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Research Note

A very fast method for guaranteed generation of one facet for 0-1 knapsack polyhedron

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Abstract. The 0–1 knapsack polyhedron as the most basic relaxation of a 0–1 integer program has attracted the attention of many researchers over the years. We present a very fast method that is guaranteed to generate one facet for the 0–1 knapsack polyhedron. Unlike lifting of cover inequities, our method does not require an initial minimal cover or a predetermined lifting sequencing, and its worst-case complexity is linear in some variables. Therefore, with minimal computational burden, it can be used to generate a potentially strong valid inequality based on any 0–1 relaxation of a general (Mixed) Integer Program (M)IP. Such valid inequalities can be added to the (M)IP problem prior to solving, or given their low computational cost, can be generated during solving the (M)IP, checked to see if they separate the incumbent fractional solution, and added to the problem if they do.

1. Introduction

The 0–1 knapsack polyhedron as the most basic relaxation of a 0–1 Integer Program (IP) has attracted the attention of many researchers over the years. In particular, developing facets for the 0–1 knapsack polyhedron has been extensively addressed over the past several decades see [1–16] among many others. Most of the work in this direction has been focused on the characterization of facets arising from lifting of the so-called minimal cover inequalities. Properties of the formal lifting procedure presented in [1,3,6,17–20] and its resulting facets have been studied in several of the aforementioned references. The lifting procedure is dependent on an initial minimal cover inequality, and in most cases, the sequence of the variables chosen for lifting. Zemel [5] showed that for a 0–1 knapsack problem with n variables, given a minimal cover inequality of s variables, and a sequence of the n–s variables to lift, this lifting procedure can be performed in $O(ns)$ time. Del Pia et al. [21] proposed a new approach to generate valid inequalities for a special multiple knapsack set, called the Totally-Ordered Multiple Knapsack Set (TOMKS). Bazzi et al. [22] address the issue of relaxation of exponential size and obtain LP relaxations of quasi-polynomial size that are at least as strong as that given by the knapsack cover inequalities. Vitor and Easton [23] proposed

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both approximate and exact merged knapsack cover inequalities for knapsack and multiple knapsack IP. The approximate merged knapsack cover inequalities can be performed in $O(n \log n)$ time. Bienstock et al. [24] show an algorithm for efficiently separating inequalities with coefficients in $\{0, 1, ..., \pi\}$ for any fixed π up to an arbitrarily small error. The main contribution of the study is developing approximate separation oracles for valid inequalities and generalizing for MinKnap. Shim et al. [25] identified strong facets defining inequalities for the master knapsack polytope. Their computational experiments interestingly show that $1/k$-facets for small values of k ($k \leq 4$) are strong facets for the knapsack polytope. Chopera et al. [26] also focus on $1/k$-inequalities for k dividing 6 or 8. They obtain a concise characterization of the superadditive version of knapsack inequalities which allows us to efficiently separate the inequalities. Letchford and Souli [27] show how one of the earliest lifting procedures, due to Balas, can be improved to yield both stronger and more general lifted cover inequalities. Furthermore, Letchford and Souli [28] used lifting and presented two approximate procedures based on mixed-integer rounding and superadditivity. In addition, the proposed procedures that can yield non-trivial facet-defining inequalities. From the practical perspective of solving 0–1 programs, a fast method that is guaranteed to generate a facet-defining inequality for any given 0–1 knapsack set, and is free of choosing an initial cover inequality or the lifting sequence, is of significant interest. In this paper, we present such a method. Our method is very fast (its worst case complexity is $O(n)$), and therefore, with minimal computational burden, can be used to generate a potentially strong valid inequality based on any 0–1 relaxation of a general (Mixed) Integer Program (M)IP. Such valid inequalities can be added to the (M)IP problem prior to solving, or given their low computational cost, can be generated during solving the (M)IP, checked to see if they separate the incumbent fractional solution, and added to the problem if they do.

2. Facet generation method

Let X be the set of 0–1 points in the knapsack problem defined as:

$$X = \left\{(x_1, x_2, ..., x_n) : \sum\nolimits_{j=1}^{n} a_j x_j \geq b; \quad x_j = 0, 1,\ j = 1, 2, ..., n\right\}. \tag{1}$$

Without loss of generality, we assume $0 < b < \sum_{j=1}^{n} a_j$, and $0 < a_1 \leq a_2 \leq ... \leq a_n$. Note if the 0–1 knapsack set is defined by a $\leq$ constraint, it can be converted to the set X.

Now, let k_1 and k_2 be the minimum numbers satisfying $\sum_{j=1}^{k_1} a_j \geq b$ and $\sum_{j=n-k_2+1}^{n} a_j \geq b$. If $k_1 = k_2 = k$, then the sum of every k of coefficients a_j, $j = 1, ..., n$, is equal to or greater than b and hence the set X can also be written as $\{(x_1, x_2, ..., x_n): \sum_{j=1}^{n} x_j \geq k; x_j = 0, 1, 2, ..., n\}$. In other words, in this case, the inequality $\sum_{j=1}^{n} a_j x_j \geq b$ can be replaced with the cover and stronger inequality $\sum_{j=1}^{n} x_j \geq k$. In addition, $conv(X) = \{(x_1, x_2, ..., x_n) : \sum_{j=1}^{n} x_j \geq k; 0 \leq x_j \leq 1, j = 1, ..., n\}$, where $conv(X)$ denotes the convex hull of X. As a result, the inequality $\sum_{j=1}^{n} x_j \geq k$ is the only non-trivial facet of $conv(X)$.

However, in general $k_1 > k_2$. In this case, we present a method which uses the defining inequality:

$$\sum\nolimits_{j=1}^{k_1} a_j \geq b. \tag{2}$$

To find at least one facet-defining cover inequality of X. Define $\bar{x}_j, j = 1, ..., n$, as complement of x_j, i.e., $\bar{x}_j = 1 - x_j$. For any $t \in \{1, ..., n\}$, the term $a_t \bar{x}_t$ can be added to the left-hand side of Inequality (2), to obtain a valid relaxation of Inequality (2), i.e., $\sum_{j=1, j \neq t}^{n} a_j x_j \geq b - a_t$ for X. Now for this new relaxed inequality, define k_1 and k_2 just as defined above. If $k_1 = k_2$ the valid cover inequality has been obtained, else we repeat this process until $k_1 = k_2$. Then setting $k = k_1 (= k_2)$ the final inequality is:

$$\sum\nolimits_{j=1, j \notin S}^{k_1} x_j \geq k, \tag{3}$$

where S is the set of t's for which $a_t \bar{x}_t$ has been added to the left-hand side of Inequality (2). It can be seen that in the worst case complexity of this method is $O(n)$; however, it may terminate after adding much fewer $a_t \bar{x}_t$ terms than n.

The terms $a_t \bar{x}_t$ can be added to the left-hand side of Inequality (2) in different orders; hence, different valid cover inequality may be obtained. For the priority rule of "smallest a_j first", we show that the resulting cover inequality defines a facet of $conv(X)$.

Proposition 1. In generation of Inequality (3), if the "smallest a_j first" priority rule is followed, then the resulting inequality either defines a facet of $conv(X)$, or is an implicit equality for X, which can be used to fix k variables.

Proof. Suppose, using this priority rule, we have obtained the inequality $\sum_{j=T+1}^{n} a_j x_j \geq b - \sum_{j=1}^{T} a_j$ and hence, the resulting valid cover inequality is:

$$\sum\nolimits_{j=T+1}^{k_1} x_j \geq k. \tag{4}$$

Note that if $k = n-T$, then Inequality (4) is satisfied at equality by all points in X, and therefore, Inequality (4)

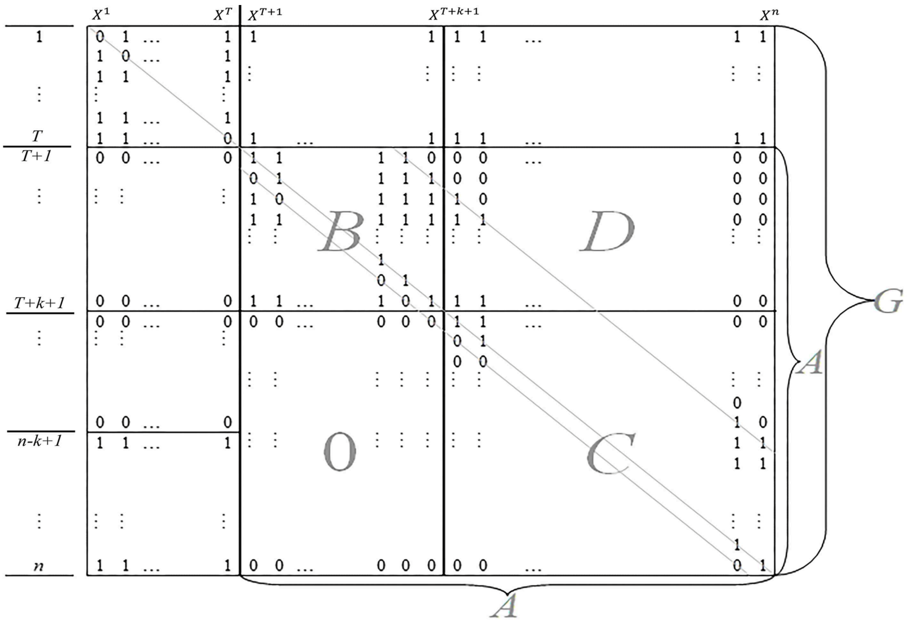

Figure 1. Matrix G.

becomes and implicit equality for X and we can fix $\bar{x}_j, j = T+1, ..., n$ and reduce the problem to a smaller problem.

Now, we consider the case where $1 \leq k \leq n-T-1$. Note that, during the process of determining k, k_1, k_2, and k_2 are non-increasing. Therefore, since $k_1 > k_2$, in the last iteration for determining k, it is k_1 that is decreased, and k_2 does not change. Therefore, we have:

$$\sum\nolimits_{j=n-k+1}^{n} a_j \geq b - \sum\nolimits_{j=1}^{T-1} a_j. \tag{5}$$

Now we derive a lower bound on the number of points in X that are on the hyperplane of cover inequality (4), i.e., the points in X that satisfy $\sum_{j=T+1}^{n} x_j = k$. To this end, consider two sets of points $(x_1, ..., x_n)$ in X: The first set contains T points. For $i = 1, ..., T$, construct the point i in this set by defining $P_i = \{1, ..., T\} \backslash \{i\}$ and then setting $x_j = 1$, $j \in P_i$; $x_j = 1$, $j = n-k+1, ..., n$; and $x_j = 0$ for all other j. For each of these points, we have $\sum_{j=T+1}^{n} x_j = k$, and also:

$$\sum\nolimits_{j=1}^{n} a_j x_j = \sum_{j \in P_i} a_j + \sum\nolimits_{j=n-k+1}^{n} a_j$$

$$\geq \sum_{j \in P_i} a_j + b - \sum\nolimits_{j=1}^{T-1} a_j, \tag{6}$$

where the last inequality is based on (5). Since $a_1 \leq a_2 \leq ... \leq a_n$, we have $\sum_{j \in P_i} a_j \geq \sum_{j=1}^{T-1} a_j$, and hence Inequality (6) implies $\sum_{j=T+1}^{n} a_j x_j \geq b$, i.e. all points in the first set are in X.

The second set of points are constructed as follows: Construct point i in this set by selecting a subset of size k of $\{T+1, ..., n\}$, named Q_i, and setting $x_j = 1$, $j = 1, ..., T$; $x_j = 1$, $j \in Q_i$; and $x_j = 0$ for all other j. Now since $k_1 = k$, we have $\sum_{j=T+1}^{T+k} a_j \geq b - \sum_{j=1}^{T} a_j$. Also, since $a_1 \leq a_2 \leq ... \leq a_n$, we have $\sum_{j \in Q_i} a_j \geq \sum_{j=T+1}^{T+k} a_j$. Therefore, for each point i in the second set we have $\sum_{j=1}^{n} a_j x_j = \sum_{j=1}^{T} a_j + \sum_{j \in Q_i} a_j \geq \sum_{j=1}^{T} a_j + b - \sum_{j=1}^{T} a_j = b$, hence the points in X. For each of there points, we also have $\sum_{j=T+1}^{n} x_j = k$. The number of points in the second set is equal to the number of Q_i's, which is $\begin{pmatrix} n-T \\ k \end{pmatrix}$. The sum of number of points in the aforementioned two sets, i.e., $\begin{pmatrix} n-T \\ k \end{pmatrix} + T$, is a lower bound on the number of points in X that are on the hyperplane $\sum_{j=T+1}^{n} x_j = k$. Since $1 \leq k \leq n-T-1$, we have $\begin{pmatrix} n-T \\ k \end{pmatrix} + T = \frac{(n-T)!}{k!(n-T-k)!} + T \geq n-T+T = n$. Therefore, we have at least n points in X on the hyperplane of the valid cover Inequality (4).

Now, since the set X is full dimensional, to prove that Inequality (4) defines a facet for $conv(X)$, it is sufficient to show that, in the above two sets of points, there are at least one set of n linearly independent points. For this purpose, define a matrix $G_{n \times n}$ in which each column is a point selected from these two sets of points, as described below: Denote the gth column of G by $X^g = [x_1^g, x_2^g, ..., x_n^g]$. The first T columns of G are all the points in the first set arranged such that $X_g^g = 0, g = 1, ..., T$. For $g = T+1, ..., n$, the vector X^g is selected from the points of the second set. Each X^g from the second set has k ones in the rows $T+1, ..., n$ and since $1 \leq k \leq n-T-1$, the point X^g has at least one 1 and one 0 in these rows. The T vectors of the first set of points and the $n-T$ selected vectors from the second set and their arrangement in G are shown in Figure 1.

We show that the n points in matrix G are linearly independent by proving that G is invertible. Consider the partitioning of shown in Figure 1. If we multiply each row i, $i = T+1, ..., n$ by $-\frac{1}{k}$ and add it to every one of the first T rows of G, we obtain the matrix $\bar{G} = \begin{bmatrix} -I & 0 \\ * & A \end{bmatrix}$. Therefore, G is invertible if A is invertible.

Based on Figure 1, is invertible if B and C are invertible. However, it is easy to see that B and C are both invertible. Therefore, G is invertible, which concludes the proof. $\square$

3. Conclusion

In this paper, we've introduced a new method for generating facets for the 0–1 knapsack polyhedron. Unlike traditional approaches, our method doesn't require initial minimal covers or predetermined sequences, simplifying the process significantly. Our method is highly efficient, with linear worst-case complexity in some variables, making it suitable for integration into solving procedures for general (Mixed) Integer Programs (MIPs). It swiftly generates potentially strong valid inequalities, enhancing problem-solving effectiveness. Through rigorous analysis, we've demonstrated the reliability and effectiveness of our method. It offers a practical solution for efficiently deriving facet-defining inequalities for 0–1 knapsack sets, addressing optimization needs effectively. Future research can explore extensions and refinements to tackle more complex problems, enhancing its utility in real-world optimization scenarios.

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Biographies

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