fences was shown as $NP-hard$ for a case where the input included polygons in two colors with n corners in total and gave a randomized 4/3.1.2965-approximation algorithm for polygons and any number of colors.

Arkin et al. investigated the problem of separating cycle [25] and aimed to find a simple tour of minimum length that separated the two points of each pair to different sides. They proved the hardness of the problem and proposed some approximation algorithms in different settings. Misra et al. [26] worked on a special case of the separation problem in which the points were on a circle and demonstrated a polynomial-time algorithm for that case.

Acharyya et al. [27] worked on different problems regarding a set of bichromatic points on a plane and proposed some in-place algorithms to compute an arbitrarily oriented monochromatic rectangle of the maximum size of R^2 as well as an axis-parallel monochromatic cuboid of the maximum size of R^3.

Different versions of the separation of points based on other objects such as L-shape and well-covered rectangles were also suggested. Seara carried out a thorough study [1]. In addition, in 2017, Moslehi and Bagheri examined the separation of two sets of bichromatic points by a minimal triangle with a fixed angle [28] and solved an $O(n \log n)$-time algorithm. The current research aimed to examine the separation of polylines using a triangle and presented an algorithm relevant to $O(N \log N)$-time; in other words, the previous solutions for polylines were generalized.

3. Preliminaries and algorithm description

First, some rudimentary definitions should be taken into account. Section 3.1 shows that the separation of line segments is different from separation of points. Given that the vertices of separating the triangle points cannot be placed anywhere in the plane, Section 3.2 determines the area suitable for them, i.e., $\overline{A}$. Since the angle of the separating triangles is fixed, Section 3.3 specifies the trajectory of their corresponding vertex which is a set of arcs called $\theta - cloud$. Section 3.4 shows the topology of the separating triangles which may change at some event points. Section 3.5 shows how to handle these events and pseudo-code of the proposed algorithm. Finally, Section 3.6 extends the results to the polylines.

3.1. Separating blue and red lines by a triangle

Let $P = \{p_1\hat{p_1}, p_2\hat{p_2}, \cdots, p_n\hat{p_n}\}$ be a set of n blue line segments and $Q = \{q_1\hat{q_1}, q_2\hat{q_2}, \cdots, q_m\hat{q_m}\}$ a set of m red line segments. The objective here is to calculate the minimal triangles with a fixed angle such that each of them would include all members of P and exclude all members of Q.

First of all, make sure whether this problem can be reduced to a bichromatic point set separation problem. An important question may arise here: Is it possible to only consider the end points of the red and blue lines and distinguish these two-point sets to separate the red from the blue lines? According to the findings, the mentioned idea is wrong. As shown in Figure 1, the red-end points are separated from the blue ones by a triangle, yet they may contain some parts of the red lines.

From now on, the blue and red lines are shown by thin gray lines and thick black lines, respectively.

Since a triangle is convex and the convex hull of a collection of objects is defined as the minimum convex polygon that includes the relevant objects, the separating triangle of the relevant objects includes the convex hull of P, i.e., $CH(P)$. Of note, all three sides of the minimal separating triangle are tangential to $CH(P)$. If there is a triangular separator, $CH(P)$ will be monochromatic; otherwise, two sets of P and Q will be empty of any triangular separator resulting from the convexity of a triangle. Obviously, $CH(P)$ and its monochromatic state in $O(n \log n)$ time can be calculated. Then, we suppose $CH(P)$ as monochromatic. Figure 2 shows the convex hull of the blue lines rather than red lines.

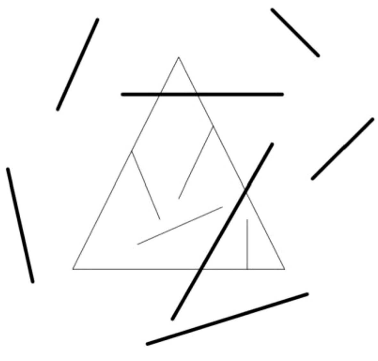

Figure 1. End points separated by a triangle while line segments not separated.

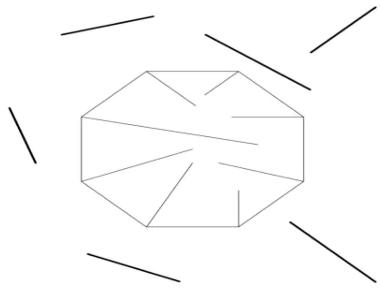

Figure 2. Separation of blue and red lines.

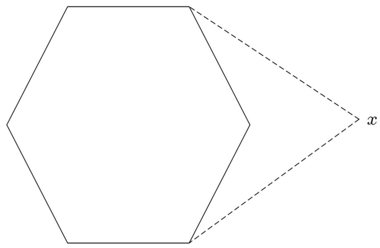

Figure 3. Supporting lines of a point.

3.2. Calculation of feasible area $\overline{A}$

This section gives three necessary definitions related to the subject under study.

Definition 1. Assume a convex polygon Y and a point x outside Y. Here, by the ***supporting lines*** of x to Y, we mean the two lines passing through x that are tangent to Y. For more information, refer to Figure 3. The supporting line can be calculated in $O(n \log n)$ time [29].

Definition 2. Assume a convex polygon Y and a line segment L outside Y. Here, by the ***supporting lines*** of L to Y, we mean the two common internal tangent lines of L and Y. For more information, refer to Figure 4.

Definition 3. The vertices of a separating triangle cannot lie in some areas defined by the red line segments as well as their extension of their supporting lines to $CH(P)$. Consider a red line segment $S_i = q_i\hat{q_i}$. The extension of its supporting lines defines the four

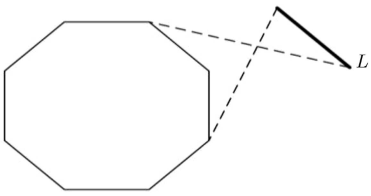

Figure 4. Supporting lines of a line.

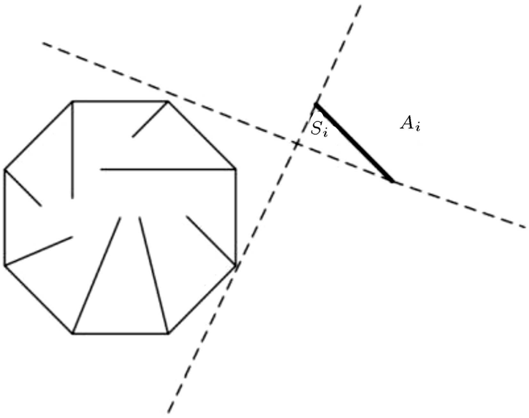

Figure 5. Forbidden area A_i.

areas in the plane where $CH(P)$ lies. The opposite area is called the ***forbidden area*** A_i of S_i.

The vertices of a separating triangle do not remain in the forbidden areas mainly because if the vertex of a separating triangle lies in the forbidden area, a part of the red line segment S_i will also lie in this area. In other words, if $A = \cup_1^m A_i$, the area allowed for vertices of a separating triangle is a complimentary area of A (see Figure 5).

The area $\overline{A}$ is star-shaped, often unlimited with borders where $CH(P)$ is located in its kernel [4]. The border of this forbidden area consists of line segments of the set Q and extension of their supporting lines (in [4], a similar idea is used) which can be calculated in $O(N \log N)$ time, where $N = n + m$ [1]. We denote it by σ.

3.3. Computing the trajectory of $\theta - vertex$ of $\theta - triangles$ around $CH(P)$

In this section, we have some definitions.

Definition 4. A $\theta - triangle$ is a triangle with an interior angle of θ.

Definition 5. The $\theta - vertex$ of a $\theta - triangle$ is a vertex such that the angle between the corresponding edges is θ.

Assume that in Figure 6, the two supporting lines to $CH(P)$ intersect at the point x, and the internal angle between them is θ. Once the tangency points of the supporting lines are fixed, the locus of the point x becomes a circular arc. In this regard, the $\theta - vertex$ of a separating $\theta - triangle$ lies at the trajectory τ. It

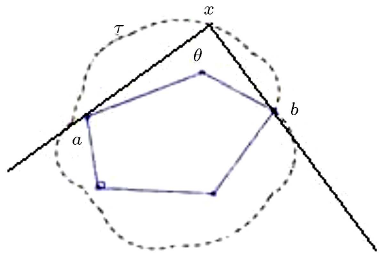

Figure 6. $\theta - cloud$ τ by circular arcs.

consists of the mentioned circular arcs and is obtained when the $\theta - vertex$ of a $\theta - triangle$ that is tangent to $CH(P)$ which rotates over a full 2π turn around $CH(P)$. For more details, see [30].

Definition 6. $\theta - cloud$ is the trajectory τ of the $\theta - vertex$ [31].

Lemma 1. The $\theta - cloud$ can be constructed from $CH(P)$ in $O(n)$ time and it contains $O(n)$ vertices and circular arcs [8].

3.4. Calculation of event points

Through determination of the $\theta - cloud$, the separating triangle is examined based on the position of the $\theta - vertex$ in τ. Clearly, the set of points belonging to τ inside the forbidden area is unusable, and it is not possible to locate the vertex of separating triangle in them. In this respect, τ is divided into intervals to determine whether or not a separating triangle is suitable for all points of the intervals, considering the position of the vertex at the points of the intervals. Two supporting lines were drawn from $\theta - vertex$ w to hit the first point in the forbidden area, considering the supporting lines as the directed vectors starting from w. The supporting line on the right side of which $CH(P)$ lies is called $l(w)$, and the other one is called $r(w)$. The first intersecting point of $l(w)$ with σ is called x (if it exists), and the first intersecting point of $r(w)$ with σ is called y (if it exists). The tangential lines of $CH(P)$ passing through x and y are called L and R, respectively (see Figure 7).

As shown in Lemma 3, the existence of the separating triangle depends on the external angle between L and R. In case the external angle between L and R is not less than 180°, the two sets P and Q of the bichromatic line segments can be separated by a minimal triangle. Once w on τ is rotating, two groups of important points called event points emerge. These event points divide τ into some intervals. The points belonging to an interval are characterized by the same properties regarding the existence of separating triangles. The two types of event points are as follows:

- **Event points of type 1:** This event occurs when R or L is changed by turning w on τ. While w moves

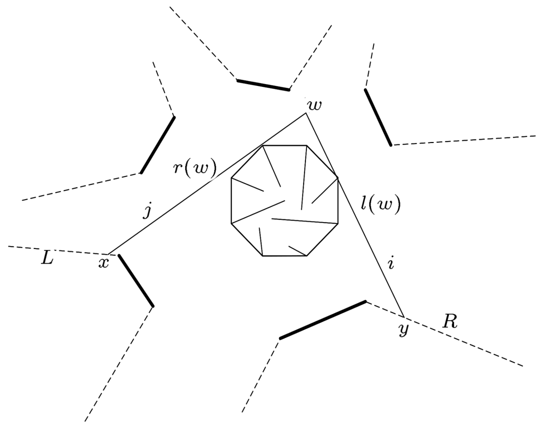

Figure 7. $l(w)$ and $r(w)$ intersecting two supporting lines.

on τ, R (followed by L) is changed after a change is made to the intersection point of the extension of $r(w)$ (followed by $l(w)$) with border σ;

- **Event points of type 2:** They are formed by intersection of τ and σ.

In order to trace the changes of L and R and test the external angle between L and R during the rotation of w on τ, it is required to consider the intersecting points of $r(w)$ and $l(w)$ with σ. Given the two intersection points on the border of the forbidden area which consists of line segments of set Q as well as the extension of their supporting lines and unlimited areas, six cases are to occur which are equal to $\binom{3+2-1}{2}$ (including two combinations of the three parts of the forbidden area with repetition).

Case A. The tangential lines $l(w)$ and $r(w)$ hit the border, σ, of the forbidden area at the extensions of some line segments of the set Q. The parameters used in Figure 7 are defined as follows:

w	The vertex of a separating triangle with the angle θ
$l(w)$	Tangential line to $CH(P)$ from the point w, on the right side of which $CH(P)$ lies
$r(w)$	Tangential line to $CH(P)$ from the point w, on the left side of which $CH(P)$ lies
L	The first line segment from the border which does not belong to Q and is extended from one of the supporting lines of one of red line segments. The extension of $l(w)$ creates an intersection at point x on the right side of which $CH(P)$ lies and L is tangential to $CH(P)$ by extension
R	The first line segment from border which does not belong to Q and is extended from one of the supporting lines of one of red line segments. The extension of $r(w)$ creates an intersection at point y, on the left side of which $CH(P)$ lies and R is tangential to $CH(P)$ by extension
i	The intersection point between L and $l(w)$
j	The intersection point between R and $r(w)$
x	The intersection point between L and $r(w)$
y	The intersection point between R and $l(w)$

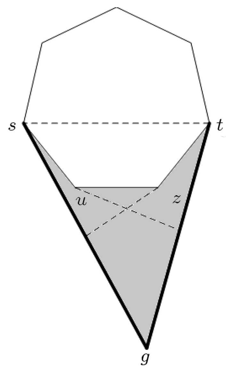

Figure 8. Induction judgment of Lemma 2.

The two edges of the hypothetical separating triangle, which is one of its vertices, are located on the extended lines of $l(w)$ and $r(w)$. The third side (in front of w) is located on the line segment $l(w)$ at one end and on the line segment $r(w)$ at the other end.

Lemma 2. Consider a convex polygon Y with n edges, a point g outside it, and two supporting lines of g and their tangency points s and t. For each point h inside the bounded area by Y and two supporting lines of g (the gray area in Figure 8), the extensions of the supporting lines of point h intersect "both" two line segments gs and gt.

Proof by Induction method. In this section, we first assumed that the polygon under study is a triangle (for $n = 3$) to examine the induction basis. In this regard, if we draw the supporting lines from a point such as h inside the bounded area (sgt) to any selected point, both s and t will be tangential to the same points. In addition, as h gets closer to st than to g, the angle $\angle sht$ will then be larger than the angle $\angle sgt$ and the supporting lines will intersect both gs and gt by extension. More details are given in Figure 9. In

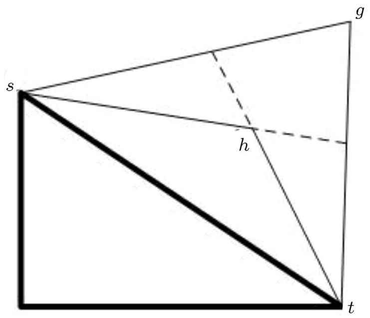

Figure 9. Induction basis of Lemma 2.

case the whole triangle is within the triangle hst, the claim is effective too.

Induction hypothesis: It is assumed to be true for $n = k$;

Induction judgment: The aforementioned statement was proved to be true for $n = k + 1$.

Consider a convex polygon with $k + 1$ edges and draw two tangential lines from a point outside it, similar to g in Figure 8 where the tangency points are denoted by s and t. In case the two points are connected to each other, a convex polygon whose edges are less than $k+1$ will be obtained. A convex polygon is divided into two polygons by drawing one of its chords. If the connecting line, st, is located on one side of the polygon, it will be the same as our induction basis, which has been already discussed. Consequently, the presupposition judgment will be proved effective for the polygon. As a result, the supporting lines of any point inside the area restricted by the main polygon with $k+1$ edges as well as the tangential lines of hu and hz will intersect both line segments of gs and gt by extension; hence, the assumption for a polygon with $k + 1$ edges will be effectively true. □

In this study, L and R are referred to as tw directed vectors such that $CH(P)$ lies on the right side of R and on the left side of L. Both external and internal angles between R and L are defined as a standard way, as shown in Figure 10.

Lemma 3. Two colored sets of P and Q are separable

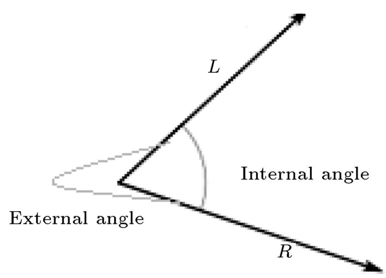

Figure 10. Internal and external angles.

by the $\theta - triangle$ if and only if the external angle, followed by the extensions of L and R, is not less than 180°.

In order to prove the lemma mentioned above, it should be considered that when the external angle of R and L is less than 180°, any line segment which intersects $l(w)$ and $r(w)$ will overlap $CH(P)$. As a result, there will be no separator triangle. Further, it should be proved that there is no red line in the restricted area by $CH(P)$, $l(w)$, $r(w)$, and xi and yj line segments.

Proof by reductio ad absurdum. According to our absurd hypothesis, there is at least one red point in the mentioned area. According to Lemma 2 and given that $l(w)$ and xi line segments are tangential to $CH(P)$, it can be concluded that in case there is a red point in the restricted area by $CH(P)$ as well as the mentioned line segments, the extensions of the supporting lines will intersect xi and $l(w)$ so that $l(w)$ hits the border σ before L line segment at another point, which is contrary to our basic hypothesis. Consequently, the absurd hypothesis was proved false while the judgment was effective. This claim is also true for yj and $r(w)$ line segments. As illustrated in Figure 7, there is no red line in the restricted area by $CH(P)$, $l(w)$, $r(w)$, xi, and yj line segments mainly because if there is any red line, its supporting line will intersect $l(w)$ and $r(w)$ which is contradictory to our basic hypothesis based on the fact that L and R are the first lines from the border σ where $l(w)$ and $r(w)$ will intersect by extension. As a result, the absurd hypothesis is proved false while the judgment is effective. In addition, any line that intersects both wx and wy can be the third side of a separating triangle if and only if the external angle of L and R is not less than 180°. In this respect, the third side does not overlap $CH(P)$ and there is no red line inside the formed separator triangle.□

Case B. In this case, one of the tangential lines, either $l(w)$ or $r(w)$, intersects σ at one of the red line segments from the set Q, while the other tangential line intersects a supporting line (of a red line segment). The parameters used in Figure 11 are defined as follows:

R	One of the supporting lines of the first line segment from border σ which belongs to Q and intersects $r(w)$ by extension such that $CH(P)$ is located on its right and will be located in front of w by extension
x	The intersection point between $l(w)$ and red line which belongs to Q

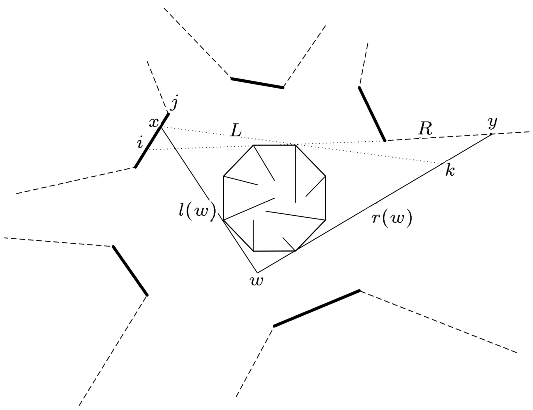

Figure 11. $l(w)$ intersecting a red line segment and $r(w)$ intersects a supporting line.

- y — The intersection point between $r(w)$ and a supporting line from the border
- L — A line passing through x which is tangential to $CH(P)$ such that $CH(P)$ is located on its left
- k — The intersection point between $r(w)$ and extension of L
- j — The end point of the red line segment opposite to w
- i — The intersection point between the red line and the other line extended by R

In this case, Lemma 3 is effective, too.

Proof. Unlike Case A, $l(w)$ intersects one of red lines by extension and any points in the red lines have their own specific L and R. In addition, unlike a supporting line extension where all the involved points have a unique line tangential to $CH(P)$, the red line has a specific tangential line to each point of it. Then, the condition of the third side of the separator triangle depends on the point where $l(w)$ or $r(w)$ hits the red line by extension.

In Figure 11, the external angle between the line passing through x (i.e., L) and R determines whether or not the third side of a separator triangle can be formed. Given that $r(w)$ in this case has already hit the supporting line by extension and all the involved points have a fixed R value, similar to Case A for L and R, the red line can be divided into two parts based on the extension of R and calculation of its intersection point with the red line. According to the external angle of R and considering the relevant L of each point after point i, which is the intersection point between R and red line, it can be concluded that the third side of the separating triangle can be formed without overlapping the $CH(P)$ area. Therefore, according to Lemma 3 in Case A, in case the intersection point of x is located on the line ij, the third side of the separating triangle will end in $l(w)$ and $r(w)$ so that it would not overlap $CH(P)$; hence, there is no red line in the area restricted by $CH(P)$, yi, and xk. In addition, in case there is no intersection point between R and red line, it lies on one side of R. However, if it lies on the right side of R, then there is a separating triangle for any x on the red line, and if it lies on the left side of R, then there is no separating triangle. □

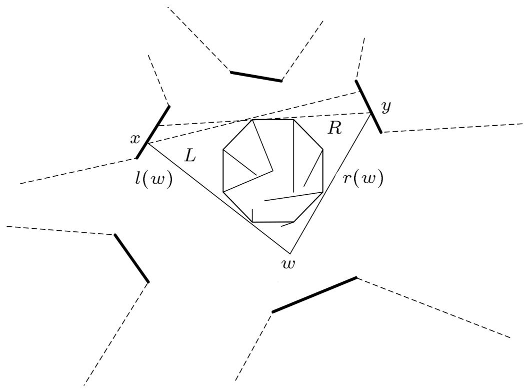

Figure 12. $l(w)$ and $r(w)$ intersecting two red line segments.

Case C. In the third case, the tangential lines to $CH(P)$, which cross over w, meet the red lines which belong to Q while hitting the border σ, (see Figure 12).

In this condition, both $l(w)$ and $r(w)$ intersect the red lines by extension, and each point on the red lines has specific L and R. Under this condition, the tangential lines must be drawn from x and y to $CH(P)$ to determine whether or not a separating triangle can be formed. If the external angle is more than 180°, there will be a separating triangle; otherwise, no separating triangle will be formed. Given that there are an infinite number of points on these two red lines and each point must be examined separately, it can be suggested that by turn of around τ, the red lines are divided into a number of intervals so that all points at the intervals are characterized by a unique feature (Figure 13).

As shown in Figure 13, if we consider the intersection point of $l(w)$ with the red line L_1 as f_2, which is fixed, and want to examine the position of the intersection point of $r(w)$ with red line L_2 as f_1 in order for a separating triangle to be formed, by drawing a tangential line to $CH(P)$ from f_2, we can define its intersection point with the red line L_2 as f_3. For a separating triangle to be formed, the external angle of L and R must not be less than 180°; thus, the other end of the third side of the triangle must be located at the interval from f_3 to point (a_1, b_1).

It can now be claimed that in general, when f_3 reaches f_1, a separating triangle is formed. Therefore, by assuming F_1 as a distance from f_1 to the beginning point of L_2 and F_3 as distance from f_3 to point (a_1, b_1),

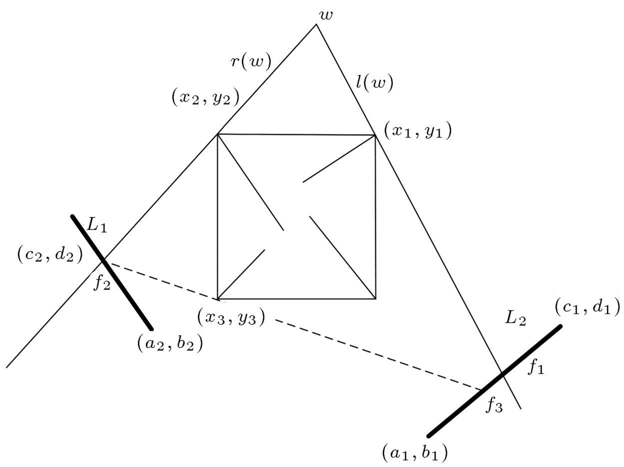

Figure 13. Turn of w around $\theta - cloud\ \tau$.

we can decide whether a triangle is formed by $F_1 - F_3$ so that by turn of w, when the amount is not positive, the separating triangle is formed. Through the beginning of a line segment, we mean the endpoint with the lower y coordinate.

Keeping the issue in mind, we assume an arc from τ around which w is turned and is related to a circle with radius one and the center of coordinate origin. Then, by applying polar coordinates of w and intersecting red lines, $l(w)$ and $r(w)$, we achieve Eq. (1) through simple high-school calculations:

$$F_1 - F_3 = \left(\frac{A_1 * \sin(\phi) + A_2 * \cos(\phi) + A_3}{\sin(\phi) + A_4 * \cos(\phi) + A_5}\right) - \left(\frac{A_6 * \sin(\phi) + A_7 * \cos(\phi) + A_8}{A_9 * \sin(\phi) + A_{10} * \cos(\phi) + A_{11}}\right). \quad (1)$$

For all: $0 \leq \theta \leq 2\pi$ and $A_i s$ as constant coefficients.

Once the phrase given above is calculated and the variable $x = \sin\theta$ is changed using the algebraic identity:

$$\sin(\theta)^2 + \cos(\theta)^2 = 1,$$

we can obtain a polynomial with utmost eight roots. Therefore, the function intersects the x-axis at utmost eight points. Here, the function mentioned above cannot be considered positive at these eight points; therefore, the red line segments are divided into eight intervals at most in a way that every point at this interval has the same L and R. Therefore, it can be stated that Lemma 3 is effective.

Case D. In the fourth case, the extension of one of the tangential lines to $CH(P)$ does not hit w crossing the border σ and the other one hits a side of the border σ which does not belong to Q. The parameters used in Figure 14 are defined as follows:

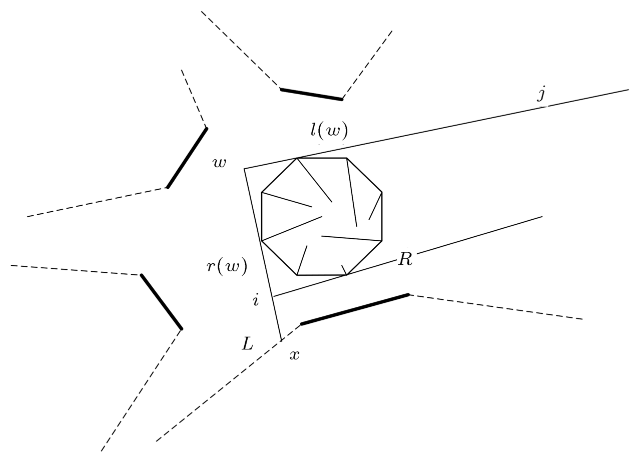

Figure 14. $r(w)$ intersecting a supporting line and $l(w)$ does not hit border.

- L The first line segment from the border σ which does not belong to Q; instead, it is an extension of one of the supporting lines of a red line segment where the extension of $l(w)$ creates an intersection at x in a way that $CH(P)$ is located on its right
- x The intersection point between $l(w)$ and L
- R A line parallel to $r(w)$ and tangential to $CH(P)$ in a way that R and $r(w)$ are located on the opposite sides of $CH(P)$
- j The intersection point of a line extended by a tangential line to $CH(P)$ which passes through x and $r(w)$
- i The intersection point between $l(w)$ and R

Lemma 4. For each convex polygon such as Y and both parallel lines including R and L which are tangential to Y and any point such as g in the area restricted by R and L and outside Y, if we draw two tangential lines to Y which cross over g, the lines will intersect both R and L by extension (see Figure 15).

Proof by induction method. According to the findings, for $n = 3$, the assumption is true when the convex polygon is a triangle, as shown in Figure 16.

Induction hypothesis: We assume the judgment to be true for $n = k$;

Induction judgment: In addition, the judgment was proved to be true for $n = k + 1$.

Given that the polygon is convex, if we connect points

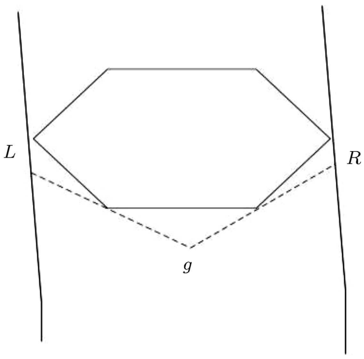

Figure 15. Tangential lines crossing over g intersecting both R and L.

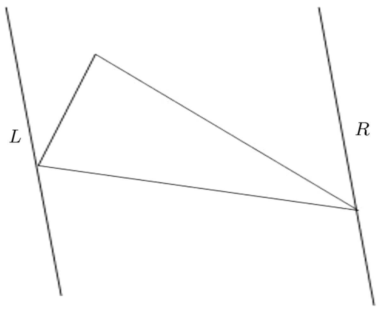

Figure 16. Induction basis of Lemma 4.

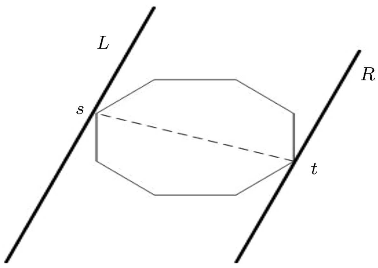

Figure 17. Induction judgment of Lemma 4.

s and t, a convex polygon is obtained; and if we draw a chord in a convex polygon, it will be divided into two convex polygons whose edges are less than $n+1$. As a result, the judgment can be proven effective for this polygon according to our induction judgment. Therefore, the extensions of both supporting lines of each point inside the area restricted by the main polygon and parallel tangential lines intersect both parallel tangential lines; hence, judgment is found effective in the case of a polygon with $n+1$ edges (Figure 17). □

In this case, Lemma 3 is also effective as proved in the following.

Proof by reductio ad absurdum. Similar to the previous sections, if the external angle of R and L is less than 180°, then any line segment that ends in $l(w)$ and $r(w)$ will overlap; hence, no triangle will be formed. Of note, there is also no red line on the area restricted by $CH(P)$ and $l(w)$ and line segment xj in addition to the area restricted by $CH(P)$, $r(w)$, and R. In this regard, both Lemmas 2 and 4 were considered to prove this statement.

According to Lemma 2, $l(w)$ and xj are tangential to $CH(P)$; therefore, if there is even one red point inside the restricted area, its supporting lines will intersect both $l(w)$ and xj by extension. This is contrary to our basic hypothesis where L is the first line segment from the border which is intersected by $l(w)$. Then, the absurd hypothesis is false and the judgment is effective. □

According to Lemma 4, given that R and $r(w)$ are parallel and tangential to $CH(P)$, if there is even one red point inside the restricted area, its supporting lines will intersect R and $r(w)$ by extension. This is contrary to our basic hypothesis where -$r(w)$ does not intersect with any border in the forbidden area. Then, the absurd hypothesis is false and the judgment is effective. As a result, no red line is found inside the area restricted by $r(w)$, $CH(P)$, and R.

Case E. In the fifth case, two tangential lines to $CH(P)$, which cross over w, do not hit border by extension. The parameters used in Figure 18 are defined as follows:

L	A line parallel to $l(w)$ and tangential to $CH(P)$ in a way that L and $l(w)$ are located on opposite sides of $CH(P)$
R	A line parallel to $r(w)$ and tangential to $CH(P)$ in a way that R and $r(w)$ are both located on sides of $CH(P)$
i	The intersection point between $l(w)$ and extension of R
j	The intersection point between $r(w)$ and extension of L

In this case, Lemma 2 can be effective.

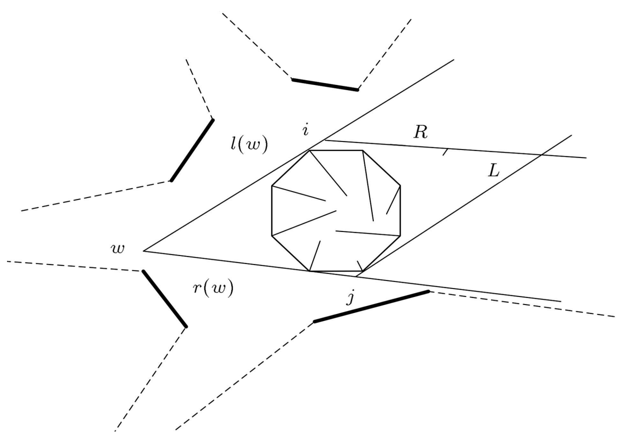

Figure 18. $l(w)$ and $r(w)$ do not intersect the border.

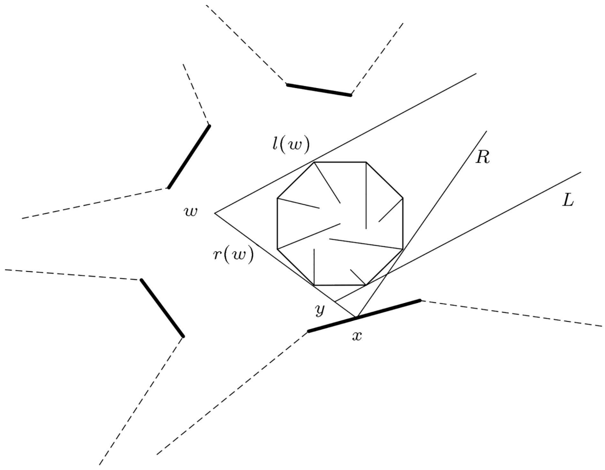

Figure 19. $r(w)$ intersects a red line segment and $l(w)$ does not hit the border.

Proof. Similar to the previous part and according to Lemma 4, Lemma 3 holds. Only the definition of R and L is different, which has no effect on Lemma 3. □

Case F. In the sixth case, one of the tangential lines to $CH(P)$ which cross over w does not hit the border by extension, while the other tangential lines to $CH(P)$ hit a side of border which belongs to Q. The parameters used in Figure 19 are defined as follows:

x	The intersection point between $r(w)$ and red line
R	The tangential line to $CH(P)$ while crossing x
L	A line parallel to $l(w)$ and tangential to $CH(P)$ in a way that $l(w)$ and L are located on opposite sides of $CH(P)$
y	The intersection point between $r(w)$ and the extension of L.

In this this case, Lemma 3 is effective, too.

Proof. Similar to the Case B, $r(w)$ hits one of the red lines by extension, and each point in the red lines has its own specific L or R mainly because unlike the supporting line where all points have a unique tangential line to $CH(P)$, any red line has a specific tangential line depending on its different points. In this respect, the formation of the third side of the triangle is defined based on the point hit by $r(w)$. The external angle between L and tangential line crossing R determines whether or not the third side of the separating triangle is formed. In this case, the extension of $l(w)$ does not hit the forbidden area where all points have a fixed L value. If y exists and $y \neq x$, then the third side of the separating triangle can be formed without overlapping $CH(P)$. In accordance with the previous lemmas, there is no red line in the area restricted by $CH(P)$, $l(w)$, $r(w)$, l and the tangential line to $CH(P)$ crossing x. □

The event points can be examined based on all of possible cases relevant to the extension of $l(w)$ and $r(w)$. Given that the external angle between the lines extended by R and L determines whether or not the set of line segments is separable by $\theta - triangle$ when w turns around $CH(P)$ at τ, all points have the same condition: There is either $\theta - triangle$ for each of them or no $\theta - triangle$ for any of them as long as R and L do not change; in other words, the sides of border σ which $l(w)$ and $r(w)$ intersect do not change. Depending on where the extensions of $r(w)$ and $l(w)$ lie, L and R are updated and in each case, it is determined whether or not there is any separating triangle.

Therefore, τ is divided into a set of intervals which makes it feasible to examine one point at the interval to decide whether a separating $\theta - triangle$ can be formed.

Based on the previous assumptions, we can define two events as:

The event of *R*, *L* change. This event occurs when R or L changes by turning w on τ. While w moves on τ, R (followed by L) is changed when the intersection point of the extension of $r(w)$ (followed by $l(w)$) with border σ is changed, which occurs in six Cases A to F.

Based on previous considerations, some points in τ at which R and L are changed are reported as the event points. However, if the intersection point of the extension of $l(w)$ and $r(w)$ with border σ is located in the third case, the red line will be divided into 8 intervals at most so that each interval has the same R and L. Then, 8 events are reported.

The event of departure. This event occurs when σ and τ are intersected. For each point in τ which is located inside the forbidden area A, there is not any $\theta - triangle$. Then, the intersection point between τ and σ is reported as the point of departure.

Lemma 5. Given two sets of blue and red lines P and Q of total size N, the total number of events of Types 1 and 2 is $O(N)$ and can be calculated in $O(N)$ time.

Proof. The total number of events is corresponding to the number of the sides of σ. The sides consist of red lines and supporting lines and as we draw two supporting lines for each red line, the total number of the sides is at most 3 times the number of red lines. Then, the total number of events is $O(N)$. When we extend any side of σ, it may intersect τ at many points. The last intersection point of the extension of each side of σ with τ is a point related to event of Type 1. The intersection point of each side of σ with τ is related to the event of departure. We remind that σ is star shaped and $CH(P)$ is like its kernel. We can define sort of order for the sides of σ. The intersection point of σ and τ obeys the order, too. As we have already calculated the order of sides of σ, all points can be reported in $O(N)$ time. □

3.5. Handling the events

Based on the previous considerations, we need to determine whether a separating $\theta - triangle$ is formed for an arbitrary point except the end-points at each interval and generalize the results to all points of the interval. If, for any intervals of τ, no separating triangle is formed, then the set of lines is not separable by $\theta - triangle$. We know that all of event points are saved in event queue depending on their order in τ. While the event queue is not empty, we can put them out of the queue in order to process each.

The event of departure. It is clear that if the two sides related to a concave vertex from σ hit τ, two events of departure can be determined. These two consecutive sides specify two consecutive events of departure. There is no $\theta - triangle$ between the two events. Therefore, none of the events between the two events mentioned are processed.

The event of *R*, *L*. When we encounter this event, we update R, L as to decide whether a separating $\theta - triangle$ is formed for each of the points between the present event point and the next one. It must be noted that these two events are processed in $O(1)$ time.

The algorithm. In this section, the pseudo-code of our algorithm, as a whole, is given regarding the previous sections (see Algorithm 1). The algorithm has four main steps. First of all, the monochromatic state of $CH(P)$ is checked. If $CH(P)$ is not monochromatic, there is no separating triangle; otherwise, we need to measure the star-shaped structure and the $\theta - cloud$ around $CH(P)$ as in Sections 3.2 and 3.3. In Section 3.4, as the third main step, some points in τ, at which R and L are changed or σ and τ intersect, are reported as the event points. Finally, we handle those event points, as in Section 3.5.

Theorem 1. The algorithm Report_Minimal_ Separating_Triangles takes $O(N \log N)$ time to specify $\theta - triangle$ separability of two given bichromatic sets of lines P and Q of total size N and give a report on all of the separating triangles.

Proof. We examine all of the algorithm steps to analyze the time complexity of the algorithm. In Step 1, computing the convex hull of P and its monochromatic state takes $O(N \log N)$ time and in Steps 2 to 6, we must gain a star-shaped polygon in order to examine the separability of lines with $\theta - triangle$ in $O(N \log N)$ time [1] and also, to calculate τ in $O(N \log N)$ time [8]. We can specify two events of change and departure by σ and τ. According to Lemma 3, all events are calculated in $O(N)$ time. Given that the number of events is $O(N)$, we can enlist the events in order in $O(N \log N)$ time. We know that each event is processed within a fixed time. As there are $O(N)$ events, all events need $O(N)$ time to be processed. Therefore, the algorithm is carried out in $O(N \log N)$ time.

3.6. Extending the results to polylines

Let P be a set of blue polylines with n blue segments and Q be a set of red polylines with m red segments. Within $O(n \log m)$ time, it is possible to whether $CH(P)$ is monochromatic is not; thus, the problem of triangle separability for polylines is easily reduced to the problem of the triangle separability for line segments described in Section 3.1. If $CH(P)$ is

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Input: A finite set P of blue line segments and a finite set Q of red line segments in the plane and an angle $0 < \theta < \pi$.

Output: All combinatorially different minimal triangles containing all line segments of P, avoiding any line segment of Q and having an angle θ.

1. Compute the convex hull of P. If $CH(P)$ is not monochromatic, we don't have any separating triangle and return empty.
2. Compute the star-shaped structure and denote it by σ (refer to sec. 4).
3. Compute the $\theta - cloud$ around $CH(P)$ and denote it by τ (refer to sec. 5).
4. Let w be an arbitrary point on τ and maintain $r(w), l(w), L, R, x, y, i$ and j as defined in sec. 5.
5. Move the apex w clockwise along τ and collect all of two types of event points described in section. 6.
6. For each event point computed in 5, handle it according to sec. 7 and store the separating $\theta - triangle$ between each pair of consecutive points.

Algorithm 1. Report, minimal, separating, triangles.

monochromatic, consider the set of line segments of the polylines of P and Q and run the algorithm described in Section 3.5. This can be done at $O(N \log N)$ where $N = n + m$.

4. Conclusion

This study solved the problem of separating two sets of polylines in a plane with minimal triangles. The proposed algorithm reports all combinatorially different minimal separating $\theta - triangles$ of two sets P and Q of segments and polylines and it ran in $O(N \log N)$ time. An interesting problem that can be studied is finding the minimal rectangles.

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Biographies

Mohammad Hosseinzadeh Moghaddam received his BS degree in Applied Mathematics from Islamic Azad University at Tabriz and his MS degree in Computer Engineering from Islamic Azad University, Qazvin. Currently, he is a PhD candidate in Software Engineering at North Tehran branch of Islamic Azad University. His research interests include computational geometry, graph algorithms, and image processing.

Alireza Bagheri received his BS and MS degrees in Computer Engineering from Sharif University of Technology, Tehran. He received his PhD degree in Computer Science from Amirkabir University of Technology (Tehran Polytechnic), Tehran. Currently, he is an Associate Professor at the Computer Engineering Department, Amirkabir University of Technology (Tehran Polytechnic), Tehran. His research interests include computational geometry, graph algorithms, and social network analysis.