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Solution of an economic production quantity model using the generalized Hukuhara derivative approach

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Abstract
In this study, an Economic Production Quantity (EPQ) model with deterioration is developed where the production rate is stock dependent and the demand rate is unit selling price and stock dependent. The low unit selling price and more stocks correspond high demand but more stock corresponds to slow production because of the avoidance of unnecessary stocks. First of all, we develop the production model by solving some ordinary differential equations having deterministic profit function under some specific assumptions. Later, we develop the fuzzy model by solving the fuzzy differential equations using generalized Hukuhara (gH) derivative. In fact, the differential equation of the model has been split into two parts namely gH(L-R) and gH(R-L) on the basis of left (L) and right (R) α-cuts of fuzzy numbers for which the problem itself is transformed into multi-objective EPQ problem. A new formula of aggregation of several objective values obtained at different aspiration levels has been discussed to defuzzify the fuzzy multi-objective problems. We solve the crisp and fuzzy models using LINGO software. Numerical and graphical illustrations confirm that the model under gH derivative of (R-L) type contributes more profit which is one of the basic novelties of the proposed approach.

1. Introduction

The basic objective of the supply chain modelling is to make sure the uninterrupted service or flow of goods from manufacturer or dealer to consumer through all possible. Also, we know the production rate and demand rate are two basic components related to the study of the inventory control problem. Two popular approaches for describing the inventory control problems are Economic Order Quantity (EOQ) model introduced by Harris [1] and Economic Production Quantity (EPQ) model formulated by Taft [2]. The main objective is to find the optimal production quantity or optimal order quantity of the model that minimizes the cost objective function or maximizes the profit function with respect to some real constraints. Traditionally, all the models are assumed to be deterministic because the associated parameters are deterministic in nature. But, in reality, some of the parameters may be flexible (non-random uncertainty) in nature.

The parameter like demand rate is a vital component in the theory of inventory control problems. It would be a matter of easiness to the decision maker to control the inventory problem if the information regarding the demand pattern is available in crystal clear form. But in practice, the demand of certain product in the market fluctuates within finite specific range. Also, various costs and revenues related to the production and marketing procedures may fluctuate depending upon several factors on which the decision maker has no control. So, uncertain decision-making policies are coming into the situation.

Also, the earliest trends were to assume the constant demand with no shortage to develop the lot sizing modelling [3,4]. Later, the literature regarding EOQ and EPQ modelling gradually enriched through incorporating deterioration, partial and fully backlogged shortage, and credit-linked demand [5,6] respectively were discussed. The present article has solved the following research problems:

(i) What is the optimal production-marketing strategy of a deteriorating inventory with profit maximization objective function when demand or consumption rate depends upon the unit selling price and the displayed stock?
(ii) If the non-random uncertainty associated with parameters and decision variables of a model is not

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ignored, in what extent the fuzzy counterpart be the best fitted approach of modelling?
(iii) How much the Fuzzy Differential Equation (FDS) be helpful for the complicated and realistic model via new defuzzification aggregation method in optimization?

Motivating from the above research problems, the proposed model of inventory control management is developed under some very realistic assumptions. The demand rate is assumed to be a function of unit selling price and real time stock of the items and the production rate is also real time stock dependent. Generally, low selling price increases the demand pattern of the customer in a developing country like India [7-9]. Although, big size of the inventory in the showrooms increases enthusiasm and attraction of the customers towards purchasing the products. Indeed, to grow a sustainable network of supply to fulfil the customers' demand aiming for maximum profitability, the control on the production rate is made such that no items are left unsold. However, to analyse the non-random uncertainty of the various parameters of the model we have gone through the FDE approach under generalized Hukuhara (gH) derivative of two different types (L-R & R-L) is adopted to describe the fuzzy model. A new defuzzification method in term of aggregation of several objective values obtained at different aspiration level has been formulated to score the numerical results of the fuzzy model with interval representation.

The organisation of the remaining part of the paper is described as follows: The brief literature review related to the proposed research objectives is carried out in Section 2. After that, a detailed discussion on general overview on FDE and Lagrange's multiplier method to solve differential equation is represented in Section 3. The notations and assumptions are explained in Section 4. Section 5 includes the crisp and fuzzy mathematical model, solution algorithm etc. In Section 6, numerical illustrations and in Section 7, graphical illustrations are done. Finally, a concluding remark is given in Section 8 followed by a scope of future work.

2. Literature review

In this paper, a theoretical accumulation of different research domains has been carried out for a meaningful managerial perspective. Following the questions mentioned in the introduction section, the present section is going through a brief review on three different research disciplines, namely, popular lot-sizing modelling (with a special concern on the key words price, stock, deterioration), inventory modelling under uncertainty, FDE and its application on inventory control problem. Thus, the present section contains five different subsections presenting the literature review of three different disciplines, the sense of accumulation of the ideas and the major contribution of the current article.

2.1. Popular lot-sizing models

In reality, the demand of the produced item depends on several factors, the unit selling price of items is one of such important issues involved in the production and retailing business. Considering the demand as a function of unit selling price [10], the subsequent worthy works on deterioration [11], fully backlogged shortage [12], no shortage [13] and discount policy with the price depended characteristic of demand rate [14] etc. have been studied in modelling rigorously. Another vital issue is the stock of the product in the inventory cycle. Arbitrary large amount stock may result to the ultimate loss of the retailer due to unsellable items. Moreover, the presence of moderate number of displayed stocks in showroom makes a positive result on demand, creating more interest of the customers towards those particular products. Incorporating these facts in inventory modelling researchers like Giri et al. [15], Mondal et al. [16] etc. studied the inventory models with stock dependent Later, more improvements in this regard have been done by incorporating the sense of dependent damage rate [17], shortage [18] and time varying holding cost [19] along with the presence of stock depended demand exclusively. The study of the joint impact of stock and price on demand is also considered by Datta and Pal [20] and Teng and Chang [21] considered deterioration of items in this context. Sana [22] discussed the negative influence of uncontrolled large stock on the demand under the consideration of stock and price dependent demand of deteriorating item. Khan et al. [23] addressed the price discount facility for advance payment in the study of an EOQ model of deteriorating item with price and stock dependent demand allowing partial backlogged shortage. Indeed, some works on recent trends in supply chain models for Single Set up Single Delivery (SSSD) [24] and Single Set up Multi Delivery (SSMD) [25-27] may be considered over here.

2.2. Inventory model under fuzzy uncertainty

We know, non-random uncertainty of facts can be described by fuzzy set theory [28] that is being used frequently recent times. Bellman and Zadeh [29] advocated for the fuzzy decision making as very fruitful application of the proposed theory. Park [30] was the pioneer to study the lot-sizing problem under the fuzzy decision-making phenomena. In learning theory for decision making, some interesting works are dense fuzzy set [31], dense fuzzy Neutrosophic set [32], Lock fuzzy set [33], Moonsoon fuzzy set [34] etc. The applications of the experience-based learning approaches in the study of lot-sizing problem were addressed by Maity et al. [35-37], Karmakar et al. [38,39] in the light of the theory of dense and lock fuzzy number. Rahaman et al. [40] contributed a study to find out the joint impact of memory and experience-based learning on the decision of optimization for an EOQ model. Also, De and Mahata [41,42] explore the sense of cloudy fuzzy sets and its application on the inventory problems. Very recently, a MCGDM problem regarding sustainable transport investment selection with respect to knowledge measure and generalized entropy has been discussed by Aghamohagheghi et al. [43] in interval valued Pythagorean fuzzy phenomena. Also, Guleria and Bajaj [44] contributed a very relevant study on the theoretical properties of the T-spherical soft set and its application on decision making problems.

2.3. FDE in inventory management problem

The topics on the FDEs have been rapidly grown in recent years. There are many approaches to solve the FDEs. In this context, the most important job was to introduce the definition of fuzzy derivative. The concepts of the fuzzy derivative were first initiated by Chang and Zadeh [45], whereas the concept of FDE was first formulated by Keleva [46]. In FDE, all the derivatives are characterized by either Hukuhara or generalized derivatives. The Hukuhara derivative has some limitations because the solution turns into imprecise as time goes on. Bede et al. [47] exhibited that a large class of boundary value problem has no solution if Hukuhara derivative is applied. To remove this difficulty and deficiency the concept of gH derivative was developed [48,49] and FDE is utilized using this concept. Some researchers transformed the FDE into the corresponding fuzzy integral equation and solved it [50]. Another common approach to solve the FDE is Zadeh's extension principle [51,52]. To solve the linear FDE, Allahviranloo and Ahmadi [53] used Laplace transformation approach. Mondal and Roy [54] solved the linear FDE by Lagrange multiplier method using Hukuhara derivative. Recently, Rahaman et al. [55] has added a new literature exploring a new method of solving difference equation under Gaussian fuzzy environment.

For the application of inventory management problems, the presence of fuzzy demand rate leads to FDE for instantaneous state of the inventory level. In comparison, till now the FDE is of little use to formulate and to solve the various fuzzy inventory model. Das et al. [56] gave two methods of solution of an initial valued first order FDE and described its application on a fuzzy EOQ model using fuzzy extension principle and centroid formula for defuzzification. Guchhait et al. [57] formulated a production inventory model with fuzzy demand and production rate in an imperfect production process using the FDEs with the interval valued genetic algorithm approach. A production recycling model is formulated and solved by Mondal et al. [58]. An EPQ model with partial trade credit policy in fuzzy environment was studied as an application of gH derivative approach of FDE by Majumder et al. [59]. Mondal [60] described a solution of the basic inventory model in fuzzy and interval environments with FDE and Inter Differential Equation (IDE) approach. Debnath et al. [61] introduced a sustainable fuzzy EPQ model with the demand as type-2 fuzzy number using gH derivative of FDE. Recently, Rahaman et al. [62] have studied a memory motivated fuzzy EPQ model in fuzzy fractional differential equation under Riemann-Liouville sense of fractional derivative.

2.4. Research gaps and motivations

Completing a comprehensive survey of existing literature in the earlier mentioned research domain, the following Completing a comprehensive survey of existing literature in the earlier mentioned research domain, the following lacks are spotted which are tried to fulfill in the present study:

(i) A vast literature on the stock and price dependent demand consideration to construct the EOQ models are available.
Up to the author's knowing, only one article [63] on the study of an EPQ model of deteriorating items with stock and price dependent demand and stock dependent production rate is available. But that article emphasized on the memory effect related outcomes in a deterministic phenomenon through fractional calculus. In this paper, the model is developed under the same assumptions on the demand and production rate. But here the objective is quite different from the existing one. The main goal of the present article is to adopt an intelligent decision-making using FDE.

(ii) There are huge collections of literature on the fuzzy inventory models. But, most of them were developed on various defuzzification techniques avoiding the rate of changes. Thus, the paper related to the fuzzy differential approach to solve the inventory problem is little rare in the existing literature. There are only few papers (described in Table 1) on inventory with FDE.

(iii) GH derivative approach to solve the FDE is very meaningful way to quantify the changes of dependent fuzzy variables with respect to the dependent variables. But, up to the author's knowledge, very few works are identified yet.

So, we consider our proposed EPQ model in fuzzy environment utilizing fuzzy differential equation under gH derivative. Dealing with fuzzy variables and fuzzy calculus, this article has been reduced to a multi-objective decision-making problem and finally the problem is solved with the help of new defuzzification rule.

2.5. Major contribution

The basic novelty is that all the cost components, the deterioration rate and the unit selling price associated to the model assume triangular fuzzy numbers. However, we adopt the FDE approach with the help of the extension of Lagrange's method to describe fuzzy mathematical problem and solve the fuzzy model via gH derivatives of two kinds namely L-R and R-L types. Also, a new defuzzification technique is studied with some aggregation rules to find the crisp equivalent problem of the more complex fuzzy objective function of the proposed model. A comparative analysis over the numerical results of the crisp model and the fuzzy model under different circumstances is done that focuses the managerial insights as well.

3. Preliminaries

3.1. Fuzzy sets and fuzzy calculus

Definition 1. A fuzzy set $\tilde{A}$ on a crisp set A is an ordered pair given by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}$, where x is the element of A and $\mu_{\tilde{A}}(x)$ is the corresponding member function and $\mu_{\tilde{A}}(x) \in [0,1]$ for all $x \in A$.

Definition 2. The α-cut of the fuzzy set $\tilde{A}$ of X is given by $A^{\alpha} = \{x: \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\}$. By definition the α-cut is a crisp set. This is also called the interval of confidence, α-level set etc.

Definition 3. The fuzzy number is a fuzzy set given by $F: R \to [0,1]$ which satisfies the following properties:
(i) F is upper semi-continuous;
(ii) $F(x) = 0$ for $x < \gamma$ and $x > \delta$ for some γ, δ;
(iii) There exist two real numbers α, β such that $\gamma \leq \alpha \leq \beta \leq \delta$ such that:
(a) $F(x)$ is monotonic increasing on $[\gamma, \alpha]$;
(b) $F(x)$ is monotonic decreasing on $[\beta, \delta]$;
(c) $F(x) = 1$ for $\alpha \leq x \leq \beta$.

Definition 4. In parametric form a fuzzy number $F(x)$ is given by the pair (F_1, F_2) of functions $F_1(r), F_2(r)$, $0 \leq$

Table 1. Major contribution on FDE in inventory management problems.

Refs.	**Model type**	**Demand rate**	**Production rate**	**Deterioration**	**Solution approach**
Das et al. [56]	Bi- level EOQ	Time dependent fuzzy L-R type	--	Yes	Fast and Elitist Multi-Objective Genetic Algorithm (MOGA) and Interactive fuzzy decision making
Guchhait et al. [57]	EPQ	Time and selling price dependent fuzzy TFN	Fixed fuzzy TFN	--	Interval Compared Genetic Algorithm
Mondal et al. [58]	EPQ	Displayed inventory dependent	Fixed fuzzy	--	Modified Graded Mean Integration Value (MGMIV) and Fuzzy Preference Ordering of Interval (FPOI)
Majumder et al. [59]	EPQ	Decreasing function of time	Fuzzy constant	Yes	Generalized Hukuhara derivative approach
Mondal [60]	EOQ	Fuzzy constant	--	--	Fuzzy Differential and Interval Differential approach
Debnath et al. [61]	SFEPQ	Stock and Production Price dependent type-2 fuzzy number	Linearly dependent on demand	--	Generalized Hukuhara derivative approach
Rahaman et al. [62]	Fractional EPQ	Fuzzy constant	Fuzzy constant	Yes	Riemann-Liouville fractional differential equation under fuzzy uncertainty
This paper	EPQ	Stock and price dependent	Stock dependent	Yes	Generalized Hukuhara derivative approach

$r \le 1$, where the functions $F_1(r)$ and $F_2(r)$ satisfying the following conditions:

1. $F_1(r)$ is a bounded, monotonic increasing and left continuous function;
2. $F_2(r)$ is a bounded, monotonic decreasing and right continuous function;
3. $F_1(r) \le f_2(r);\ 0 \le r \le 1$.

Obviously, a crisp number, say x as particular case of the fuzzy number can be written in parametric form as $x = (x, x)$.

Properties of the fuzzy numbers

Let $\zeta = (\zeta_1(r), \zeta_2(r))$ and $\eta = (\eta_1(r), \eta_2(r))$ be two fuzzy numbers. Then, the arithmetic operations are given as follows:

I. $\zeta = \eta$ if $\zeta_1(r) = \eta_1(r)$ and $\zeta_2(r) = \eta_2(r)$,
II. $\zeta + \eta = (\zeta_1(r) + \eta_1(r), \zeta_2(r) + \eta_2(r))$,
III. $\zeta - \eta = (\zeta_1(r) - \eta_1(r), \zeta_2(r) - \eta_2(r))$,
IV. $\kappa\zeta = (\kappa\zeta_1(r), \kappa\zeta_2(r))$ for $k > 0$, and
$\kappa\zeta = (\kappa\zeta_2(r), \kappa\zeta_1(r))$ for $k < 0$.

Definition 5. Let θ, ϕ be two fuzzy numbers. If there exists a fuzzy number ψ such that $\theta = \phi + \psi$, then ψ is called the Hukuhara difference of two fuzzy numbers θ and ϕ and symbolically this is denoted by $\psi = \theta\Theta\phi$. Here, one important thing to remember is that:

$\theta\Theta\phi \ne \theta + (-1)\phi$.

Definition 6. Let θ and ϕ be two fuzzy numbers. Then, the gH difference of these two fuzzy numbers is given as follows:

$$\theta\Theta_g\phi = \psi \Leftrightarrow \begin{cases} (i)\ \theta = \phi \oplus \psi, \\ \text{or } (ii)\ \phi = \theta \oplus (-)\psi. \end{cases}$$

Then:

$\psi_L(\alpha) = \min\{\theta_L(\alpha) - \phi_L(\alpha), \theta_R(\alpha) - \phi_R(\alpha)\}$,

and

$\psi_R(\alpha) = \max\{\theta_L(\alpha) - \phi_L(\alpha), \theta_R(\alpha) - \phi_R(\alpha)\}$,

where in the parametric form, a fuzzy valued function f on $[a, b]$ is expressed by:

$[f(t)]_\alpha = [f_L(t, \alpha), f_R(t, \alpha)], t \in [a, b], \alpha \in [0,1]$.

Definition 7. Let f be a fuzzy value function defined on (a, b). Then the gH derivative of the function f at t_0 is defined as:

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h)\Theta g\ f(t_0)}{h}.$$

Now, there are two different types of gH derivative. Suppose in the parametric form, a fuzzy valued function f on $[a, b]$ is expressed by:
$[f(t)]_\alpha = [f_L(t, \alpha), f_R(t, \alpha)], t \in [a, b], \alpha \in [0,1]$. Then:
(1) If $[f'(t_0)]_\alpha = [f_L'(t_0, \alpha), f_R'(t_0, \alpha)]$, then $f(t)$ is (i)-gH (L-R) differentiable at t_0;

(2) If $[f'(t_0)]_\alpha = [f'_R(t_0,\alpha), f'_L(t_0,\alpha)]$, then $f(t)$ is (ii)-gH(R-L) differentiable at t_0.

3.2. Solution of the differential equations using the Lagrange's multiplier method

Let, the homogeneous differential equations of first order are given by:

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y\\ \frac{dy}{dt} &= a_2x + b_2y\end{aligned}\right\}. \quad (1)$$

After adjusting with λ, Eq. (1) gives:

$$\frac{d(x+\lambda y)}{dt} = (a_1+\lambda a_2)x + (b_1+\lambda b_2)y = (a_1+\lambda a_2)\left(x + \frac{b_1+\lambda b_2}{a_1+\lambda a_2}y\right). \quad (2)$$

Choose the number λ so that:

$$\frac{b_1+\lambda b_2}{a_1+\lambda a_2} = \lambda, \quad (3)$$

which gives two roots, say λ_1 and λ_2.

Then, Eq. (2) reduces to an equation linear in $x+\lambda y$:

$$\frac{d(x+\lambda y)}{dt} = (a_1+\lambda a_2)(x+\lambda y).$$

That gives:

$$x + \lambda y = Ce^{(a_1+\lambda a_2)t}. \quad (4)$$

So, for two distinct roots λ_1 and λ_2 of Eq. (3), Eq. (4) gives a system of simultaneous equations:

$$\begin{cases} x + \lambda_1 y = Ce^{(a_1+\lambda_1 a_2)t}\\ x + \lambda_2 y = Ce^{(a_1+\lambda_2 a_2)t}\end{cases} \quad (5)$$

which gives value of x and y, that is the solution of the system of the ODE.

3.3. Solution of the differential equations using the Lagrange's multiplier method

Here we do slight modification of the above theory. Let the system of differential equations is given by:

$$\left.\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y + c_1\\ \frac{dy}{dt} &= a_2x + b_2y + c_2\end{aligned}\right\}. \quad (6)$$

Now, as per Eq. (3) and approach of Eq. (2) we write:

$$\frac{d(x+\lambda y)}{dt} = (a_1+\lambda a_2)x + (b_1+\lambda b_2)y + (c_1+\lambda c_2).$$

On simplification and taking $\lambda = \lambda_1, \lambda_2$, we obtain:

$$x + \lambda_1 y + \frac{c_1+\lambda_1 c_2}{a_1+\lambda_1 a_2} = Ce^{(a_1+\lambda_1 a_2)t},$$

$$x + \lambda_2 y + \frac{c_1+\lambda_2 c_2}{a_1+\lambda_2 a_2} = Ce^{(a_1+\lambda_2 a_2)t}, \quad (7)$$

which gives value of x and y that is the solution of the system of the ODE.

3.4. New defuzzification formula

Let a fuzzy multi-objective problem having lower objective functions $\{f_1, f_2, \dots, f_n\}$ and that of upper objective functions $\{g_1, g_2, \dots, g_n\}$ obtained from fuzzification of a crisp problem. Also let the individual optimal values of the above objective functions at m aspiration level are $\{f_1^*, f_2^*, \dots, f_n^*\}$ and $\{g_1^*, g_2^*, \dots, g_n^*\}$ with crisp optimal f_0, where,

$$f_1^* = \{f_{11}^*, f_{12}^*, \dots, f_{1m}^*\}, f_2^* = \{f_{21}^*, f_{22}^*, \dots, f_{2m}^*\}, \dots, f_n^* = \{f_{n1}^*, f_{n2}^*, \dots, f_{nm}^*\},$$

and

$$g_1^* = \{g_{11}^*, g_{12}^*, \dots, g_{1m}^*\}, g_2^* = \{g_{21}^*, g_{22}^*, \dots, g_{2m}^*\}, \dots, g_n^* = \{g_{n1}^*, g_{n2}^*, \dots, g_{nm}^*\}.$$

respectively. Now, the individual aggregated value of the lower fuzzy objective functions can be defined as:

$$\bar{l} = \frac{\sum_{i=1}^m \alpha_i \otimes f_{ji}}{\sum_{i=1}^m f_{ji}}, \qquad j = 1.2, \dots, n,$$

Similarly, the individual aggregated value of the lower fuzzy objective functions can be defined as:

$$\bar{r} = \frac{\sum_{i=1}^m \alpha_i \otimes g_{ji}}{\sum_{i=1}^m g_{ji}}, \qquad j = 1,2, \dots, n.$$

Therefore, the relative change in optimal values is $\frac{\bar{r}-\bar{l}}{f_0}$.

Noting that for increasing objective function $\bar{r} > \bar{l}$ and that of decreasing function $\bar{r} < \bar{l}$. Hence, the aggregation formulas for fuzzy multi-objective functions defined by $\bar{f}$ are:

a) If it is maximization function then:

$$\bar{f} = \begin{cases} f_0 + \frac{\bar{r}-\bar{l}}{f_0}, & \text{when } \bar{r} > \bar{l}\\ f_0 - \frac{\bar{r}-\bar{l}}{f_0}, & \text{when } \bar{r} < \bar{l}\end{cases}$$

b) If it is minimization function then:

$$\bar{f} = \begin{cases} f_0 - \frac{\bar{r}-\bar{l}}{f_0}, & \text{when } \bar{r} > \bar{l}\\ f_0 + \frac{\bar{r}-\bar{l}}{f_0}, & \text{when } \bar{r} < \bar{l}\end{cases}$$

4. Notations and assumptions

To describe our proposed model, we use the following notation and assumptions.

4.1. Notations

c_h: Holding cost per unit product ($)
c_0: Set up cost per cycle ($)
c_p: Production cost per unit product ($)
p: Selling price per unit product ($)
K: Production rate (Units) per month
D: Annual demand (Units)

T: Total cycle time (months) (dependent decision variable)

t_1: Production time (months) (independent decision variable)

Q: Highest inventory level (Units) (dependent decision variable)

θ_1: Rate of deterioration in $[0, t_1]$

θ_2: Rate of deterioration in $[t_1, T]$

TAP: Total Average Profit ($/Cycle)

4.2. Assumptions

The following assumptions have been considered to develop the proposed model:

a) The production rate depends on the stock or on hand inventory. Generally, the rate of the production (K) follows a decreasing function, $q(t)$ $K = m - nq(t)$,

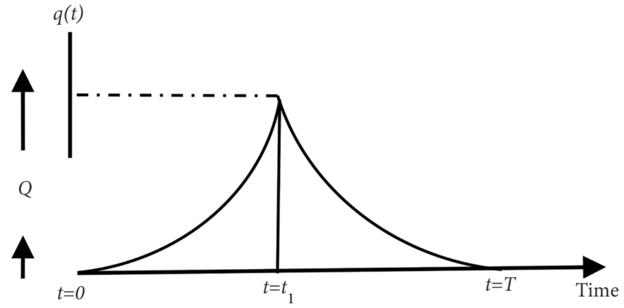

Figure 1. Visualization of the proposed model.

where m, n are positive constants and $q(t)$ is the on-hand inventory or stock;

b) Demand of the produced items depends on price and stock. When selling price is low then the demand increases. Also, presence of the lot of stock makes increasing demand, so, $D = a - bp + cq(t)$, where a, b, c are positive constants and p, is the unit selling price of the product;

c) No shortage is allowed;

d) Both the replenishment rate and lot size are finite;

e) The time horizon is infinite;

f) Deterioration rate is constant and it is θ_1, when $t \in [0, t_1]$ and θ_2, when $t \in [t_1, T]$;

g) Lead time is zero.

5. Formulation of crisp EPQ model

Let, a manufacturing farm starts with the production rate K. At the same time the system is meeting up the demand rate D and facing a deterioration rate θ_1. At time $t = t_1$, the farm stops the production after reaching the sufficient stock of the product. Then, the stock gradually decreases meeting up the demand of the customers and deterioration rate θ_2 during the interval $[t_1, T]$. Figure 1 describes the production model graphically.

5.1. Crisp EPQ model

The governing differential equations of the production-consumption process are given below:

$$\frac{dq(t)}{dt} + \theta_1 q(t) = m - nq(t) - \{a - bp + cq(t)\}, \quad \text{for } 0 \le t \le t_1, \tag{8}$$

$$\frac{dq(t)}{dt} + \theta_2 q(t) = -\{a - bp + cq(t)\}, \quad \text{for } t_1 \le t \le T. \tag{9}$$

The initial, intermediate and terminating information about stock level are given by:

$$\{q(0) = 0, q(t_1) = Q, q(T) = 0. \tag{10}$$

Solving Eqs. (8) and (9) and using Eq. (10), the stock levels at productive and non-productive phases are obtained as:

$$q(t) = \frac{m-(a-bp)(1-e^{-k_1 t})}{k_1}, \quad 0 \le t \le t_1, \tag{11}$$

and

$$q(t) = \frac{(a-bp)}{k_2}\{e^{k_2(T-t)} - 1\}, \quad t_1 \le t \le T. \tag{12}$$

Also, the maximum level of stock at the end of the productive phase is obtained as:

$$Q = \frac{m-(a-bp)}{k_1}\{1 - e^{-k_1 t_1}\}. \tag{13}$$

The values of k_1 and k_2 are given by:

$$\begin{cases} k_1 = \theta_1 + n + c, \\ k_2 = \theta_2 + c. \end{cases} \tag{14}$$

Also, using the continuity conditions given in Eqs. (11) and (12), the relationship between the independent variable and dependent variable is established as:

$$T = t_1 + \frac{1}{k_2} ln\left[1 + \left\{\frac{k_2(m-a+bp)}{k_1(a-bp)}\right\}(1 - e^{-k_1 t_1})\right]. \tag{15}$$

The total holding cost in the time interval $[0, T]$ is given by

$$HC = c_h\left[\int_0^{t_1} q(t)dt + \int_{t_1}^{T} q(t)dt\right]$$

$$= c_h\begin{bmatrix} \frac{m-(a-bp)}{{k_1}^2}\{e^{-k_1 t_1} + k_1 t_1 - 1\} \\ + \frac{(a-bp)}{k_2^2}\{e^{k_2(T-t_1)} - k_2(T - t_1) - 1\} \end{bmatrix}. \tag{16}$$

The Sales Revenue (SR) during the entire circle is given by:

$$SR = p\begin{bmatrix} \int_0^{t_1}\{a - bp + cq(t)\}dt \\ + \int_{t_1}^{T}\{a - bp + cq(t)\}dt \end{bmatrix}$$

$$= p(a - bp)T + cp\frac{HC}{c_h}. \tag{17}$$

The production cost during whole cycle is given by:

$$PC = c_p \int_0^{t_1}\{m - nq(t)\}dt$$

$$= c_p m t_1 - \frac{m-(a-bp)}{k_1} c_p n\left\{t_1 + \frac{(e^{-k_1 t_1} - 1)}{k_1}\right\}. \tag{18}$$

Therefore, the average profit of the production system during the entire circle is given by:

$$TAP = \frac{SR - c_0 - HC - PC}{T}. \tag{19}$$

So, the optimization problem is given by:

$$\begin{cases} \text{Maximize} & TAP = \frac{SR - C_0 - HC - PC}{T}, \\ \text{Subject to} & Q = \frac{m-(a-bp)}{k_1}\{1 - e^{-k_1 t_1}\}, \\ T = t_1 + \frac{1}{k_2} ln\left[1 + \left\{\frac{k_2(m-a+bp)}{k_1(a-bp)}\right\}(1 - e^{-k_1 t_1})\right]. \end{cases} \tag{20}$$

5.2. Fuzzy EPQ model

Assuming the entire cost coefficients, deterioration rate and unit selling price as fuzzy numbers, the governing differential equation of the model can be put as follows:

$$\frac{d\tilde{q}(t)}{dt} + \tilde{\theta}_1 \tilde{q}(t) = \{m - n\tilde{q}(t)\} - \{a - b\tilde{p} + c\tilde{q}(t)\}, \quad 0 \le t \le t_1, \tag{21}$$

$$\frac{d\tilde{q}(t)}{dt} + \tilde{\theta}_2 \tilde{q}(t) = -\{a - b\tilde{p} + c\tilde{q}(t)\}, \quad t_1 \le t \le T. \tag{22}$$

Also, the fuzzy valued stock level at the starting and stopping time are given by:

$$\tilde{q}(0) = \tilde{q}(T) = 0. \tag{23}$$

Suppose the parametric representation of fuzzy valued stock, deterioration rate and selling price are given as:

$$\begin{cases}\tilde{q}(t) = [q_L(t,\alpha), q_R(t,\alpha)],\\ \tilde{\theta}_1 = [\theta_{1L}(\alpha), \theta_{1R}(\alpha)],\\ \tilde{\theta}_2 = [\theta_{2L}(\alpha), \theta_{2R}(\alpha)],\\ \tilde{p} = [p_L(\alpha), p_R(\alpha)].\end{cases} \quad (24)$$

Here, the notion of gH derivative is applied to solve Eqs. (21) and (22) under two different cases of gH differentiability of the fuzzy valued function $\tilde{q}(t)$ as follows:

Case 1. when $\tilde{q}(t)$ is (i)-gH(L-R) differentiable.
Then, the FDE given by Eq. (21) is turned into a system of differential equations as bellow:

$$\begin{cases} q_L'(t,\alpha) + \theta_{1L}(\alpha) q_L(t,\alpha) = \\ m - n q_R(t,\alpha) - a + b p_L(\alpha) - c q_R(t,\alpha) \\ q_R'(t,\alpha) + \theta_{1R}(\alpha) q_R(t,\alpha) = \\ m - n q_L(t,\alpha) - a + b p_R(\alpha) - c q_L(t,\alpha) \\ \text{with } q_L(0,\alpha) = q_R(0,\alpha) = 0 \end{cases} \quad (25)$$

Similarly, from Eq. (22) we get:

$$\begin{cases} q_L'(t,\alpha) + \theta_{2L}(\alpha) q_L(t,\alpha) = -a + b p_L(\alpha) - c q_R(t,\alpha), \\ q_R'(t,\alpha) + \theta_{2R}(\alpha) q_R(t,\alpha) = -a + b p_R(\alpha) - c q_L(t,\alpha), \\ \quad \text{with } q_L(T,\alpha) = q_R(T,\alpha) = 0. \end{cases} \quad (26)$$

In Eqs. (25) and (26) and the rest of the paper, $q_L'(t,\alpha)$ and $q_R'(t,\alpha)$ represents the first derivative of $q_L(t,\alpha)$ and $q_R(t,\alpha)$ respectively with respect to t i.e., $q_L'(t,\alpha) \equiv \frac{d(q_L(t,\alpha))}{dt}$ and $q_R'(t,\alpha) \equiv \frac{d(q_R(t,\alpha))}{dt}$. After simplification, the Eq. (25) is reduced as:

$$\begin{cases} q_L'(t,\alpha) = -a_1 q_L(t,\alpha) - b_1 q_R(t,\alpha) + c_1, \\ q_R'(t,\alpha) = -a_2 q_L(t,\alpha) - b_2 q_R(t,\alpha) + c_2, \\ \quad \text{with } q_L(0,\alpha) = q_R(0,\alpha) = 0. \end{cases} \quad (27)$$

The values of $a_1, a_2, b_1, b_2, c_1, c_2$ are given by:

$$\begin{cases} a_1 = \theta_{1L}(\alpha), b_1 = n + c, \\ c_1 = m - a + b p_L(\alpha), \\ a_2 = n + c, b_2 = \theta_{1R}(\alpha), \\ c_2 = m - a + b p_R(\alpha). \end{cases} \quad (28)$$

The system of differential equations given by Eq. (27) is solved using Lagrange's multiplier method in the following way:

$$\frac{d\big(q_L(t,\alpha) + \lambda q_R(t,\alpha)\big)}{dt} =$$

$$-(a_1 + \lambda a_2) q_L(t,\alpha) - (b_1 + \lambda b_2) q_R(t,\alpha) + (c_1 + \lambda c_2),$$

i.e.,

$$\frac{d\big(q_L(t,\alpha) + \lambda q_R(t,\alpha)\big)}{dt} =$$

$$-(a_1 + \lambda a_2)\left\{\begin{matrix} q_L(t,\alpha) + \\ \frac{(b_1+\lambda b_2)}{(a_1+\lambda a_2)} q_R(t,\alpha) - \frac{(c_1+\lambda c_2)}{(a_1+\lambda a_2)} \end{matrix}\right\}. \quad (29)$$

Then, we choose a constant λ such that:

$$\frac{(b_1+\lambda b_2)}{(a_1+\lambda a_2)} = \lambda. \quad (30)$$

Using Eq. (30), Eq. (29) is reduced as:

$$\frac{dz(t)}{dt} = -(a_1 + \lambda a_2) z(t). \quad (31)$$

For simplification $z(t)$ is assumed as:

$$z(t) = q_L(t,\alpha) + \lambda q_R(t,\alpha) - \frac{(c_1+\lambda c_2)}{(a_1+\lambda a_2)}. \quad (32)$$

Therefore, solving Eq. (31) and putting the value of $z(t)$ from Eq. (32), the result is obtained as:

$$q_L(t,\alpha) + \lambda q_R(t,\alpha) - \frac{(c_1+\lambda c_2)}{(a_1+\lambda a_2)} = A e^{-(a_1+\lambda a_2)t}. \quad (33)$$

In Eq. (33), A is a constant of integration which has to be determined using the initial conditions. Using the initial conditions, Eq. (33) is reduced to:

$$q_L(t,\alpha) + \lambda q_R(t,\alpha) = \frac{(c_1+\lambda c_2)}{(a_1+\lambda a_2)} \{1 - e^{-(a_1+\lambda a_2)t}\}. \quad (34)$$

Also, the relation described by Eq. (30) is actually a quadratic equation having two roots, say, λ_1 and λ_2. For different values of λ_1 and λ_2, Eq. (34) is again turned into a system of simultaneous equations:

$$\begin{cases} q_L(t,\alpha) + \lambda_1 q_R(t,\alpha) = \frac{(c_1+\lambda_1 c_2)}{(a_1+\lambda_1 a_2)} \{1 - e^{-(a_1+\lambda_1 a_2)t}\}. \\ q_L(t,\alpha) + \lambda_2 q_R(t,\alpha) = \frac{(c_1+\lambda_2 c_2)}{(a_1+\lambda_2 a_2)} \{1 - e^{-(a_1+\lambda_2 a_2)t}\}, \end{cases}$$

So,

$$\begin{cases} q_L(t,\alpha) + \lambda_1 q_R(t,\alpha) = A_1(1 - e^{-B_1 t}). \\ q_L(t,\alpha) + \lambda_2 q_R(t,\alpha) = A_2(1 - e^{-B_2 t}), \end{cases} \quad (35)$$

The values of the constants A_1, A_2, B_1 and B_2 are given by:

$$\begin{cases} A_1 = \frac{(c_1+\lambda_1 c_2)}{(a_1+\lambda_1 a_2)}, \; B_1 = a_1 + \lambda_1 a_2. \\ A_2 = \frac{(c_1+\lambda_2 c_2)}{(a_1+\lambda_2 a_2)}, \; B_2 = a_1 + \lambda_2 a_2, \end{cases} \quad (36)$$

The solution of the system of simultaneous equation given by Eq. (35) is obtained as follows:

Eq. (35) is obtained as follows:

$$\begin{cases} q_{1L}(t,\alpha) = \frac{A_1\lambda_2(1-e^{-B_1 t}) - A_2\lambda_1(1-e^{-B_2 t})}{\lambda_2-\lambda_1}. \\ q_{1R}(t,\alpha) = \frac{A_1(1-e^{-B_1 t}) + A_2(e^{-B_2 t}-1)}{\lambda_1-\lambda_2}. \\ \quad 0 \le t \le t_1, \end{cases} \quad (37)$$

On the other hand, the system of differential equations described by Eq. (26) can be simplified as:

$$\begin{cases} q_L'(t,\alpha) = -f_1 q_L(t,\alpha) - g_1 q_R(t,\alpha) - h_1. \\ q_R'(t,\alpha) = -f_2 q_L(t,\alpha) - g_2 q_R(t,\alpha) - h_2. \\ q_L(T,\alpha) = q_R(T,\alpha) = 0. \\ \text{where } f_1 = \theta_{2L}(\alpha).\; g_1 = c.\; h_1 = a - b p_L(\alpha). \\ \; f_2 = c,\; g_2 = \theta_{2R}(\alpha).\; h_2 = a - b p_R(\alpha), \end{cases} \quad (38)$$

The system of differential equations described in Eq. (39) can be written as:

$$q_L(t,\alpha)+\mu q_R(t,\alpha)=\frac{(h_1+\mu h_2)}{(f_1+\mu f_2)}\{e^{(f_1+\mu f_2)(T-t)}-1\}. \quad (39)$$

The constant μ is chosen in such a manner that the following relationship holds:

$$\mu=\frac{g_1+\mu g_2}{f_1+\mu f_2}. \quad (40)$$

The above relationship is actually a quadratic equation which produces two roots, say, μ_1 and μ_2.Then, proceeding as the productive phase, a system of simultaneous equations is obtained as:

$$\begin{cases} q_{2L}(t,\alpha)=\frac{A_3\mu_2(e^{B_3(T-t)}-1)-A_4\mu_1(e^{B_4(T-t)}-1)}{\mu_2-\mu_1}. \\ q_{2R}(t,\alpha)=\frac{A_3(e^{B_3(T-t)}-1)-A_4(e^{B_4(T-t)}-1)}{\mu_1-\mu_2}. \\ \text{where } A_3=\frac{(h_1+\mu_1 h_2)}{(f_1+\mu_1 f_2)}.\ B_3=f_1+\mu_1 f_2. \\ A_4=\frac{(h_1+\mu_2 h_2)}{(f_1+\mu_2 f_2)}.\ B_4=f_1+\mu_2 f_2. \\ \qquad t_1\le t\le T, \end{cases} \quad (41)$$

Several relevant costs and revenue associated with the model are computed in parametric representation as follows:

(i) The set-up cost $c_0=[c_{0L}(\alpha),c_{0R}(\alpha)]$.

(ii) $\tilde{c}_h=[c_{hL}(\alpha),c_{hR}(\alpha)]=$ Unit holding cost per unit product.

Therefore, the holding cost $=[HC_L(\alpha),HC_R(\alpha)]$ given by:

$$HC_L(\alpha)=c_{hL}\left[\int_0^{t_1}q_{1L}(t,\alpha)dt+\int_{t_1}^{T}q_{2L}(t,\alpha)dt\right]$$

$$=c_{hL}\begin{bmatrix}\int_0^{t_1}\left\{\frac{A_1\lambda_2(1-e^{-B_1t})-A_2\lambda_1(1-e^{-B_2t})}{\lambda_2-\lambda_1}\right\}dt+ \\ \int_{t_1}^{T}\left\{\frac{A_3\mu_2(e^{B_3(T-t)}-1)-A_4\mu_1(e^{B_4(T-t)}-1)}{\mu_2-\mu_1}\right\}dt\end{bmatrix}$$

$$=c_{hL}\{I_1+I_2\}, \quad (42)$$

In Eq. (42), the values of I_1 and I_2 are assumed as:

$$\begin{cases} I_1=\frac{\frac{A_1\lambda_2}{B_1}(e^{-t_1B_1}+B_1t_1-1)-\frac{A_2\lambda_1}{B_2}(e^{-t_1B_2}+B_2t_1-1)}{\lambda_2-\lambda_1}, \\ I_2=\frac{\frac{A_3\mu_2}{B_3}(e^{(T-t_1)B_3}+(t_1-T)B_3-1)-\frac{A_4\mu_1}{B_4}(e^{(T-t_1)B_4}+(t_1-T)B_4-1)}{\mu_2-\mu_1}, \end{cases} \quad (43)$$

$$HC_R(\alpha)=c_{hR}\left[\int_0^{t_1}q_{1R}(t,\alpha)dt+\int_{t_1}^{T}q_{2R}(t,\alpha)dt\right]$$

$$=c_{hR}\begin{bmatrix}\int_0^{t_1}\left\{\frac{A_1(1-e^{-B_1t})+A_2(e^{-B_2t}-1)}{\lambda_1-\lambda_2}\right\}dt \\ +\int_{t_1}^{T}\left\{\frac{A_3(e^{B_3(T-t)}-1)-A_4(e^{B_4(T-t)}-1)}{\mu_1-\mu_2}\right\}dt\end{bmatrix}$$

$$=c_{hR}\{J_1+J_2\}. \quad (44)$$

In Eq. (44), the values of J_1 and J_2 are assumed as:

$$\begin{cases} J_1=\frac{\frac{A_1}{B_1}(B_1t_1+e^{-t_1B_1}-1)-\frac{A_2}{B_2}(B_2t_1+e^{-t_1B_2}-1)}{\lambda_1-\lambda_2}. \\ J_2=\frac{\frac{A_3}{B_3}(e^{(T-t_1)B_3}-B_3(T-t_1)-1)-\frac{A_4}{B_4}(e^{(T-t_1)B_4}-B_4(T-t_1)-1)}{\mu_1-\mu_2}, \end{cases} \quad (45)$$

(iii) The $\tilde{p}=[p_L(\alpha),p_R(\alpha)]=$ The SR per unit product.

Therefore, total SR, $SR=[SR_L(\alpha),SR_R(\alpha)]$ during the entire cycle is given by:

$$SR_L(\alpha)$$

$$=p_L(\alpha)\begin{bmatrix}\int_0^{t_1}\{a-bp_R(\alpha)+cq_{1L}(t,\alpha)\}dt \\ +\int_{t_1}^{T}\{a-bp_R(\alpha)+cq_{2L}(t,\alpha)\}dt\end{bmatrix}$$

$$=p_L(\alpha)[\{a-bp_R(\alpha)\}T+c\{I_1+I_2\}]. \quad (46)$$

and

$$SR_R(\alpha)$$

$$=p_R(\alpha)\begin{bmatrix}\int_0^{t_1}\{a-bp_L(\alpha)+cq_{1R}(t,\alpha)\}dt \\ +\int_{t_1}^{T}\{a-bp_L(\alpha)+cq_{2R}(t,\alpha)\}dt\end{bmatrix}$$

$$=p_R(\alpha)[\{a-bp_L(\alpha)\}T+c\{J_1+J_2\}], \quad (47)$$

(iv) $\tilde{c}_p=[c_{pL}(\alpha),c_{pR}(\alpha)]=$ The unit production cost per unit product.

Therefore, total production cost $PC=[PC_L(\alpha),PC_R(\alpha)]$ during the entire cycle is given by:

$$PC_L(\alpha)=c_{pL}\left[\int_0^{t_1}\{m-nq_{1R}(t,\alpha)\}dt\right]$$
$$=c_{pL}[mt_1-nJ_1], \quad (48)$$

and

$$PC_R(\alpha)=c_{pR}\left[\int_0^{t_1}\{m-nq_{1L}(t,\alpha)\}dt\right]$$
$$=c_{pR}[mt_1-nI_1], \quad (49)$$

The total average profit, $TAP=[TAP_{1L}(\alpha),TAP_{1R}(\alpha)]$ during the entire cycle is given by:

$$\begin{cases} TAP_{1L}(\alpha) \\ =\frac{(SR_L(\alpha)-PC_R(\alpha)-HC_R(\alpha)-c_{0R}(\alpha))}{T}. \\ TAP_{1R}(\alpha) \\ =\frac{(SR_R(\alpha)-PC_L(\alpha)-HC_L(\alpha)-c_{0L}(\alpha))}{T}, \end{cases} \quad (50)$$

Therefore, mathematically the optimization problem can be written in form:

$$\begin{cases} \text{Maximize} \quad TAP_{1L}(\alpha), \\ \text{Maximize} \quad TAP_{1R}(\alpha), \\ \text{Subject to } (15),\ (37) \text{ and } (41), \\ \qquad 0\le\alpha\le 1. \end{cases} \quad (51)$$

Case 2. when $\tilde{q}(t)$ is (ii)-gH(R-L) differentiable.

Then, the FDE given by Eq. (21) is transformed into a system of differential equation as following:

$$\begin{cases} q_R'(t,\alpha)+\theta_{1L}(\alpha)q_L(t,\alpha) \\ =m-nq_R(t,\alpha)-a+bp_L(\alpha)-cq_R(t,\alpha). \\ q_L'(t,\alpha)+\theta_{1R}(\alpha)q_R(t,\alpha) \\ =m-nq_L(t,\alpha)-a+bp_R(\alpha)-cq_L(t,\alpha). \\ q_L(0,\alpha)=q_R(0,\alpha)=0, \end{cases} \quad (52)$$

On the other hand, the FDE given by Eq. (22) is transformed into a system of differential equation as following:

$$\begin{cases} q_R'(t,\alpha)+\theta_{2L}(\alpha)q_L(t,\alpha)=-a+bp_L(\alpha)-cq_R(t,\alpha). \\ q_L'(t,\alpha)+\theta_{2R}(\alpha)q_R(t,\alpha)=-a+bp_R(\alpha)-cq_L(t,\alpha). \\ \qquad q_L(T,\alpha)=q_R(T,\alpha)=0, \end{cases} \quad (53)$$

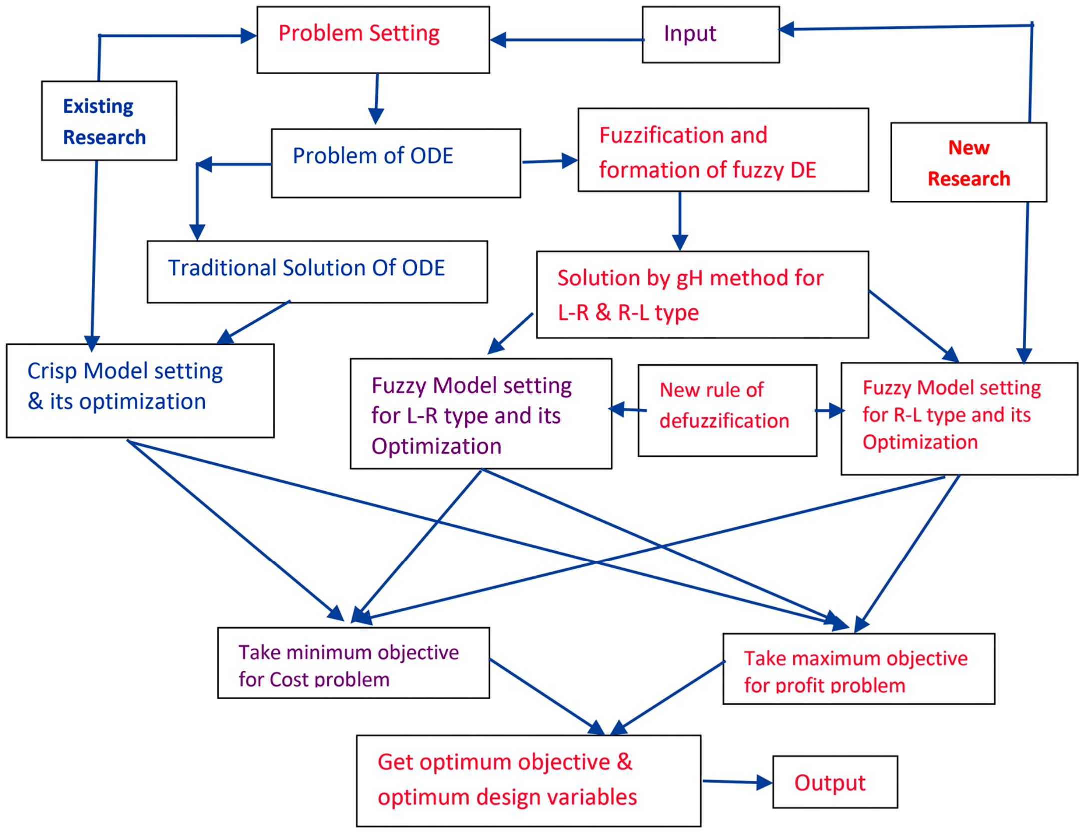

Figure 2. Schematic diagram of the proposed model.

Utilizing the procedure developed in Case 1, the parametric values of the total average profit, $TAP = [TAP_{2L}(\alpha), TAP_{2R}(\alpha)]$ is given by (For more details see Appendix A):

$$TAP_{2L}(\alpha) = \frac{(SR_L(\alpha) - PC_R(\alpha) - HC_R(\alpha) - c_{0R}(\alpha))}{T}, \quad (54)$$

$$TAP_{2R}(\alpha) = \frac{(SR_R(\alpha) - PC_L(\alpha) - HC_L(\alpha) - c_{0L}(\alpha))}{T}. \quad (55)$$

Therefore, the optimization problem can be reduced to Eq. (56):

$$\begin{cases} \text{Maximize} & TAP_{2L}(\alpha). \\ \text{Maximize} & TAP_{2R}(\alpha). \\ \text{Subject to Constraints (15), } (A,2) \text{ and } (A,4). \\ & 0 \le \alpha \le 1, \end{cases} \quad (56)$$

However, the schematic diagram of the proposed model is given by Figure 2.

5.3. Solution algorithm of the proposed model

Here we develop a solution algorithm to solve the proposed model.

Step1: Solve the crisp problem using LINGO software or any programming language,

Step 2: Covert the crisp problem into two fuzzy problems (Cases 1 and 2) for different scenarios of α-cuts of the FDEs,

Step 3: Find maximum value of lower and upper objective functions in each case for different cycle time under different aspiration levels,

Step 4: Defuzzify the results using the definite formula stated in Subsection 2.1,

Step 5: Compare the numerical out puts of each case with respect to crisp result and get the optimum average profit of the original problem,

Step 6: Record the optimum decision variables.

6. Numerical and graphical illustrations

In this section we take some numerical study over crisp and fuzzy models. Moreover, we perform some graphical illustrations on the basis of these numerical data.

6.1. Numerical results of crisp model

For numerical study of the crisp model, we take the following values of the input parameters:

$a = 100$, $b = 0.1$, $c = 0.14$, $m = 180$, $n = 0.8$, $p = 95$, $c_h = 3.25$, $c_p = 40$, $c_0 = 300$, $\theta_1 = 0.05$, $\theta_2 = 0.07$.

Then, using the LINGO 17.0 software the optimum results are obtained and presented in Table 2.

Table 2 shows that average profit of the model gets maximum value \$5041.09 for the cycle time 2.5 months and the production run time 1.742 months with order quantity 74.29 units respectively. Beyond this, if the cycle time increases or decreases then the average profit of the model is also decreasing.

Table 2. Crisp optimal solution.

T	t_1^*	Q^*	Z^*
1	0.582	39.57	4860.22
1.5	0.933	54.50	4988.65
2	1.322	65.97	5035.41
2.5	1.742	74.29	5041.09
3	2.19	80.05	5025.49
3.5	2.653	83.87	5000.21
4	3.13	86.33	4971.93

6.2. Numerical results of crisp model

Let the deterioration rate (θ_1, θ_2), unit holding cost (c_h), ordering cost (c_0), unit production cost (c_p) and unit selling price (p) assume as triangular fuzzy numbers. Then the left and right α-cuts of each triangular fuzzy number are respectively given by:

$$\theta_{1L} = 0{,}03 + 0{,}02\alpha. \quad \theta_{1R} = 0{,}07 - 0{,}02\alpha.$$

$$\theta_{2L} = 0.05 + 0.02\alpha, \quad \theta_{2R} = 0.09 - 0.02\alpha,$$

$$c_{hL} = 2.25 + 1\alpha, \; c_{hR} = 4.25 - 1\alpha, c_{pL} = 36 + 4\alpha,$$

$$c_{pR} = 44 - 4\alpha, \; p_L = 90 + 5\alpha, \; p_R = 100 - 5\alpha,$$

$$c_{oL} = 280 + 20\alpha, \; c_{oR} = 320 - 20\alpha,$$

Now, Tables 3 and 4 represent the values of different decision variables and objective function in different level of aspirations in the cases of (i)-gH(L-R) differentiability and (ii)-gH(R-L) differentiability respectively.

Now, utilizing Subsection 3.4 on the numerical values of Table 3 we get, $\bar{l} = 0.6772$ and $\bar{r} = 0.8652$ and then optimum cycle time is 2.5 months, production run time is 1.74 months and the average profit= Crisps value of the objective $\times \left(1 + \bar{r} - \bar{l}\right) = \$5988.81$ and order quantity = 74.96 units.

Now, utilizing Section 3.4 on the numerical values of Table 3 we get, $\bar{l} = 0.8845$ and $\bar{r} = 1.1933$ and then optimum cycle time is 2.5 months, production run time is 1.74 months and the average profit=$ Crisps value of the objective $\times \left(1 + \bar{r} - \bar{l}\right) = \$6597.78$ and order quantity = 74.58 units respectively.

6.3. Graphical illustrations

We shall draw several graphs using the data set obtained from Tables 2–4. Figure 3 shows that in crisp model the average profit function gets minimum value in compared to fuzzy models. On the other hand, Case 2 gives the maximum average profit of the proposed model. From Figure 4 we see that at cycle time $T = 1$ month the profit of the inventory model assumes lower value. Then profit function began to increase and reaches its maximum at $T =$ 2.5 months. After that the profit function is going to decrease. In Figure 5 it is seen that the multi-objective functions obtained from fuzzy problem (Case 2) are getting more and closer to the crisp value (near $5000) whenever the aspiration level is going to increase from 0.1 to 0.9. From Figure 6 we see that lower objective function (TAP_L) plane like surface gets maximum value at cycle time 3 months and aspiration level 0.9 whereas Figure 7 indicate that the upper objective function attained maximum value at cycle time 3 months and aspiration level 0.1. Figure 8 explores that the span of lower objective functions due to different cycle times is gradually increasing with respect to the change of aspiration level. But for the upper objective function it began to decrease and they are going to intersect near the aspiration level 0.9 keeping the profit value near $5000.

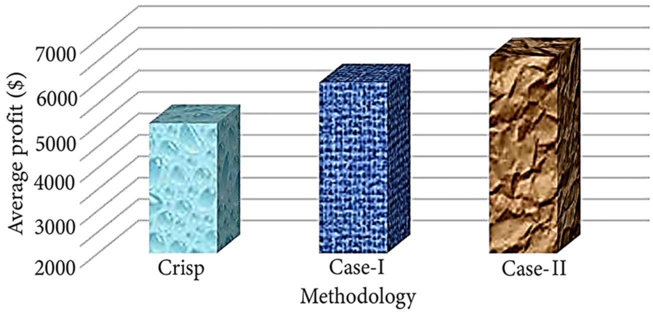

Figure 3. Optimal solution of crisp and fuzzy models.

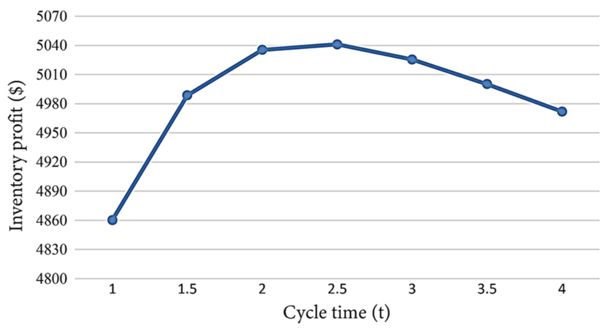

Figure 4. Inventory profit vs cycle time

Table 3. Fuzzy optimal solution for Case 1 (L-R type gH method).

α		t_1^*	q_L	q_R	TAP_L	TAP_R
0.1	[1, 2.5]	[0.58, 1.74]	[39.58, 73.95]	[39.59, 75.09]	[1730.46, 3013.21]	[4663.61, 4763.77]
0.2	[1, 2.5]	[0.58, 1.74]	[39.55, 73.98]	[39.57, 75.00]	[1853.56, 3113.29]	[4460.75, 4633.35]
0.3	[1, 2.5]	[0.58, 1.74]	[39.55, 74.03]	[39.58, 74.92]	[1976.81, 3213.38]	[4258.07, 4528.83]
0.4	[1, 2.5]	[0.58, 1.74]	[39.55, 74.06]	[39.58, 74.84]	[2100.24, 3313.51]	[4055.58, 4441.00]
0.5	[1, 2.5]	[0.58, 1.74]	[39.55, 74.10]	[39.58, 74.84]	[2223.88, 3413.68]	[3853.30, 4353.22]
0.6	[1, 2.5]	[0.58, 1.74]	[39.56, 74.14]	[39.58, 74.65]	[2347.67, 3513.87]	[3651.19, 4265.49]
0.7	[1, 2.5]	[0.58, 1.74]	[39.56, 74.18]	[39.58, 74.56]	[2471.67, 3614.09]	[3449.31, 4177.80]
0.8	[1, 2.5]	[0.58, 1.74]	[39.57, 74.23]	[39.58, 74.47]	[2595.85, 3714.36]	[3247.62, 4090.15]
0.9	[1, 2.5]	[0.58, 1.74]	[39.57, 74.27]	[39.58, 74.38]	[2720.17, 3814.64]	[3046.10, 4002.53]

Table 4. Fuzzy optimal solution for Case 2 (R-L type gH method).

α	T	t_1^*	q_L	q_R	TAP_L	TAP_R
0.1	[1, 3]	[0.58, 2.18]	[39.55, 80.18]	[39.58, 80.62]	[3892.99, 4036.27]	[5439.30, 6838.66]
0.2	[1, 3]	[0.58, 2.18]	[39.55, 80.16]	[39.58, 80.56]	[3984.97, 4141.53]	[5359.42, 6632.37]
0.3	[1, 3]	[0.58, 2.18]	[39.55, 80.15]	[39.58, 80.49]	[4076.96, 4246.98]	[5279.58, 6426.31]
0.4	[1, 3]	[0.58, 2.18]	[39.55, 80.13]	[39.58, 80.43]	[4168.99, 4352.62]	[5199.78, 6220.49]
0.5	[1, 3]	[0.58, 2.18]	[39.56, 80.12]	[39.58, 80.37]	[4261.06, 4458.45]	[5120.03, 6014.88]
0.6	[1, 3]	[0.58, 2.18]	[39.56, 80.10]	[39.58, 80.36]	[4353.16, 4564.46]	[5040.31, 5809.53]
0.7	[1, 3]	[0.58, 2.18]	[39.56, 80.09]	[39.58, 80.24]	[4445.28, 4670.67]	[4960.64, 5604.39]
0.8	[1, 3]	[0.58, 2.18]	[39.57, 80.07]	[39.58, 80.17]	[4537.44, 4777.06]	[4880.99, 5399.50]
0.9	[1, 3]	[0.58, 2.18]	[39.57, 80.06]	[39.57, 80.11]	[4629.62, 4883.64]	[4801.40, 5194.84]

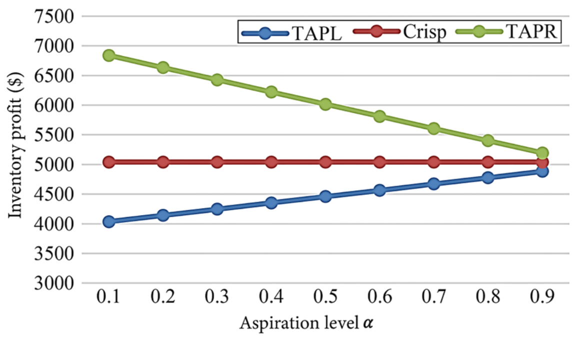

Figure 5. Fuzzy optimal model.

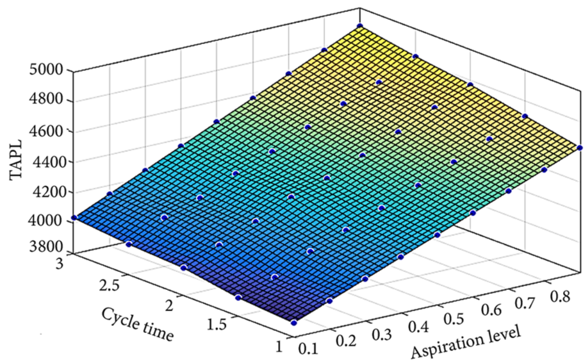

Figure 6. TAP$_L$ vs aspiration level and cycle time.

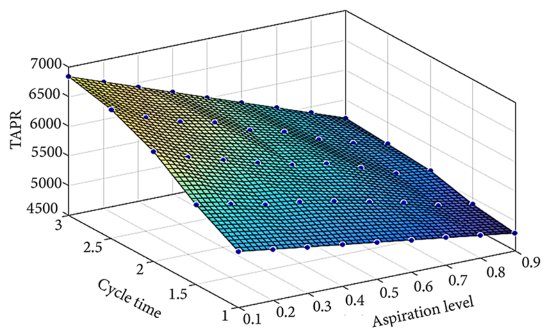

Figure 7. TAP$_R$ vs aspiration level and cycle time.

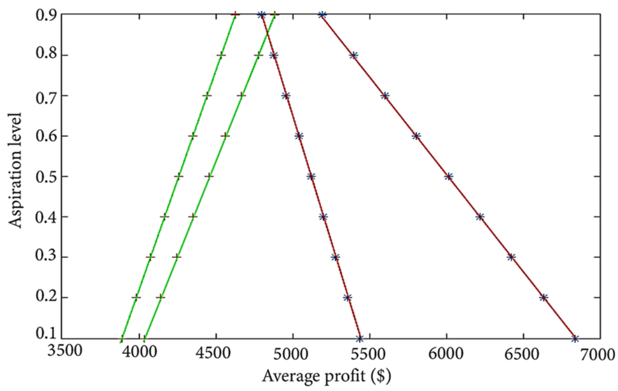

Figure 8. Aspiration level vs average profit.

7. Merits and demerits of the proposed approach

In this paper, an EPQ model is studied under fuzzy uncertainty with FDE as a working tool. The definition of gH derivative is taken to make sense of the fuzzy derivative. The approach adopted in this article has some advantages:

(i) Very frequently, the researchers solve ordinary differential equations to obtain a crisp model they use defuzzification techniques for fuzzy valued parameters and variables. But, doing so, they ignore the need of FDEs. In this proposed approach, anybody can control and estimate the defuzzified value of the fuzzy parameters in more concrete way;

(ii) New defuzzification formula in terms of aggregation of fuzzy outcomes is very much helpful towards definite decision making that involves multi-valued objective functions.

On the other hand, a drawback of the proposed approach cannot be over looked. The FDEs are solved by two methods one of them is L-R type gH method and the other one is R-L type gH method. So, the proposed approach will take much computational time for more complicated problems.

8. Conclusion and future research scope

The present study discusses a production inventory model considering the demand function which is depending on unit selling price and on hand stock of the inventory. The profit function of the proposed model has been analysed under crisp and fuzzy environment because of several parametric flexibility. In the process of constructing the fuzzy profit function we have utilized gH derivative which are directly associated with the α-cuts of fuzzy numbers. However, we have developed a new defuzzyfication method for the fuzzy mathematical model to capture the multi-objective optimization problem. This new defuzzification method is nothing but the aggregation of several objective values obtained at different aspiration levels. Aggregation formula is used as because the spans of lower and upper objective functions are dissimilar and hence, they carry more uncertainty as a whole. In fact, the model has been studied in two different derivative approaches where the α-cuts of the fuzzified function assume increasing (Case-I, L-R type gH method) and decreasing (Case-II, R-L type gH method) values respectively. Our numerical result shows that the crisp solution is quite inferior than both the solutions obtained from fuzzy problem. Sensitivity analysis of the fuzzy problem has been made by varying different aspiration level α. The comparative study reveals that the profit function gets higher value 30.88% more for Case-II and that for Case-I, it becomes 18.8% more profit keeping the optimum inventory cycle time and production run time unaltered with respect to crisp model. Moreover, the optimum order quantity becomes 74.58 units for Case-II and that for Case-I it is 74.96 units which are slightly higher than that for crisp model. The basic managerial insight as well as the novelty of the current study is focused focused from the fuzzy model that utilizes fuzzy R-L gH derivatives and the newly defined defuzzification aggregation method exclusively.

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this article.

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Appendix A

Simplifying the system of differential equations described by Eq. (52):

$$\begin{cases} q_R'(t,\alpha) = -a_3 q_R(t,\alpha) - b_3 q_L(t,\alpha) + c_3, \\ q_L'(t,\alpha) = -a_4 q_R(t,\alpha) - b_4 q_L(t,\alpha) + c_4, \\ \text{with} \quad q_L(0,\alpha) = q_R(0,\alpha) = 0, \\ \text{where} \quad a_3 = n + c, b_3 = \theta_{1L}(\alpha), \\ c_3 = m - a + bp_L(\alpha), \\ a_4 = \theta_{1R}(\alpha), b_4 = n + c, \\ c_4 = m - a + bp_R(\alpha). \end{cases} \tag{A.1}$$

The system of differential equations described by Eq. (A.1) can be written as:

$$q_R(t,\alpha) + \lambda q_L(t,\alpha) = \frac{(c_3+\lambda c_4)}{(a_3+\lambda a_4)}\{1 - e^{-(a_3+\lambda a_4)t}\}. \tag{A.2}$$

The constant λ is chosen in such a manner that the following relationship holds:

$$\frac{(b_3+\lambda b_4)}{(a_3+\lambda a_4)} = \lambda. \tag{A.3}$$

The above relationship is actually a quadratic equation which produces two roots, say, λ_1 and λ_2.

Then, proceeding as the productive phase of Case 1, a system of simultaneous equations is obtained as:

$$\begin{cases} q_{1R}(t,\alpha) = \frac{A_5\lambda_4(1-e^{-B_5 t}) - A_6\lambda_3(1-e^{-B_6 t})}{\lambda_4 - \lambda_3}, \\ q_{1L}(t,\alpha) = \frac{A_5(1-e^{-B_5 t}) - A_6(1-e^{-B_6 t})}{\lambda_3 - \lambda_4}, \\ \text{where } A_5 = \frac{(c_3+\lambda_3 c_4)}{(a_3+\lambda_3 a_4)}, B_5 = a_3 + \lambda_3 a_4, \\ \quad A_6 = \frac{(c_3+\lambda_4 c_4)}{(a_3+\lambda_4 a_4)}, B_6 = a_3 + \lambda_4 a_4, \\ \quad 0 \le t \le t_1. \end{cases} \tag{A.4}$$

Simplifying the system of differential equations described by Eq. (53):

$$\begin{cases} q'_L(t,\alpha) = -f_4 q_R(t,\alpha) - g_4 q_L(t,\alpha) - h_4, \\ q'_R(t,\alpha) = -f_3 q_R(t,\alpha) - g_3 q_L(t,\alpha) - h_3, \\ q_L(T,\alpha) = q_R(T,\alpha) = 0, \\ \text{where } f_3 = c, g_3 = \theta_{2L}(\alpha), h_3 = a - bp_L(\alpha), \\ f_4 = \theta_{2R}(\alpha), g_4 = c, h_4 = a - bp_R(\alpha). \end{cases} \quad \text{(A.5)}$$

The system of differential equations described by Eq. (A.5) can be written as:

$$q_R(t,\alpha) + \mu q_L(t,\alpha) = \frac{(h_3+\mu h_4)}{(f_3+\mu f_4)}\{e^{(f_3+\mu f_4)(T-t)} - 1\}. \quad \text{(A.6)}$$

The constant μ is chosen in such a manner that the following relationship holds:

$$\frac{(g_3+\mu g_4)}{(f_3+\mu f_4)} = \mu. \quad \text{(A.7)}$$

Then, proceeding as the non-productive phase of Case 1, a system of simultaneous equations is obtained as:

$$\begin{cases} q_{2L}(t,\alpha) = \frac{A_7\{e^{B_7(T-t)}-1\} - A_8\{e^{B_8(T-t)}-1\}}{\mu_3-\mu_4}. \\ q_{2R}(t,\alpha) = \frac{A_7\mu_4\{e^{B_7(T-t)}-1\} - A_8\mu_3\{e^{B_8(T-t)}-1\}}{\mu_4-\mu_3}. \\ \text{where } A_7 = \frac{(h_3+\mu_3 h_4)}{(f_3+\mu_3 f_4)}.\ B_7 = f_3 + \mu_3 f_4. \\ A_8 = \frac{(h_3+\mu_4 h_4)}{(f_3+\mu_4 f_4)}. B_8 = f_3 + \mu_4 f_4. \\ t_1 \le t \le T, \end{cases} \quad \text{(A.8)}$$

Some relevant fuzzy valued costs and revenue in parametric form:

(i) The set-up cost $\tilde{c}_0 = [c_{0L}(\alpha), c_{0R}(\alpha)]$.

(ii) $\tilde{c}_h = [c_{hL}(\alpha), c_{hR}(\alpha)]$ the holding cost per unit product.

Therefore, the holding cost, $HC = [HC_L(\alpha), HC_R(\alpha)]$ is given by:

$$HC_L(\alpha) = c_{hL}\left[\int_0^{t_1} q_{1L}(t,\alpha)dt + \int_{t_1}^{T} q_{2L}(t,\alpha)dt\right]$$

$$= c_{hL}\left[\begin{array}{l}\int_0^{t_1}\left\{\frac{A_5(1-e^{-B_5 t}) - A_6(1-e^{-B_6 t})}{\lambda_3-\lambda_4}\right\}dt \\ +\int_{t_1}^{T}\left\{\frac{A_7\{e^{B_7(T-t)}-1\} - A_8\{e^{B_8(T-t)}-1\}}{\mu_3-\mu_4}\right\}dt\end{array}\right]$$

$$= c_{hL}\{I_3 + I_4\}. \quad \text{(A.9)}$$

The values of I_3 and I_4 are assumed as:

$$\begin{cases} I_3 = \frac{\frac{A_5}{B_5}(e^{-t_1 B_5} + t_1 B_5 - 1) - \frac{A_6}{B_6}(e^{-t_1 B_6} + t_1 B_6 - 1)}{\lambda_3-\lambda_4}. \\ I_4 = \frac{\frac{A_7}{B_7}(e^{(T-t_1)B_7} - B_7(T-t_1) - 1) - \frac{A_8}{B_8}(e^{(T-t_1)B_8} - B_8(T-t_1) - 1)}{\mu_3-\mu_4}. \end{cases} \quad \text{(A.10)}$$

$$HC_R(\alpha) = c_{hR}\left[\int_0^{t_1} q_{1R}(t,\alpha)dt + \int_{t_1}^{T} q_{2R}(t,\alpha)dt\right]$$

$$= c_{hR}\left[\begin{array}{l}\int_0^{t_1}\left\{\frac{A_5\lambda_4(1-e^{-B_5 t}) - A_6\lambda_3(1-e^{-B_6 t})}{\lambda_4-\lambda_3}\right\}dt \\ +\int_{t_1}^{T}\left\{\frac{A_7\mu_4\{e^{B_7(T-t)}-1\} - A_8\mu_3\{e^{B_8(T-t)}-1\}}{\mu_4-\mu_3}\right\}dt\end{array}\right]$$

$$= c_{hR}\{J_3 + J_4\}, \quad \text{(A.11)}$$

The values of J_3 and J_4 are assumed as:

$$\begin{cases} J_3 = \frac{\frac{A_5\lambda_4}{B_5}(e^{-t_1 B_5} + B_5 t_1 - 1) - \frac{A_6\lambda_3}{B_6}(e^{-t_1 B_6} + B_6 t_1 - 1)}{\lambda_4-\lambda_3}. \\ J_4 = \frac{\frac{A_7\mu_4}{B_7}(e^{(T-t_1)B_7} - B_7(T-t_1) - 1) - \frac{A_8\mu_3}{B_8}(e^{(T-t_1)B_8} - B_8(T-t_1) - 1)}{\mu_4-\mu_3}, \end{cases}$$

(iii) $\tilde{p} = [p_L(\alpha), p_R(\alpha)]$ = The SR per unit product.

Therefore, $SR = [SR_L(\alpha), SR_R(\alpha)]$ during the entire cycle is given by:

$$SR_L(\alpha) = p_L(\alpha)\left[\begin{array}{l}\int_0^{t_1}\{a - bp_R(\alpha) + cq_{1L}(t,\alpha)\}dt \\ +\int_{t_1}^{T}\{a - bp_R(\alpha) + cq_{2L}(t,\alpha)\}dt\end{array}\right]$$

$$= p_L(\alpha)[\{a - bp_R(\alpha)\}T + c\{I_3 + I_4\}], \quad \text{(A.12)}$$

and

$$SR_R(\alpha) = p_R(\alpha)\left[\begin{array}{l}\int_0^{t_1}\{a - bp_L(\alpha) + cq_{1R}(t,\alpha)\}dt \\ +\int_{t_1}^{T}\{a - bp_L(\alpha) + cq_{2R}(t,\alpha)\}dt\end{array}\right]$$

$$= p_R(\alpha)[\{a - bp_L(\alpha)\}T + c\{J_3 + J_4\}]. \quad \text{(A.13)}$$

(iv) $\tilde{c}_p = [c_{pL}(\alpha), c_{pR}(\alpha)]$ = The production cost per unit product.

Therefore, the production cost $= [PC_L(\alpha), PC_R(\alpha)]$ during the entire cycle is given by:

$$PC_L(\alpha) = c_{pL}\left[\int_0^{t_1}\{m - nq_{1R}(t,\alpha)\}dt\right]$$

$$= c_{pL}[mt_1 - nJ_3]. \quad \text{(A.14)}$$

$$PC_R(\alpha) = c_{pR}\left[\int_0^{t_1}\{m - nq_{1L}(t,\alpha)\}dt\right]$$

$$= c_{pR}[mt_1 - nI_3]. \quad \text{(A.15)}$$

Biographies

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