A new alternative unit-Lindley distribution with increasing failure rate

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Abstract

In this paper, a new one-parameter distribution is proposed by unitizing the Lindley distribution through the hyperbolic tangent transformation. The goal is to map the functionality of the Lindley distribution on the unit interval, with the perspective of offering a new modeling option for treating unit data. In the first part, we provide the motivations and some mathematical properties of the new distribution. Two truncated moments and hazard rate functions are used to characterize the distribution. The emphasis is then switched to its statistical characteristics. Several methods are used to discuss the point estimation of the parameter. The related bias and mean squared error behavior is tested using Monte Carlo simulations for a range of sample sizes. To demonstrate the ability of the model to fit real data, distributional analyses are given.

Keywords: Characterization, data analysis, hyperbolic tangent function, Lindley distribution, point estimation, unit distribution, unit-Lindley distribution.

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1 Introduction

The Lindley distribution is introduced by [1] with the cumulative distribution function (cdf) and probability density function (pdf) specified by

$$F_L(x;\theta) = 1 - \left(1 + \frac{\theta x}{1+\theta}\right) e^{-\theta x},$$

and

$$f_L(x;\theta) = \frac{\theta^2}{1+\theta}\,(1+x)\,e^{-\theta x}, \quad x > 0,$$

respectively, where $\theta > 0$ is an adjustment parameter.

The Lindley distribution has attracted the attention of many researchers. In particular, statistical properties of the Lindley distribution are given in [2]. In addition, there are many

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extensions, generalizations, and discretizations of the Lindley distribution in the literature. Among them, we list the generalized Lindley distribution [3], exponentiated Lindley distribution [4], discrete Lindley distribution [5], binomial discrete Lindley distribution [6], extended Lindley distribution [7], exponentiated power Lindley distribution [8], and odd log logistic Lindley distribution [9].

We use the Lindley distribution for other purposes in our analysis, such as modeling bounded phenomena on the unit interval. Although the literature contains a large number of real data sets on the unit interval for proportion and percentage, the number of models remains small. The beta and Kumaraswamy distributions are well-known to derive models on the unit interval. These distributions, however, have been found to be inadequate in data modeling in some cases. Consequently, several authors suggest creating new unit distributions, especially by mapping lifetime distributions with well-known properties on the unit interval. Some of the most recent unit interval distributions that have been proposed are as follows: [10] obtained the unit-gamma distribution using the random variable transformation $X = \exp(-Y)$, where Y has the gamma distribution, [11] obtained the unit-Weibull distribution using the transformation $X = \exp(-Y)$, where Y has the Weibull distribution, [12] obtained the unit-Birnbaum-Saunders distribution using the transformation $X = \exp(-Y)$, where Y has the Birnbaum-Saunders distribution, [13] obtained the unit-inverse Gaussian distribution using the transformation $X = \exp(-Y)$, where Y has the inverse Gaussian distribution, [14] obtained the unit-Lindley distribution using the transformation $X = Y/(1+Y)$, where Y has the Lindley distribution and [15] obtained the unit-Lindley distribution using the transformation $X = 1/(1+Y)$, where Y has the Lindley distribution. Some other unit distributions obtained in recent years can be reffered to as follows: [16]-[26].

This paper intends to introduce a new unit distribution alternative to the current unit distributions in literature such as the beta distribution, Kumaraswamy distribution, unit-Lindley distribution introduced by [14] and unit-Weibull distribution introduced by [11]. It is different since it employs the hyperbolic tangent transformation, which is an understudied field of unit distributions. This technique, to our knowledge, appears only in [27] with complete success in terms of the implementations. The goal is to map the workability of the Lindley distribution in an original way for new viewpoints of applications dealing with unit data.

The following is how the paper is structured: The new unit distribution is proposed in Section 2, along with theoretical studies of its key functions. In Section 3, a range of distributional properties are discussed. In Section 4, we present characterizations based on two truncated moments as well as in terms of the hazard rate function. In Section 5, several different point estimation methods for the parameter are discussed. Simulation studies are carried out in Section 6 to compare the capability of these methods. The paper concludes with distribution modeling studies based on actual data in Section 7. Concluding remarks are given in Section 8.

2 A new unit-Lindley distribution

The following description is used to add another unit form of the Lindley distribution. If a random variable X has the probability density function $g(x)$, then the pdf of the random variable $Y = \tanh(X)$ is

$$f(y) = \frac{1}{1-y^2} g\left(\operatorname{arctanh}(y)\right),$$

for $y \in (0,1)$, and $f(y) = 0$ for $y \notin (0,1)$. For the sake of clarity, we recall that $\tanh(x) = \sinh(x)/\cosh(x) = (\exp(x) - \exp(-x))/(\exp(x) + \exp(-x))$ $\operatorname{arctanh}(x) = (1/2)\log[(1+x)/(1-x)]$, $x \in (-1,1)$, being the inverse hyperbolic tangent function. The goal is to combine the joint functionalities of the tanh transformation and Lindley distribution for new statistical perspectives on the unit interval, following [27] with the Weibull distribution as the baseline distribution.

Here, a random variable Y has a new unit-Lindley (NwUL) distribution with parameter θ if its pdf is given by

$$f(y;\theta) = \frac{1}{1-y^2} f_L\left(\operatorname{arctanh}(y);\theta\right) = \frac{\theta^2}{(\theta+1)\left(1-y^2\right)}\left(1+\operatorname{arctanh}(y)\right) e^{-\theta \operatorname{arctanh}(y)}, \tag{1}$$

for $y \in (0,1)$, and $f(y;\theta) = 0$ for $y \notin (0,1)$.

The corresponding cdf is provided by standard manipulations as

$$F(y;\theta) = 1 - \left(1 + \frac{\theta \operatorname{arctanh}(y)}{1+\theta}\right) e^{-\theta \operatorname{arctanh}(y)}, \tag{2}$$

with $F(y;\theta) = 0$ for $y \leq 0$ and $F(y;\theta) = 1$ for $y \geq 1$.

Observe that using the logarithmic expression of $\operatorname{arctanh}(x)$, the alternative form of the cdf of the NwUL distribution can also be obtained as

$$F(y;\theta) = 1 - \left[1 + \frac{\theta}{2(1+\theta)} \log\left(\frac{1+y}{1-y}\right)\right]\left(\frac{1-y}{1+y}\right)^{\theta/2},$$

where $y \in (0,1)$. In what follows, we will prefer the expression in (2) for some technical reasons.

The hazard rate function (hrf) is obtained as

$$h(y;\theta) = \frac{\theta^2\left(1+\operatorname{arctanh}(y)\right)}{\left(1-y^2\right)\left(1+\theta+\theta \operatorname{arctanh}(y)\right)}, \tag{3}$$

for $y \in (0,1)$, and $h(y;\theta) = 0$ for $y \notin (0,1)$. The notation $NwUL(\theta)$ will sometimes be used to denote the newly described distribution with parameter θ.

For $y \in (0,1)$, the first-order derivatives of the pdf and hrf are given, respectively, by

$$\frac{\partial f(y;\theta)}{\partial y} = \frac{\theta^2}{(1+\theta)(1-y^2)^2} e^{-\theta \operatorname{arctanh}(y)} \left\{1 - \theta + 2y + (2y-\theta)\operatorname{arctanh}(y)\right\} \tag{4}$$

and

$$\frac{\partial h(y;\theta)}{\partial y} = \frac{2\theta^2\left(y\theta\left[\operatorname{arctanh}(y)\right]^2 + y\left(2\theta+1\right)\operatorname{arctanh}(y) + 1/2 + y(1+\theta)\right)}{\left(1-y^2\right)^2\left(1+\theta+\theta \operatorname{arctanh}(y)\right)^2}. \tag{5}$$

It is clear that, since the first-order derivatives of the hrf is always positive for all $y \in (0,1)$ and $\theta > 0$, the hrf is an increasing function. On the other hand, analytically, the shapes of the pdf can be challenging to identify the signs of the Equation (4). So, we focus on the plots of the pdf and hrf for some selected parameter values θ to see their potential shapes.

From Figure 1, it is observed that the NwUL distribution can be increasing or decreasing as well as its hrf can be increasing shaped for all parameter values. Further, Figure 2 displays shape regions of the pdf for $\theta \in (0,5)$ interval. In view of Figure 2, according to numerical results, the pdf is increasing for $\theta \in (0,1]$, U-shaped for $\theta \in (1, 2.207]$, inverse N-shaped for $\theta \in (2.207, 2.264]$ and decreasing for $\theta > 2.264$.

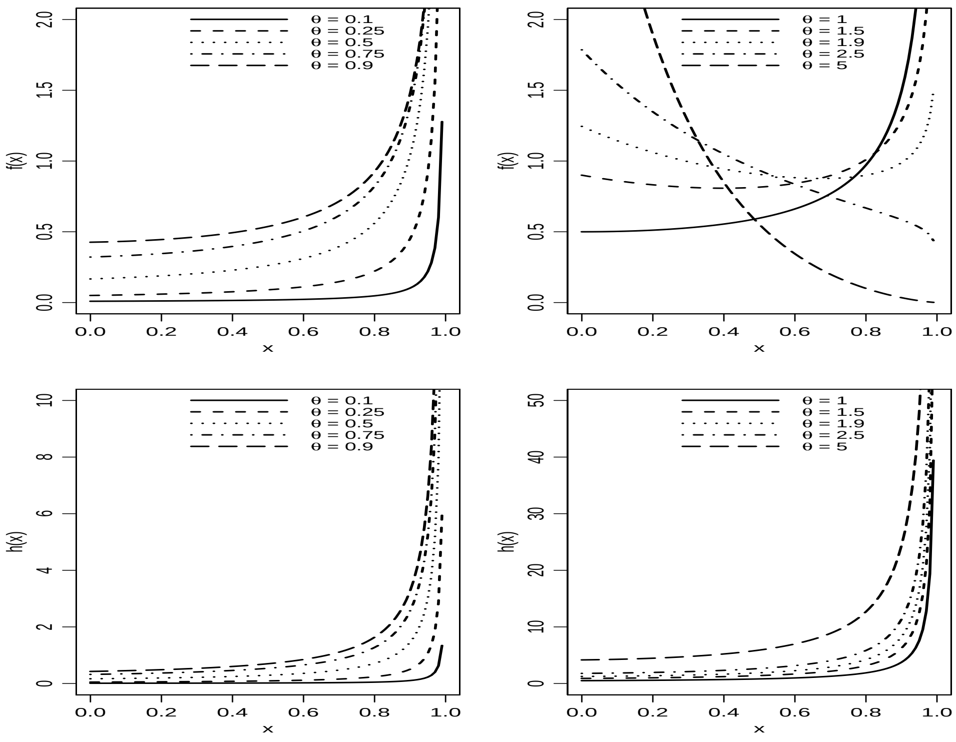

Figure 1: Possible pdf and hrf plots of the NwUL distribution

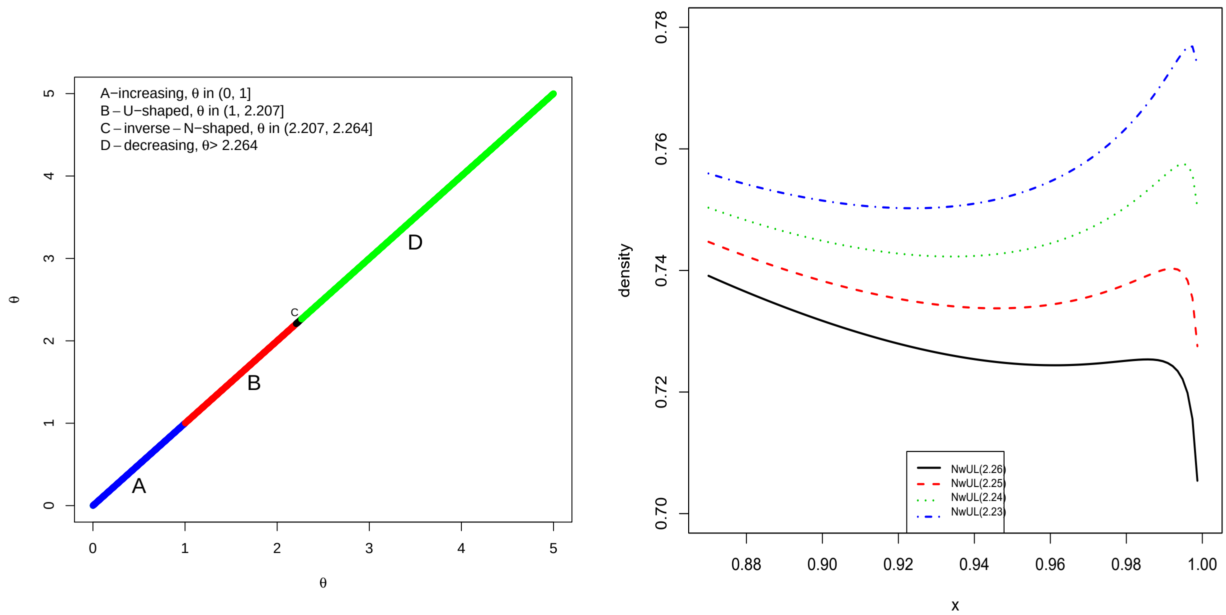

Figure 2: The pdf regions (left) and N-shaped pdf shapes (right) of the NwUL distribution

3 Technical results

This section compiles a list of technical results that the NwUL distribution has generated. The first makes use of the stochastic ordering principle. The aim of stochastic ordering in distribution theory is to compare the action of random variables, we refer to [28] for more detail on stochastic orderings in various applications. Likelihood ratio ordering is shortly defined as follows: We say that a random variable X is larger than another random variable Y in the likelihood ratio order sense, i.e., $Y \leq_{lr} X$, if the ratio of the corresponding pdfs given as $f_X(x)/f_Y(x)$ is an increasing function with respect to x over the union of the supports of X and Y.

In the context of the NwUL distribution, the following stochastic ordering holds.

Theorem 1. *If $X \sim NwUL(\theta_1)$ and $Y \sim NwUL(\theta_2)$ and $\theta_2 > \theta_1$, then $Y \leq_{lr} X$.*

Proof. For any $y \in (0,1)$, the ratio of the corresponding pdfs is given by

$$g(y) = \frac{\theta_1^2\,(1+\theta_2)}{\theta_2^2\,(1+\theta_1)} e^{(\theta_2-\theta_1)\,\mathrm{arctanh}(y)}.$$

Since the function $\mathrm{arctanh}(y)$ is increasing for $y \in (0,1)$, by composition, $g(y)$ is an increasing function with respect to y if and only if $\theta_2 > \theta_1$. This completes the proof. □

The quantile function is described in the next result as an essential function of the NwUL distribution.

Theorem 2. *The quantile function of the NwUL distribution is given as*

$$Q(u;\theta) = \tanh\left[-\frac{1}{\theta} - 1 - \frac{1}{\theta} W_{-1}\left((1+\theta)(u-1)e^{-(1+\theta)}\right)\right], \quad u \in (0,1),$$

where $W_{-1}(x)$ refers to the negative branch of the classic Lambert function.

Proof. The quantile function of the NwUL distribution is obtained by the inverse of $F(y;\theta)$. Now, we recall a random variable Y with the NwUL distribution can be written as $Y = \tanh(X)$, where X denotes a random variable with the Lindley distribution with parameter θ. We also recall that the quantile function of the Lindley distribution is specified by $Q_L(u;\theta) = -(1/\theta) - 1 - (1/\theta)W_{-1}\left((1+\theta)(u-1)e^{-(1+\theta)}\right)$, $u \in (0,1)$ (see [29]). Therefore, the following equivalence holds: For any $u \in (0,1)$, we have

$$\begin{aligned} &F(y;\theta) = u \;\Leftrightarrow\; P(Y \leq y) = u \;\Leftrightarrow\; P(\tanh(X) \leq y) = u \;\Leftrightarrow\; P(X \leq \mathrm{arctanh}(y)) = u \\ &\;\Leftrightarrow\; \mathrm{arctanh}(y) = Q_L(u;\theta) \;\Leftrightarrow\; y = \tanh[Q_L(u;\theta)]. \end{aligned}$$

This implies the desired result. □

As usual, the quartile of the NwUL distribution are obtained by setting $u = 1/4$, $u = 1/2$ and $u = 3/4$, respectively. It also allows to define various measures of asymmetry and kurtosis, and generates numerical values from the NwUL distribution.

The following result concerns the ordinary moments of the NwUL distribution, focusing on simple inequalities and a possible series expression.

Theorem 3. *The s^{th} ordinary moment of the NwUL distribution exists and satisfies the following inequalities:*

$$0 \leq m_s \leq \min\left[1, \frac{s!(\theta+s+1)}{\theta^s(\theta+1)}\right].$$

Moreover, m_s can be expressed as

$$m_s = \frac{\theta^2}{1+\theta} \sum_{k=0}^{s} \sum_{\ell=0}^{+\infty} \binom{s}{k} \binom{-k}{\ell} (-2)^k \frac{1+\theta+2(k+\ell)}{[\theta+2(k+\ell)]^2}.$$

Proof. We recall that a random variable Y with the NwUL distribution can be written as $Y = \tanh(X)$, where X denotes a random variable with the Lindley distribution with parameter θ. Therefore, we can write

$$m_s = E(Y^s) = E([\tanh(X)]^s).$$

It follows from the standard inequalities: $0 \leq \tanh(x) \leq \min(x,1)$ for $x \geq 0$ and the expression of the s^{th} ordinary moment of X reported in [2] that

$$0 \leq m_s \leq \min\left[1, E(X^s)\right] = \min\left[1, \frac{s!(\theta+s+1)}{\theta^s(\theta+1)}\right].$$

Now, for any $x > 0$, by applying the binomial formula, we have

$$[\tanh(x)]^s = \left(1 - 2\frac{e^{-2x}}{1+e^{-2x}}\right)^s = \sum_{k=0}^{s} \binom{s}{k} (-2)^k e^{-2kx}(1+e^{-2x})^{-k}$$
$$= \sum_{k=0}^{s} \sum_{\ell=0}^{+\infty} \binom{s}{k} \binom{-k}{\ell} (-2)^k e^{-2(k+\ell)x}.$$

Therefore, by virtue of the dominated convergence theorem, we obtain

$$m_s = \sum_{k=0}^{s} \sum_{\ell=0}^{+\infty} \binom{s}{k} \binom{-k}{\ell} (-2)^k E(e^{-2(k+\ell)X}).$$

Now, using the Lindley distribution, it follows that

$$E(e^{-2(k+\ell)X}) = \int_0^{+\infty} e^{-2(k+\ell)x} f_L(x;\theta)dx = \frac{\theta^2}{1+\theta} \int_0^{+\infty} (1+x)\, e^{-[\theta+2(k+\ell)]x} dx$$
$$= \frac{\theta^2}{1+\theta} \times \frac{1+\theta+2(k+\ell)}{[\theta+2(k+\ell)]^2}.$$

By combining the above equalities, the proof is completed. □

From the ordinary moments, we can derive the mean and variance of the NwUL distribution, given as m_1 and $V = m_2 - m_1^2$. Also, the s^{th} global coefficient is defined by

$$G_s = V^{-s/2} \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} m_k m_1^{s-k}.$$

The coefficient of skewness of the NwUL distribution is given as G_3, and the coefficient of kurtosis by G_4. We calculate these measurements based on the selected values of θ in Table 1.

Table 1: Mean, variance, coefficients of skewness and kurtosis for different values of parameter

θ	m_1	V	G_3	G_4
0.1	0.9907	0.0044	-9.6489	105.3227
0.25	0.9540	0.0212	-4.1135	20.6124
0.5	0.8694	0.0541	-2.0567	6.3139
0.75	0.7817	0.0785	-1.2592	3.3565
1	0.7014	0.0921	-0.7938	2.3073
1.5	0.5700	0.0974	-0.2240	1.7191
1.7855	0.5108	0.0936	0.0006	1.7063
2	0.4725	0.0893	0.1420	1.7649
2.23	0.4364	0.0839	0.2747	1.8682
3	0.3438	0.0658	0.6213	2.3722
4	0.2660	0.0472	0.9340	3.1450
5	0.2151	0.0345	1.1553	3.8990

We can deduce from this table that the NwUL distribution can model various types of data sets in terms of skewness and kurtosis.

The incomplete moments of the NwUL distribution are now discussed.

Theorem 4. *For any $t \in [0,1]$, the s^{th} incomplete moment of the NwUL distribution at t is obtained as*

$$m_s(t) = \frac{\theta^2}{1+\theta}\sum_{k=0}^{s}\sum_{\ell=0}^{+\infty}\binom{s}{k}\binom{-k}{\ell}(-2)^k\frac{1+\theta+2(k+\ell)}{[\theta+2(k+\ell)]^2}\times$$
$$\left[1-\left(1+\frac{[\theta+2(k+\ell)]\operatorname{arctanh}(t)}{1+\theta+2(k+\ell)}\right)e^{-[\theta+2(k+\ell)]\operatorname{arctanh}(t)}\right].$$

Proof. By following the first lines of the Proof of Theorem 3, we obtain

$$\begin{aligned} m_s(t) &= E[Y^s I(Y\le t)] = E\left\{[\tanh(X)]^s I(X\le \operatorname{arctanh}(t))\right\} \\ &= \sum_{k=0}^{s}\sum_{\ell=0}^{+\infty}\binom{s}{k}\binom{-k}{\ell}(-2)^k E\left[e^{-2(k+\ell)X}I(X\le \operatorname{arctanh}(t))\right]. \end{aligned}$$

Now, we have

$$\begin{aligned} E\left[e^{-2(k+\ell)X}I(X\le \operatorname{arctanh}(t))\right] &= \int_0^{\operatorname{arctanh}(t)} e^{-2(k+\ell)x}f_L(x;\theta)dx \\ &= \frac{\theta^2}{1+\theta}\int_0^{\operatorname{arctanh}(t)}(1+x)\,e^{-[\theta+2(k+\ell)]x}dx \\ &= \frac{\theta^2}{1+\theta}\times\frac{1+\theta+2(k+\ell)}{[\theta+2(k+\ell)]^2}F_L(\operatorname{arctanh}(t);\theta+2(k+\ell)) \\ &= \frac{\theta^2}{1+\theta}\times\frac{1+\theta+2(k+\ell)}{[\theta+2(k+\ell)]^2}\left[1-\left(1+\frac{[\theta+2(k+\ell)]\operatorname{arctanh}(t)}{1+\theta+2(k+\ell)}\right)e^{-[\theta+2(k+\ell)]\operatorname{arctanh}(t)}\right]. \end{aligned}$$

By combining the above equalities, we end the proof. □

Taking $s = 1$, for example, yields the first incomplete moment of the NwUL distribution. We may extract various mean deviations and essential curves such as the Lorenz and Bonferroni curves used in various income analysis.

The stress-strength reliability measure of the NwUL distribution is presented below.

Theorem 5. *If $X \sim NwUL(\theta_1)$ and $Y \sim NwUL(\theta_2)$ with X and Y independent. Then*

$$R = P(Y < X) = 1 - \frac{\theta_1^2[\theta_1(\theta_1+1) + \theta_2(\theta_1+1)(\theta_1+3) + \theta_2^2(2\theta_2+3) + \theta_2^3]}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^3}.$$

Proof. Since the function $\tanh(x)$ is increasing, writing X and Y as $X = \tanh(A)$ and $Y = \tanh(B)$, where A and B denote two independent random variables with the Lindley distribution with parameters θ_1 and θ_2, respectively, we obtain

$$\begin{aligned} R &= P(Y < X) = P(\tanh(B) < \tanh(A)) = P(B < A) \\ &= \int_0^{+\infty} F_L(x;\theta_2) f_L(x;\theta_1) dx \\ &= \frac{\theta_2^2[\theta_2(\theta_2+1) + \theta_1(\theta_2+1)(\theta_2+3) + \theta_1^2(2\theta_1+3) + \theta_1^3]}{(\theta_1+1)(\theta_2+1)(\theta_1+\theta_2)^3}. \end{aligned}$$

The detail of the calculus can be found in [30]. This ends the proof. □

We now present some results on the order statistics of the NwUL distribution. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the NwUL distribution and $Y_{(1)} \leq Y_{(2)} \ldots \leq Y_{(n)}$ denote the corresponding order statistics. Adopting general notations, the cdf and pdf of the r^{th} order statistic, that is $Y_{(r)}$, are easily given by

$$F_{Y_{(r)}}(y;\theta) = \sum_{i=r}^{n} \binom{n}{i} F(y;\theta)^i \left\{1 - F(y;\theta)\right\}^{n-i} = \sum_{i=r}^{n}\sum_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} F(y;\theta)^{i+j}$$

and

$$\begin{aligned} f_{Y_{(r)}}(y;\theta) &= \frac{1}{B(r, n-r+1)} F(y;\theta)^{r-1} \left\{1 - F(y;\theta)\right\}^{n-r} f(y;\theta) \\ &= \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} F(y;\theta)^{r+i-1} f(y;\theta), \end{aligned}$$

respectively, where $B(\cdot,\cdot)$ is the beta function and $r = 1, 2, \ldots, n$. So, the cdf and pdf of the r^{th} order statistic of the NwUL distribution are given also by

$$F_{Y_{(r)}}(y;\theta) = \sum_{i=r}^{n}\sum_{j=0}^{n-i} (-1)^j \binom{n}{i}\binom{n-i}{j} \left\{1 - \left[1 + \frac{\theta \operatorname{arctanh}(y)}{1+\theta}\right] e^{-\theta \operatorname{arctanh}(y)}\right\}^{i+j}$$

and

$$\begin{aligned} f_{Y_{(r)}}(y;\theta) &= \frac{\theta^2 \left(1 + \operatorname{arctanh}(y)\right) e^{-\theta \operatorname{arctanh}(y)}}{(\theta+1)\left(1-y^2\right) B\left(r, n-r+1\right)} \\ &\quad \times \sum_{i=0}^{n-r} (-1)^i \binom{n-r}{i} \left\{1 - \left[1 + \frac{\theta \operatorname{arctanh}(y)}{1+\theta}\right] e^{-\theta \operatorname{arctanh}(y)}\right\}^{r+i-1}. \end{aligned}$$

It is clear that for $r = 1$ and $r = n$, the cdf and pdf of $Y_{(1)} = \min\{Y_1, Y_2, \ldots, Y_n\}$ and $Y_{(n)} = \max\{Y_1, Y_2, \ldots, Y_n\}$ are obtained, respectively.

4 Characterizations

To understand the behavior of data obtained from a particular experiment, its approximate probability distribution can be described. However, this necessitates the creation of conditions that control the selected probability distribution.

This section is dedicated to the characterizations of the NwUL distribution based on: (i) a simple relationship between two truncated moments; (ii) the hrf and (iii) conditional expectation of a single function of the random variable.

4.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of the NwUL distribution in terms of a simple relationship between two truncated moments. We will employ [31, Theorem 1]. As shown in [32], this characterization is stable in the sense of weak convergence.

Proposition 1. *Let $Y : \Omega \to (0,1)$ be a continuous random variable and let*

$$q_1(y) = 1, \quad q_2(y) = e^{-\theta \operatorname{arctanh}(y)}$$

for $y \in (0,1)$. Then Y has the pdf specified in (1) if and only if the function $\eta(x)$ defined in [31, Theorem 1] is of the form

$$\eta(y) = \frac{1}{2} e^{-\theta \operatorname{arctanh}(y)}, \quad y \in (0,1).$$

Proof. If Y has the pdf given by (1), then

$$(1 - F(y;\theta)) E[q_1(Y)|Y \geq y] = \left(1 + \frac{\theta \operatorname{arctanh}(y)}{1+\theta}\right) e^{-\theta \operatorname{arctanh}(y)}, \quad y \in (0,1)$$

and

$$(1 - F(y;\theta)) E[q_2(Y)\ |Y \geq y] = \frac{1}{2}\left(1 + \frac{\theta \operatorname{arctanh}(y)}{1+\theta}\right) e^{-2\theta \operatorname{arctanh}(y)}, \quad y \in (0,1),$$

and hence

$$\eta(y) = \frac{1}{2} e^{-\theta \operatorname{arctanh}(y)}.$$

For any $y \in (0,1)$, we also have

$$\eta(y) q_1(y) - q_2(y) = -\frac{1}{2} e^{-\theta \operatorname{arctanh}(y)} < 0.$$

Conversely, if $\eta(y)$ is of the above form, then

$$s'(y) = \frac{\eta'(y) q_1(y)}{\eta(y) q_1(y) - q_2(y)} = \frac{\theta}{1 - y^2}, \quad y \in (0,1),$$

and

$$s(y) = \theta \operatorname{arctanh}(y).$$

Now, according to [31, Theorem 1], Y has the pdf specified in (1). $\square$

Corollary 1. *Suppose Y is a continuous random variable. Let $q_1(y)$ be as in Proposition 1. Then Y has the pdf (1) if and only if there exist functions $q_2(y)$ and $\eta(y)$ defined in [31, Theorem 1] for which the following first order differential equation holds*

$$\frac{\eta'(y)q_1(y)}{\eta(y)\,q_1(y)-q_2(y)}=\frac{\theta}{1-y^2},\quad y\in(0,1).$$

Corollary 2. *The differential equation in Corollary 1 has the following general solution*

$$\eta(y)=e^{\theta\operatorname{arctanh}(y)}\left[-\int_0^1\frac{\theta}{1-y^2}e^{-\theta\operatorname{arctanh}(y)}\left(q_1(y)\right)^{-1}q_2(y)dy+D\right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 1 with $D=0$. Clearly, there are other triplets $(q_1(y),q_2(y),\eta(y))$ satisfying the conditions of [31, Theorem 1].

4.2 Characterization based on hrf

The hrf $h_F(x)$ of a twice differentiable cdf $F(x)$ with pdf $f(x)$, satisfies the following trivial differential equation:

$$\frac{f'(x)}{f(x)}=\frac{h'_F(x)}{h_F(x)}-h_F(x).$$

Based on the hrf, the following result establishes a non-trivial characterization of the NwUL distribution.

Proposition 2. *Suppose that Y is a continuous random variable, Y has the pdf (1) if and only if its hrf $h(y)$ satisfies the following first order differential equation:*

$$h'(y)-\frac{2y}{1-y^2}h(y)=\frac{\theta^2}{1-y^2}\frac{d}{dy}\left\{\frac{1+\operatorname{arctanh}(y)}{1+\theta+\theta\operatorname{arctanh}(y)}\right\},\quad y\in(0,1).$$

Proof. The proof is straightforward and hence omitted. □

5 Different methods of estimation

Six different methods for estimating the parameter θ have been established in this section. The following are the descriptions.

1. Maximum likelihood estimation

   The method of maximum likelihood estimate (MLE) based on the NwUL distribution is discussed in this paragraph. Let $Y_1,Y_2,\dots,Y_n$ be a random sample from the NwUL distribution with observed values $y_1,y_2,\dots,y_n$. Then, the log-likelihood function is

$$\ell(\theta)=2n\log\theta-n\log(\theta+1)-\sum_{i=1}^n\log\left(1-y_i^2\right)+\sum_{i=1}^n\log\left(1+\operatorname{arctanh}(y_i)\right)-\theta\sum_{i=1}^n\operatorname{arctanh}(y_i).\tag{6}$$

   The MLE, say $\hat{\theta}_{MLE}$, is obtained by maximizing $\ell(\theta)$ with respect to θ. It can be calculated using the following equations as a standard procedure:

$$\frac{\partial\ell(\theta)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{\theta+1}-\sum_{i=1}^n\operatorname{arctanh}(y_i)=0.\tag{7}$$

From equation (7), we have

$$\hat{\theta}_{MLE} = \left(2\bar{h}\right)^{-1}\left[1 - \bar{h} + \sqrt{1 + 6\bar{h} + \bar{h}^2}\right],$$

where $\bar{h}$ is the mean of the following n values: $h_i = \operatorname{arctanh}(y_i)$, $i = 1, 2, \ldots, n$.

Hence, the Fisher information matrix is given by

$$I = -E\left(\frac{\partial \ell^2(\theta)}{\partial \theta^2}\right) = \left(n\theta^2 + 4n\theta + 2n\right)\left[\theta^2\left(\theta^2 + 1\right)\right]^{-1}.$$

Under the regularity conditions, $\hat{\theta}_{MLE}$ has the normal distribution with mean θ and variance $I^{-1} = \theta^2\left(\theta^2 + 1\right)\left(n\theta^2 + 4n\theta + 2n\right)^{-1}$. Finally, the $(1 - \varsigma)\%$ confidence interval estimation of the θ is

$$\left[\hat{\theta}_{MLE} - z_{\varsigma/2}\sqrt{\frac{\hat{\theta}^2_{MLE}\left(\hat{\theta}^2_{MLE} + 1\right)}{n\hat{\theta}^2_{MLE} + 4n\hat{\theta}_{MLE} + 2n}}; \hat{\theta}_{MLE} + z_{\varsigma/2}\sqrt{\frac{\hat{\theta}^2_{MLE}\left(\hat{\theta}^2_{MLE} + 1\right)}{n\hat{\theta}^2_{MLE} + 4n\theta + 2n}}\right],$$

where $z_{\varsigma/2}$ is the upper $(\varsigma/2)^{th}$ percentile of the standard normal distribution.

2. Maximum product spacing estimation

Cheng and Amin [33] proposed the maximum product spacing (MPS) approach as a possible alternative to MLE. The principle behind this approach is that variations (spacings) between the values of the cdf at consecutive data points should be uniformly distributed. Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ be the ordered statistics from the NwUL distribution with sample size n, and $y_{(1)}, y_{(2)}, \ldots, y_{(n)}$ be the observed ordered values. Then, the geometric mean (GM) of the differences is given as

$$GM(\theta) = \sqrt[n+1]{\prod_{i=1}^{n+1}\left[F(y_{(i)}; \theta) - F(y_{(i-1)}; \theta)\right]},$$

where $F(y_{(0)}; \theta) = 0$ and $F(y_{(n+1)}; \theta) = 1$. The maximum product spacing estimate (MPSE) of θ is obtained by maximizing GM with respect to θ. Taking logarithm of the above expression, we have

$$MPS(\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F(y_{(i)}; \theta) - F(y_{(i-1)}; \theta)\right]. \tag{8}$$

The MPSE can be obtained as the solution of the following equation:

$$\frac{\partial MPS(\theta)}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_{\theta}(y_{(i)}; \theta) - F'_{\theta}(y_{(i-1)}; \theta)}{F(y_{(i)}; \theta) - F(y_{(i-1)}; \theta)}\right] = 0,$$

where

$$F'_{\theta}(y; \theta) = \theta\left(\theta + 2 + (\theta + 1)\operatorname{arctanh}(y)\right)e^{-\theta\operatorname{arctanh}(y)}\left(\theta + 1\right)^{-2}\operatorname{arctanh}(y).$$

3. Least squares estimation

By minimizing the following equation according to θ, the least square estimate (LSE) of θ is obtained:

$$LSE(\theta) = \sum_{i=1}^{n} \left(F\left(y_{(i)};\theta\right) - E\left[F(Y_{(i)};\theta)\right] \right)^2, \tag{9}$$

where $E\left[F(Y_{(i)};\theta)\right] = i/(n+1)$. Then, it is also the solution of the following equation:

$$\frac{\partial LSE(\theta)}{\partial\theta} = 2\sum_{i=1}^{n} F_{\theta}^{'}(y_{(i)};\theta)\left(F(y_{(i)};\theta) - \frac{i}{n+1} \right) = 0,$$

where $F_{\theta}^{'}(y_{(i)};\theta)$ is mentioned before.

4. Weighted least squares estimation

   By minimizing the following equation with respect to θ, the weighted least square estimate (WLSE) of θ is obtained:

$$WLSE(\theta) = \sum_{i=1}^{n} \frac{\left(F(y_{(i)};\theta) - E\left[F(Y_{(i)};\theta)\right]\right)^2}{V\left[F(Y_{(i)};\theta)\right]}, \tag{10}$$

   where $E\left[F(Y_{(i)};\theta)\right] = i/(n+1)$ and $V\left[F(Y_{(i)};\theta)\right] = i(n-i+1)/[(n+2)(n+1)^2]$. Then, it is solution of the following equation:

$$\frac{\partial WLSE(\theta)}{\partial\theta} = 2\sum_{i=1}^{n} F_{\theta}^{'}(y_{(i)};\theta)\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(F(y_{(i)};\theta) - \frac{i}{n+1} \right) = 0.$$

5. Anderson-Darling estimation

   The Anderson-Darling (AD) minimum distance estimate of θ is obtained by minimizing

$$AD\left(\theta\right) = -n - \sum_{i=1}^{n} \frac{2i-1}{n}\left[\log F(y_{(i)};\theta) + \log\left\{1 - F(y_{(n+1-i)};\theta)\right\}\right], \tag{11}$$

   according to θ. Therefore, it can be obtained as the solution of the following equation:

$$\frac{\partial AD(\theta)}{\partial\theta} = -\sum_{i=1}^{n} \frac{2i-1}{n}\left[\frac{F_{\theta}'(y_{(i)};\theta)}{F(y_{(i)};\theta)} - \frac{F_{\theta}'(y_{(n+1-i)};\theta)}{1 - F(y_{(n+1-i)};\theta)}\right] = 0.$$

6. The Cramér-von Mises estimation

   By minimizing the following function according to θ, the Cramér-von Mises (CVM) minimum distance estimate of θ is obtained:

$$CVM\left(\theta\right) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(y_{(i)};\theta) - \frac{2i-1}{2n}\right]^2, \tag{12}$$

   Therefore, the $\hat{\theta}_{CVM}$ can be obtained as the solution of the following equation:

$$\frac{\partial CVM(\theta)}{\partial\theta} = 2\sum_{i=1}^{n}\left(F(y_{(i)};\theta) - \frac{2i-1}{2n} \right) F_{\theta}'(y_{(i)};\theta) = 0.$$

Since, with the exception of the ML method, all estimating equations include non-linear functions, it is not possible to obtain explicit forms of all estimates directly. As a result, computational methods such as the Newton-Raphson and quasi-Newton algorithms must be used to solve them. In addition, Equations (6), (8), (9), (10), (11) and (12) can also be optimized directly by using the `R` (`constrOptim`, `optim` and `maxLik` functions), `SAS` (`PROC NLMIXED`) or `Ox` program (sub-routine `MaxBFGS`) to numerically optimize the non-linear functions $\ell(\theta)$, $MPS(\theta)$, $LSE(\theta)$, $WLSE(\theta)$, $AD(\theta)$ and $CVM(\theta)$.

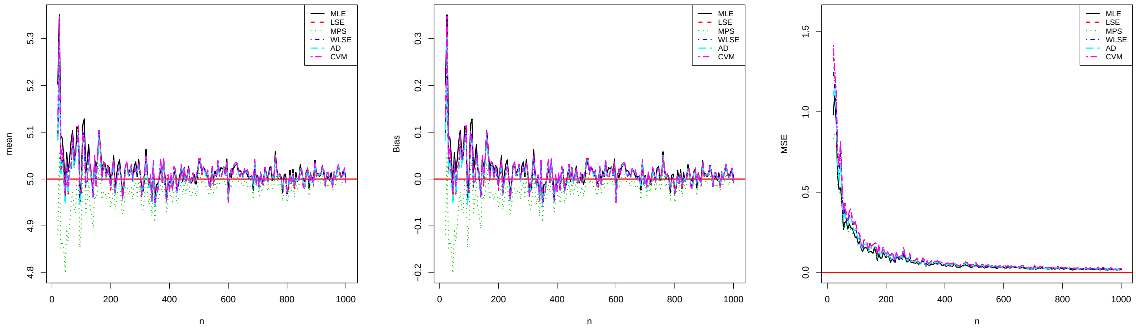

Figure 3: The results obtained regarding the parameter θ in the simulation study

6 Simulation experiments

This section depicts the results of the simulation analysis graphically in order to see how the above estimates of the NwUL distribution performed when the sample size n changed. Based on the actual parameter values, we produce $N = 1000$ samples of size $n = 20, 25, \ldots, 1000$ from the NwUL distribution. For the simulation analysis, we use $\theta = 5$. The random numbers generation is obtained by the relation $y = \tanh x$, where x is the random number from the Lindley distribution with parameter θ. The `constrOptim` routine in the R program is used to obtain all the estimates based on the presented estimation methods. For comparisons between these methods, we measure the empirical mean, bias, and mean squared error (MSE) of the estimates based on the following formulas:

$$Em_{\theta}(n) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i, \quad Bias_{\theta}(n) = \frac{1}{N}\sum_{i=1}^{N}\left(\theta - \hat{\theta}_i\right), \quad MSE_{\theta}(n) = \frac{1}{N}\sum_{i=1}^{N}\left(\theta - \hat{\theta}_i\right)^2,$$

respectively. When the MSEs and biases are close to zero, the analytical means should be close to true values. Figure 3 depicts the effects of this simulation analysis.

As shown in Figure 3, all of the estimates are consistent since the MSEs and biasedness decrease to zero as sample size increases, as expected. All of the projections are therefore asymptotically unbiased. On changing sample sizes, the amount of biases and MSEs are very similar. For small sample sizes, the amount of biases and MSEs of all methods are large at first. When the sample size is increased, however, these amounts become closer. For different parameter settings, similar results can be obtained. We also include a simulation analysis based on the above MLE findings for the efficacy of the 95% confidence intervals. The output of the MLEs is evaluated using the average length (AL) criterion. The AL is the difference between the upper bound and the lower bound of the estimated confidence interval. This property is only available to MLEs. Hence, they are related to the consistency properties of the MLEs. If they are consistent when the same size increases, the AL measurement will decrease to zero. This is the reason that the empirical means converge to the true parameter value. That's why, the AL decreases when the sample size increases. In order to observe these consistency properties, we present some simulation studies. The approximate AL is calculated as follows:

$$AL_{\theta}(n) = \frac{3.919928}{N}\sum_{i=1}^{N}\sqrt{\frac{\theta_i^2\left(\theta_i^2+1\right)}{n\theta_i^2+4n\theta_i^2+2n}}\Bigg|_{\theta=\hat{\theta}_{MLE}},$$

Figure 4 displays the simulation results for the obtained ALs.

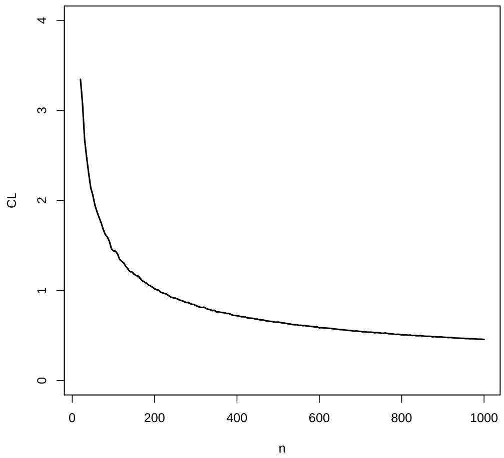

Figure 4: The estimated ALs for the selected parameter vector.

As the sample size grows larger, the ALs for each parameter decrease as predicted.

7 Real data example

The Better Life Index (BLI) data set, which can be found at https://stats.oecd.org/, is used to demonstrate the effectiveness of the NwUL model. The BLI data set includes 11 metrics, including housing, wages, employment, culture, education, climate, civic participation, health, life satisfaction, safety, work-life balance, and 24 variables. Apart from Brazil, Russia, and South Africa, which are Non-OECD countries, BLI data are also effective in classifying OECD countries. We use the long-term unemployment rate metric in this report, which refers to the number of people who have been out of work for a year or more as a percentage of the labor force (the total number of working and unemployed people). Concretely, the following data are used: 0.0131, 0.0184, 0.0354, 0.0077, 0.0079, 0.0104, 0.0131, 0.0192, 0.0213, 0.0400, 0.0157, 0.1565, 0.0172, 0.0026, 0.0323, 0.0049, 0.0659, 0.0103, 0.0005, 0.0335, 0.0269, 0.0235, 0.0007, 0.0197, 0.0074, 0.0066, 0.0152, 0.0443, 0.0478, 0.0317, 0.0766, 0.0112, 0.0182, 0.0239, 0.0113, 0.0066, 0.0159, 0.1646.

We fit the NwUL model to a long-term unemployment rate data set using the MLE method to compare some distributions. The following are the pdfs of the distributions described in the interval $(0, 1)$ that we use for comparison:

• Kumaraswamy distribution:

$$f_{Kw}\left(y;\alpha,\beta\right)=\alpha\beta y^{\alpha-1}\left(1-y^{\alpha}\right)^{\beta-1}, \quad \alpha,\beta>0.$$

• Beta distribution:

$$f_{Beta}\left(y;\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}y^{\alpha-1}\left(1-y\right)^{\beta-1}, \quad \alpha,\beta>0.$$

• Topp-Leone distribution ([34]):

$$f_{Topp-Leone}\left(y;\alpha\right)=\alpha\left(2-2y\right)\left(2y-y^{2}\right)^{\alpha-1}, \quad \alpha>0.$$

• Unit Lindley(UL2) distribution ([15]):

$$f_{UL2}(y;\theta) = \frac{\theta^2}{y^3(1+\theta)} \exp\left(-\theta\left(\frac{1-y}{y}\right)\right), \quad \theta > 0.$$

- Unit Lindley(UL3) distribution ([14]):

$$f_{UL3}(y;\theta) = \frac{\theta^2}{(1-y)^3(1+\theta)} \exp\left(-\frac{\theta y}{1-y}\right), \quad \theta > 0.$$

Log-likelihood value $\left(\widehat{\ell}\right)$, Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (KS), Cramér-von-Mises (W^*) and Anderson–Darling (A^*) goodness-of-fit statistics and related p-values for all the models are calculated to determine the best model for long-term unemployment rate data set. It is well-known that one can choose best model according to the smaller values of the AIC, BIC, KS, A^* and W^* and the larger values of $\widehat{\ell}$ and p-value of the related goodness-of-fit statistics. The results are indicated in Table 2.

Table 2: MLEs, standard errors of the estimates (in parentheses), $\widehat{\ell}$, AIC, BIC and goodness-of-fit statistic and related p-value (in parentheses) for long-term unemployment rate data set

Model	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\ell}$	AIC	BIC	KS	A^*	W^*
NwUL	36.0927			97.3318	-192.6637	-191.0261	0.1160	0.7839	0.1231
	(5.7052)						(0.6858)	(0.4918)	(0.4845)
Kw		0.9406	28.1413	97.1479	-190.2959	-187.0207	0.1169	0.8367	0.1317
		(0.1179)	(11.4976)				(0.6763)	(0.4549)	(0.4521)
Beta		0.9798	33.2394	97.0288	-190.0577	-186.7825	0.1223	0.8569	0.1365
		(0.1975)	(8.5587)				(0.6200)	(0.4409)	(0.4347)
Topp-Leone		0.2898		71.2670	-140.5340	-138.8964	0.3918	8.4404	1.7432
		(0.0470)					(0.0000)	(0.0000)	(0.0000)
UL2	0.0124			61.4598	-120.9196	-119.2820	0.4221	16.7631	2.7746
	(0.0014)						(0.0000)	(0.0000)	(0.0000)
UL3	33.5018			96.5834	-191.1669	-189.5293	0.1349	0.9808	0.1606
	(5.2858)						(0.4929)	(0.3669)	(0.3597)

Based on Table 2, From all criteria and goodness-of-fit statistics for the long-term unemployment rate data collection, the best model is chosen as the NwUL model.

Let us mention that the estimated inverse of the Fisher information matrix related to the NwUL model is given by $I^{-1} = 32.5493$. Also, the 95% confidence interval for the related parameter θ is [13.7287, 58.4566].

The fitted pdf and cdf of the NwUL model are shown in Figure 5.

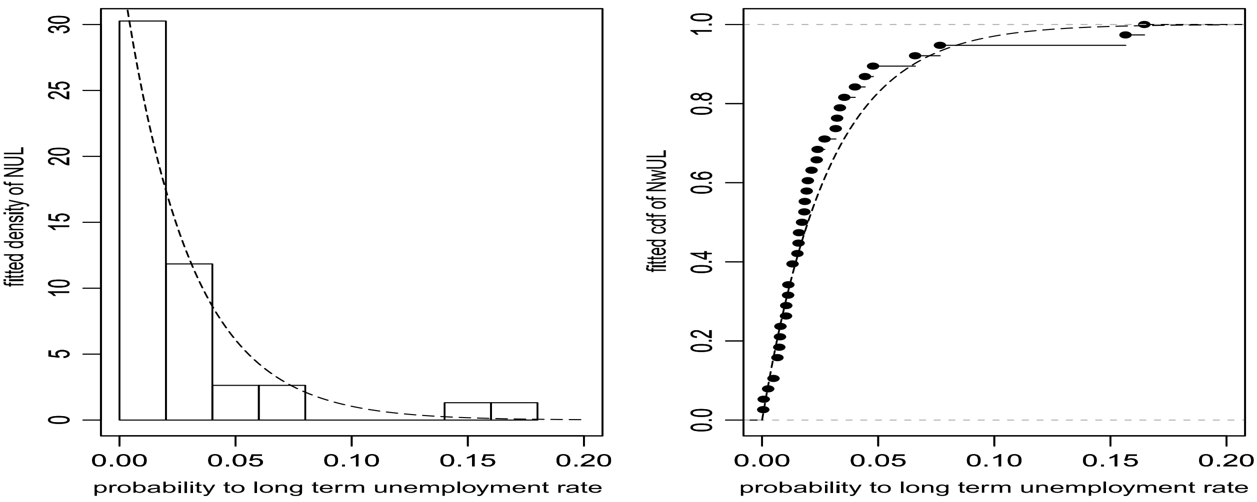

Figure 5: Estimated pdf and cdf of the NwUL model for long-term unemployment rate data set

The obtained fits, as shown in this figure, are of very good quality.

8 Concluding remarks

In this paper, the hyperbolic tangent function transformation of the Lindley distribution is used to create a unit distribution. The perspective of new modelling are discussed, supported by diverse mathematical, numerical and graphical results. Characterizations are described based on two truncated moments and hrf. The model parameters are estimated using six different point estimation methods. The simulation analysis is used to test the accuracy of all of the estimates in terms of biases, MSEs, and ALs. The MLE method reveals to be a suitable choice. Finally, a real data set about the BLI demonstrates the utility of the new unit model. For this data collection, the new generation model consistently outperforms other competitive models.

Disclosure statement

No potential competing interest was reported by the authors.

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