ROBUSTNESS OF SHAPE PARAMETER FOR ERLANG AND WEIBULL BAYESIAN ACCEPTANCE SAMPLING PLANS

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Abstract. This article assesses the robustness of shape parameter for Bayesian acceptance sampling plans assuming Erlang and Weibull distributions. In particular, the prior information on the parameter is combined assuming different loss functions to derive different sampling plans. The cost model for the group sampling plans is studied by satisfying the constraints of producer's and consumer's risks for the Weibull sampling. The single sampling plan is compared with the group sampling plan and the results suggest that the group sampling plan performs better than the single sampling in terms of cost. It is noticed that the shape parameters of Erlang and Weibull distributions are not robust as claimed in the literature.
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1. Introduction

In any manufacturing systems, planning of inspection is crucial part as it decides whether a product is conforming or nonconforming. A product is said to be nonconforming if it consists of one or more defects, otherwise the product is declared as a the conforming one. Therefore, acceptance sampling plan is needed for determining the range in which the goods have to be inspected before delivering to the customers. The acceptance sampling is an important field of the statistical quality control (SQC) [1]. It is the sampling inspection procedure in which consumer decide either to reject or accept the lot of goods, which are shipped by producer, on the basis of random sample [2]. Dodge and Romig[3] gave a comprehensive account to develop acceptance sampling plans.

Bayesian acceptance sampling technique is related to the use of prior knowledge of process history to describe the random variations, which are involved in the acceptance sampling. Basically, the prior distribution is the expected distribution of the lot quality on which a sampling plan is operated. The combination of both, i.e., the prior information which is represented by the prior distribution and empirical information which is based on the sample, may lead to a better decision for the lots. Thus, the main objective of the study is to construct a Bayesian acceptance sampling plan for lot consisting of M units, where the number of defects in a unit can be defined by Erlang or Weibull distribution to assess the effect of the shape parameter. Furthermore, we compare the effect of different priors, loss functions, and single versus group acceptance sampling plans. In the acceptance sampling, from the lot of size M a sample of size n is randomly selected. If the number of defects in the sample is less than the acceptance number c, the lot is accepted otherwise it is rejected. The expected total cost (ETC) for the acceptance sampling depends whether a lot is rejected or accepted. In case a lot is accepted, the remaining of the lot is not examined. For any defect in a lot, we assume the step-loss and the quadratic loss functions for the calculation of ETC. We further assume that the cost components are independent of each other [4].

In the literature, Tang et al.[5] developed Bayesian multi-attribute acceptance sampling schemes for the determination of optimal sample size. An efficient repetitive algorithm was developed to find the best near multiattribute sampling plans possessing a large number of attributes. In Bayesian inspection models, the basic assumption is the prior information about the number of defects. Usually this prior information can be illustrated as a probability distribution. Chun and Sumichrast[6] proposed three conditions for the determination of prior distribution for a defective product. Reviewing several probability distribution, they noticed that the negative binomial distribution satisfies the desired conditions only. By using the negative binomial as a prior distribution, they showed that the effect of undetected errors. Kwon[7] considered the Bayesian life test sampling plans for the products assuming Weibull distribution with known shape parameter.

"Fallahnezhad and Aslam[8] proposed a acceptance sampling model on the basis of a cost function. To update the probability distribution function of the proportion of defective, Bayesian inference is used. Furthermore, backward induction along with the Bayesian inference is used to estimate the expected total cost for the various decisions. The sensitivity analysis is carried out for the parameters of the proposed methodology to analyse the optimal solution for various decisions. Fallahnezhad and Babadi[9] developed acceptance sampling plan in the presence of inspection error using the decision tree approach.

Following Moskowitz and Tang[4], Fallahnezhad and Saredorahi[10] proposed a Bayesian acceptance sampling plan on the basis of smallest proportion of a lot which should be inspected in the presence of inspection error. Gonzalez and Palomo[11] derived Bayesian acceptance sampling plans for the number of defect to minimize the expected total cost (ETC) of quality. For the calculation of acceptance sampling cost, two loss functions are considered for the Poisson distribution. In manufacturing industries, the decision either to reject or accept a product is generally made on the basis of the measurement information. As this information is seldom complete, in general, it is not possible to be completely sure about the measured values. Lira[12] studied the probabilities of incorrectly rejecting or accepting the product using Bayesian statistics. Adibfar et al.[13] proposed a sampling scheme assuming Bayesian methods. The Bayesian risks for both consumer and producer provide a better understanding for decision making than the traditional ones. The results of sensitivity analysis show that lot size, the cost of inspection, and the cost of one defective items are the key factors in the sampling design. The lot tolerance proportion defective, the acceptable quality level and the Bayesian risks also influence the sampling policy. On frequentist side, we refer to [2, 14, 15, 16, 17, 18] for sampling plans assuming different truncated distribution. For group acceptance sampling plans, we refer to [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34], and references cited therein.

Hsu[35] proposed an economic model for determining optimal sampling plan which minimizes the producer's total cost by satisfying both the producer's and consumer's risks. For variable acceptance sampling plan, Schmidt et al.[36, 37] presented cost models. Tagaras[38] developed an economic model for acceptance sampling plan for variables assuming normal distribution. Taguchi loss function is used when the quality characteristics deviate from the target value. Aslam et al.[39] proposed economic reliability test plans by taking into account the lifespan of the submitted products assuming the Pareto distribution of second kind. For different acceptance number (c), sample size (n), producer's risk and the minimum test termination time are obtained. Fallahnezhad and Fakhrzad[40] proposed a new sampling plans for the defective proportion of the batch. To measure the deviations between the proportion of defective and the acceptance quality level, a continuous loss function is used. A sensitivity analysis is performed for the desired values of sample size, which allows practitioners to plan optimum inspection plan.

The remainder of the study is organized as follows: Section 2 discusses the expected total cost for the Erlang distribution. The Weibull acceptance sampling plan is discussed in Section 3. Some concluding remarks are given in Section 4.

2. Expected Total Cost assuming Erlang Sampling

Let X denotes the number of defects per unit of the product, where X follows the Erlang distribution with parameter θ. For the Bayesian analysis, the prior for parameter θ is required and here, we consider two different prior distributions. The first one is the gamma prior $f(\theta) = \frac{1}{\Gamma(a)} b^a \theta^{a-1} e^{-b\theta}$, where a and b are the shape and rate parameters, respectively. The second prior is the noninformative prior $f(\theta) = K\theta^{-1}$, where $K > 0$ is a positive constant. Let $l(x)$ denotes the loss due to the presence of defects per-unit X in the accepted lot. This study consider the quadratic and step loss functions [11]. The quadratic loss function is defined as $l(x) = hx^2$, where $h > 0$ is a positive constant while the step-loss function is defined as

$$l(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq p \\ S & \text{if } x > p \end{cases} \tag{1}$$

where $S > 0$ and $p > 0$ are positive constants. For a given value of θ, the per-unit conditional expected loss is given by

$$L(\theta) = \sum_{x_i=0}^{\infty} l(x_i) P(x_i|\theta) \tag{2}$$

where $l(x_i)$ denotes the loss incurred having x defects. For the parameter θ, $P(x_i|\theta)$ denotes the probability of defects. For the quadratic loss function, the conditional expected loss is

$$L(\theta) = h\frac{\theta^n}{(n-1)!} \sum_{x_i=0}^{\infty} x^{n+1} e^{-x\theta} \tag{3}$$

Similarly, for the step loss function, we have

$$L(\theta) = S\left[1 - \frac{\theta^n}{(n-1)!} \sum_{x_i=0}^{p} x^{n-1} e^{-x\theta}\right] \tag{4}$$

In fact, the expected total cost is composed of three independent components [4, 11, 36]. The important one is the *cost of acceptance*, which is incurred due to defective items in the accepted lots. Let X be the number of defects in a sample of n units, having Erlang distribution with parameter θ. The probability of acceptance of a lot for a given value of θ is

$$P(acceptance|\theta) = P(X \leq c|\theta) = \sum_{r=0}^{c} \frac{1}{(n-1)!} \theta^{n-1} r^{n-1} e^{-r\theta} \tag{5}$$

The marginal probability of acceptance can be found as

$$P(acceptance) = P(X \leq c) = \sum_{r=0}^{c} \int_0^{\infty} \frac{1}{(n-1)!} \theta^n r^{n-1} e^{-r\theta} f(\theta) d(\theta) \tag{6}$$

and hence, the expected cost of acceptance (ECA) is

$$ECA = M \int_0^{\infty} L(\theta) P(acceptance|\theta) f(\theta) d(\theta) \tag{7}$$

The ECA for both cases, i.e., quadratic and step loss functions, can be obtained by substituting Equation 6 in Equation 7. Four different quantities of interest can be considered, which depends on the prior distribution of θ and the loss functions. To this end, we calculate the expected cost of rejection (ECR), that is the cost associated with the disposition of rejected lots obtained by

$$ECR = M \times R \times P(rejection) \tag{8}$$

where R represents the per-unit cost of rejection. The probability of rejection of a lot is

$$P(rejection) = 1 - P(acceptance) = 1 - \sum_{r=0}^{c} \int_0^\infty \frac{1}{(n-1)!} \theta^n r^{n-1} e^{-r\theta} f(\theta) d(\theta) \tag{9}$$

Next, to compute the expected cost of inspection (ECI), let J represents the per-unit inspection cost. Then, $ECI = nJ$. Finally, the expected total cost (ETC) becomes $ETC = ECA + ECR + ECI$, thus by minimizing the ETC, the optimum sampling plan can be obtained.

2.1. **Optimum Sampling Plans.** To determine the sampling plan (n, c), we minimize the ETC using different loss functions and prior distributions of θ. To this end, we present the sampling plans for the quadratic and step loss functions assuming gamma and noninformative priors for θ.

2.1.1. *ETC Quadratic Loss Function and Gamma Prior.* The expected cost of acceptance ECA for the quadratic loss function by using the gamma prior is discussed in this section. To this end, the ECA is

$$ECA = Mh \frac{b^a}{\Gamma(a)(n-1)!(n-1)!} \sum_{x_i=0}^{\infty} \sum_{r=0}^{\infty} x^{n+1} r^{n-1} \frac{\Gamma(2n+a)}{(r+x+b)^{2n+a}} \tag{10}$$

Similarly, the ECR is obtained as

$$ECR = MR \left[1 - \sum_{r=0}^{c} \frac{b^a}{\Gamma(a)(n-1)!} r^{n-1} \frac{\Gamma(a+n)}{(b+r)^{a+n}} \right] \tag{11}$$

The expected cost of inspection is $ECI = nJ$, which is the same for different sampling plans discussed in this study. Consequently, the ETC takes the form

$$\begin{aligned} ETC =& Mh \frac{b^a}{\Gamma(a)(n-1)!(n-1)!} \sum_{x_i=0}^{\infty} \sum_{r=0}^{\infty} x^{n+1} r^{n-1} \frac{\Gamma(2n+a)}{(r+x+b)^{2n+a}} + \\ & MR \left[1 - \sum_{r=0}^{c} \frac{b^a}{\Gamma(a)(n-1)!} r^{n-1} \frac{\Gamma(a+n)}{(b+r)^{a+n}} \right] + nJ \end{aligned} \tag{12}$$

For the determination of the optimum values (n,c) the ETC is minimized. As the derivative of the function is very complicated, the optimum values are computed numerically.

2.1.2. *ETC using Quadratic Loss Function and Noninformative Prior.* The expected cost of acceptance ECA for the quadratic loss function by using noninformative prior is obtained as follows

$$ECA = \frac{hkM}{(n-1)!(n-1)!} \sum_{x_i=0}^{\infty} \sum_{r=1}^{c} r^{n-1} x^{n+1} \frac{\Gamma(2n)}{(x+r)^{2n}} \tag{13}$$

Similarly, the ECR is obtained as

$$ECR = MR \left[1 - k \sum_{r=1}^{c} \frac{1}{r} \right] \tag{14}$$

Therefore, the final form of the ETC is given as

$$ETC = \frac{hkM}{(n-1)!(n-1)!} \sum_{x_i=0}^{\infty} \sum_{r=1}^{c} r^{n-1} x^{n+1} \frac{\Gamma(2n)}{(x+r)^{2n}} + MR \left[1 - k \sum_{r=1}^{c} \frac{1}{r} \right] + nJ \tag{15}$$

which needs to be solved numerically for optimizing n and c.

2.1.3. *ETC using Step Loss Function and Gamma Prior.* The ECA for the step loss function by using the gamma prior is

$$ECA = MS\frac{b^a}{(n-1)!\Gamma(a)}\sum_{r=0}^{c} r^{n-1}\left[\frac{\Gamma(a+n)}{(b+r)^{a+n}} - \frac{\Gamma(2n+a)}{(n-1)!}\sum_{x_i=0}^{p}\frac{x^{n-1}}{(b+r+x)^{2n+a}}\right] \tag{16}$$

and ECR is obtained as

$$ECR = MR\left[1 - \frac{b^a}{\Gamma(a)(n-1)!}\sum_{r=0}^{c} r^{n-1}\frac{\Gamma(a+n)}{(b+r)^{a+n}}\right] \tag{17}$$

Therefore, the final form of the ETC is

$$\begin{aligned} ETC =& MS\frac{b^a}{(n-1)!\Gamma(a)}\sum_{r=0}^{c} r^{n-1}\left[\frac{\Gamma(a+n)}{(b+r)^{a+n}} - \frac{\Gamma(2n+a)}{\Gamma(n)}\sum_{x_i=0}^{p}\frac{x^{n-1}}{(b+r+x)^{2n+a}}\right] \\ &+ MR\left[1 - \frac{b^a}{\Gamma(a)(n-1)!}\sum_{r=0}^{c} r^{n-1}\frac{\Gamma(a+n)}{(b+r)^{a+n}}\right] + nJ \end{aligned} \tag{18}$$

The optimal parameters are calculated numerically.

2.1.4. *ETC using Step Loss Function and Noninformative Prior.* The ECA for the step loss function by using the noninformative prior is obtained as

$$ECA = \frac{MSk}{(n-1)!}\sum_{r=1}^{c} r^{n-1}\left[\frac{(n-1)!}{r^n} - \frac{\Gamma(2n)}{(n-1)!}\sum_{x_i=0}^{p}\frac{x^{n-1}}{(r+x)^{2n}}\right] \tag{19}$$

whereas the ECR is obtained as

$$ECR = MR\left[1 - k\sum_{r=1}^{c}\frac{1}{r}\right] \tag{20}$$

Therefore, the final form of the ETC is

$$ETC = \frac{MSk}{(n-1)!}\sum_{r=1}^{c} r^{n-1}\left[\frac{(n-1)!}{r^n} - \frac{\Gamma(2n)}{(n-1)!}\sum_{x_i=0}^{p}\frac{x^{n-1}}{(r+x)^{2n}}\right] + MR\left[1 - k\sum_{r=1}^{c}\frac{1}{r}\right] + nJ \tag{21}$$

For the determination of the optimum values (n,c) the ETC is again minimized numerically and the results are discussed in the next section.

2.2. **Prior Robustness.** Suppose that the incoming lot of size $M = 10000$ to be inspected. To test a unit, it cost $J = 4.5$, and $h = 5$. Furthermore, the cost associated with the rejected lot per unit is $R = 2.5$. In the case of step loss function, the values of parameters are assumed $S = 50$ and $p = 1$. Furthermore, it is supposed that the number of defects per unit follows the gamma distribution with parameters $a = 1.25$ and $b = 0.25$ which are used by Gonzalez and Palomo[11].

Table 1 Here

Table 1 lists the results for the quadratic and step loss functions. The ETC for the aforementioned specifications is 473252, and the optimum rule draws 21 units and the lot is accepted when number of defects is less than 6 otherwise rejected. For the quadratic loss function, the expected total cost ETC associated with the best decision without inspection is

$$\min\left(R, E\left[L(\theta)\right]\right)$$

which becomes

$$\min\left(R, \frac{hb^a\Gamma(a+n)}{\Gamma(a)\Gamma(n)}\sum_{x_i=0}^{\infty}\frac{x^{n+1}}{(b+x)^{a+n}}\right).$$

Similarly, it can be shown that for the step loss function, the expected total cost ETC associated with the best decision without inspection is

$$\min\left(R, E[L(\theta)]\right)$$

which becomes

$$\min\left(R, \frac{hb^a\Gamma(a+n)}{\Gamma(a)\Gamma(n)}\sum_{x_i=0}^{p}\frac{x^{n-1}}{(b+x)^{a+n}}\right).$$

When the step-loss function is used the ETC is 266504, and the optimum rule draws 25 units and the lot is accepted when number of defects are less than 7 otherwise the lot is rejected. When the quadratic loss function is used and the prior distribution is gamma, the sampling plan obtain from noninformative will result 18% increase in ETC. Similarly, when the step loss function is used with gamma prior, the sampling plan will result 40% increase in ETC.

Table 2 Here

Table 2 presents the percentage increment in ETC when the prior standard deviation remains unchanged and prior mean differs from the true mean. For example, when the quadratic loss function is used and the prior distribution is gamma, the sampling plan computed by the gamma prior with 20% greater mean than the true one will result 20.88% increase in the ETC. Similarly, the percentage increase in the ETC when the prior standard deviation changes and prior mean remains unchanged is presented in Table 2. From the table it is noticed that the percentages of ETC are smaller with the misspecification of mean as compared to the standard deviation, which shows that sampling plans are robust with respect to the prior mean than the standard deviation.

Figure 1 Here

In Figure 1, the ETC computed by using the gamma prior for quadratic and step loss functions is plotted against sample size. It is evident from the graph that as the sample size increases, the ETC gradually decrease.

3. Economic Design of the Group Acceptance Sampling assuming Weibull Distribution

The probability of accepting a poor quality lot is called the consumer risk and the probability of rejecting a good quality lot is called the producer risk. It is of the great interest of producers to use a sampling plan that make sure protection from the risk of rejecting a good quality lot and thus a producer always wants to use a sampling plan that allows the inspection of the lot at the optimal cost, as inspection of the product need for a laboratory equipped with testers, time, labors, etc. A single acceptance sampling plan established upon on the inspection of product by putting a single item in a single tester. In this situation, the number of items selected in the sample is equal to the number of testers. When more than one item can be put in a single tester, the group sampling plans are used to reduce the inspection cost.

If the quality level of the product is higher than the specified level, the product is said to be of a good quality. Generally, these levels are determined by using the percentile ratios t_q/t_{q_0}, where t_q represents the qth percentile life of the product and t_{q_0} represents the specified percentile life. Therefore, the most important goal of the acceptance sampling is to accept a lot of goods when $t_q \geq t_{q_0}$, otherwise reject the lot. Aslam et al.[24] showed that it performs better than the existing group sampling plan in terms of average sample numbers. The algorithm of the group sampling plan is

- Take a random sample of size n from the lot of size M and assign r items to g groups, i.e., $n = rg$, for the time duration t_0.

- The lot is accepted if the total number of failures from the g groups is smaller than or equal to c, otherwise the lot is rejected before the experiment time t_0.

Suppose that the observations lifespan follows a Weibull distribution with the following probability density function.

$$f(t;\sigma,\lambda) = \begin{cases} \frac{\lambda}{\sigma}(\frac{t}{\sigma})^{\lambda-1}e^{-(t/\sigma)^{\lambda}} & ;\ t \geq 0 \\ 0 & ;\ t < 0 \end{cases} \tag{22}$$

where "σ" denotes the scale parameter and "λ" denotes the shape parameter of the Weibull distribution. Contrary to previous studies, this study assumes known scale parameter and unknown shape parameter. The cumulative distribution function (CDF) of the Weibull distribution is given as follows

$$F(t) = 1 - \exp\left(-(t/\sigma)^{\lambda}\right) \qquad t \geq 0 \tag{23}$$

and the qth percentile life of the product is given by

$$t_q = \theta\left[\ln\left(\frac{1}{1-q}\right)\right]^{\frac{1}{\lambda}} \tag{24}$$

Under the group sampling plan, the probability of the acceptance of a lot [2] is given as follows

$$L(p) = \sum_{i=0}^{c}\binom{rg}{i}p^i(1-p)^{rg-i} \tag{25}$$

where p denotes the probability of failure before the termination time t_0, which can be calculated from the CDF of the Weibull distribution. The experimentation time is a multiple of the percentile life $t_0 = mt_{q0}$, where m is a fixed constant which is called the termination time. The probability of failure p can be written as

$$p = 1 - \exp\left[-m^{\lambda}(t_q/t_{q0})^{\lambda}\ln\left(\frac{1}{1-q}\right)\right] \tag{26}$$

3.1. **Minimization of Total Cost Model.** Let the per-unit inspection cost is denoted by C_i, the internal failure cost (i.e., reparation, rework, and restoration of the failed products) by C_f, the outgoing defective cost by C_0, and the setup cost per group by C_g. Then, the total cost for the group acceptance sampling plan is considered as follows

$$TC = C_i(ATI) + C_f(D_d) + C_o(D_n) + g(C_g) \tag{27}$$

where ATI denotes the average total inspection, D_d is the number of defective products detected and D_n represents the number of defective products not detected. Thus

$$ATI = rg + (1 - L(p))(M - rg) \tag{28}$$

$$D_d = rgp + (1 - L(p))(M - rg)p \tag{29}$$

$$D_n = L(p)(M - rg)p \tag{30}$$

where r denotes the group size, g is the number of groups and M denotes the lot size. The average outgoing quality (AOQ) is a measure of rectifying inspected items, i.e., the quality of the lot that result from the application of rectifying inspection. The AOQ can be obtained over a long sequence of lots by a process defective fraction (p). It can be obtained as

$$AOQ = \frac{pL(p)(M - rg)}{M} \tag{31}$$

3.2. **Bayesian Group Design.** As p is unknown and to obtain the plan parameters, Hsu[35] prefixed the values of p. Contrary to Hsu[35], we use the Bayesian approach to estimate the unknown values of p which is function of λ. To this end, the prior distribution of λ is assumed to follow a gamma distribution with the shape parameter $\gamma > 0$ and the scale parameter $\delta > 0$ using the PDF $f(\lambda) = \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}$. Hence, the ATI and AOQ can be written as

$$ATI = \int_0^\infty [rg + (1 - L(p))(M - rg)] \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}d\lambda \tag{32}$$

$$Dd = \int_0^\infty [rgp + (1 - L(p))(M - rg)p] \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}d\lambda \tag{33}$$

$$Dn = \int_0^\infty [L(p)(M - rg)p] \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}d\lambda \tag{34}$$

$$AOQ = \int_0^\infty \frac{L(p)(M - rg)p}{M} \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}d\lambda \tag{35}$$

Due to complexity of integration, the results can be calculated by numerical integration. For the hyperparameters, we assume $\gamma = 1.25$, $\delta = 0.25$ and obtain the following equations.

$$\begin{aligned} ATI =& rg + (M - rg) \int_0^\infty \sum_{i=c+1}^{n} \binom{rg}{i} \left(1 - \exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^i \times \\ & \left(\exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{rg-i} \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}e^{-\lambda\delta}d\lambda \end{aligned} \tag{36}$$

$$\begin{aligned} Dd =& rg \int_0^\infty \left(1 - \exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right) \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}\exp(-\lambda\delta)d\lambda \\ & + (M - rg) \int_0^\infty \sum_{i=c+1}^{n} \binom{rg}{i} \left(1 - \exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{i+1} \times \\ & \left(\exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{rg-i} \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}\exp(-\lambda\delta)d\lambda \end{aligned} \tag{37}$$

$$\begin{aligned} Dn =& (M - rg) \int_0^\infty \sum_{i=0}^{c} \binom{rg}{i} \left(1 - \exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{i+1} \times \\ & \left(\exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{rg-i} \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}\exp(-\lambda\delta)d\lambda \end{aligned} \tag{38}$$

$$\begin{aligned} AOQ =& \frac{(M - rg)}{M} \int_0^\infty \sum_{i=0}^{c} \binom{rg}{i} \left(1 - \exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{i+1} \times \\ & \left(\exp\left[-m^\lambda (t_q/t_{q_0})^\lambda \ln\left(\frac{1}{1-q}\right)\right]\right)^{rg-i} \frac{\delta^\gamma}{\Gamma(\gamma)}\lambda^{\gamma-1}\exp(-\lambda\delta)d\lambda \end{aligned} \tag{39}$$

Since acceptance sampling plans are associated with producer and consumer risks denoted by α, and β respectively, (1-α) denotes the producer and (1-β) consumer confidence levels, respectively. The producer wishes the acceptance chance of the items batch to be greater than the confidence level (1-α), and the consumer desires that it should be less than the β risk. Let p_1 and p_2 denote the probability of failure of a product before the termination time t_0 at α and β, respectively. Then, we have to minimize the following cost model[26]

$$Minimize\ TC = C_i(ATI) + C_f(D_d) + C_o(D_n) + g(C_g) \tag{40}$$

subject to the constraints

$$L(p_2) \leq \beta \tag{41}$$

$$L(p_1) \geq 1 - \alpha \tag{42}$$

For the calculation of the optimal parameters and to minimize the total cost, we use the following values: $C_i = 1.0$, $C_g = 3$, $C_f = 2.0$, and $C_0 = 1.0$ [26, 35]. In addition, we considered $r = 5$ and 10 as group sizes, $a = 0.5$, 1.0 as the experiment time ratio, $q = 0.5$, $M = 1000$ as the lot size, $\alpha = 0.05$, and $t_q/t_{q0} = 2, 4, 6, 8$ as the percentile ratio. The results are listed in Tables 3-6.

Tables 3-6 Here

For $r = 5$, it is observed that the optimal TC increases by increasing the percentile ratio or experiment time ratio m. Similarly, for $r = 10$, we observe a similar trend as noticed for $r = 5$.

Figures 2-3 Here

From Figures 2 and 3, it is clear that as the percentile ratio (t_q/t_{q_0}) increases, the total cost also increases. Thus, increase in the quality may also increase the total cost of inspection when the shape parameter is unknown. Similarly, the average total inspection remains constant for ratio from 2 to 4 and as the percentile ratio (t_q/t_{q_0}) increased, the average total inspection cost decreased. Hence, the shape parameter has a significant impact on the Bayesian design and cannot be treated fixed as considered by Aslam et al.[26].

3.3. **Comparison of Group Sampling Plan and Single Sampling Plan.** Here, we present a comparison of single acceptance sampling plan to the group acceptance sampling plan. The group sampling plan reduces to single sampling plan when $r = 1$, i.e., the single sampling plan is the special case of group sampling plan, however the setup cost for the group sampling plan is larger than the single sampling plan. For example, Aslam et al.[26] pointed out that the setup cost for the group sampling plan is $C_g = 3$ while $C_g = 1.5$ for the single sampling plan.

Table 7 lists the total cost for both single sampling plan and group sampling plan. From the table, one can conclude that the group sampling plan will perform better than the single sampling plan in terms of total cost, i.e., the total cost associated with group sampling plan is smaller than the single sampling plan assuming unknown shape parameter of the Weibull distribution.

Table 7 Here

4. Conclusion

In this article, Bayesian acceptance sampling plans are derived under Erlang and Weibull distributions to assess the robustness of the shape parameters which is mainly ignored in the previous studies. For the Erlang distribution, we used two loss functions while Bayesian acceptance group and single sampling plans are discussed for the Weibull distribution. The robustness analysis of the Erlang sampling plan with respect to the prior distribution and misspecification of the variance and mean of the process average is analyzed. For the group acceptance sampling plan a cost model is used. We compared a single sampling plan with the group sampling plan and showed that the group sampling plan will perform better than the single sampling plan in terms of the total cost. The Weibull distribution is assumed because of its importance in quality control. In future, the techniques presented here can also be extended to design two-stage group sampling plan. Furthermore, other distributions, priors, and loss functions can also be considered. A recent literature on neutrosophic statistics can be used to extend the present work [41, 42, 43].

References

1. Montgomery, D. C., "*Statistical quality control: a modern introduction*". John Wiley (2009).
2. Balakrishnan, N., Leiva, V., and Lopez, J., "Acceptance sampling plans from truncated life tests based on the generalized Birnbaum–Saunders distribution", *Communications in Statistics—Simulation and Computation*, **36**(3), pp. 643–656 (2007).
3. Dodge, H. F. and Romig, H. G., "*Sampling inspection tables: single and double sampling*". Wiley Series in Probability and Statistics - Applied Probability and Statistics Section Series, Wiley (1959).
4. Moskowitz, H. and Tang, K., "Bayesian variables acceptance-sampling plans: quadratic loss function and step-loss function", *Technometrics*, **34**(3), pp. 340–347 (1992).
5. Tang, K., Plante, R., and Moskowitz, H., "Multiattribute Bayesian acceptance sampling plans under nondestructive inspection", *Management Science*, **32**(6), pp. 739–750 (1986).
6. Chun, Y. H. and Sumichrast, R. T., "Bayesian inspection model with the negative binomial prior in the presence of inspection errors", *European Journal of Operational Research*, **182**(3), pp. 1188–1202 (2007).
7. Kwon, Y. I., "A bayesian life test sampling plan for products with Weibull lifetime distribution sold under warranty", *Reliability Engineering & System Safety*, **53**(1), pp. 61–66 (1996).
8. Fallahnezhad, M. S. and Aslam, M., "A new economical design of acceptance sampling models using Bayesian inference", *Accreditation and Quality Assurance*, **18**(3), pp. 187–195 (2013).
9. Fallahnezhad, M. S. and Babadi, A. Y., "A new acceptance sampling plan using Bayesian approach in the presence of inspection errors", *Transactions of the Institute of Measurement and Control*, **37**(9), pp. 1060–1073 (2015).
10. Fallahnezhad, M. S. and Saredorahi, F., "A new policy for designing acceptance sampling plan based on Bayesian inference in the presence of inspection errors", *Communications in Statistics-Theory and Methods*, **47**(21), pp. 5307–5318 (2018).
11. Gonzalez, C. and Palomo, G., "Bayesian acceptance sampling plans following economic criteria: an application to paper pulp manufacturing", *Journal of Applied Statistics*, **30**(3), pp. 319–333 (2003).
12. Lira, I., "A Bayesian approach to the consumer's and producer's risks in measurement", *Metrologia*, **36**(5), p. 397 (1999).
13. Adibfar, S., Fallahnezhad, M. S., and Jafari, R., "Optimal acceptance sampling policy considering Bayesian risks", *Communications in Statistics-Theory and Methods*, **46**(11), pp. 5228–5237 (2017).
14. Baklizi, A. and El Masri, A. El. Q., "Acceptance sampling based on truncated life tests in the Birnbaum Saunders model", *Risk Analysis: An International Journal*, **24**(6), pp. 1453–1457 (2004).
15. Rosaiah, K., Kantam, R. R. L., and Kumar, S., "Reliability test plans for exponentiated log-logistic distribution", *Economic Quality Control*, **21**(2), pp. 279–289 (2006).
16. Tsai, Tzong-Ru. and Wu, Shuo-Jye., "Acceptance sampling based on truncated life tests for generalized Rayleigh distribution", *Journal of Applied Statistics*, **33**(6), pp. 595–600 (2006).
17. Lio, Y. L. and Tsai, Shuo-Jye., Tzong-Ru.and Wu, "Acceptance sampling plans from truncated life tests based on the Burr type XII percentiles", *Journal of the Chinese institute of Industrial Engineers*, **27**(4), pp. 270–280 (2010).
18. Aslam, M., Kundu, D., and Ahmad, M., "Time truncated acceptance sampling plans for generalized exponential distribution", *Journal of Applied Statistics*, **37**(4), pp. 555–566 (2010).

19. Jun, Chi-Hyuck., Balamurali, S., and Lee, Sang-Ho., "Variables sampling plans for Weibull distributed lifetimes under sudden death testing", *IEEE Transactions on Reliability*, **55**(1), pp. 53–58 (2006).
20. Aslam, M. and Jun, Chi-Hyuck., "A group acceptance sampling plans for truncated life tests based on the inverse Rayleigh and Log-logistic distributions.", *Pakistan Journal of Statistics*, **25**(2), pp. 333–340 (2009).
21. Aslam, M. and Jun, Chi-Hyuck., "A group acceptance sampling plan for truncated life test having Weibull distribution", *Journal of Applied Statistics*, **36**(9), pp. 1021–1027 (2009).
22. Aslam, M., Jun, Chi-Hyuck., and Ahmad, M., "A group sampling plan based on truncated life test for Gamma distributed items", *Pakistan Journal of Statistics*, **25**(3), pp. 333–340 (2009).
23. Rao, G. S., "A group acceptance sampling plans for lifetimes following a generalized exponential distribution", *Economic Quality Control*, **24**(1), pp. 75–85 (2009).
24. Aslam, M., Jun, Chi-Hyuck., Lee, H., Ahmad, M., and Rasool, M., "Improved group sampling plans based on time-truncated life tests", *Chilean Journal of Statistics*, **2**(1), pp. 85–97 (2011).
25. Aslam, M., Pervaiz, M. K., and Jun, Chi-Hyuck., "An improved group sampling plan based on time-truncated life tests", *Communications for Statistical Applications and Methods*, **17**(3), pp. 319–326 (2010).
26. Aslam, M., Azam, M., Balamurali, S., and Jun, Chi-Hyuck, "An economic design of a group sampling plan for a Weibull distribution using a Bayesian approach", *Journal of Testing and Evaluation*, **43**(6), pp. 1497–1503 (2014).
27. Sina, S. and Fallahnezhad, M. S., "Variable sampling plan for resubmitted lots based on process capability index and Bayesian approach", *International Journal of Advanced Manufacturing Technologies*, **88**, pp. 2547–2555 (2017).
28. Li, X., Chen, W., Sun, F., Liao, H., Kang, R., and Li, R., "Bayesian accelerated acceptance sampling plans for a lognormal lifetime distribution under type-i censoring", *Reliability Engineering & System Safety*, **171**, pp. 78 – 86 (2018).
29. Fallahnezhad, M. S. and Seifi, S., "Repetitive group sampling plan based on the process capability index for the lot acceptance problem", *Journal of Statistical Computation and Simulation*, **87**(1), pp. 29–41 (2017).
30. Al-Omari, A. I., "The transmuted generalized inverse Weibull distribution in acceptance sampling plans based on life tests", *Transactions of the Institute of Measurement and Control*, **40**(16), pp. 4432–4443 (2018).
31. d, M. S. F. and Seifi, S., "Designing optimal double-sampling plan based on process capability index", *Communications in Statistics - Theory and Methods*, **46**(13), pp. 6624–6634 (2017).
32. Hu, M. and Gui, W., "Acceptance sampling plans based on truncated life tests for Burr type X distribution", *Journal of Statistics and Management Systems*, **21**(3), pp. 323–336 (2018).
33. Balamurali, S., Jeyadurga, P., and Usha, M., "Designing of multiple deferred state sampling plan for generalized inverted exponential distribution", *Sequential Analysis*, **36**(1), pp. 76–86 (2017).
34. Balamurali, S., Jeyadurga, P., and Usha, M., "Optimal designing of a multiple deferred state sampling plan for Weibull distributed life time assuring mean life", *American Journal of Mathematical and Management Sciences*, **36**(2), pp. 150–161 (2017).
35. Hsu, Jia-Tzer, "Economic design of single sample acceptance sampling plans", *Journal of Hungkuang University*, **97**, pp. 108–122 (2009).
36. Schmidt, J. W., Case, K. E., and Bennett, G. K., "The choice of variables sampling plans using cost effective criteria", *AIIE Transactions*, **6**(3), pp. 178–184 (1974).

37. Schmidt, J. W., Bennett, G. K., and Case, K.E., "A three action cost model for acceptance sampling by variables", *Journal of Quality Technology*, **12**(1), pp. 10–18 (1980).
38. Tagaras, G., "Economic acceptance sampling by variables with quadratic quality costs", *IIE Transactions*, **26**(6), pp. 29–36 (1994).
39. Aslam, M., Mughal, A. R., Hanif, M., and Ahmad, M., "Economic reliability group acceptance sampling based on truncated life tests using Pareto distribution of the second kind", *Communications for Statistical Applications and Methods*, **17**(5), pp. 725–731 (2010).
40. Fallahnezhad, M. S. and Fakhrzad, M. B., **"Determining an economically optimal (n, c) design via using loss functions",**, pp. 197–201 (2012).
41. Aslam, M., "A variable acceptance sampling plan under neutrosophic statistical interval method", *Symmetry*, **11**(1), p. 365–370 (2019).
42. Aslam, M., "A new attribute sampling plan using neutrosophic statistical interval method", *Complex & Intelligent Systems*, **5**, pp. 365–370 (2019).
43. Aslam, M., "Testing average wind speed using sampling plan for Weibull distribution under indeterminacy", *Scientific Reports*, **11**, p. 7532 (2021).

List of Figures and Tables

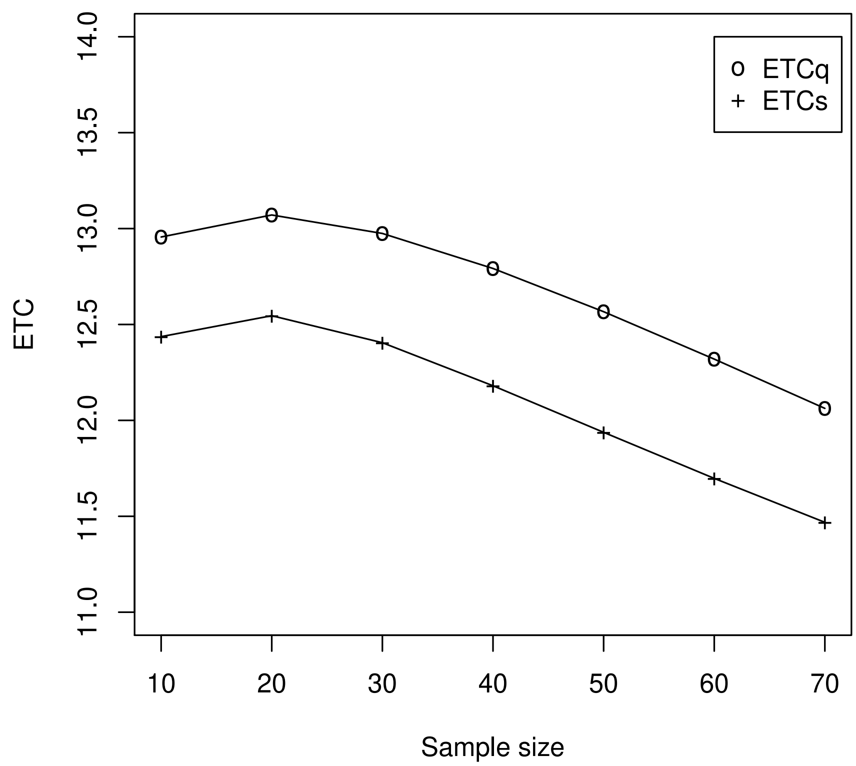

FIGURE 1. Variations of the ETC verses the sample size

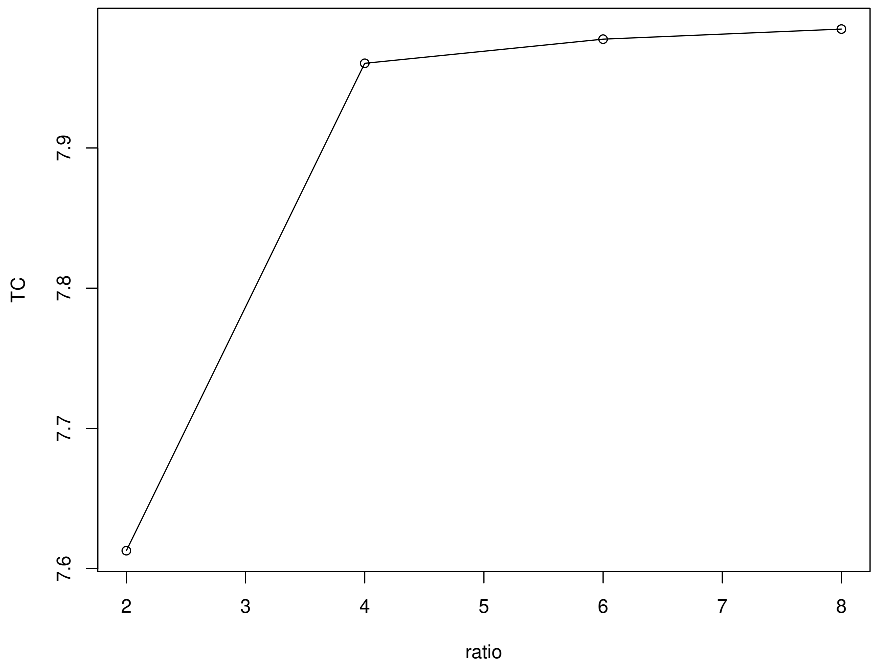

FIGURE 2. Total cost assuming unknown shape parameter with $m = 0.5$ and $r = 5$ for different percentile ratios

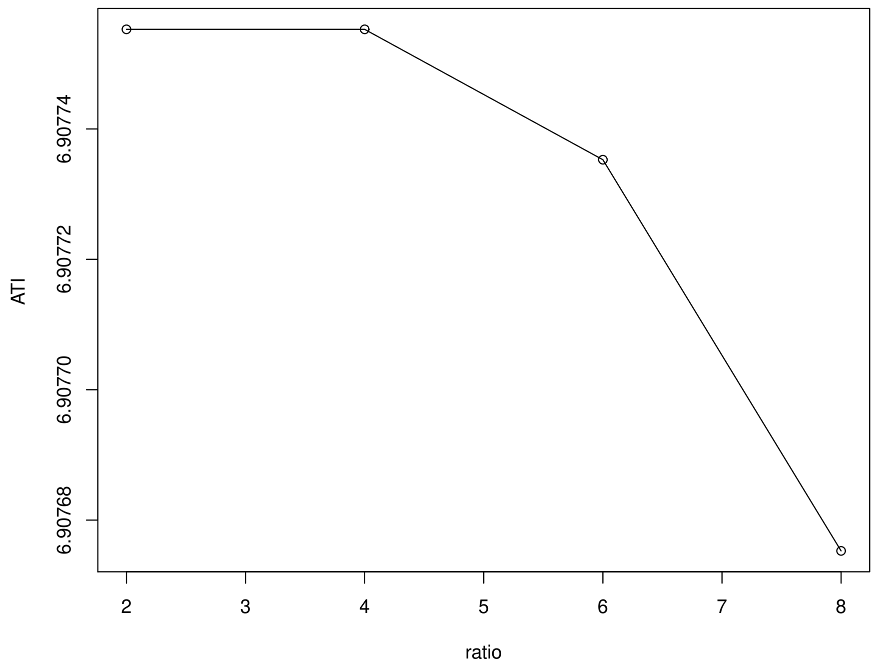

Figure 3. Average total inspection assuming unknown shape parameter with $m = 0.5$ and $r = 5$ for different percentile ratios

Table 1. Bayesian sampling plans assuming different loss functions and priors

Prior distribution	n	c	ETC	percentage increment	percentage saving
			Quadratic loss function		
$\Gamma(a,b)$	21	6	473252	18	34.9
$K\theta^{-1}$	28	9	2.58052×10^6	–	
			Step loss function		
$\Gamma(a,b)$	25	7	266504	40	34.9
$K\theta^{-1}$	32	10	660656	–	

Table 2. Misspecification of gamma prior mean and standard deviation for the quadratic and step loss functions

Misspecification	−30%	−20%	−10%	10%	20%	30%
			Quadratic loss function			
Mean	12.76	14.83	16.72	19.61	20.47	20.88
SD	24.20	21.97	20.05	16.85	15.54	14.41
			Step loss function			
Mean	26.46	31	35.7	44.82	49	52.88
SD	50.30	47.53	43.58	37.29	34.43	31.86

Table 3. The optimal parameters of the Bayesian plan for the Weibull distribution for r=5 and m=0.5

β	t_q/t_{q_0}	ATI	Dd	Dn	AOQ	g,c	TC
0.25	2	1000	500	0.00143765	1.43765×10^{-6}	8,4	2024
	4	1000	926.56	0.000579062	5.79062×10^{-7}	4,1	2865.12
	6	999.99	954.44	0.0221231	0.0000221	2,0	2914.89
	8	999.99	964.95	0.0166504	0.0000166504	2,0	2935.91
0.10	2	1000	530	1.53962×10^{-7}	1.53962×10^{-10}	12,5	2096
	4	999.97	925.05	0.000017051	1.7054×10^{-8}	5,1	2865.07
	6	1000	954.45	9.67503×10^{-6}	9.67503×10^{-9}	5,1	2923.89
	8	1000	964.96	0.00033018	3.3018×10^{-7}	3,0	2938.91
0.05	2	1000	500	3.34929×10^{-10}	3.34929×10^{-13}	15,6	2045
	4	1000	925.06	5.07223×10^{-7}	5.07223×10^{-10}	6,1	2868.13
	6	1000	954.446	2.87355×10^{-7}	2.87355×10^{-10}	6,1	2926.89
	8	1000	964.96	2.15432×10^{-7}	2.15432×10^{-10}	6,1	2947.91
0.01	2	1000	500	1.64815×10^{-15}	1.64815×10^{-18}	21,8	2063
	4	1000	925.06	1.77339×10^{-11}	1.77339×10^{-14}	10,2	2880.13
	6	1000	954.45	2.58459×10^{-10}	2.58459×10^{-13}	8,1	2932.89
	8	1000	964.96	1.93645×10^{-10}	1.93645×10^{-13}	8,1	2953.91

Table 4. The optimal parameters of the Bayesian plan for the Weibull distribution for r=5 and m=0.1

β	t_q/t_{q_0}	ATI	Dd	Dn	AOQ	g,c	TC
0.25	2	–	–	–	–	–	–
	4	999.973	964.943	0.113803	0.000113803	3,2	2938.97
	6	999.918	973.886	0.223859	0.000223859	2,1	2953.91
	8	999.932	978.063	0.186395	0.000186395	2,1	2962.24
0.10	2	–	–	–	–	–	–
	4	999.973	964.943	0.113803	0.000113803	3,2	2938.97
	6	999.918	976.886	0.223859	0.000223859	2,1	2953.91
	8	999.932	978.063	0.186395	0.000186395	2,1	2962.24
0.05	2	–	–	–	–	–	–
	4	999.973	964.943	0.113803	0.000113803	3,2	2938.97
	6	999.918	973.886	0.223859	0.000223859	2,1	2953.91
	8	999.932	978.063	0.186395	0.000186395	2,1	2962.244
0.01	2	–	–	–	–	–	–
	4	999.973	964.943	0.113803	0.000113803	3,2	2938.97
	6	999.918	973.886	0.223859	0.000223859	2,1	2953.91
	8	999.998	978.1	0.00518982	5.18982×10^{-6}	3,1	2965.20

Table 5. The optimal parameters of the Bayesian plan for the Weibull distribution for r=10 and m=0.5

β	t_q/t_{q_o}	ATI	Dd	Dn	AOQ	g,c	TC
0.25	2	1000	500	1.53962×10^{-7}	1.53962×10^{-7}	6,5	2018
	4	1000	925.063	0.000579062	5.79062×10^{-7}	2,1	2856.13
	6	999.99	954.44	0.0219018	0.0000219018	1,0	2911.89
	8	999.99	964.953	0.0166504	0.0000166504	1,0	2932.91
0.10	2	1000	500	1.53962×10^{-7}	1.53962×10^{-10}	6,5	2018
	4	1000	925.063	5.07223×10^{-7}	5.07223×10^{-10}	3,1	2859.13
	6	1000	954.446	2.87355×10^{-7}	2.87355×10^{-10}	3,1	2917.89
	8	1000	964.953	2.15432×10^{-7}	2.15432×10^{-10}	3,1	2938.91
0.05	2	1000	500	3.27347×10^{-10}	3.27347×10^{-13}	8,7	2024
	4	1000	925.063	5.07223×10^{-7}	5.07223×10^{-10}	3,1	2859
	6	1000	954.446	2.87355×10^{-7}	2.87355×10^{-10}	3,1	2917.89
	8	1000	964.953	2.15432×10^{-7}	2.15432×10^{-10}	3,1	2938.91
0.01	2	1000	500	7.51059×10^{-17}	7.51059×10^{-20}	11,8	2033
	4	1000	925.063	1.77339×10^{-11}	1.77339×10^{-14}	5,2	2865.13
	6	1000	954.446	2.58459×10^{-10}	2.58459×10^{-13}	4,1	2920.89
	8	1000	964.957	1.93645×10^{-10}	1.93645×10^{-13}	4,1	2941.91

Table 6. The optimal parameters of the Bayesian plan for the Weibull distribution for r=10 and m=0.1

β	t_q/t_{q_o}	ATI	Dd	Dn	AOQ	g,c	TC
0.25	2	–	–	–	–	–	–
	4	1000	964.957	1.83807×10^{-10}	1.83807×10^{-13}	6,5	2947.94
	6	1000	973.932	4.03624×10^{-6}	4.03624×10^{-9}	3,2	2956.86
	8	1000	978.101	0.000149585	1.49585×10^{-7}	2,1	2962.20
0.10	2	–	–	–	–	–	–
	4	1000	964.957	1.83807×10^{-10}	1.83807×10^{-13}	6,5	2947.91
	6	1000	973.932	4.03624×10^{-6}	4.03624×10^{-9}	3,2	2956.86
	8	1000	978.101	0.000149585	1.49585×10^{-7}	2,1	2962.20
0.05	2	–	–	–	–	–	–
	4	1000	964.957	1.83807×10^{-10}	1.83807×10^{-13}	6,5	2947.91
	6	1000	973.932	4.03624×10^{-6}	4.03624×10^{-9}	3,2	2956.86
	8	1000	978.101	0.000149585	1.49585×10^{-7}	2,1	2962.20
0.01	2	–	–	–	–	–	–
	4	1000	964.957	1.83807×10^{-10}	1.83807×10^{-13}	6,5	2947.91
	6	1000	973.932	4.03624×10^{-6}	4.03624×10^{-9}	3,2	2956.86
	8	1000	978.101	0.000149585	1.49585×10^{-7}	2,1	2962.20

Table 7. Comparison of group and single sampling plans for the Weibull distribution

β	t_q/t_{q_o}	Single plan	Group plan when r=10
0.25	2	7956.25	2018
	4	2862.9972	2856.13
0.10	2	10153.34	2018
	4	2861.79	2859.13
	6	2918.77	2917.89
	8	2939.57	2938.91
0.05	2	15020.27	2018
	4	2861.43	2859
	6	2919.21	2917.89
	8	2940.26	2938.91
0.01	2	35085.74	2033
	4	2868.13	2865.13
	6	2921.25	2920.89
	8	2942.19	2941.91

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