UNIT NADARAJAH AND HAGHIGHI DISTRIBUTION: PROPERTIES AND APPLICATIONS IN QUALITY CONTROL

Ismail Shah[1], Brikhna Iqbal[2], Muhammad Farhan Akram[3], Sajid Ali[4,*], and Sanku Dey[5]

[1,2,3,4] *Department of Statistics, Quaid-i-Azam University, Islamabad 45320, Pakistan*
[5] *Department of Statistics, St. Anthony's College, Shillong, India*
[1] *ishah@qau.edu.pk,* [2] *khannasarkhang555@gmail.com,* [3] *fanjum566@gmail.com,*
[4] *sajidali.qau@hotmail.com,* [5] *sankud66@gmail.com*
Corresponding Author Phone: 00925190642185

Abstract. In practice, the data related to rates and proportion may have excess of ones wherein the beta distribution does not fit well. To deal with the inflation of ones, this article introduces unit Nadarajah and Haghighi distribution. Besides deriving statistical properties of the proposed distribution, several estimation methods are discussed. In particular, maximum likelihood estimation, least squares estimation, weighted least squares estimation, maximum product of spacing, minimum spacing absolute distance estimation, minimum spacing absolute log-distance estimation, Cramér-Von-Mises, Anderson-Darling method and right-tail Anderson-Darling method are considered. Using real data sets, it is shown that the new distribution outperforms some well-known existing distributions. Furthermore, the application of the proposed distribution in quality control is also discussed. A control chart using unit Nadarajah and Haghighi distribution is constructed and its performance is evaluated using the average run length.
Keywords: Anderson-Darling method; Control chart, Cramér-Von-Mises Estimation Method; Maximum Likelihood Estimation, Root mean squared error; Weighted Least Squared Estimation.
AMS 2010 Classification: 62F10, 62P10.

1. Introduction

Recently many distributions have been introduced in statistics to accommodate natural phenomena arising from diverse fields. In lifetime data analysis, Weibull distribution has a special significance and considered as the benchmark model. Depending on the shape parameter, the Weibull distribution is flexible to model increasing, decreasing, and constant hazard function. In addition, its closed form cumulative distribution function also exists. However, to deal the data with range between zero and one, beta distribution is more appropriate and many absolutely continuous distributions have been used to generate flexible distributions to accommodate the data of proportion. For example, Mazucheli et al.[1] introduced the unit-Weibull distribution and showed its flexibility over the beta distribution. Similarly, the unit-gamma distribution [2], unit logistic distribution [3], unit Lindley distribution [4], unit Gompertz distribution [5], Topp-Leone generated distributions [6], reflected generalized Topp-Leone power series distribution [7], etc., are introduced to deal proportion data.

Nadarajah and Haghighi[8] introduced a new extension of the exponential distribution, known as the Nadarajah and Haghighi (NH) distribution, to deal with the inflation of zeros in absolutely continuous data. Motivated by the application of NH distribution, the aim of this article is to introduce unit Nadarajah and Haghighi (UNH) distribution. A distinct feature of UNH distribution is that it is not constructed by taking into account the positive part of the real line and neither includes special functions nor additional parameters in the formulation but it is constructed in the unit interval. As a consequence, very few distributions with unit interval/finite support are available in the literature. However, while considering real life data sets concerning percentages, proportions or fractions, etc., one needs to consider values in a limited range [9]. Likewise, survival time of units/items/subjects of interest are normally greater than zero and also the lifetime of units/items/subjects of interest cannot arrive at infinite point. In such cases, it is necessary to use a bounded model [10, 11]. Similarly, there are many random variables and random processes that appear in real life applications whose values are bounded both at the lower and upper ends [12, 13, 14, 15, 16]. Besides, in the context of reliability measurement, Genç[17] stated that to get plausible results of reliability, it is better to have models defined on the unit interval.

In the premise of the above, the UNH distribution is suitable to handle the inflation of ones in the proportion data. For example, let compare the mean proportion of days out of 30 wherein people do some physical exercises for at least 30 minutes. If people do exercise 30 out of 30 days, then data will have inflation of one and the response will be highly skewed. In such situation, beta distribution cannot be used because it does not accommodate the occurrence of one. Similarly, comparing the proportion of rain in two cities can also lead to inflation of one when both cities have the same amount of rain in a given time. Besides introducing UNH, we estimate the parameters of the UNH using nine different methods, including maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least square estimation (WLSE), maximum product of spacing (MPS), minimum spacing absolute distance estimation (MSADE), minimum spacing absolute log-distance estimation (MSALDE), Cramér-Von-Mises (CVM), Anderson-Darling method (AD) and right-tail Anderson-Darling method (RAD). In addition to estimation of the parameters of the model, we also construct control charts using UNH distribution to show its practical application for monitoring data.

The rest of the article is organized as follows. Section 2 presents the derivation of the unit Nadarajah and Haghighi distribution while properties including quantile function, moments, entropies, order statistic are discussed in Section 3. Section 4 discusses different estimation methods to estimate the unknown parameters of the proposed distribution. The simulation study is presented in Section 5. Control charts and their performance assessment are presented in Section 6. Real data applications are presented in Section 7, whereas concluding remarks are given in Section 8.

2. Unit Nadarajah and Haghighi Distribution

The main aim of the proposed model is to deal with the inflation of ones. To this end, the probability density function and cumulative distribution function of the NH distribution with two parameters α, λ are defined as

$$f(x;\alpha,\lambda) = \alpha\lambda(1+\lambda x)^{\alpha-1}\exp\left[1-(1+\lambda x)^{\alpha}\right]; x>0, \alpha, \lambda>0 \tag{1}$$

$$F(x;\alpha,\lambda) = 1-\exp\left[1-(1+\lambda x)^{\alpha}\right], x>0, \alpha, \lambda>0. \tag{2}$$

Now, using the transformation $Y=\exp(-X)$, we obtain the following probability density function

$$f(y;\alpha,\lambda) = \frac{\alpha\lambda}{y}(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\ 0<y<1, \tag{3}$$

Figure 1 Here

Figure 1a represents the shape of the UNH distribution which is decreasing and increasing for different values of the parameters. The parameters $\alpha, \lambda > 0$ are non-negative where α is the shape parameter and λ is the rate parameter.

The expression of the CDF of the unit Nadarajah-Haghighi distribution is

$$F(y;\alpha,\lambda) = \exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\ 0<y<1, \tag{4}$$

whereas the graphical depiction is given in Figure 1b.

The survival function is a function that provides the probability that a particular object will survive after a specific time. The term survival function is extensively used in human mortality to show the survival time of a patient beyond a specific time. In reliability, it is used to show the performance of electric devices beyond a specific time. The survival function is given by

$$S(y;\alpha,\lambda) = 1-\exp\left[1-(1-\lambda\ln y)^{\alpha}\right],\ 0<y<1, \tag{5}$$

The hazard function is the ratio of probability density function and survival function. For the UNH, it is obtained as

$$h(y) = \frac{\alpha\lambda(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right]}{y(1-\exp\left[1-(1-\lambda\ln y)^{\alpha}\right])} \tag{6}$$

Figure 1d depicts the hazard function of the UNH distribution where it can be noticed that the distribution has decreasing, increasing-decreasing hazard function for different choices of the parameters. This shows the flexibility of the UNH distribution.

The cumulative hazard is the sum of all the hazard values to a particular time. The cumulative hazard function of the UNH is given by

$$H(y;\alpha,\lambda) = \int_0^y \frac{\alpha\lambda(1-\lambda\ln y)^{\alpha-1}\exp\left[1-(1-\lambda\ln y)^{\alpha}\right]}{y\left[1-\exp(1-(1-\lambda\ln y)^{\alpha})\right]}dy \tag{7}$$

Similarly, the reversed hazard function (RHF) is a important tool in reliability. The reversed hazard function of the UNH distribution is defined as

$$r(y;\alpha,\lambda) = \frac{f(y;\alpha,\lambda)}{F(y;\alpha,\lambda)} \tag{8}$$

and depicted in Figure 1e.

3. Statistical Properties

This section derives some important statistical properties of the UNH distribution.

3.1. **Quantile Function.** The quantile function of the UNH is obtained by $F(y) = u$, where $u \sim Uniform(0,1)$, that is, $u = \exp(1-(1-\lambda\log(y))^{\alpha})$. The simplified form of the quantile function of the UNH is given by

$$y = \exp\frac{1}{\lambda}(1-(1-\ln(u))^{\frac{1}{\alpha}}) \tag{9}$$

The p^{th} quantile function of UNH distribution is defined as

$$y_p = \exp\left(\frac{1}{\lambda}(1-(1-\ln(p))^{\frac{1}{\alpha}})\right) \tag{10}$$

Using $p = 0.5$ in Equation 10, one can obtain the median of the unit UNH distribution as under

$$y_{0.5} = \exp\left(\frac{1}{\lambda}(1-(1-\log(0.5))^{\frac{1}{\alpha}})\right) \tag{11}$$

3.2. **The Moments.** In this section, we derive the rth moment for the UNH distribution. The first fourth moments are the most important to describe the shape of the distribution. Suppose the random variable Y follows the UNH (λ, α), then the rth moment is given as

$$u_r' = \int_0^1 y^r f(y,\alpha,\lambda)dy = \int_0^1 \alpha\lambda y^{r-1}(1-\lambda\log y)^{\alpha-1}\exp(1-(1-\lambda\log y)^{\alpha}) \tag{12}$$

Using the binomial expansion on $[1-F(y)]^i$, i.e.,

$$[1-F(y;\alpha,\lambda)]^i = \sum_{k=0}^{i}\binom{i}{k}(-1)^k[F(y)]^k. \tag{13}$$

we obtain

$$(1-\lambda\log y)^{\alpha-1} = \sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(\lambda\log y)^i \tag{14}$$

$$u_r' = \alpha\lambda\int_0^1 y^{r-1}\sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}(\lambda\log y)^i\exp(1-(1-\lambda\log y)^{\alpha})dy \tag{15}$$

$$u_r' = \sum_{i=1}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\alpha\lambda\int_0^1 y^{r-1}(\lambda\log y)^i\exp(1-(1-\lambda\log y)^{\alpha})dy \tag{16}$$

The rth moment of the UNH distribution cannot be expressed analytically further but can be solved numerically.

3.3. **Rényi Entropy.** The Rényi entropy measures uncertainty of a random variable and defined as

$$R_\upsilon(y) = \frac{1}{1-\upsilon}\log\left[\int_0^1 (f(y))^\upsilon dy\right] \tag{17}$$

$$R_\upsilon(y) = \frac{1}{1-\upsilon}\log\left[\int_0^1 \frac{(\alpha\lambda)^\upsilon}{(y)^\upsilon}(1-\lambda\ln y)^{\upsilon(\alpha-1)}\exp\left(\upsilon(1-(1-\lambda\ln y))^{\alpha}\right)\right]dy \tag{18}$$

Using the Taylor series

$$\exp\left(\upsilon(1-(1-\lambda\ln y)^{\alpha})\right) = \sum_{i=0}^{1}\frac{(\upsilon)^i(1-(1-\lambda\ln y)^{\alpha})^i}{i!} \tag{19}$$

and

$$[1-(1-\lambda\ln y)^{\alpha}]^i = \sum_{k=0}^{i}\binom{i}{k}(-1)^k[(1-\lambda\ln y)^{\alpha}]^k.$$

we get

$$\begin{aligned}
R_\upsilon(y) &= \frac{1}{1-\upsilon}\ln\left[\int_0^1 \frac{(\alpha\lambda)^\upsilon}{(y)^\upsilon}(1-\lambda\ln y)^{\upsilon(\alpha-1)}\sum_{i=0}^{1}\frac{(\upsilon)^i(1-(1-\lambda\ln y)^{\alpha})^i}{i!}\right]dy\\
&= \frac{1}{1-\upsilon}\ln\left[\sum_{i=0}^{1}\frac{(\upsilon)^i}{i!}\left(\int_0^1 \frac{(\alpha\lambda)^\upsilon}{(y)^\upsilon}(1-\lambda\ln y)^{\upsilon(\alpha-1)}(1-(1-\lambda\ln y)^{\alpha})^i\right)\right]dy\\
&= \frac{1}{1-\upsilon}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{(\upsilon)^i}{i!}\left(\int_0^1 \frac{(\alpha\lambda)^\upsilon}{(y)^\upsilon}(1-\lambda\ln y)^{\upsilon(\alpha-1)}[(1-\lambda\ln y)^{\alpha}]^k\right)\right]dy\\
&= \frac{1}{1-\upsilon}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\frac{(\upsilon)^i}{i!}\left(\int_0^1 \frac{(\alpha\lambda)^\upsilon}{(y)^\upsilon}(1-\lambda\ln y)^{(\upsilon(\alpha-1)-k\alpha)}\right)\right]dy
\end{aligned} \tag{20}$$

Again using

$$(1-\lambda\ln y)^{(\upsilon(\alpha-1)-k\alpha)} = \sum_{s=0}^{\upsilon(\alpha-1)-k\alpha}\binom{\upsilon(\alpha-1)-k\alpha}{s}(-1)^s[\lambda\log y]^s \tag{21}$$

we obtain

$$R_v(y) = \frac{1}{1-v}\ln\left[\sum_{i=0}^{1}\sum_{k=0}^{i}\sum_{s=0}^{v(\alpha-1)-k\alpha}\binom{v(\alpha-1)-k\alpha}{s}\binom{i}{k}\frac{(v)^i}{i!}(-1)^s(-1)^k(\alpha\lambda)^v\left(\int_0^1 (y)^{-v}(\lambda\ln y)^s\right)\right]dy \tag{22}$$

3.4. **Stress and Strength Modeling.** Suppose Y_1 and Y_2 are two independent continuous random variables, where $Y_1 \sim UNH(\alpha_1, \lambda_1)$ and $Y_2 \sim UNH(\alpha_2, \lambda_2)$. Then, the stress and strength, denoted by R, is determined as

$$R = P(y_1 > y_2) = \int_{-\infty}^{\infty} f_{y_1}(y)F_{y_2}(y)dy. \tag{23}$$

$$R = P(y_1 > y_2) = \int_0^1 \frac{\alpha_1\lambda_1}{y}(1-\lambda_1\ln y)^{\alpha_1-1}\exp(1-(1-\lambda_1\ln y)^{\alpha_1})\exp(1-(1-\lambda_2\ln y)^{\alpha_2})dy.$$

$$= \int_0^1 \frac{\alpha_1\lambda_1}{y}(1-\lambda_1\ln y)^{\alpha_1-1}\exp(2-(1-\lambda_1\ln y)^{\alpha_1}-(1-\lambda_2\ln y)^{\alpha_2})dy. \tag{24}$$

Using

$$[1-\lambda_1\ln y]^{\alpha_1-1} = \sum_{k=0}^{\alpha_1-1}\binom{\alpha_1-1}{k}(-1)^k(\lambda_1\ln y)^k. \tag{25}$$

$$R = P(y_1 > y_2) = \alpha_1\lambda_1\sum_{k=0}^{\alpha_1-1}\binom{\alpha_1-1}{k}(-1)^k\int_0^1 y^{-1}(\lambda_1 ly)^k\exp(2-(1-\lambda_1\ln y)^{\alpha_1}-(1-\lambda_2\ln y)^{\alpha_2})dy.$$

Since

$$\exp(2-(1-\lambda_1\ln y)^{\alpha_1}-(1-\lambda_2\ln y)^{\alpha_2}) = \sum_{i=0}^{1}\frac{(2-(1-\lambda_1\ln y)^{\alpha_1}-(1-\lambda_2\ln y)^{\alpha_2})^i}{i!} \tag{26}$$

$$R = P(Y_1 > Y_2) = \frac{\alpha_1\lambda_1}{i!}\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i}{k}(-1)^k\int_0^1 y^{-1}(\lambda_1\ln y)^k(2^i-(1-\lambda_1\ln y)^{i\alpha_1}-(1-\lambda_2\ln y)^{i\alpha_2})dy. \tag{27}$$

Again using

$$[1-\lambda_1\log y]^{i\alpha_1} = \sum_{j=0}^{i\alpha_1}\binom{i\alpha_1}{j}(-1)^j(\lambda_1\ln y)^j.$$

$$R = P(y_1 > y_2) = (-1)^j(-1)^k(-1)^j\frac{\alpha_1\lambda_1}{i!}\sum_{j=0}^{i\alpha_1}\sum_{j=0}^{i\alpha_2}\sum_{i=0}^{1}\sum_{k=0}^{i}\binom{i\alpha_1}{j}\binom{i\alpha_2}{j}\binom{i}{k}\left(2^i{\lambda_1}^k\int_0^1 y^{-1}(\ln y)^k\,dy\right)$$

$$-\left({\lambda_2}^{k+j}\int_0^1 y^{-1}(\ln y)^{k+j}\,dy\right)-\left({\lambda_1}^k{\lambda_2}^j\int_0^1 y^{-1}(\ln y)^{k+j}dy\right). \tag{28}$$

3.5. **Order Statistics.** In this section, we define the probability density function of the ith order statistic of the UNH distribution. Suppose a sample of size k, $Y_{(1)}, \cdots, Y_{(k)}$, be the order statistic obtained from a random sample $Y_1, \cdots, Y_k$ of size k from a continuous population with distribution function $F(y;\varphi)$ and probability density function $f(y;\varphi)$. Then, the probability density function of $y_{(i)}$ is given by

$$f_{Y(i)}(y) = \frac{k!}{(i-1)!(k-i)!}f_Y(y)[F(y;\varphi)]^{i-1}[1-F(y;\varphi)]^{k-i} \tag{29}$$

for $i = 1, 2, \cdots, k$. For the UNH distribution, we have

$$f_{Y_{(i)}}(y) = \frac{k!}{(i-1)!(k-i)!}\left(\frac{\alpha\lambda}{y}(1-\lambda\ln y)^{\alpha-1}\exp(1-(1-\lambda\ln y)^{\alpha})\right)[\exp(1-(1-\lambda\ln y)^{\alpha})]^{i-1}[1-\exp(1-(1-\lambda\ln y)^{\alpha})]^{k-i} \tag{30}$$

while the probability density function of the largest order statistic $y_{(k)}$ is given by

$$f_{Y_{(k)}}(y) = \frac{\alpha\lambda k}{y}(1-\lambda\ln y)^{\alpha-1}\exp(1-(1-\lambda\ln y)^{\alpha})[\exp(1-(1-\lambda\ln y)^{\alpha})]^{k-1} \tag{31}$$

and the probability density function of the smallest order statistic $y_{(1)}$ is given by

$$f_{Y(1)}(y) = \frac{\alpha\lambda k}{y}(1-\lambda\ln y)^{\alpha-1}\exp(1-(1-\lambda\ln y)^{\alpha})[1-\exp(1-(1-\lambda\ln y)^{\alpha})]^{k-1} \tag{32}$$

4. Estimation of Parameters

In this section, we discuss the unknown parameters estimation of the UNH distribution using the maximum likelihood estimation, ordinary least squares, percentile estimation, maximum product of spacing, minimum spacing absolute distance estimator, minimum spacing absolute log distance estimator, Cramér-Von-Mises, Anderson-Darling (AD) and right-tail Anderson-Darling methods [18, 19].

4.1. **Maximum Likelihood Estimation.** Suppose $Y_1, Y_2, \cdots, Y_n$ be a simple random sample from the UNH distribution. Then, the likelihood function is given by

$$L(\lambda,\alpha,\mathbf{y}) = \prod_{i=1}^{n} f(y_i,\lambda,\alpha) = \prod_{i=1}^{n}\frac{\alpha\lambda}{y_i}(1-\lambda\log y_i)^{\alpha-1}\exp(1-(1-\lambda\log y_i)^{\alpha}) \tag{33}$$

The log-likelihood function is given by

$$\ln L(\lambda,\alpha,\mathbf{y}) = n\ln(\lambda\alpha) - \sum_{i=1}^{n}\ln(y_i) + (\alpha-1)\sum_{i=1}^{n}\ln(1-\lambda\ln y_i) + n - \sum_{i=1}^{n}(1-\lambda\ln y_i)^{\alpha} \tag{34}$$

It follows that the maximum likelihood estimators MLEs of the parameters are obtained by differentiating the log-likelihood function with respect to the parameters λ and α and then equating the resulting equations to zero.

$$\frac{\partial\ln L(\lambda,\alpha,\mathbf{y})}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-\lambda\ln y_i) - \sum_{i=1}^{n}(1-\lambda\ln y_i)^{\alpha}\ln(1-\lambda\ln y_i) = 0 \tag{35}$$

$$\frac{\partial\ln L(\lambda,\alpha,\mathbf{y})}{\partial\lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{(\ln y_i)}{(1-\lambda\ln y_i)} + \alpha\sum_{i=1}^{n}(\ln y_i)(1-\lambda\ln y_i)^{\alpha-1} = 0 \tag{36}$$

The MLEs of the UNH distribution cannot be obtained in closed forms. Thus, it needs to be solved numerically for the parameters λ and α.

4.2. **Ordinary Least Squares Estimators.** Let $Y_1, \cdots, Y_n$ is a random sample of size n from the distribution function $F(.)$ and $Y_{(i)} < \cdots < Y_{(n)}$ denote the corresponding order sample. The ordinary least squares estimators can be obtained by minimizing

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}[F(y_{(i)}) - E(F(y_{(i)}))]^2. \tag{37}$$

Using

$$E(F(Y_{(i)})) = \frac{i}{n+1}, \tag{38}$$

we get

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[F(Y_{(i)}) - \frac{i}{n+1}\right]^2. \tag{39}$$

Therefore, in the case of the UNH distribution, the ordinary least squares estimators of λ and α, say λ_{OLS} and α_{OLS}, respectively, can be obtained by minimizing

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]^2 \tag{40}$$

Differentiate Equation 40 with respect to the unknown parameters λ and α and equating the resulting equations to zero, one can get the OLS estimators.

$$\frac{\partial Z(\lambda,\alpha)}{\partial\lambda} = 2\sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]\exp(1-(1-\lambda\ln y_{(i)})^{\alpha})\alpha(1-\lambda\ln y_{(i)})^{\alpha-1}\ln y_{(i)} = 0 \tag{41}$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial\alpha} = 2\sum_{i=1}^{n}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]\exp(1-(1-\lambda\ln y_{(i)})^{\alpha})(1-\lambda\ln y_{(i)})^{\alpha}\ln(1-\lambda\ln y_{(i)}) = 0 \tag{42}$$

As these equations cannot be solved analytically, the non-linear equations need to be solved numerically. The weighted least squares estimators of the unknown parameters can be obtained to minimizing

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i[F(Y_{(i)}) - E(F(Y_{(i)}))]^2. \tag{43}$$

Using

$$E(F(Y_{(i)})) = \frac{i}{n+1}, \tag{44}$$

we get

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i\big[F(Y_{(i)}) - \frac{i}{n+1}\big]^2. \tag{45}$$

The weight w_i are equal to

$$\frac{1}{V(y_{(i)})} = \frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in the case of the UNH distribution, the weighted least squares estimators of λ and α, say $\hat{\lambda}_{WLSE}$ and $\hat{\alpha}_{WLSE}$, respectively, can be obtained by minimizing

$$Z(\lambda,\alpha) = \sum_{i=1}^{n} w_i\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]^2, \tag{46}$$

that is, differentiate with respect to the unknown parameters λ and α and equating to zero, we get the following equations.

$$\frac{\partial Z(\lambda,\alpha)}{\partial\lambda} = 2\sum_{i=1}^{n} w_i\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]\exp(1-(1-\lambda\ln y_{(i)})^{\alpha})\alpha(1-\lambda\ln y_{(i)})^{\alpha-1}\ln y_{(i)} = 0 \tag{47}$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial\alpha} = 2\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[\exp(1-(1-\lambda\ln y_{(i)})^{\alpha}) - \frac{i}{n+1}\right]\exp(1-(1-\lambda\ln y_{(i)})^{\alpha})(1-\lambda\ln y_{(i)})^{\alpha}\ln(1-\lambda\ln y_{(i)}) = 0 \tag{48}$$

The above equations need to be solved numerically.

4.3. **Percentile Estimation (PCE) Method.** If the cumulative distribution function have a closed form, then one can estimate the unknown parameter by fitting a straight line to the percentile points. In our case,

$$F(y;\alpha,\lambda) = \exp\left(1-(1-\lambda\log y)^{\alpha}\right), \tag{49}$$

therefore

$$y = \exp\left(\frac{1}{\lambda}(1-(1-\log(u))^{\frac{1}{\alpha}})\right) \tag{50}$$

Let $Y_1, \cdots, Y_n$ is a random sample of size n from the distribution function F(.) and $Y_{(i)} < \cdots < Y_{(n)}$ denote the corresponding ordered sample. The estimate of λ and α can be obtained by minimizing

$$Z(\lambda,\alpha) = \sum_{i=1}^{n}\left[y_{(i)} - \exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)\right]^2 \tag{51}$$

that is, differentiate with respect to α and λ.

$$\frac{\partial Z(\lambda,\alpha)}{\partial \alpha} = \sum_{i=1}^{n}\left[y_{(i)} - \exp\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right]\frac{1}{\lambda}\exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)(1-\ln(u_i))^{\frac{1}{\alpha}}\ln(1-\ln(u_i)) = 0 \tag{52}$$

$$\frac{\partial Z(\lambda,\alpha)}{\partial \lambda} = \sum_{i=1}^{n}\left[y_{(i)} - \exp\left(\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\right)\right]\exp\frac{1}{\lambda}(1-(1-\ln(u_i))^{\frac{1}{\alpha}})\frac{1}{(\lambda)^2}(1-(1-\ln(u_i))^{\frac{1}{\alpha}}) = 0 \tag{53}$$

where $u_i = \frac{i}{n+1}$.

4.4. **Maximum Product Spacing (MPS) Method.** For the method of maximum product of spacing (MPS) [20, 21], we define

$$D_j(\alpha,\lambda) = F(y_{j:k}|\alpha,\lambda) - F(y_{j-1:k}|\alpha,\lambda),\ j = 1,2,\cdots,k, \tag{54}$$

Let $\hat{\alpha}_{MPS}$ and $\hat{\lambda}_{MPS}$ are the estimators obtained using the maximum product of spacings for the UNH distribution parameters α and λ. The geometric mean of the spacings is defined as

$$G(\alpha,\lambda) = \left[\prod_{j=1}^{k+1} D_j(\alpha,\lambda)\right]^{\frac{1}{k+1}} \tag{55}$$

or maximizing the function

$$H(\alpha,\lambda) = \frac{1}{k+1}\sum_{j=1}^{k+1}\ln D_j(\alpha,\lambda) \tag{56}$$

$$\frac{\partial H(\alpha,\lambda)}{\partial \alpha} = \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{D_j(\alpha,\lambda)}\left[\omega_1(y_{j:k}|\alpha,\lambda) - \omega_1(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{57}$$

$$\frac{\partial H(\alpha,\lambda)}{\partial \lambda} = \frac{1}{k+1}\sum_{j=1}^{k+1}\frac{1}{D_j(\alpha,\lambda)}\left[\omega_2(y_{j:k}|\alpha,\lambda) - \omega_2(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{58}$$

$$\omega_1(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}) \tag{59}$$

$$\omega_2(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}) \tag{60}$$

Maximizing H(α,λ) is as efficient as the maximum likelihood estimation and the maximum product of spacing estimators are consistent under more common conditions than the MLE estimators.

4.5. **Minimum Spacing Absolute Distance Estimation (MSADE) Method.** The method of minimum spacing absolute distance estimator (MSADE) [22] and the authors showed that parameters estimation by MSADE is as efficient as MLE estimation. Furthermore, the MSADE estimators are consistent under more flexible condition than the MLE estimators. We define

$$D_j(\alpha,\lambda) = F(y_{j:k}|\alpha,\lambda) - F(y_{j-1:k}|\alpha,\lambda),\ j = 1,2,\cdots,k. \tag{61}$$

Then, $\hat{\alpha}_{MSADE}$ and $\hat{\lambda}_{MSADE}$, are the UNH distribution parameters α and λ are obtained by minimizing the following function with respect to α and λ.

$$T(\alpha,\lambda) = \sum_{j=1}^{k+1}\left|\ D_j(\alpha,\lambda) - \frac{1}{n+1}\ \right| \tag{62}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial \alpha} = \sum_{j=1}^{k+1}\frac{D_j(\alpha,\lambda) - \frac{1}{n+1}}{|\ D_j(\alpha,\lambda) - \frac{1}{n+1})\ |}\left[\omega_1(y_{j:k}|\alpha,\lambda) - \omega_1(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{63}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\lambda} = \sum_{j=1}^{k+1} \frac{D_j(\alpha,\lambda) - \frac{1}{n+1}}{\mid D_j(\alpha,\lambda) - \frac{1}{n+1}) \mid} \left[\omega_2(y_{j:k}|\alpha,\lambda) - \omega_2(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{64}$$

where

$$\omega_1(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}) \tag{65}$$

$$\omega_2(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}) \tag{66}$$

4.6. **Minimum Spacing Absolute Log Distance Estimation (MSALDE) Method.** The minimum spacing absolute log distance estimators (MSALDE) are obtained by minimizing T(α,λ) as follows:

$$T(\alpha,\lambda) = \sum_{j=1}^{k+1} \mid \ln D_j(\alpha,\lambda) - \ln\frac{1}{n+1} \mid \tag{67}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\alpha} = \sum_{j=1}^{k+1} \frac{\ln D_j(\alpha,\lambda) - \ln\frac{1}{n+1}}{\mid \ln D_j(\alpha,\lambda) - \ln\frac{1}{n+1}) \mid} \frac{1}{D_j(\alpha,\lambda)} \left[\omega_1(y_{j:k}|\alpha,\lambda) - \omega_1(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{68}$$

$$\frac{\partial T(\alpha,\lambda)}{\partial\lambda} = \sum_{j=1}^{k+1} \frac{\ln D_j(\alpha,\lambda) - \ln\frac{1}{n+1}}{\mid \ln D_j(\alpha,\lambda) - \ln\frac{1}{n+1}) \mid} \frac{1}{D_j(\alpha,\lambda)} \left[\omega_2(y_{j:k}|\alpha,\lambda) - \omega_2(y_{j-1:k}|\alpha,\lambda)\right] = 0 \tag{69}$$

where

$$\omega_1(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}) \tag{70}$$

$$\omega_2(y_{j:k}|\alpha,\lambda) = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}) \tag{71}$$

4.7. **Cramér-Von-Mises Method.** To encourage our decision of Cramér Von-Mises estimators, MacDonald[23] presented an empirical proof that the bias of these estimators is smaller than the other small distance type estimators. The Cramér-von-Mises estimators $\hat{\alpha}_{CME}$ and $\hat{\lambda}_{CME}$ of the UNH distribution parameters α and λ are obtained by minimizing the following function

$$C(\alpha,\lambda) = \frac{1}{12n} + \sum_{j=1}^{n}\left(F(y_{j:n|\alpha,\lambda}) - \frac{2j-1}{2n}\right)^2 \tag{72}$$

These estimators can also be obtained by solving the following non-linear equations.

$$\frac{\partial C(\alpha,\lambda)}{\partial\alpha} = \sum_{i=1}^{n}\left(\exp(1-(1-\lambda\ln y_{j:k})^{\alpha}) - \frac{2j-1}{2n}\right)(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}) = 0 \tag{73}$$

$$\frac{\partial C(\alpha,\lambda)}{\partial\lambda} = \sum_{i=1}^{n}\left(\exp(1-(1-\lambda\ln y_{j:k})^{\alpha}) - \frac{2j-1}{2n}\right)\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}) = 0 \tag{74}$$

4.8. **Anderson-Darling (AD) and Right-tail Anderson-Darling (RTADE) Methods.** In this section, we define the method of Anderson-Darling (AD) estimation for the UNH distribution as

$$A(\alpha,\lambda) = -k - \frac{1}{k}\sum_{j=1}^{k}(2j-1)\{\ln F(y_{j:k}|\alpha,\lambda) + \ln\bar{F}(y_{k+1-j:k}|\alpha,\lambda)\}. \tag{75}$$

These estimators can also be obtained by solving the following non-linear equations

$$\begin{aligned}\frac{\partial A(\alpha,\lambda)}{\partial\alpha} =& \sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln y_{j:k})^{\alpha})} \\ &- \sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{(1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha}))} = 0,\end{aligned} \tag{76}$$

$$\frac{\partial A(\alpha,\lambda)}{\partial\lambda} = \sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}\ln(y_{j:k})}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$

$$-\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y))^{\alpha-1}\ln(y_{j:k})}{(1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha}))} = 0, \quad (77)$$

$$\frac{\partial F(\alpha,\lambda)}{\partial\alpha} = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})(1-\lambda\ln y_{j:k})^{\alpha}\ln(1-\lambda\ln y_{j:k}) \quad (78)$$

$$\frac{\partial F(\alpha,\lambda)}{\partial\lambda} = \exp(1-(1-\lambda\ln y_{j:k})^{\alpha})\alpha(1-\lambda\ln y_{j:k})^{\alpha-1}(\ln y_{j:k}) \quad (79)$$

Similarly, the right tail Anderson-Darling (RTADE) estimators $\hat{\alpha}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the UNH parameters α and λ are obtained by minimizing

$$R(\alpha,\lambda) = \frac{k}{2} - 2\sum_{j=1}^{k}\ln F(y_{j:k}|\alpha,\lambda) - \frac{1}{k}\sum_{j=1}^{k}(2j-1)\ln\bar{F}(y_{k+1-j:k}|\alpha,\lambda). \quad (80)$$

These estimators can also be obtained by solving the following non-linear equations.

$$\frac{\partial R(\alpha,\lambda)}{\partial\alpha} = -2\sum_{j=1}^{k}\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$

$$+\frac{1}{k}\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})(1-\lambda\ln(y_{j:k}))^{\alpha}\ln(1-\lambda\ln(y_{j:k}))}{1-\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})} = 0 \quad (81)$$

$$\frac{\partial R(\alpha,\lambda)}{\partial\lambda} = -2\sum_{j=1}^{k}\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y_{j:k}))^{\alpha-1}\ln(y_{j:k}))}{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})}$$

$$+\frac{1}{k}\sum_{j=1}^{k}(2j-1)\frac{\exp(1-(1-\lambda\ln(y_{j:k}))^{\alpha})\alpha(1-\lambda\ln(y_{j:k}))^{\alpha-1}\ln(y_{j:k}))}{(1-\exp(1-(1-\lambda ln(y_{j:k}))^{\alpha}))} = 0 \quad (82)$$

5. Simulation Study

Tables 1-6 Here

The performance of ten different estimation methods is compared using a comprehensive simulation study. For all methods, we computed biases, mean squared errors, average absolute difference between the theoretical and empirical estimate of the distribution functions (Dabs), and the maximum absolute difference between the theoretical and empirical distribution functions (Dmax). The experiments were repeated N=10000 times by taking samples of sizes n= 20 , 40 , 60 , 80 and 100, with (α,λ) = (0.5,0.5), (0.5,2.0), (1.5,2.0), (1.5,0.5), (3.5,2.0), (3.0,0.5).

It is noticed from Tables 1-6 that the biases and RMSE of α and λ decrease when sample size increased for all methods of estimation. The average absolute difference between the theoretical and empirical estimate of the distribution functions (Dabs) is smaller than the maximum absolute difference between the theoretical and empirical distribution functions (Dmax) for all methods of estimation. The simulation results suggest that the WLS estimators perform better in terms of biases and RMSEs. The second better performing estimators is the MPS estimators. Moreover, the WLS, MPS, MLE, AD, CVM, PCE estimators are among the good estimators for the UNH distribution. The LS estimator does not perform well. It is also confirmed that the performance of the MLE and PC estimators are the same, as expected, and the performance of the CVM and AD estimators is the same.

6. TBE Control Chart and Performance Assessment

Time-between-events (TBE) control charts are frequently used in reliability and other system related applications. A TBE chart monitors the inter-arrival times so it does not require sampling intervals [24]. The defects or nonconforming items from a manufacturing system are generally modeled by a Poisson process and Poisson cumulative sum (CUSUM) and Shewhart c charts are the examples of such control charts. Alternatively, we could use control charts that are based on inter-arrival times. These inter-arrival times are assumed to be independent and identically distributed exponential random variables. The exponential CUSUM chart and exponential chart are the two examples of these type of charts [25]. The exponential chart is preferred because one does not have to wait for the fixed time period as it plots the information immediately as soon it is obtained. A comprehensive overview of these charts is provided by Ali et al.[26].

The aim of this section is to introduce control charts to monitor the TBE data measured between zero and one scale. Moreover, as the UNH provides better fit in the case of inflation of ones in the data, the proposed TBE chart is also suitable to monitor such data. The recent contributions to monitor data of rates and proportion can be seen in [27, 28, 29, 30, 31] and the references cited therein.

Let β denotes the false alarm probability. To derive the control limits of the proposed chart, we equate $F(x) = \beta/2$ and $1 - \beta/2$ to obtain the two-sided control chart. Similarly, equate $F(x) = \beta$ or $1 - \beta$ to obtain the lower or upper-sided control limit of the one-sided chart. The simplified expressions of the ARL and control limits for the one-sided charts are given as

$$\begin{aligned} LCL &= \exp((1/\lambda_0)(1 - (1 - \log\beta)^{(1/\alpha_0)})) \\ ARL_L &= 1/\exp(1 - (1 - \lambda\log(LCL))^{\alpha}) \\ UCL &= \exp((1/\lambda_0)(1 - (1 - \log(1-\beta))^{(1/\alpha_0)})) \\ ARL_U &= 1/(1 - \exp(1 - (1 - \lambda\log(UCL))^{\alpha})). \end{aligned} \tag{83}$$

Similarly, the control limits and ARL expressions for the two-sided control charts are given as

$$\begin{aligned} LCL &= \exp((1/\lambda_0)(1 - (1 - \log(\beta/2))^{(1/\alpha_0)})) \\ UCL &= \exp((1/\lambda_0)(1 - (1 - \log(1 - (\beta/2)))^{(1/\alpha_0)})) \\ ARL_{L\cup U} &= 1/(\exp(1 - (1 - \lambda\log(LCL))^{\alpha}) + 1 - \exp(1 - (1 - \lambda\log(UCL))^{\alpha})) \end{aligned} \tag{84}$$

The most common measure to access the performance of a control chart is the average run length (ARL). It is defined to be the average number of points (samples) plotted until we observe a signal indicating that the process is out-of-control. The in-control ARL (ARL_0) and the out-of-control ARL (ARL_1) are the two types of ARL. Ideally, we should have a large value of (ARL_0) so that we do not have to make unnecessary adjustments to the process while a small value of (ARL_1) so that a shift in the process may be detected quickly. Further, for the Shewhart structure, the ARL is known to have geometric distribution and thus $ARL = 1/p$, where "p" is the parameter of geometric distribution which represents the probability of shift detection.

Although the ARL is widely used for performance evaluation, it is to be noted that the variance of the ARL distribution is large and in some cases, nearly equal to the mean. This implies that there would be large fluctuations in the frequencies of false alarms. To overcome this drawback, the coefficient of variation (CV) of the run length distribution can be utilized because of the fact that the CV values do not fluctuate drastically with the increasing/decreasing magnitude of shifts. In addition, the CV values can directly be compared especially when the ARL values do not differ greatly from each other.

We conducted the ARL analysis of UNH distribution for different values of shape and scale parameters along with some additional quantities including CV, first, second, and third quartile (Q1, Q2 and Q3). It is worth mentioning that the ARL_0 value for all combinations of in-control rate (λ_0) and shape (α_0) parameters, assuming level of significance to be 0.0027, is 370.370. Furthermore, we computed the ARL values of upper, lower and two-sided control charts for all the considered combination of in-control values of the parameters. To be more specific, in our study, we used λ_0=2.5 in combination with three different values of α_0, i.e., $\alpha_0 \in (0.75, 1, 1.50)$. Thus, we have three combinations of in-control parameters $(\lambda_0, \alpha_0) = \{(2.50, 0.75), (2.50, 1.00), (2.50, 1.50)$. For these in-control, three cases we assumed $\lambda_1 \in (0.1, 0.4, 0.5, 0.6, 0.9, 1, 1.3, 1.5, 2, 2.5, 2.7, 3)$ and $\alpha_1 \in (0.1, 0.4, 0.5, 0.6, 0.75, 0.9, 1, 1.3, 1.5)$ to represent the out-of-control situation.

6.1. **Performance Analysis assuming** $\lambda_0 = 2.5, \alpha_0 = 0.75$**.** From Tables 7, 8 and 9 and additional Tables A.1-A.6, given in the appendix, it is quite clear that when we fix the value of the shape parameter α, the two-sided control chart is the quickest to detect the downward shift in the rate parameter λ. Furthermore, for fixed α, the ARL has an increasing pattern in the lower-sided chart but an opposite pattern for the upper-sided chart. The same pattern is observed for lower and upper sided charts when we fix the value of λ. The two-sided control chart, however, behaves differently; for fixed α, its ARL values increase till the nominal value of α and when $\alpha > 0.75$, the ARL has increasing trend till $\lambda < 2$ and beyond that the ARL decreases. It can also be seen that the lower-sided control chart performs poorly for $\alpha > 0.75$ (upward shift in the shape parameter) as compared to $\alpha < 0.75$. The performance of two-sided control chart also deteriorates for $\alpha > 0.75$ but not as much as it does for the lower-sided chart. On the other hand, the upper-sided control chart performs better for $\alpha > 0.75$ than the lower-sided chart. It is also noticed that the behavior of ARL for some combination of parameters is biased, i.e., $ARL_1 > ARL_0$, and we left those cells blank in the tables.

The CV analysis of Table 7 shows a decreasing pattern when we fix the value of the rate parameter λ for downward shifts and increasing pattern for upward shifts. This suggests that the lower-sided control chart is efficient for detecting large-size shifts in downward direction only. A similar behavior is observed when we fix the value of shape parameter α, that is, the chart is only efficient in detecting large-size shifts in the downward direction. For upper-sided chart, when we fix the value of λ, the CV values decrease for $\alpha > 0.75$ and increase for $\alpha < 0.75$ which implies that the chart can efficiently be used for detection of large size shifts in upward direction.

The quartile analysis of the Table 7 shows that, for fixed λ, the ARL value is greater than the third quartile (Q3) or lies between second and third quartile (Q2 and Q3). This means that the ARL distribution is either highly or moderately skewed (positively). Similarly, fixing the value of α, the ARL distribution is observed highly skewed for large downward shift in λ and less skewed for comparatively small downward or upward shift in λ. The two-sided control chart shows similar characteristics. The upper-sided chart shows that for fixed λ, the distribution of ARL is moderately skewed as all the ARL values lie between Q2 and Q3. For a fixed α, the ARL distribution shows a similar pattern as it does for fixed value of λ. Similarly, one can compare the results listed in Tables A.1-A.6, which are given in the appendix.

Tables 7-9 Here

7. Real Data Analysis

This section presents two real data applications to show the suitability of the proposed distribution and its application in quality control.

7.1. **Rainfall Data.** The first data set has taken from Nadarajah and Haghighi[8], which is the daily rainfall (in mm) in the January for a location in Florida from 1907-2000. The mode of the original data set is zero. We transformed the data using $Y = \exp(-X)$ and the resulted data set is listed in Table 10, which represents the proportion of daily rainfall.

Table 10 Here

Figure 2 Here

We compare the proposed UNH model with some other distributions, such as Kumaraswamy distribution [32],

$$f(y;\alpha,\lambda) = \alpha\lambda y^{\alpha-1}(1-y^{\alpha})^{\lambda-1},\ y \in (0,1) \tag{85}$$

Topp-Leone distribution [17],

$$f(y;\alpha,\lambda) = 2\alpha y^{\alpha-1}(1-y)(2-y)^{\alpha-1},\ y \in (0,1) \tag{86}$$

reflected Generalized Topp-Leone (rGTL) distribution [7],

$$f(y;\alpha,\lambda) = 2\alpha y^{\alpha-1}(1-y)(2-y)^{\alpha-1},\ y \in (0,1) \tag{87}$$

Beta distribution,

$$f(y;\alpha,\lambda) = \frac{1}{B(\alpha,\lambda)} y^{\alpha-1}(1-y)^{\lambda-1},\ y \in (0,1) \tag{88}$$

Tables 11-12 Here

The values of the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hanan Quinn information criterion (HQIC), MLEs with their standard errors, Kolmogorov-Smirnov (K-S) statistic p-values are listed in Tables 11 and 12, showed that the UNH distribution fits better than the other distributions. From Figure 2, it is clear that the proposed chart can effectively be used for monitoring the rainfall data.

7.2. **Anxiety Data Analysis.** The second data have been obtained from Bourguignon et al.[33], which is about the anxiety test performed in a group of 180 "normal" women, i.e., outside of a pathological clinic Townsville, Queensland, Australia. The data set is reproduced in Table 13.

Table 13 Here

Figure 3 Here

Tables 14-15 Here

The values of AIC, CAIC, BIC, HQIC, MLEs with their standard errors, Kolmogorov-Smirnov (K-S) statistic p-values are listed in Tables 14 and 15. From the tables, it is evident that the UNH distribution outperformed the other distributions. Furthermore, the UNH distribution has the lowest AIC and BIC values. Figure 3 indicates that anxiety level of many women fall on the lower limit of the proposed chart. This implies that these women need psychological therapy to improve their mind health.

8. Conclusion

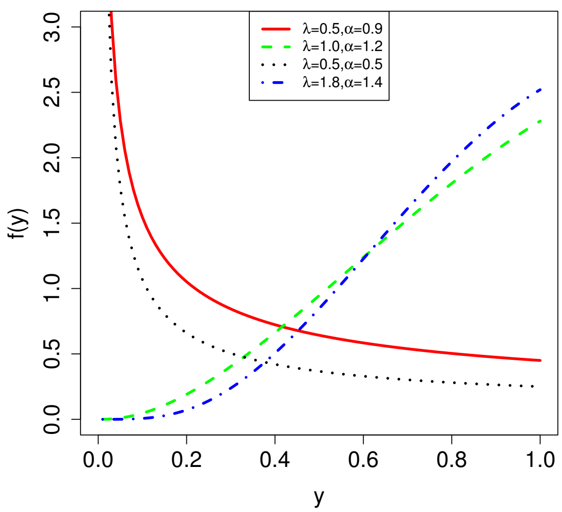
In this article, a new distribution to accommodate the inflation of the ones is proposed. Furthermore, different properties and estimation methods are discussed in detail. From the simulation results using different methods of estimation, it is clear that the MPS, MLE, AD, CVM, and PCE perform better in terms of RMSE than the rest of the methods. In addition to estimation methods, control charts are also proposed and their performance is studied using the ARL criterion. Two-real data applications to show the practicality of the proposed distribution and utilization in process monitoring are also discussed. From the ARL study, it is noticed that for some combination of parameters, the $ARL_1 > ARL_0$ and hence, unbiased design of the control chart may be studied in the future.

References

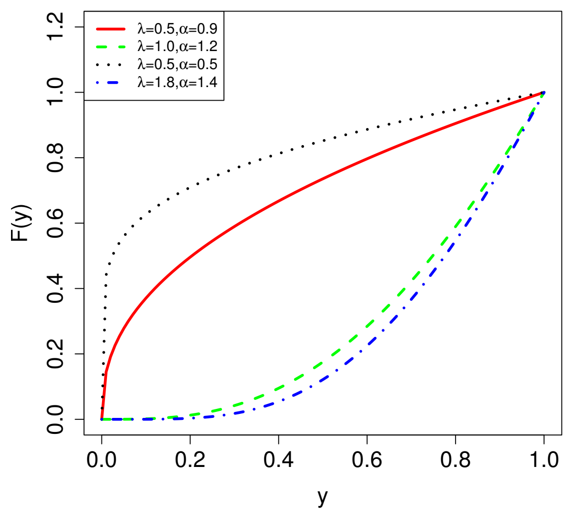
1. Mazucheli, J., Menezes, A. F. B., Fernandes, L. B., de Oliveira, R. P., and Ghitany, M. E., "The unit-Weibull distribution as an alternative to the Kumaraswamy distribution for the modeling of quantiles conditional on covariates", *Journal of Applied Statistics*, **47**(6), pp. 954–974 (2020).
2. Mazucheli, J., Menezes, A.F.B., and Dey, S., "Improved maximum-likelihood estimators for the parameters of the unit-gamma distribution", *Communications in Statistics-Theory and Methods*, **47**(15), pp. 3767–3778 (2018).
3. Menezes, A. F. B., Mazucheli, J., and Dey, S., "The unit-logistic distribution: Different methods of estimation", *Pesquisa Operacional*, **38**(3), pp. 555–578 (2018).
4. Mazucheli, J., Menezes, A.F.B., and Chakraborty, S., "On the one parameter unit- Lindley distribution and its associated regression model for proportion data", *Journal of Applied Statistics*, **46**(4), pp. 700–714 (2019).
5. Mazucheli, J., Menezes, A.F.B., and Dey, S., "Unit-Gompertz distribution with applications", *Statistica*, **79**(1), pp. 25–43 (2019).
6. Sangsanit, Y. and Bodhisuwan, W., "The Topp-Leone generator of distributions: properties and inferences.", *Songklanakarin Journal of Science & Technology*, **38**(5) ((2016)).
7. Condino, F. and Domma, F., "A new distribution function with bounded support: the reflected generalized Topp-Leone power series distribution", *Metron*, **75**(1), pp. 51–68 (2017).
8. Nadarajah, S. and Haghighi, F., "An extension of the exponential distribution", *Statistics*, **45**(6), pp. 543–558 (2011).
9. Marshall, A.W. and Olkin, I., "*Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families*". Springer Series in Statistics, Springer New York (2007).
10. Aban, I. B., Meerschaert, M. M., and Panorska, A. K., "Parameter estimation for the truncated pareto distribution", *Journal of the American Statistical Association*, **101**(473), pp. 270–277 (2006).
11. Zhang, T. and Xie, M., "On the upper truncated Weibull distribution and its reliability implications", *Reliability Engineering & System Safety*, **96**(1), pp. 194 – 200 (2011).
12. Papke, L.E. and Wooldridge, J.M., "Econometric methods for fractional response variables with an application to 401(k) plan participation rates", *Journal of Applied Econometrics*, **11**(6), pp. 619–632 (1996).
13. Fletcher, S.G. and Ponnambalam, K., "Estimation of reservoir yield and storage distribution using moments analysis", *Journal of Hydrology*, **182**(1), pp. 259 – 275 (1996).
14. Seifi, A., Ponnambalam, K., and Vlach, J., "Maximization of manufacturing yield of systems with arbitrary distributions of component values", *Annals of Operations Research*, **99**, p. 373–383 (2000).
15. Gangi, A., K., Ponnambalam., D., Khalili., and Karamouz, M., "Grain yield reliability analysis with crop water demand uncertainty", *Stochastic Environmental Research and Risk Assessment*, **20**(4), pp. 259 – 277 (2006).
16. Cook, D. O., Kieschnick, R., and McCullough, B. D., "Regression analysis of proportions in finance with self selection", *Journal of Empirical Finance*, **15**(5), pp. 860 – 867 (2008).
17. Genc, A.I., "Estimation of $p(x > y)$ with Topp–Leone distribution", *Journal of Statistical Computation and Simulation*, **83**(2), pp. 326–339 (2013).

18. Ali, S., Dey, S., Tahir, M.H., and Mansoor, M., "Two-parameter logistic-exponential distribution: Some new properties and estimation methods", *American Journal of Mathematical and Management Sciences*, **39**(3), pp. 270–298 (2020).
19. Ali, S., Dey, S., Tahir, M. H., and Mansoor, M., "A comparison of different methods of estimation for the flexible Weibull distribution", *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, **69**(1), pp. 794 – 814 (2020).
20. Cheng, R. C. H. and Amin, N. A. K., "Maximum product of spacings estimation with application to the lognormal distribution", tech. rep., (1979).
21. Cheng, R. C. H. and Amin, N. A. K., "Estimating parameters in continuous univariate distributions with a shifted origin", *Journal of the Royal Statistical Society. Series B (Methodological)*, **45**(3), pp. 394–403 (1983).
22. Torabi, H., "A general method for estimating and hypotheses testing using spacings", *Journal of Statistical Theory and Practice*, **8**(2), pp. 163–168 (2008).
23. MacDonald, P. D. M., "Comment on "an estimation procedure for mixtures of distributions" by Choi and Bulgren", *Journal of the Royal Statistical Society. Series B (Methodological)*, **33**(2), pp. 326–329 (1971).
24. Shamsuzzaman, M., Xie, X., Goh, N.T., and Zhang, H., "Integrated control chart system for time-between-events monitoring in a multistage manufacturing system", *The International Journal of Advanced Manufacturing Technology*, **40**(3-4), pp. 373–381 (2009).
25. Zhang, C.W., Xie, M., Liu, J.Y., and Goh, T.N., "A control chart for the gamma distribution as a model of time between events", *International Journal of Production Research*, **45**(23), pp. 5649–5666 (2007).
26. Ali, S., Pievatolo, A., and Göb, R., "An overview of control charts for high quality processes", *Quality and Reliability Engineering International*, **32**(7), pp. 2171–2189 (2016).
27. Linda, L. H., Fernandes, F.H., and Bourguignon, M., "Control charts to monitor rates and proportions", *Quality and Reliability Engineering International*, **35**(1), pp. 74–83 (2019).
28. Cruz, F. R. B., Quinino, R. C., and Ho., Linda L., "Control charts for traffic intensity monitoring of Markovian multiserver queues", *Quality and Reliability Engineering International*, **36**(1), pp. 354–364 (2020).
29. Lima-Filho, L. M. de A., Pereira, T.L., de Souza, T. C., and Bayer, F.M., "Inflated beta control chart for monitoring double bounded processes", *Computers & Industrial Engineering*, **136**, pp. 265 – 276 (2019).
30. Lima-Filho, L.M. de A. and Bayer, F. M., "Kumaraswamy control chart for monitoring double bounded environmental data", *Communications in Statistics - Simulation and Computation*, **50**(9), pp. 2513–2528 (2021).
31. Chukhrova, N. and Johannssen, A., "Improved control charts for fraction non-conforming based on hypergeometric distribution", *Computers & Industrial Engineering*, **128**, pp. 795 – 806 (2019).
32. Lemonte, A.J., "Improved point estimation for the Kumaraswamy distribution", *Journal of Statistical Computation and Simulation*, **81**(12), pp. 1971–1982 (2011).
33. Bourguignon, M., Ghosh, I., and Cordeiro, G.M., "General results for the transmuted family of distributions and new models", *Journal of Probability and Statistics*, **2016**, pp. 1–12 (2016).

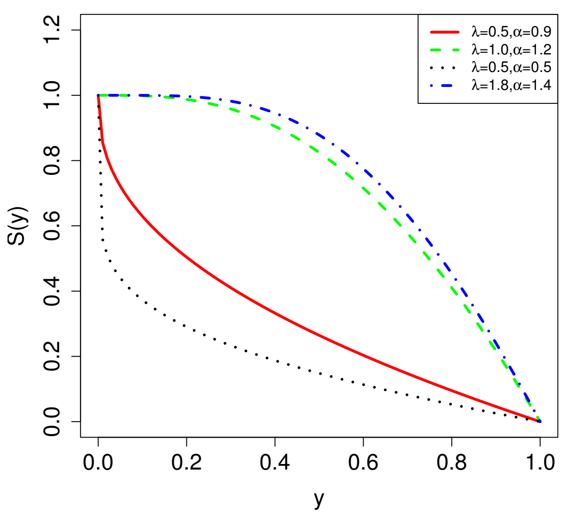
List of Figures and Tables



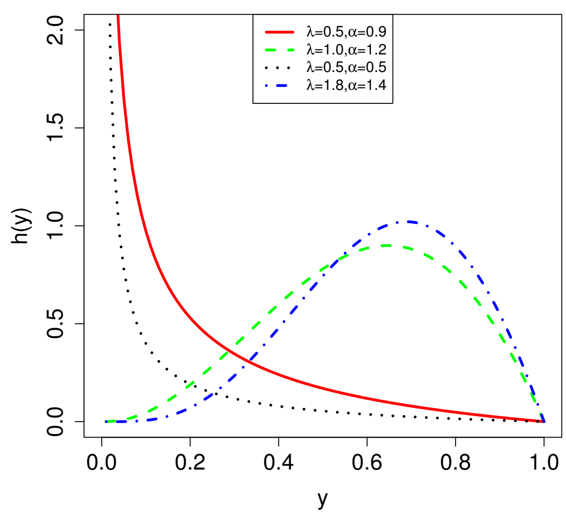
(A) PDF



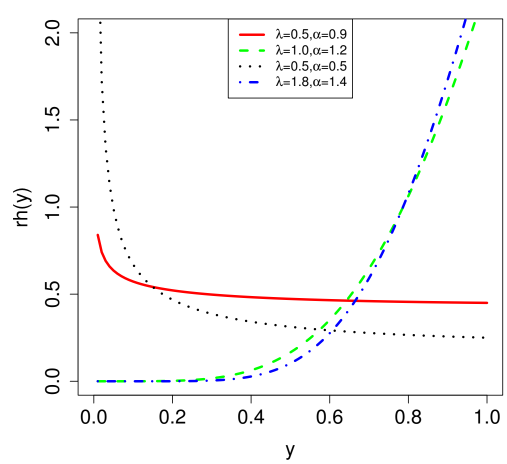
(B) CDF



(C) Survival function

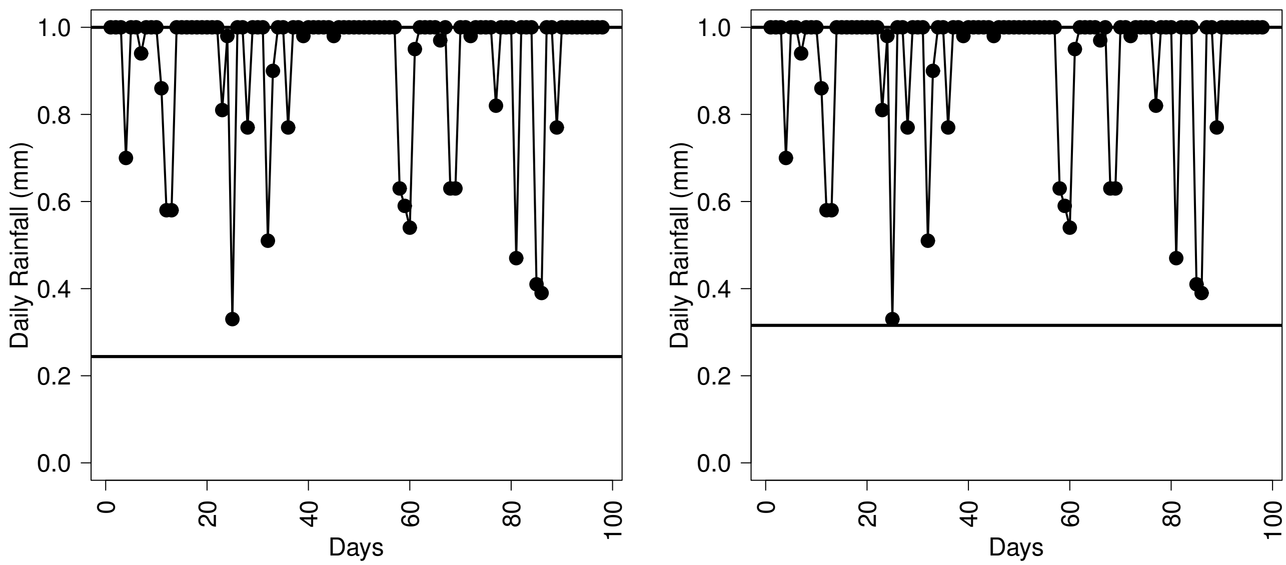


(D) Hazard function



(E) Reversed hazard function

FIGURE 1. PDF, CDF, Survival, hazard, and reverse hazard function of the UNH Distribution



(A) UNH chart

(B) Beta chart

FIGURE 2. Control Charts for the rainfall data assuming UNH and beta distributions

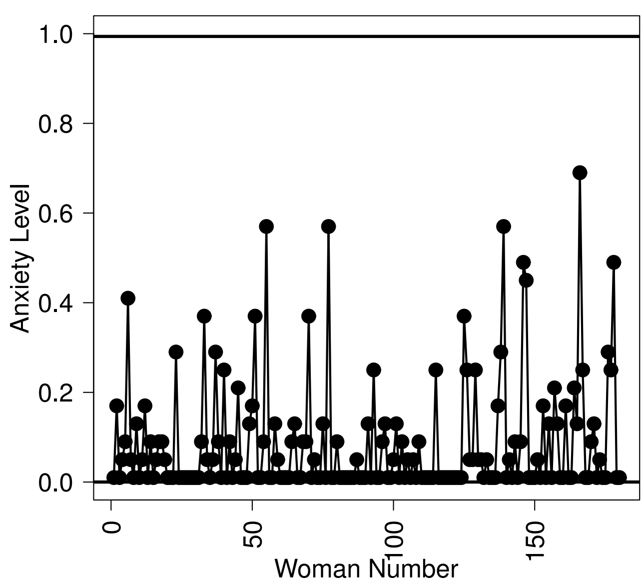


FIGURE 3. Control Chart for Anxiety Data

Table 1. Simulation results for α=0.5 and λ=0.5

n	**Est.**	**MLE**	**LSE**	**WLS**	**PCE**	**MPS**	**MSADE**	**MSALDE**	**CVM**	**AD**	**RAD**
20	Bias(α)	0.766^{7}	0.400^{4}	0.094^{2}	-0.385^{3}	-0.085^{1}	1.787^{8}	7.415^{10}	0.513^{5}	0.647^{6}	3.388^{9}
	$RMSE(\alpha)$	2.409^{6}	0.400^{3}	0.219^{1}	0.712^{4}	0.221^{2}	3.050^{7}	8.825^{9}	1.017^{5}	3.919^{8}	11.528^{10}
	Bias(λ)	0.395^{2}	167.377^{10}	0.813^{7}	-0.400^{3}	-0.043^{1}	1.3234^{8}	7.892^{9}	0.546^{4}	0.666^{5}	0.792^{6}
	$RMSE(\lambda)$	0.957^{2}	177.911^{10}	1.520^{5}	0.400^{1}	1.943^{6}	2.706^{7}	9.206^{9}	1.302^{3}	1.406^{4}	2.760^{8}
	Dabs	0.166^{1}	0.201^{7}	0.168^{4}	0.310^{8}	0.167^{2}	0.448^{9}	0.657^{10}	0.168^{5}	0.168^{3}	0.169^{6}
	$Dmax$	0.266^{4}	0.311^{7}	0.249^{1}	0.719^{8}	0.250^{2}	0.784^{9}	0.958^{10}	0.270^{5}	0.260^{3}	0.277^{6}
	Total	22^{3}	41^{7}	20^{2}	$27^{4.5}$	14^{1}	48^{9}	57^{10}	$27^{4.5}$	29^{6}	45^{8}
40	Bias(α)	0.246^{4}	$-0.400^{6.5}$	0.115^{2}	$-0.400^{6.5}$	-0.0964^{1}	2.009^{9}	8.752^{10}	0.295^{5}	0.190^{3}	0.808^{8}
	RMSE(α)	0.637^{7}	$0.400^{3.5}$	0.195^{2}	$0.400^{3.5}$	0.140^{1}	3.639^{8}	10.085^{10}	0.585^{6}	0.572^{5}	3.915^{9}
	Bias(λ)	0.378^{1}	162.444^{10}	0.541^{7}	-0.400^{2}	-0.506^{6}	1.629^{8}	9.381^{9}	0.426^{3}	0.489^{4}	0.502^{5}
	RMSE(λ)	0.654^{2}	167.240^{10}	0.837^{5}	0.400^{1}	1.056^{7}	3.181^{8}	10.531^{9}	0.8014^{3}	0.812^{4}	1.032^{6}
	Dabs	0.166^{2}	0.199^{7}	0.168^{5}	0.310^{8}	0.165^{1}	0.460^{9}	0.659^{10}	0.168^{4}	0.167^{3}	0.168^{6}
	Dmax	0.256^{4}	$0.312^{7.0}$	0.250^{2}	0.761^{8}	0.247^{1}	0.835^{9}	0.973^{10}	0.262^{5}	0.254^{3}	0.266^{6}
	Total	20^{2}	44^{8}	23^{4}	29^{6}	17^{1}	51^{9}	$58^{1}0$	26^{5}	22^{3}	40^{7}
60	Bias(α)	0.169^{4}	$-0.400^{7.5}$	0.120^{2}	$-0.400^{7.5}$	-0.096^{1}	2.258^{9}	$9.535^{1}0$	0.217^{5}	0.149^{3}	0.335^{6}
	RMSE(α)	0.289^{4}	$0.400^{6.5}$	0.182^{2}	$0.400^{6.5}$	0.122^{1}	3.995^{9}	10.811^{10}	0.398^{5}	0.250^{3}	1.391^{8}
	Bias(λ)	0.374^{1}	161.036^{10}	0.462^{6}	-0.400^{3}	-0.638^{7}	1.875^{8}	10.258^{9}	0.395^{2}	0.439^{4}	0.439^{5}
	RMSE(λ)	0.555^{2}	164.130^{10}	0.660^{5}	0.400^{1}	0.910^{7}	3.528^{8}	11.294^{9}	0.654^{4}	0.652^{3}	0.785^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.165^{1}	0.473^{9}	0.660^{10}	0.167^{5}	0.167^{3}	0.168^{6}
	$Dmax$	0.253^{4}	0.312^{7}	0.250^{2}	0.777^{8}	0.246^{1}	0.859^{9}	0.979^{10}	0.258^{5}	0.252^{3}	0.261^{6}
	Total	17^{1}	48^{8}	21^{4}	34^{6}	18^{2}	52^{9}	58^{10}	26^{5}	19^{3}	37^{7}
80	Bias(α)	0.145^{4}	$-0.400^{7.5}$	0.123^{2}	$-0.400^{7.5}$	-0.094^{1}	2.915^{9}	9.744^{10}	0.188^{5}	0.138^{3}	0.255^{6}
	RMSE(α)	0.208^{3}	$0.400^{6.5}$	0.175^{2}	$0.400^{6.5}$	0.113^{1}	4.437^{9}	10.955^{10}	0.316^{5}	0.209^{4}	1.029^{8}
	Bias(λ)	0.371^{1}	159.932^{10}	0.421^{6}	-0.400^{4}	-0.707^{7}	2.468^{8}	10.459^{9}	0.372^{2}	0.409^{5}	0.395^{3}
	RMSE(λ)	0.507^{2}	162.206^{10}	0.571^{5}	0.400^{1}	0.875^{7}	4.043^{8}	11.430^{9}	0.569^{4}	0.568^{3}	0.650^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.166^{1}	0.522^{9}	0.661^{10}	0.167^{5}	0.167^{3}	0.167^{6}
	Dmax	0.251^{4}	0.312^{7}	0.251^{2}	0.786^{8}	0.246^{1}	0.890^{9}	0.982^{10}	0.256^{5}	0.256^{3}	0.259^{6}
	Total	17^{1}	48^{8}	$21^{3.5}$	$35^{6.5}$	18^{2}	52^{9}	58^{10}	26^{5}	$21^{3.5}$	$35^{6.5}$
100	Bias(α)	0.137^{4}	$-0.400^{7.5}$	0.124^{2}	$-0.400^{7.5}$	-0.092^{1}	3.545^{9}	9.836^{10}	0.172^{5}	0.133^{3}	0.207^{6}
	RMSE(α)	0.186^{3}	$0.400^{6.5}$	0.169^{2}	$0.400^{6.5}$	0.108^{1}	4.761^{9}	11.036^{10}	0.272^{5}	0.191^{4}	0.479^{8}
	Bias(λ)	0.369^{2}	159.557^{10}	0.401^{6}	-0.400^{5}	-0.746^{7}	2.986^{8}	10.484^{9}	0.362^{1}	0.395^{4}	0.377^{3}
	RMSE(λ)	0.482^{2}	161.409^{10}	0.528^{5}	0.400^{1}	0.872^{7}	4.324^{8}	11.452^{9}	0.525^{3}	0.527^{4}	0.590^{6}
	Dabs	0.167^{2}	0.199^{7}	0.167^{4}	0.310^{8}	0.166^{1}	0.571^{9}	0.660^{10}	0.167^{5}	0.167^{3}	0.167^{6}
	Dmax	0.251^{3}	0.312^{7}	0.250^{2}	0.791^{8}	0.246^{1}	0.916^{9}	0.982^{10}	0.255^{5}	0.251^{4}	0.257^{6}
	Total	16^{1}	48^{8}	21^{3}	36^{7}	18^{2}	52^{9}	58^{10}	24^{5}	22^{4}	35^{6}

Authors' Biographies

Ismail Shah received the master's degree from Lund University, Sweden, and the Ph.D. degree from the University of Padova, Italy. He is currently an Assistant Professor with the Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan. He is also working as an editor for the journal of Quantitative methods. His research interests include functional data analysis, time series analysis, regression analysis, energy economics, applied and industrial statistics.

Brikhna Iqbal completed her MPhil in Statistics from Quaid-i-Azam University (QAU), Islamabad, Pakistan. Her research interests are focused on construction of probability distribution and applied statistics.

Muhammad Farhan Akram completed his MPhil in Statistics from Quaid-i-Azam University (QAU), Islamabad, Pakistan. His research interests are: statistical quality control, probability distributions and applied statistics.

Table 2. Simulation results for α=0.5 and λ=2.0

n	Est.	MLE	LSE	WLS	PCE	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias(α)	3.218^{7}	-1.400^{5}	-0.116^{1}	25.879^{10}	0.648^{2}	-1.356^{4}	-1.328^{3}	2.342^{6}	5.638^{8}	1.088^{9}
	$RMSE(\alpha)$	6.599^{7}	1.400^{2}	0.785^{1}	27.146^{10}	3.002^{5}	1.492^{3}	1.584^{4}	4.908^{6}	18.195^{8}	25.055^{9}
	Bias(λ)	0.698^{4}	626.252^{10}	1.649^{7}	-0.400^{3}	1.649^{8}	-0.362^{1}	-0.331^{1}	1.202^{5}	1.373^{6}	1.865^{9}
	$RMSE(\lambda)$	1.646^{4}	656.253^{10}	2.924^{8}	0.400^{3}	2.898^{7}	0.690^{2}	0.894^{3}	2.608^{4}	2.866^{8}	6.561^{9}
	Dabs	0.169^{5}	0.209^{7}	0.168^{2}	0.386^{10}	0.169^{6}	0.328^{8}	0.329^{9}	0.168^{3}	0.168^{4}	0.167^{1}
	$Dmax$	0.267^{6}	0.326^{7}	0.251^{1}	0.589^{8}	0.252^{2}	0.786^{9}	0.786^{10}	0.265^{4}	0.260^{3}	0.260^{5}
	Total	33^{6}	41^{8}	20^{1}	$42^{9.5}$	$30^{4.5}$	28^{2}	$30^{4.5}$	29^{3}	35^{7}	$42^{9.5}$
40	Bias(α)	1.376^{4}	-1.400^{7}	-0.084^{1}	27.044^{10}	0.125^{2}	-1.397^{5}	-1.399^{6}	1.339^{3}	1.675^{8}	5.221^{9}
	RMSE(α)	4.233^{7}	1.400^{2}	0.666^{1}	27.693^{10}	1.826^{5}	1.410^{4}	1.402^{3}	3.566^{6}	8.117^{8}	14.934^{9}
	Bias(λ)	0.649^{4}	611.574^{10}	1.076^{7}	-0.400^{3}	1.171^{9}	-0.397^{1}	-0.399^{2}	0.915^{5}	0.985^{6}	1.149^{8}
	RMSE(λ)	1.134^{4}	625.250^{10}	1.651^{6}	0.400^{1}	1.681^{8}	0.443^{3}	0.406^{2}	1.651^{7}	1.626^{5}	2.271^{9}
	Dabs	0.166^{1}	0.207^{7}	0.167^{5}	0.400^{10}	0.168^{6}	0.328^{8}	0.328^{9}	0.167^{3}	0.167^{4}	0.167^{2}
	Dmax	0.259^{4}	0.327^{7}	0.253^{2}	0.613^{8}	0.251^{1}	0.841^{10}	0.841^{9}	0.261^{5}	0.257^{3}	0.262^{6}
	Total	24^{2}	$43^{9}.5$	22^{1}	42^{8}	31^{5}	31^{5}	31^{5}	29^{3}	34^{7}	$43^{9.5}$
60	Bias(α)	0.589^{3}	$-1.400^{7.5}$	-0.091^{2}	27.524^{10}	-0.072^{1}	-1.400^{6}	$-1.400^{7.5}$	0.794^{5}	0.596^{4}	2.919^{9}
	RMSE(α)	2.534^{6}	$1.400^{4.5}$	0.597^{1}	27.964^{10}	1.217^{2}	1.400^{3}	$1.400^{4.5}$	2.704^{7}	4.364^{8}	10.090^{9}
	Bias(λ)	0.641^{4}	607.577^{10}	0.918^{7}	-0.400^{2}	1.014^{9}	-0.399^{1}	-0.400^{3}	0.833^{5}	0.878^{6}	0.986^{8}
	RMSE(λ)	0.963^{4}	616.405^{10}	1.297^{5}	0.400^{1}	1.333^{7}	0.412^{3}	0.400^{2}	1.347^{8}	1.298^{6}	1.713^{9}
	Dabs	0.166^{1}	0.207^{7}	0.167^{4}	0.405^{10}	0.168^{6}	0.327^{9}	0.327^{8}	0.167^{2}	0.167^{5}	0.167^{3}
	$Dmax$	0.256^{4}	0.327^{7}	0.253^{2}	0.621^{8}	0.250^{1}	0.864^{9}	0.864^{10}	0.258^{5}	0.255^{3}	0.260^{6}
	Total	22^{2}	46^{10}	21^{1}	41^{8}	26^{3}	31^{4}	35^{7}	$32^{5.5}$	$32^{5.5}$	44^{9}
80	Bias(α)	0.257^{4}	-1.400^{7}	-0.096^{1}	27.738^{10}	-0.161^{2}	-1.400^{8}	-1.398^{6}	0.506^{5}	0.218^{3}	1.861^{9}
	RMSE(α)	1.639^{6}	1.400^{3}	0.550^{1}	28.074^{10}	0.857^{2}	1.400^{4}	1.410^{5}	2.162^{7}	2.713^{8}	7.719^{9}
	Bias(λ)	0.642^{4}	604.034^{10}	0.837^{7}	-0.400^{2}	0.938^{9}	-0.400^{3}	-0.398^{1}	0.781^{5}	0.817^{6}	0.881^{8}
	RMSE(λ)	0.884^{4}	610.504^{10}	1.125^{5}	0.400^{1}	1.171^{7}	0.400^{2}	0.437^{3}	1.174^{8}	1.128^{6}	1.417^{9}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.408^{10}	0.167^{6}	0.328^{8}	0.328^{9}	0.167^{3}	0.167^{5}	0.167^{4}
	Dmax	0.254^{4}	0.327^{7}	0.253^{2}	0.625^{8}	0.250^{1}	0.879^{9}	0.879^{10}	0.257^{5}	0.254^{3}	0.258^{6}
	Total	23^{2}	44^{9}	18^{1}	41^{8}	27^{3}	$34^{6.5}$	$34^{6.5}$	33^{5}	31^{4}	45^{10}
100	Bias(α)	0.128^{3}	-1.400^{8}	-0.102^{2}	27.901^{10}	-0.185^{4}	-1.400^{8}	-1.400^{8}	0.336^{5}	0.095^{1}	1.263^{6}
	RMSE(α)	1.193^{3}	1.400^{5}	0.518^{1}	28.176^{10}	0.676^{2}	1.400^{5}	1.400^{5}	1.803^{7}	2.024^{8}	6.022^{9}
	Bias(λ)	0.640^{4}	602.929^{10}	0.798^{7}	-0.400^{1}	0.887^{9}	$-0.400^{2.5}$	$-0.400^{2.5}$	0.758^{5}	0.786^{6}	0.839^{8}
	RMSE(λ)	0.840^{4}	608.204^{10}	1.038^{5}	0.400^{1}	1.079^{7}	$0.400^{2.5}$	$0.400^{2.5}$	1.085^{8}	1.045^{6}	1.286^{9}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.410^{10}	0.167^{6}	$0.327^{8.5}$	$0.327^{8.5}$	0.167^{3}	0.167^{4}	0.167^{5}
	Dmax	0.254^{4}	0.327^{7}	0.253^{2}	0.628^{8}	0.250^{1}	$0.888^{9.5}$	$0.888^{9.5}$	0.256^{5}	0.254^{3}	0.257^{6}
	Total	$19^{1.5}$	47^{10}	$19^{1.5}$	40^{8}	29^{4}	$36^{6.5}$	$36^{6.5}$	33^{5}	28^{3}	43^{9}

Sajid Ali is currently Assistant Professor at the Department of Statistics, Quaid-i-Azam University (QAU), Islamabad, Pakistan. He graduated (PhD Statistics) from Bocconi University, Milan, Italy. His research interests include time series analysis, Bayesian inference, construction of new flexible probability distributions, change point detection, and process monitoring.

Sanku Dey PhD: An Associate Professor in the Department of Statistics, St. Anthony's College, Shillong, Meghalaya, India. He has to his credit more than 220 research papers in journals of repute. He is a reviewer and associate editors of reputed international journals. He has a good number of contributions in almost all fields of Statistics viz., distribution theory, discretization of continuous distribution, reliability theory, multi-component stress-strength reliability, survival analysis, Bayesian inference, Record Statistics, Statistical quality control, order statistics, lifetime performance index based on classical and Bayesian approach as well as different types of censoring schemes etc.

TABLE 3. Simulation results for α=1.5 and λ=2.0

n	**Est.**	**MLE**	**LSE**	**WLS**	**PCE**	**MPS**	**MSADE**	**MSALDE**	**CVM**	**AD**	**RAD**
20	Bias(α)	4.041^{5}	-1.400^{3}	0.273^{1}	9.274^{7}	1.882^{4}	-1.399^{2}	13.684^{9}	4.110^{6}	9.352^{8}	13.755^{10}
	$RMSE(\alpha)$	9.070^{6}	1.400^{2}	0.952^{1}	9.447^{7}	4.608^{4}	1.401^{3}	15.056^{8}	6.740^{5}	21.421^{9}	25.657^{10}
	Bias(λ)	-1.140^{6}	674.605^{10}	-0.249^{2}	-1.689^{7}	-0.256^{3}	-1.898^{8}	12.48^{9}	-0.680^{5}	-0.547^{4}	-0.113^{1}
	$RMSE(\lambda)$	1.694^{1}	711.801^{10}	2.095^{5}	1.694^{2}	2.034^{4}	1.903^{3}	13.948^{9}	2.123^{6}	2.247^{7}	5.756^{8}
	Dabs	0.163^{2}	0.243^{8}	0.165^{4}	0.077^{1}	0.164^{3}	0.661^{10}	0.350^{9}	0.167^{6}	0.165^{5}	0.169^{7}
	$Dmax$	0.251^{6}	0.392^{8}	0.243^{2}	0.152^{1}	0.245^{3}	0.982^{10}	0.894^{9}	0.250^{5}	0.248^{4}	0.252^{7}
	Total	26^{4}	41^{8}	15^{1}	25^{3}	21^{2}	36^{6}	53^{10}	33^{5}	37^{7}	43^{9}
40	Bias(α)	3.288^{5}	-1.400^{3}	0.390^{1}	9.027^{9}	1.177^{2}	2.301^{4}	14.483^{10}	3.255^{6}	4.949^{7}	8.958^{8}
	RMSE(α)	6.782^{6}	1.400^{2}	0.892^{1}	9.334^{7}	3.317^{3}	6.290^{5}	15.630^{9}	5.717^{4}	13.220^{8}	18.185^{10}
	Bias(λ)	-1.160^{7}	660.486^{10}	-0.766^{3}	-0.050^{1}	-0.700^{2}	$2.321\ ^{8}$	14.396^{9}	-0.964^{6}	$-\ 0.905^{5}$	-0.781^{4}
	RMSE(λ)	1.404^{3}	681.658^{10}	1.291^{2}	2.238^{7}	1.258^{1}	6.427^{8}	15.361^{9}	1.518^{5}	1.435^{4}	1.863^{6}
	Dabs	0.165^{3}	0.241^{8}	0.165^{1}	0.172^{7}	0.165^{2}	$0.495^{1}0$	0.341^{9}	0.166^{5}	0.165^{4}	0.167^{6}
	Dmax	0.251^{4}	0.394^{7}	0.245^{1}	0.401^{8}	0.246^{2}	0.894^{9}	0.927^{10}	0.251^{5}	0.249^{3}	0.252^{6}
	Total	28^{3}	40^{7}	9^{1}	39^{6}	12^{2}	44^{9}	56^{10}	$31^{4.5}$	$31^{4.5}$	$40^{7.5}$
60	Bias(α)	1.185^{3}	$-1.400^{7.5}$	0.433^{2}	11.598^{10}	0.840^{1}	3.757^{6}	$14.823^{7.5}$	2.707^{5}	3.090^{4}	6.693^{9}
	RMSE(α)	7.820^{6}	$1.400^{4.5}$	0.853^{1}	11.715^{10}	2.564^{2}	5.926^{3}	$16.090^{4.5}$	5.008^{7}	9.006^{8}	14.483^{9}
	Bias(λ)	-1.190^{4}	659.309^{10}	-0.917^{7}	4.788^{2}	-0.834^{9}	3.788^{1}	14.995^{3}	-1.054^{5}	-1.007^{6}	-0.942^{8}
	RMSE(λ)	1.348^{4}	674.518^{10}	1.185^{5}	5.370^{1}	1.127^{7}	5.905^{3}	15.969^{2}	1.388^{8}	1.300^{3}	1.526^{6}
	Dabs	0.168^{6}	0.239^{7}	0.165^{2}	0.308^{8}	0.164^{1}	0.387^{10}	0.349^{9}	0.166^{4}	0.165^{3}	0.166^{5}
	$Dmax$	0.254^{6}	0.392^{7}	0.246^{2}	0.853^{9}	0.246^{1}	0.846^{8}	0.940^{10}	0.251^{4}	0.249^{3}	0.252^{5}
	Total	31^{5}	40^{7}	10^{2}	49^{9}	9^{1}	45^{8}	57^{10}	27^{4}	26^{3}	36^{6}
80	Bias(α)	-2.010^{5}	$-1.400^{3.5}$	0.463^{1}	16.188^{10}	0.655^{2}	$-1.400^{3.5}$	15.092^{9}	2.337^{7}	2.189^{6}	5.437^{8}
	RMSE(α)	11.790^{7}	$1.400^{2.5}$	$0.831\ ^{1}$	16.272^{9}	2.078^{4}	$1.400^{2.5}$	16.400^{10}	4.480^{5}	6.640^{6}	12.201^{8}
	Bias(λ)	-1.239^{6}	658.692^{10}	-0.995^{2}	8.496^{8}	-0.898^{1}	-1.900^{7}	15.299^{9}	-1.109^{5}	-1.063^{4}	-1.046^{3}
	RMSE(λ)	1.366^{5}	670.145^{10}	1.166^{2}	8.638^{8}	1.091^{1}	1.900^{7}	16.294^{9}	1.339^{4}	1.258^{3}	1.411^{6}
	Dabs	0.172^{6}	0.238^{7}	0.165^{2}	0.328^{8}	0.165^{1}	0.662^{10}	0.354^{9}	0.167^{5}	0.166^{3}	0.166^{4}
	Dmax	0.260^{6}	0.391^{7}	0.247^{2}	0.946^{8}	0.246^{1}	0.985^{10}	0.948^{9}	0.252^{5}	0.250^{3}	0.252^{4}
	$Total$	35^{6}	$40^{7.5}$	$10^{1.5}$	$51^{1.5}$	$10^{1.5}$	$40^{7.5}$	55^{10}	31^{4}	25^{3}	33^{5}
100	Bias(α)	-1.687^{4}	-1.400^{3}	-0.481^{1}	20.687^{10}	0.579^{2}	5.231^{8}	15.342^{9}	2.062^{6}	1.795^{5}	4.617^{7}
	RMSE(α)	10.846^{8}	1.400^{2}	0.821^{1}	20.758^{10}	1.773^{3}	6.741^{6}	16.672^{9}	4.050^{4}	5.555^{5}	10.645^{7}
	Bias(λ)	-1.229^{6}	657.475^{10}	-1.036^{2}	11.465^{8}	-0.941^{1}	5.451^{7}	15.610^{9}	-1.135^{5}	-1.092^{4}	-1.090^{3}
	RMSE(λ)	1.335^{5}	667.818^{10}	1.166^{2}	11.553^{8}	1.088^{1}	6.805^{7}	16.588^{9}	1.317^{4}	1.244^{3}	1.373^{6}
	Dabs	0.170^{6}	0.238^{7}	0.165^{2}	0.331^{9}	0.165^{1}	0.319^{8}	0.358^{10}	0.166^{5}	0.166^{3}	0.166^{4}
	Dmax	0.258^{6}	0.391^{7}	0.248^{2}	0.962^{10}	0.246^{1}	0.830^{8}	0.953^{9}	0.252^{4}	0.250^{3}	0.252^{5}
	$Total$	35^{6}	39^{7}	10^{2}	$55^{9.5}$	$9^{1.0}$	44^{8}	$55^{9.5}$	28^{4}	23^{3}	32^{5}

Appendix

This appendix contains the additional ARL tables.

TABLE 4. Simulation results for α=3.5 and λ=0.5

n	Est.	**MLE**	**LSE**	**WLS**	**PCE**	**MPS**	**MSADE**	**MSALDE**	**CVM**	**AD**	**RAD**
20	Bias(α)	3.759^{5}	-3.400^{3}	-0.912^{1}	5.960^{7}	1.008^{4}	-3.400^{2}	-3.400^{9}	4.157^{6}	9.003^{8}	12.338^{10}
	$RMSE(\alpha)$	8.115^{6}	3.400^{2}	2.282^{1}	6.074^{7}	6.680^{4}	3.400^{3}	3.400^{8}	9.325^{5}	25.405^{9}	27.828^{10}
	Bias(λ)	1.482^{6}	1509.383^{10}	3.258^{2}	0.288^{7}	3.292^{3}	-0.400^{8}	-0.400^{9}	2.514^{5}	2.922^{4}	4.143^{1}
	$RMSE(\lambda)$	3.207^{1}	1598.423^{10}	5.411^{5}	0.342^{2}	5.238^{4}	0.400^{3}	0.400^{9}	4.814^{6}	5.990^{7}	14.414^{8}
	Dabs	0.172^{2}	0.210^{8}	0.168^{4}	0.408^{1}	0.170^{3}	0.331^{10}	0.331^{9}	0.168^{6}	0.169^{5}	0.166^{7}
	$Dmax$	0.267^{6}	0.338^{8}	0.252^{2}	0.605^{1}	0.253^{3}	0.798^{10}	0.798^{9}	0.261^{5}	0.259^{4}	0.259^{7}
	Total	33^{6}	$39^{8.5}$	19^{1}	33^{6}	29^{2}	$31^{3.5}$	$31^{3.5}$	33^{6}	$39^{8.5}$	43^{10}
40	Bias(α)	1.774^{4}	-3.400^{6}	-0.954^{3}	-0.707^{2}	-0.182^{1}	$-3.400^{7.5}$	$-3.400^{7.5}$	2.500^{5}	3.512^{9}	6.515^{10}
	RMSE(α)	6.390^{7}	3.400^{2}	2.064^{1}	4.403^{5}	4.936^{6}	$3.400^{3.5}$	$3.400^{3.5}$	7.747^{8}	15.072^{9}	18.595^{10}
	Bias(λ)	1.311^{3}	1479.768^{10}	2.200^{6}	4.754^{9}	2.323^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.973^{4}	2.061^{5}	2.545^{8}
	RMSE(λ)	2.192^{3}	1529.969^{10}	3.353^{6}	5.745^{9}	3.295^{4}	$0.400^{1.5}$	$0.400^{1.5}$	3.357^{7}	3.346^{5}	4.810^{8}
	Dabs	0.167^{3}	0.209^{7}	0.167^{4}	NaN^{10}	0.168^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{2}	0.168^{5}	0.166^{1}
	Dmax	0.260^{6}	0.339^{7}	0.254^{2}	NaN^{10}	0.252^{1}	$0.854^{8.5}$	$0.854^{8.5}$	0.258^{5}	0.256^{3}	0.258^{4}
	Total	26^{3}	42^{9}	22^{1}	45^{10}	25^{2}	31^{5}	31^{5}	31^{5}	36^{7}	41^{8}
60	Bias(α)	0.627^{2}	-3.400^{7}	-1.049^{4}	-0.358^{1}	-0.756^{3}	$-3.400^{8.5}$	$-3.400^{8.5}$	1.416^{6}	1.308^{5}	3.815^{10}
	RMSE(α)	4.889^{7}	3.400^{3}	1.951^{2}	1.529^{1}	3.867^{6}	$3.400^{4.5}$	$3.400^{4.5}$	6.509^{8}	9.887^{9}	13.866^{10}
	Bias(λ)	1.266^{3}	1470.299^{10}	1.878^{6}	11.397^{9}	2.002^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.778^{7}	1.819^{6}	2.167^{8}
	RMSE(λ)	1.861^{3}	1509.415^{10}	2.647^{5}	11.710^{9}	2.617^{4}	$0.400^{1.5}$	$0.400^{1.5}$	2.788^{7}	2.662^{6}	3.612^{8}
	Dabs	0.166^{1}	0.209^{7}	0.167^{4}	0.585^{10}	0.168^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	$Dmax$	0.257^{4}	0.340^{7}	0.254^{2}	0.842^{8}	0.251^{1}	$0.878^{9.5}$	$0.878^{9.5}$	0.257^{6}	0.255^{3}	0.257^{5}
	Total	20^{1}	44^{10}	23^{2}	38^{8}	27^{3}	34^{6}	34^{6}	34^{6}	33^{4}	43^{9}
80	Bias(α)	-0.057^{1}	-3.400^{9}	-1.109^{5}	1.133^{6}	-1.061^{4}	-3.400^{9}	-3.400^{9}	0.743^{3}	0.239^{2}	2.438^{7}
	RMSE(α)	4.084^{7}	3.400^{5}	1.875^{2}	1.411^{1}	3.183^{3}	3.400^{5}	3.400^{5}	5.619^{8}	7.174^{9}	11.097^{10}
	Bias(λ)	1.247^{3}	1466.142^{10}	1.713^{6}	16.729^{9}	1.846^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.657^{4}	1.684^{5}	1.924^{8}
	RMSE(λ)	1.710^{3}	1497.992^{10}	2.295^{4}	16.729^{9}	2.299^{5}	$0.400^{1.5}$	$0.400^{1.5}$	2.435^{7}	2.310^{6}	2.982^{8}
	Dabs	0.165^{1}	0.209^{7}	0.167^{4}	0.637^{10}	0.167^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	Dmax	0.255^{4}	0.339^{7}	0.253^{2}	0.938^{10}	0.251^{1}	$0.894^{8.5}$	$0.894^{8.5}$	0.256^{6}	0.255^{3}	0.256^{5}
	$Total$	19^{1}	48^{10}	23^{2}	45^{9}	26^{3}	$34^{6.5}$	$34^{6.5}$	31^{5}	30^{4}	40^{8}
100	Bias(α)	-0.296^{3}	-3.400^{9}	-1.156^{4}	2.411^{7}	-1.197^{5}	-3.400^{9}	-3.400^{9}	0.283^{2}	-0.243^{1}	1.580^{6}
	RMSE(α)	3.559^{7}	3.400^{5}	1.826^{1}	2.465^{2}	2.766^{3}	3.400^{5}	3.400^{5}	4.972^{8}	5.848^{9}	9.347^{10}
	Bias(λ)	1.246^{3}	1460.600^{10}	1.630^{6}	21.344^{9}	1.7441^{7}	$-0.400^{1.5}$	$-0.400^{1.5}$	1.599^{4}	1.618^{5}	1.823^{8}
	RMSE(λ)	1.631^{3}	1491.261^{10}	2.116^{4}	21.372^{9}	2.117^{5}	$0.400^{1.5}$	$0.400^{1.5}$	2.250^{7}	2.138^{6}	2.708^{8}
	Dabs	0.166^{1}	0.209^{7}	0.167^{4}	0.650^{10}	0.167^{6}	$0.331^{8.5}$	$0.331^{8.5}$	0.167^{3}	0.167^{5}	0.167^{2}
	Dmax	0.255^{4}	0.340^{7}	0.253^{2}	0.964^{10}	0.252^{1}	$0.903^{8.5}$	$0.903^{8.5}$	0.256^{5}	0.254^{3}	0.256^{6}
	$Total$	$21^{1.5}$	48^{10}	$21^{1.5}$	$47^{9.0}$	$27^{3.0}$	$34^{6.5}$	$34^{6.5}$	$29^{4.5}$	$29^{4.5}$	40^{8}

Table 5. Simulation results for α=3.0 and λ=2.0

n	**Est.**	**MLE**	**LSE**	**WLS**	**PCE**	**MPS**	**MSADE**	**MSALDE**	**CVM**	**AD**	**RAD**
20	Bias(α)	10.138^{5}	-2.900^{3}	1.365^{1}	6.236^{7}	6.548^{4}	3.103^{2}	12.668^{9}	9.851^{6}	33.335^{8}	36.329^{10}
	$RMSE(\alpha)$	12.855^{6}	2.900^{2}	2.768^{1}	6.383^{7}	10.747^{4}	8.059^{3}	13.666^{8}	13.018^{5}	53.221^{9}	54.659^{10}
	Bias(λ)	-1.181^{6}	1575.716^{10}	-0.044^{2}	-1.143^{7}	-0.174^{3}	2.699^{8}	13.598^{9}	-0.564^{5}	0.467^{4}	0.357^{1}
	$RMSE(\lambda)$	1.917^{1}	1648.607^{10}	2.877^{5}	1.157^{2}	2.672^{4}	7.126^{3}	14.580^{9}	2.815^{6}	3.290^{7}	10.011^{8}
	Dabs	0.146^{2}	0.248^{8}	0.163^{4}	0.092^{1}	0.158^{3}	0.489^{10}	0.328^{9}	0.165^{6}	0.162^{5}	0.171^{7}
	$Dmax$	0.219^{6}	0.404^{8}	0.236^{2}	0.166^{1}	0.233^{3}	0.876^{10}	0.865^{9}	0.241^{5}	0.238^{4}	0.248^{7}
	Total	26^{6}	40$^{8.5}$	17^{1}	16^{6}	21^{2}	42$^{3.5}$	52$^{3.5}$	34^{6}	37$^{8.5}$	45^{10}
40	Bias(α)	10.418^{7}	-2.900^{2}	1.887^{1}	10.409^{6}	6.619^{4}	4.002^{3}	13.810^{8}	10.020^{5}	28.382^{9}	31.530^{10}
	RMSE(α)	13.553^{7}	2.900^{2}	2.833^{1}	10.423^{4}	10.438^{5}	7.248^{3}	14.831^{8}	12.946^{6}	44.788^{9}	46.911^{10}
	Bias(λ)	-1.363^{6}	1546.786^{10}	-0.816^{2}	7.052^{8}	-0.851^{3}	3.882^{7}	14.876^{9}	-1.064^{4}	-1.090^{5}	-0.784^{1}
	RMSE(λ)	1.625^{3}	1589.241^{10}	1.595^{1}	7.068^{8}	1.610^{2}	6.620^{7}	15.810^{9}	1.880^{5}	1.848^{4}	2.574^{6}
	Dabs	0.152^{1}	0.246^{7}	0.161^{3}	0.319^{8}	0.159^{2}	0.421^{10}	0.336^{9}	0.165^{5}	0.162^{4}	0.169^{6}
	Dmax	0.229^{1}	0.405^{7}	0.237^{3}	0.877^{9}	0.235^{2}	0.862^{8}	0.908^{10}	0.244^{5}	0.241^{4}	0.249^{6}
	Total	25^{3}	38$^{6.5}$	11^{1}	43^{9}	18^{2}	38$^{6.5}$	53^{10}	30^{4}	35^{5}	39^{8}
60	Bias(α)	10.653^{6}	-2.900^{2}	2.170^{1}	16.781^{8}	6.689^{4}	6.132^{3}	14.456^{7}	10.167^{5}	25.991^{9}	29.023^{10}
	RMSE(α)	13.761^{6}	2.900^{2}	2.900^{1}	16.792^{8}	10.250^{4}	7.771^{3}	15.439^{7}	12.919^{5}	40.115^{9}	42.550^{10}
	Bias(λ)	-1.443^{6}	1538.080^{10}	-1.047^{1}	11.393^{8}	-1.076^{2}	5.616^{7}	15.638^{9}	-1.250^{4}	-1.291^{5}	-1.084^{3}
	RMSE(λ)	1.590^{3}	1571.552^{10}	1.387^{1}	11.403^{8}	1.447^{2}	7.116^{7}	16.477^{9}	1.682^{5}	1.662^{4}	3.957^{6}
	Dabs	0.155^{1}	0.246^{7}	0.161^{3}	0.327^{8}	0.160^{2}	0.329^{9}	0.342^{10}	0.166^{5}	0.163^{4}	0.168^{6}
	$Dmax$	0.233^{1}	0.406^{7}	0.238^{3}	0.931^{10}	0.237^{2}	0.814^{8}	0.927^{9}	0.246^{5}	0.242^{4}	0.249^{6}
	Total	23^{3}	38^{7}	10^{1}	50^{9}	16^{2}	37^{6}	51^{10}	29^{4}	35^{5}	41^{8}
80	Bias(α)	10.215^{5}	-2.900^{2}	2.368^{1}	23.055^{8}	6.771^{3}	6.888^{4}	15.092^{7}	10.433^{6}	24.631^{9}	27.870^{10}
	RMSE(α)	13.109^{6}	2.900^{1}	2.961^{2}	23.065^{8}	10.158^{4}	8.117^{3}	15.903^{7}	13.009^{5}	37.454^{9}	40.367^{10}
	Bias(λ)	-1.479^{6}	1527.869^{10}	-1.163^{1}	15.606^{8}	-1.186^{2}	6.227^{7}	16.283^{9}	-1.362^{4}	-1.402^{5}	-1.272^{3}
	RMSE(λ)	1.583^{3}	1557.936^{10}	1.352^{1}	15.615^{8}	1.423^{2}	7.410^{7}	16.952^{9}	1.626^{5}	1.625^{4}	1.759^{6}
	Dabs	0.157^{1}	0.246^{7}	0.162^{3}	0.330^{9}	0.161^{2}	0.309^{8}	0.342^{10}	0.166^{5}	0.164^{4}	0.168^{6}
	Dmax	0.236^{1}	0.407^{7}	0.240^{3}	0.954^{10}	0.239^{2}	0.811^{8}	0.938^{9}	0.246^{5}	0.244^{4}	0.250^{6}
	Total	22^{3}	37$^{6.5}$	11^{1}	51$^{9.5}$	15^{2}	37$^{6.5}$	51$^{9.5}$	30^{4}	35^{5}	41^{8}
100	Bias(α)	10.324^{5}	-2.900^{2}	2.489^{1}	29.303^{10}	6.879^{3}	7.226^{4}	15.440^{7}	10.537^{6}	24.063^{8}	27.167^{9}
	RMSE(α)	13.370^{6}	2.900^{1}	2.998^{2}	29.313^{8}	10.126^{4}	8.386^{3}	16.171^{7}	13.034^{5}	36.194^{9}	38.876^{10}
	Bias(λ)	-1.505^{6}	1531.350^{10}	-1.224^{1}	19.781^{9}	-1.257^{2}	6.514^{7}	16.658^{8}	-1.425^{4}	-1.460^{5}	-1.361^{3}
	RMSE(λ)	1.588^{3}	1556.335^{10}	1.352^{1}	19.790^{9}	1.430^{2}	7.671^{7}	17.235^{8}	1.617^{4}	1.624^{5}	1.709^{6}
	Dabs	0.158^{1}	0.245^{7}	0.162^{3}	0.331^{9}	0.161^{2}	0.304^{8}	0.344^{10}	0.166^{5}	0.164^{4}	0.168^{6}
	Dmax	0.237^{1}	0.405^{7}	0.240^{3}	0.966^{10}	0.240^{2}	0.815^{8}	0.946^{9}	0.246^{5}	0.244^{4}	0.250^{6}
	Total	22^{3}	37$^{6.5}$	11^{1}	55^{10}	15$^{2.0}$	37$^{6.5}$	49^{9}	29^{4}	35^{5}	40^{8}

TABLE 6. Simulation results for α=1.5 and λ=0.5

n	**Est.**	**MLE**	**LSE**	**WLS**	**PCE**	**MPS**	**MSADE**	**MSALDE**	**CVM**	**AD**	**RAD**
20	Bias(α)	3.219^{7}	-1.400^{5}	-0.116^{1}	10.128^{9}	0.632^{2}	-1.368^{4}	-1.292^{3}	2.291^{6}	5.582^{8}	11.301^{10}
	$RMSE(\alpha)$	6.600^{7}	1.400^{2}	0.785^{1}	10.337^{8}	2.971^{5}	1.479^{3}	1.890^{4}	4.850^{6}	18.081^{9}	25.595^{10}
	Bias(λ)	0.698^{4}	588.098^{10}	1.648^{7}	-0.243^{1}	1.694^{8}	-0.369^{3}	-0.283^{2}	1.202^{5}	1.374^{6}	1.863^{9}
	$RMSE(\lambda)$	1.646^{4}	635.295^{10}	2.920^{8}	0.270^{1}	2.898^{7}	0.670^{2}	1.466^{3}	2.606^{5}	2.866^{6}	6.559^{9}
	Dabs	0.169^{5}	0.209^{7}	0.168^{2}	0.385^{10}	0.169^{6}	0.327^{8}	0.328^{9}	0.168^{3}	0.168^{4}	0.167^{1}
	$Dmax$	0.267^{6}	0.342^{7}	0.251^{1}	0.584^{8}	0.252^{2}	0.786^{9}	0.786^{10}	0.265^{4}	0.260^{3}	0.266^{5}
	Total	33^{6}	41^{9}	20^{1}	37^{8}	30^{4}	$29^{2.5}$	31^{5}	$29^{2.5}$	36^{7}	44^{10}
40	Bias(α)	1.735^{4}	-1.400^{6}	-0.084^{1}	12.951^{10}	0.117^{2}	1.969^{8}	-1.396^{5}	1.312^{3}	1.734^{7}	5.270^{9}
	RMSE(α)	4.231^{6}	1.400^{2}	0.666^{1}	13.138^{9}	1.801^{4}	6.633^{7}	1.424^{3}	3.527^{5}	8.483^{8}	15.045^{10}
	Bias(λ)	0.649^{3}	564.159^{10}	1.076^{6}	-0.292^{1}	1.171^{8}	$2.493\ ^{9}$	-0.396^{2}	0.914^{4}	0.985^{5}	1.148^{7}
	RMSE(λ)	1.134^{3}	600.154^{10}	1.651^{5}	0.300^{1}	1.681^{7}	6.308^{9}	0.486^{2}	1.651^{6}	1.627^{4}	2.271^{8}
	Dabs	0.166^{1}	0.208^{7}	0.167^{4}	0.399^{9}	0.168^{6}	0.431^{10}	0.328^{8}	0.167^{5}	0.167^{3}	0.167^{2}
	Dmax	0.259^{4}	0.350^{7}	0.253^{2}	0.609^{8}	0.251^{1}	0.860^{10}	0.841^{9}	0.261^{5}	0.257^{3}	0.262^{6}
	Total	21^{2}	$42^{8.5}$	19^{1}	38^{7}	$28^{3.5}$	53^{10}	29^{5}	$28^{3.5}$	30^{6}	$42^{8.5}$
60	Bias(α)	0.589^{3}	$-1.400^{7.5}$	-0.091^{2}	11.197^{10}	-0.077^{1}	-1.400^{6}	$-1.400^{7.5}$	0.770^{5}	0.604^{4}	2.985^{9}
	RMSE(α)	2.536^{6}	$1.400^{4.5}$	0.597^{1}	11.408^{10}	1.191^{2}	1.400^{3}	$1.400^{4.5}$	2.657^{7}	4.494^{8}	10.405^{9}
	Bias(λ)	0.641^{3}	553.105^{10}	0.918^{6}	3.993^{9}	1.014^{8}	-0.399^{1}	-0.400^{2}	0.832^{4}	0.878^{5}	0.986^{7}
	RMSE(λ)	0.963^{3}	587.759^{10}	1.297^{4}	5.085^{9}	1.333^{6}	0.412^{2}	0.400^{1}	1.347^{7}	1.298^{5}	1.713^{8}
	Dabs	0.166^{1}	0.208^{7}	0.167^{4}	0.581^{10}	0.168^{6}	0.327^{9}	0.327^{8}	0.167^{3}	0.167^{5}	0.167^{2}
	$Dmax$	0.256^{4}	0.355^{7}	0.253^{2}	0.852^{8}	0.250^{1}	0.864^{9}	0.864^{10}	0.258^{5}	0.255^{3}	0.260^{6}
	Total	20^{2}	46^{9}	19^{1}	56^{10}	24^{3}	$30^{4.5}$	33^{7}	31^{6}	$30^{4.5}$	41^{8}
80	Bias(α)	0.257^{4}	-1.400^{7}	0.096^{1}	8.634^{10}	0.161^{2}	4.728^{9}	-1.398^{6}	0.499^{5}	0.214^{3}	1.884^{8}
	RMSE(α)	1.641^{5}	1.400^{3}	$0.550\ ^{1}$	8.794^{10}	0.854^{2}	7.634^{8}	1.410^{4}	2.144^{8}6	2.624^{7}	7.712^{9}
	Bias(λ)	0.642^{2}	542.228^{10}	0.837^{5}	19.553^{9}	0.938^{7}	4.598^{8}	-0.398^{1}	0.781^{3}	0.817^{4}	0.881^{6}
	RMSE(λ)	0.884^{2}	577.891^{10}	1.125^{3}	19.771^{9}	1.171^{5}	8.041^{8}	0.437^{1}	1.174^{6}	1.128^{4}	1.416^{7}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.661^{10}	0.167^{6}	0.568^{9}	0.328^{8}	0.167^{3}	0.167^{5}	0.167^{4}
	Dmax	0.254^{4}	0.360^{7}	0.253^{2}	0.983^{10}	0.250^{1}	0.912^{9}	0.879^{8}	0.257^{5}	0.254^{3}	0.258^{6}
	Total	18^{2}	44^{8}	14^{1}	58^{10}	23^{3}	51^{9}	$28^{5.5}$	$28^{5.5}$	26^{4}	40^{7}
100	Bias(α)	0.128^{3}	$-1.400^{7.5}$	-0.102^{2}	11.454^{10}	-0.185^{4}	5.831^{9}	$-1.400^{7.5}$	0.334^{5}	0.101^{1}	1.270^{6}
	RMSE(α)	1.193^{3}	$1.400^{4.5}$	0.518^{1}	11.541^{10}	0.676^{2}	7.480^{9}	$1.400^{4.5}$	1.799^{6}	2.129^{7}	6.010^{8}
	Bias(λ)	0.640^{2}	542.885^{10}	0.798^{5}	24.793^{9}	0.887^{7}	5.375^{8}	-0.400^{1}	0.758^{3}	0.786^{4}	0.839^{6}
	RMSE(λ)	0.840^{2}	576.600^{10}	1.038^{3}	24.969^{9}	1.079^{5}	6.822^{8}	0.400^{1}	1.085^{6}	1.045^{4}	1.286^{7}
	Dabs	0.166^{1}	0.207^{7}	0.167^{2}	0.664^{10}	0.167^{6}	0.609^{9}	0.327^{8}	0.167^{3}	0.167^{4}	0.167^{5}
	Dmax	0.254^{4}	0.359^{7}	0.253^{2}	0.990^{10}	0.250^{1}	0.928^{9}	0.888^{8}	0.256^{5}	0.254^{3}	0.257^{6}
	Total	$15^{1.5}$	46^{8}	$15^{1.5}$	58^{10}	$25^{4.0}$	52^{9}	30^{6}	28^{5}	23^{3}	38^{7}

Table 7. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the lower-sided chart with α_0=0.75, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.041	1.188	1.245	1.308	1.414	1.536	1.627	1.964	2.255
CV			0.199	0.397	0.444	0.485	0.541	0.591	0.621	0.701	0.746
Q1			0.089	0.156	0.177	0.199	0.234	0.273	0.302	0.404	0.491
Q2			0.215	0.376	0.427	0.479	0.564	0.658	0.727	0.974	1.183
Q3			0.429	0.751	0.853	0.959	1.129	1.317	1.454	1.948	2.366
ARL	**0.4**		1.121	1.718	2.048	2.491	3.489	5.184	7.012	21.689	58.019
CV			0.328	0.646	0.715	0.774	0.845	0.898	0.926	0.977	0.991
Q1			0.129	0.329	0.429	0.561	0.852	1.342	1.869	6.095	16.547
Q2			0.311	0.794	1.035	1.351	2.052	3.235	4.505	14.685	39.868
Q3			0.623	1.588	2.069	2.702	4.105	6.469	9.010	29.3692	79.737
ARL	**0.5**		1.140	1.893	2.347	2.994	4.586	7.659	11.411	53.154	213.829
CV			0.351	0.687	0.758	0.816	0.884	0.932	0.955	0.991	0.998
Q1			0.137	0.383	0.518	0.708	1.169	2.056	3.137	15.147	61.371
Q2			0.331	0.923	1.249	1.705	2.818	4.954	7.558	36.496	147.868
Q3			0.662	1.845	2.497	3.411	5.636	9.908	15.115	72.992	295.736
ARL	**0.6**		1.158	2.069	2.665	3.562	5.969	11.253	18.569	135.455	867.527
CV			0.369	0.719	0.790	0.848	0.912	0.955	0.973	0.996	0.999
Q1			0.144	0.436	0.612	0.873	1.569	3.091	5.197	38.828	249.428
Q2			0.348	1.050	1.474	2.103	3.781	7.448	12.521	93.543	600.977
Q3			0.695	2.100	2.947	4.206	7.561	14.896	25.043	187.086	1201.954
ARL	**0.9**		1.201	2.613	3.743	5.727	12.599	34.761	80.017	2745.419	97404.030
CV			0.409	0.786	0.856	0.909	0.959	0.986	0.994	0.999	0.999
Q1			0.161	0.596	0.926	1.499	3.479	9.855	22.875	789.664	28021.250
Q2			0.388	1.437	2.230	3.612	8.382	23.746	55.116	1902.633	67514.980
Q3			0.776	2.874	4.460	7.224	16.764	47.492	110.233	3805.266	135030
ARL	**1**		1.214	2.800	4.148	6.630	15.971	50.259	130.211	7945.941	550636.1
CV			0.419	0.802	0.871	0.922	0.968	0.990	0.996	0.999	0.999
Q1			0.166	0.651	1.043	1.759	4.449	14.315	37.315	2285.761	158408
Q2			0.399	1.569	2.513	4.239	10.720	34.489	89.909	5507.360	381671.5
Q3			0.798	3.138	5.025	8.479	21.440	68.979	179.817	11014.72	763343
ARL	**1.3**		1.247	3.382	5.514	10.016	31.651	149.313	561.104	224986.8	—
CV			0.445	0.839	0.905	0.949	0.984	0.997	0.999	0.999	1
Q1			0.178	0.821	1.438	2.735	8.961	42.811	161.276	64724.53	—
Q2			0.428	1.977	3.464	6.589	21.590	103.149	388.581	155948.6	—
Q3			0.856	3.955	6.928	13.179	43.180	206.298	777.163	311897.2	—
ARL	**1.5**		1.266	3.788	6.564	12.948	48.992	304.797	1485.845	2349055	—
CV			0.458	0.858	0.921	0.961	0.989	0.998	0.999	1	1
Q1			0.184	0.939	1.741	3.579	13.949	87.541	427.307	675780.9	—
Q2			0.444	2.262	4.194	8.624	33.611	210.922	1029.563	1628241	—
Q3			0.888	4.523	8.388	17.247	67.221	421.844	2059.125	3256481	—
ARL	**2**		1.307	4.877	9.747	23.451	138.659	1753.213	16955.01	—	—
CV			0.485	0.892	0.947	0.978	0.996	0.999	0.999	1	1
Q1			0.199	1.254	2.658	6.602	39.746	504.224	4877.509	—	—
Q2			0.479	3.021	6.403	15.906	95.764	1214.888	11751.97	—	—
Q3			0.957	6.041	12.806	31.812	191.528	2429.776	23503.94	—	—
ARL	**2.5**		1.342	6.078	13.866	40.321	370.370	9693.294	193474	—	—
CV			0.505	0.914	0.963	0.988	0.999	0.999	1	1	1
Q1			0.210	1.600	3.843	11.455	106.405	2788.443	55658.86	—	—
Q2			0.507	3.856	9.260	27.600	256.375	6718.533	134105.6	—	—
Q3			1.014	7.711	18.521	55.201	512.749	13437.07	268211.2	—	—
ARL	**2.7**		1.354	6.591	15.819	49.513	541.553	19034.65	512333	—	—
CV			0.512	0.921	0.968	0.989	0.999	0.999	1	1	1
Q1			0.215	1.748	4.405	14.099	155.651	5475.784	147388.9	—	—
Q2			0.517	4.213	10.615	33.972	375.029	13193.47	355121.8	—	—
Q3			1.034	8.426	21.229	67.944	750.059	26386.93	710243.7	—	—
ARL	**3**		1.372	7.399	19.118	66.684	946.099	51931.07	2207735	—	—
CV			0.521	0.929	0.973	0.992	0.999	1	1	1	1
Q1			0.220	1.981	5.355	19.039	272.032	14939.49	635125.6	—	—
Q2			0.531	4.774	12.902	45.874	655.439	35995.53	1530285	—	—
Q3			1.062	9.548	25.804	91.748	1310.880	71991.060	3060570	—	—

Table 8. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the upper-sided chart with α_0=0.75, λ_0=2.5, λ_1 $\in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and α_1 $\in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		69324.44	17331.109	13864.89	11554.07	9243.258	7702.715	6932.444	5332.649	4621.629
CV			1	1	1	1	1	1	1	1	0.999
Q1			19943.25	4985.706	3988.536	3323.756	2658.976	2215.789	1994.196	1533.964	1329.416
Q2			48051.690	12012.663	9610.061	8008.326	6406.592	5338.769	4804.857	3695.964	3203.123
Q3			96103.380	24025.326	19220.12	16016.65	12813.18	10677.54	9609.715	7391.928	6406.245
ARL	**0.4**		17334.86	4333.715	3466.972	2889.143	2311.315	1926.095	1733.486	1333.451	1155.657
CV			1	1	1	1	1	1	1	0.999	0.999
Q1			4986.784	1246.588	997.242	831.011	664.779	553.959	498.549	383.466	332.318
Q2			12015.260	3003.556	2402.775	2002.255	1601.735	1334.721	1201.214	923.931	800.694
Q3			24030.52	6007.112	4805.551	4004.509	3203.469	2669.441	2402.429	1847.862	1601.388
ARL	**0.5**		13868.89	3467.222	2773.777	2311.481	1849.185	1540.988	1386.889	1066.838	924.593
CV			0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999
Q1			3989.686	997.314	797.822	664.828	531.834	443.171	398.839	306.766	265.845
Q2			9612.833	2402.948	1922.289	1601.85	1281.411	1067.785	960.971	739.129	640.532
Q3			19225.67	4805.897	3844.579	3203.7	2562.821	2135.569	1921.943	1478.258	1281.064
ARL	**0.6**		11558.24	2889.559	2311.648	1926.373	1541.099	1284.249	1155.824	889.096	770.549
CV			1	1	1	1	1	0.999	0.999	0.999	0.999
Q1			3324.954	831.131	664.876	554.039	443.203	369.312	332.366	255.633	221.529
Q2			8011.214	2002.544	1601.966	1334.914	1067.862	889.827	800.809	615.927	533.758
Q3			16022.43	4005.087	3203.931	2669.827	2135.723	1779.654	1601.619	1231.855	1067.515
ARL	**0.9**		7707.158	1926.789	1541.432	1284.527	1027.621	856.351	770.716	592.859	513.811
CV			1	1	1	1	1	0.999	0.999	0.999	0.999
Q1			2217.067	554.159	443.298	369.391	295.484	246.213	221.577	170.411	147.670
Q2			5341.849	1335.202	1068.092	890.019	711.946	593.231	533.873	410.592	355.799
Q3			10683.7	2670.405	2136.185	1780.039	1423.892	1186.461	1067.746	821.183	711.599
ARL	**1**		6936.942	1734.236	1387.389	1156.157	924.926	770.772	693.695	533.611	462.463
CV			1	1	1	1	0.999	0.999	0.999	0.999	0.999
Q1			1995.49	498.765	398.983	332.462	265.941	221.593	199.419	153.367	132.899
Q2			4807.975	1201.734	961.318	801.041	640.763	533.912	480.486	369.525	320.208
Q3			9615.951	2403.468	1922.636	1602.081	1281.526	1067.823	960.972	739.049	640.417
ARL	**1.3**		5337.263	1334.316	1067.453	889.544	711.635	593.029	533.727	410.559	355.818
CV			1	1	0.999531	0.999	0.999	0.999	0.999	0.999	0.998
Q1			1535.291	383.715	306.943	255.762	204.581	170.460	153.399	117.967	102.219
Q2			3699.162	924.531	739.555	616.238	492.921	410.710	369.604	284.231	246.288
Q3			7398.324	1849.061	1479.11	1232.477	985.843	821.420	739.209	568.462	492.575
ARL	**1.5**		4626.294	1156.574	925.259	771.049	616.839	514.033	462.629	355.869	308.420
CV			1	1	0.999459	0.999	0.999	0.999	0.999	0.998	0.998
Q1			1330.758	332.582	266.037	221.673	177.309	147.734	132.946	102.233	88.583
Q2			3206.356	801.329	640.994	534.104	427.214	355.954	320.324	246.323	213.434
Q3			6412.712	1602.658	1281.988	1068.208	854.428	711.908	640.648	492.646	426.868
ARL	**2**		3470.969	867.743	694.194	578.495	462.796	385.664	347.097	266.998	231.399
CV			0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.998	0.998
Q1			998.392	249.490	199.563	166.279	132.994	110.805	99.709	76.667	66.425
Q2			2405.546	601.127	480.832	400.636	320.439	266.975	240.243	184.722	160.047
Q3			4811.092	1202.253	961.664	801.271	640.879	533.95	480.486	369.445	320.093
ARL	**2.5**		2777.775	694.444	555.555	462.963	370.370	308.642	277.778	213.676	185.186
CV			0.999	0.999	0.999	0.999	0.998	0.998	0.998	0.998	0.997
Q1			798.972	199.635	159.679	133.042	106.405	88.647	79.768	61.327	53.131
Q2			1925.06	481.005	384.735	320.555	256.375	213.588	192.194	147.762	128.014
Q3			3850.12	962.010	769.469	641.109	512.749	427.175	384.389	295.524	256.028
ARL	**2.7**		2572.383	643.096	514.477	428.731	342.985	285.821	257.239	197.877	171.493
CV			1	0.999	0.999	0.999	0.999	0.998	0.998	0.997	0.997
Q1			739.885	184.863	147.862	123.194	98.527	82.082	73.859	56.782	49.192
Q2			1782.694	445.414	356.262	296.827	237.392	197.769	177.958	136.811	118.523
Q3			3565.387	890.827	712.523	593.654	474.784	395.538	355.915	273.621	237.046
ARL	**3**		2315.644	578.911	463.129	385.941	308.753	257.295	231.565	178.127	154.377
CV			0.999	0.999	0.999	0.999	0.998	0.998	0.998	0.997	0.997
Q1			666.026	166.398	133.090	110.884	88.679	73.875	66.473	51.100	44.268
Q2			1604.736	400.9246	320.67	267.167	213.665	177.996	160.162	123.122	106.659
Q3			3209.471	801.848	641.34	534.335	427.329	355.992	320.324	246.243	213.319

Table 9. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the Two-sided chart with α_0=0.75, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.046	1.214	1.282	1.357	1.484	1.634	1.748	2.182	2.574
CV			0.211	0.420	0.469	0.513	0.571	0.623	0.654	0.736	0.782
Q1			0.092	0.166	0.189	0.215	0.257	0.304	0.339	0.469	0.585
Q2			0.222	0.399	0.458	0.519	0.619	0.732	0.816	1.131	1.409
Q3			0.445	0.799	0.915	1.038	1.237	1.464	1.633	2.261	2.818
ARL	**0.4**		1.133	1.820	2.221	2.778	4.099	6.515	9.312	36.066	117.186
CV			0.342	0.671	0.741	0.800	0.869	0.920	0.945	0.986	0.996
Q1			0.134	0.361	0.481	0.645	1.029	1.726	2.532	10.231	33.568
Q2			0.323	0.869	1.159	1.553	2.478	4.159	6.101	24.651	80.880
Q3			0.646	1.739	2.317	3.107	4.957	8.319	12.203	49.301	161.761
ARL	**0.5**		1.153	2.022	2.578	3.402	5.563	10.134	16.226	100.386	449.219
CV			0.364	0.711	0.782	0.840	0.906	0.949	0.969	0.995	0.999
Q1			0.143	0.423	0.586	0.826	1.452	2.769	4.523	28.735	129.088
Q2			0.343	1.016	1.412	1.992	3.498	6.672	10.897	69.235	311.028
Q3			0.687	2.032	2.824	3.983	6.996	13.343	21.794	138.470	622.056
ARL	**0.6**		1.172	2.226	2.960	4.117	7.465	15.641	28.183	270.085	1042.26
CV			0.383	0.742	0.814	0.870	0.931	0.968	0.982	0.998	0.999
Q1			0.149	0.482	0.698	1.034	2.000	4.354	7.963	77.555	299.696
Q2			0.361	1.162	1.682	2.491	4.819	10.491	19.186	186.862	722.093
Q3			0.722	2.325	3.363	4.983	9.638	20.983	38.372	373.723	1444.186
ARL	**0.9**		1.217	2.859	4.278	6.925	17.081	54.772	138.689	1073.366	1027.135
CV			0.423	0.806	0.875	0.925	0.970	0.991	0.996	0.999	0.999
Q1			0.167	0.668	1.081	1.845	4.769	15.613	39.754	308.644	295.345
Q2			0.402	1.610	2.603	4.445	11.489	37.617	95.785	743.654	711.609
Q3			0.805	3.221	5.207	8.889	22.979	75.234	191.569	1487.307	1423.219
ARL	**1**		1.230	3.078	4.779	8.124	22.168	81.414	223.498	1039.881	925.319
CV			0.433	0.822	0.889	0.936	0.977	0.994	0.998	0.999	0.999
Q1			0.172	0.732	1.226	2.190	6.232	23.277	64.152	299.011	266.054
Q2			0.414	1.765	2.953	5.277	15.016	56.085	154.570	720.444	641.036
Q3			0.827	3.529	5.906	10.554	30.033	112.169	309.140	1440.888	1282.072
ARL	**1.3**		1.265	3.765	6.495	12.713	46.493	236.475	610.052	821.154	711.943
CV			0.458	0.857	0.919	0.959	0.989	0.998	0.999	0.999	0.999
Q1			0.184	0.932	1.721	3.511	13.231	67.886	175.357	236.087	204.669
Q2			0.443	2.245	4.146	8.461	31.879	163.565	422.509	568.834	493.135
Q3			0.887	4.489	8.292	16.921	63.758	327.129	845.019	1137.667	986.269
ARL	**1.5**		1.285	4.247	7.831	16.769	73.504	403.125	763.132	711.979	617.062
CV			0.471	0.874	0.934	0.969	0.993	0.999	0.999	0.999	0.999
Q1			0.191	1.072	2.106	4.679	21.002	115.828	219.396	204.679	177.374
Q2			0.460	2.582	5.073	11.273	50.601	279.078	528.616	493.160	427.368
Q3			0.920	5.164	10.146	22.546	101.203	558.156	1057.232	986.320	854.736
ARL	**2**		1.328	5.551	11.940	31.541	195.962	664.877	687.592	534.092	462.879
CV			0.497	0.905	0.957	0.984	0.997	0.999	0.999	0.999	0.999
Q1			0.206	1.448	3.289	8.929	56.231	191.129	197.664	153.505	133.018
Q2			0.496	3.489	7.925	21.514	135.483	460.511	476.256	369.858	320.497
Q3			0.991	6.979	15.849	43.028	270.967	921.022	952.512	739.716	640.994
ARL	**2.5**		1.364	7.002	17.329	55.069	370.370	606.075	555.289	427.351	370.371
CV			0.517	0.926	0.971	0.991	0.999	0.999	0.999	0.999	0.999
Q1			0.218	1.867	4.840	15.698	106.405	174.213	159.603	122.797	106.405
Q2			0.525	4.498	11.662	37.824	256.375	419.759	384.551	295.870	256.375
Q3			1.049	8.996	23.324	75.647	512.749	839.506	769.101	591.740	512.749
ARL	**2.7**		1.377	7.625	19.893	67.475	426.794	567.078	514.367	395.724	342.961
CV			0.523	0.932	0.975	0.993	0.999	0.999	0.999	0.999	0.998
Q1			0.222	2.046	5.578	19.267	122.637	162.994	147.829	113.699	98.519
Q2			0.535	4.931	13.439	46.423	295.484	392.722	356.185	273.948	237.375
Q3			1.070	9.862	26.879	92.846	590.969	785.444	712.369	547.896	474.751
ARL	**3**		1.395	8.609	24.219	89.548	477.095	513.313	463.035	356.189	308.698
CV			0.532	0.940	0.979	0.994	0.999	0.999	0.999	0.999	0.999
Q1			0.228	2.329	6.823	25.617	137.108	147.527	133.063	102.326	88.663
Q2			0.549	5.614	16.439	61.723	330.350	355.455	320.605	246.545	213.626
Q3			1.099	11.227	32.877	123.446	660.700	710.910	641.209	493.090	427.253

Table 10. Daily rainfall (in mm) on the January for a location in Florida from (1907-2000)

1.00	1.00	1.00	0.70	1.00	1.00	0.94	1.00	1.00	1.00	0.86	0.58	0.58	1.00
1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.81	0.98	0.33	1.00	1.00	0.77
1.00	1.00	1.00	0.51	0.90	1.00	1.00	0.77	1.00	1.00	0.98	1.00	1.00	1.00
1.00	1.00	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00	0.90	0.63	0.59	0.54	0.95	1.00	1.00	1.00	1.00	0.97	1.00	0.63	0.63
1.00	1.00	0.98	1.00	1.00	1.00	1.00	0.82	1.00	1.00	1.00	0.47	1.00	1.00
1.00	0.41	0.39	1.00	1.00	0.77	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
1.00													

Table 11. AIC, BIC, CAIC, and HQIC computed after fitting different distributions on for Rainfall Data

Statistic	**UNH**	Kumaraswamy	Topp-Leon	rGTL	beta
AIC	-400.864	-329.714	-107.605	-99.080	-331.849
CAIC	-400.739	-329.589	-107.605	-98.955	-331.724
BIC	-395.674	-324.524	-105.01	-93.890	-326.659
HQIC	-398.764	-327.614	-106.555	-96.980	-329.749

Table 12. Maximum likelihood estimates with their standard errors (in parenthesis) and K-S test p-value for Rainfall Data

Model	**MLEs**	**K-S**
UNH(α, λ)	$\hat{\alpha} = 0.513, \hat{\lambda}$=36.317 (0.039, 5.657)	0.717
Kumaraswamy(α, β)	$\hat{\alpha} = 5.045, \hat{\beta}$=0.428 (0.869, 0.050)	0.441
Topp-Leon(α)	$\hat{\alpha} = 8.568$ (0.861)	0.426
rGTL(α, υ)	$\hat{\alpha} = 0.443, \hat{\upsilon}$= 4.430 (0.147, 0.614)	0.920
beta(α, λ)	$\hat{\alpha} = 4.512, \hat{\lambda}$= 0.439 (0.798, 0.051)	0.438

Table 13. Anxiety Data Set

0.01	0.17	0.01	0.05	0.09	0.41	0.05	0.01	0.13	0.01	0.05	0.17	0.01	0.09
0.01	0.05	0.09	0.09	0.05	0.01	0.01	0.01	0.29	0.01	0.01	0.01	0.01	0.01
0.01	0.01	0.01	0.09	0.37	0.05	0.01	0.05	0.29	0.09	0.01	0.25	0.01	0.09
0.01	0.05	0.21	0.01	0.01	0.01	0.13	0.17	0.37	0.01	0.01	0.09	0.57	0.01
0.01	0.13	0.05	0.01	0.01	0.01	0.01	0.09	0.13	0.01	0.01	0.09	0.09	0.37
0.01	0.05	0.01	0.01	0.13	0.01	0.57	0.01	0.01	0.09	0.01	0.01	0.01	0.01
0.01	0.01	0.05	0.01	0.01	0.01	0.13	0.01	0.25	0.01	0.01	0.09	0.13	0.01
0.01	0.05	0.13	0.01	0.09	0.01	0.05	0.01	0.05	0.01	0.09	0.01	0.01	0.01
0.01	0.01	0.25	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.37	0.25
0.05	0.05	0.25	0.05	0.05	0.01	0.05	0.01	0.01	0.01	0.17	0.29	0.57	0.01
0.05	0.01	0.09	0.01	0.09	0.49	0.45	0.01	0.01	0.01	0.05	0.01	0.17	0.01
0.13	0.01	0.21	0.13	0.01	0.01	0.17	0.01	0.01	0.21	0.13	0.69	0.25	0.01
0.01	0.09	0.13	0.01	0.05	0.01	0.01	0.29	0.25	0.49	0.01	0.01		

Table 14. AIC, BIC, CAIC, and HQIC computed after fitting different distributions using Anxiety Data

Statistic	UNH	rGTL-PS	Topp-Leon
AIC	-450.782	-443.914	-430.609
$CAIC$	-450.709	-443.842	-430.585
BIC	-444.522	-437.655	-427.479
$HQIC$	-448.241	-441.374	-429.339

Table 15. Maximum likelihood estimates with their standard errors (in parenthesis) and p values of K-S test for Anxiety Data

Model	**MLE**	**K-S**
$UNH(\alpha, \lambda)$	$\hat{\alpha}$ = 8.794, $\hat{\lambda}$=0.025 (2.188,0.006)	0.356
rGTL-PS(α, υ)	$\hat{\alpha}$ = 0.537, $\hat{\upsilon}$=6.378 (0.223,1.090)	0.407
Topp-leon(α)	$\hat{\alpha}$=0.372 (0.028)	0.264

Table A.1. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability of the lower-sided chart with α_0=1.00, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.022	1.093	1.119	1.146	1.188	1.234	1.267	1.374	1.455
CV			0.146	0.291	0.326	0.356	0.398	0.436	0.459	0.522	0.559
Q1			0.075	0.117	0.128	0.139	0.156	0.173	0.185	0.221	0.248
Q2			0.179	0.281	0.309	0.336	0.376	0.417	0.445	0.533	0.596
Q3			0.359	0.562	0.618	0.672	0.753	0.834	0.890	1.066	1.193
ARL	**0.4**		1.071	1.357	1.485	1.634	1.911	2.273	2.576	3.962	5.558
CV			0.258	0.513	0.571	0.623	0.691	0.748	0.782	0.865	0.906
Q1			0.106	0.215	0.257	0.304	0.388	0.496	0.586	0.989	1.450
Q2			0.256	0.519	0.619	0.732	0.936	1.195	1.411	2.383	3.495
Q3			0.512	1.038	1.238	1.465	1.872	2.391	2.822	4.766	6.989
ARL	**0.5**		1.085	1.443	1.612	1.817	2.216	2.770	3.264	5.806	9.255
CV			0.279	0.554	0.616	0.671	0.741	0.799	0.833	0.909	0.944
Q1			0.113	0.244	0.297	0.360	0.479	0.643	0.786	1.522	2.516
Q2			0.272	0.587	0.716	0.868	1.155	1.548	1.895	3.668	6.062
Q3			0.543	1.173	1.431	1.735	2.310	3.096	3.789	7.334	12.124
ARL	**0.6**		1.097	1.528	1.743	2.012	2.559	3.369	4.135	8.618	15.854
CV			0.297	0.588	0.653	0.709	0.781	0.839	0.871	0.940	0.968
Q1			0.118	0.271	0.337	0.419	0.581	0.817	1.039	2.332	4.416
Q2			0.286	0.652	0.813	1.009	1.399	1.969	2.504	5.619	10.639
Q3			0.571	1.305	1.625	2.017	2.799	3.938	5.007	11.239	21.278
ARL	**0.9**		1.128	1.783	2.158	2.672	3.868	6.001	8.408	30.151	93.267
CV			0.337	0.663	0.732	0.791	0.861	0.913	0.939	0.983	0.995
Q1			0.132	0.349	0.462	0.614	0.962	1.578	2.272	8.529	26.687
Q2			0.319	0.842	1.113	1.478	2.317	3.802	5.474	20.551	64.301
Q3			0.638	1.684	2.226	2.967	4.635	7.604	10.949	41.102	128.601
ARL	**1**		1.138	1.868	2.304	2.919	4.415	7.251	10.653	46.718	176.786
CV			0.348	0.682	0.753	0.812	0.879	0.928	0.952	0.989	0.997
Q1			0.136	0.375	0.505	0.686	1.120	1.939	2.918	13.295	50.714
Q2			0.328	0.904	1.218	1.653	2.699	4.671	7.032	32.034	122.192
Q3			0.657	1.809	2.435	3.306	5.397	9.342	14.063	64.069	244.384
ARL	**1.3**		1.163	2.126	2.769	3.756	6.478	12.697	21.662	183.486	1377.041
CV			0.374	0.728	0.799	0.857	0.919	0.959	0.977	0.997	0.999
Q1			0.146	0.453	0.642	0.929	1.716	3.507	6.087	52.642	396.006
Q2			0.353	1.090	1.548	2.239	4.134	8.449	14.665	126.836	954.145
Q3			0.705	2.181	3.095	4.478	8.268	16.899	29.331	253.671	1908.291
ARL	**1.5**		1.178	2.300	3.104	4.399	8.287	18.344	34.768	476.096	6010.897
CV			0.388	0.752	0.823	0.879	0.938	0.972	0.986	0.999	0.999
Q1			0.152	0.504	0.739	1.116	2.237	5.132	9.858	136.820	1729.083
Q2			0.367	1.215	1.783	2.688	5.390	12.365	23.751	329.658	4166.089
Q3			0.733	2.429	3.565	5.377	10.780	24.730	47.502	659.316	8332.179
ARL	**2**		1.210	2.747	4.031	6.365	14.944	45.302	113.477	5868.797	335003.5
CV			0.417	0.797	0.867	0.918	0.966	0.989	0.996	0.999	1
Q1			0.164	0.636	1.009	1.683	4.154	12.888	32.501	1688.204	96374.357
Q2			0.396	1.531	2.431	4.055	10.008	31.053	78.309	4067.594	232206.385
Q3			0.792	3.063	4.862	8.111	20.016	62.107	156.619	8135.187	464412.77
ARL	**2.5**		1.238	3.213	5.102	8.939	26.155	109.818	370.370	85019.46	28976440
CV			0.438	0.829	0.897	0.942	0.981	0.995	0.999	1	1
Q1			0.174	0.772	1.319	2.425	7.379	31.448	106.405	24458.43	—
Q2			0.420	1.859	3.177	5.843	17.780	75.773	256.375	58930.65	—
Q3			0.840	3.718	6.354	11.686	35.561	151.545	512.749	117861.3	—
ARL	**2.7**		1.248	3.405	5.574	10.173	32.494	155.808	594.466	257929.1	—
CV			0.446	0.841	0.906	0.949	0.985	0.997	0.999	1	1
Q1			0.1779	0.828	1.454	2.780	9.203	44.679	170.873	74201.43	—
Q2			0.428	1.994	3.506	6.699	22.175	107.651	411.706	178782.5	—
Q3			0.888	3.988	7.011	13.398	44.349	215.301	823.412	357565	—
ARL	**3**		1.262	3.701	6.332	12.275	44.710	262.219	1208.825	1418697	—
CV			0.456	0.854	0.918	0.958	0.989	0.998	0.999	1	1
Q1			0.183	0.913	1.674	3.385	12.718	75.292	347.613	408133.5	—
Q2			0.441	2.200	4.032	8.157	30.643	181.409	837.547	983365.5	—
Q3			0.882	4.401	8.065	16.314	61.286	362.819	1675.094	1966731	—

Table A.2. ARL, Cv and quartiles of the run length distribution assuming 0.0027 as the false alarm probability of the upper-sided chart with α_0=1.00, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		92472.54	23118.13	18494.507	15412.089	12329.67	10274.73	9247.254	7113.272	6164.836
CV			1	1	1	1	1	1	1	1	1
Q1			26602.55	6650.529	5320.394	4433.638	3546.882	2955.711	2660.125	2046.217	1773.369
Q2			64096.73	16023.92	12819.069	10682.499	8545.93	7121.551	6409.361	4930.198	4272.792
Q3			128193.5	32047.85	25638.138	21364.999	17091.86	14243.1	12818.72	9860.396	8545.58422
ARL	**0.4**		23121.88	5780.471	4624.377	3853.647	3082.918	2569.098	2312.188	1778.607	1541.459
CV			1	1	1	1	1	1	1	1	1
Q1			6651.608	1662.794	1330.207	1108.481	886.756	738.939	665.031	511.529	443.306
Q2			16026.52	4006.371	3205.027	2670.798	2136.569	1780.416	1602.34	1232.49	1068.111
Q3			32053.04	8012.741	6410.055	5341.596	4273.139	3560.833	3204.68	2464.98	2136.223
ARL	**0.5**		18498.51	4624.627	3699.701	3083.085	2466.468	2055.39	1849.851	1422.962	1233.234
CV			1	1	1	1	1	1	1	1	1
Q1			5321.545	1330.278	1064.194	886.804	709.415	591.155	532.025	409.217	354.635
Q2			12821.84	3205.201	2564.091	2136.685	1709.278	1424.341	1281.872	985.977	854.466
Q3			25643.68	6410.401	5128.182	4273.370	3418.557	2848.682	2563.745	1971.951	1708.932
ARL	**0.6**		15416.26	3854.064	3083.251	2569.376	2055.501	1712.917	1541.626	1185.866	1027.751
CV			1	1	1	1	1	1	1	1	1
Q1			4434.837	1108.601	886.852	739.019	591.187	492.632	443.354	341.009	295.522
Q2			10685.39	2671.087	2136.800	1780.609	1424.418	1186.957	1068.227	821.633	712.036
Q3			21370.78	5342.174	4273.600	3561.218	2848.836	2373.914	2136.454	1643.266	1424.072
ARL	**0.9**		10279.17	2569.793	2055.834	1713.195	1370.556	1142.13	1027.917	790.706	685.278
CV			1	1	1	1	1	1	1	0.999	0.999
Q1			2956.989	739.139	591.283	492.713	394.141	328.427	295.569	227.328	196.998
Q2			7124.631	1780.898	1424.649	1187.149	949.651	791.318	712.151	547.729	474.652
Q3			14249.26	3561.796	2849.298	2374.299	1899.301	1582.635	1424.303	1095.457	949.304
ARL	**1**		9251.753	2312.938	1850.351	1541.959	1233.567	1027.973	925.176	711.674	616.784
CV			1	1	1	1	1	1	0.999	0.999	0.999
Q1			2661.42	665.247	532.169	443.450	354.731	295.586	266.013	204.592	177.294
Q2			6412.48	1602.86	1282.219	1068.458	854.697	712.189	640.936	492.948	427.175
Q3			12824.96	3205.72	2564.437	2136.916	1709.394	1424.379	1281.872	985.896	854.351
ARL	**1.3**		7117.886	1779.472	1423.577	1186.315	949.052	790.877	711.789	547.53	474.526
CV			1	1	1	1	0.999	0.999	0.999	0.999	0.999
Q1			2047.544	511.778	409.394	341.138	272.881	227.377	204.625	157.371	136.369
Q2			4933.396	1233.089	986.402	821.944	657.486	547.847	493.028	379.172	328.569
Q3			9866.792	2466.178	1972.804	1643.888	1314.972	1095.694	986.056	758.344	657.139
ARL	**1.5**		6169.501	1542.375	1233.900	1028.250	822.600	685.500	616.950	474.577	411.301
CV			1	1	1	1	0.999	0.999	0.999	0.999	0.999
Q1			1774.711	443.569	354.827	295.665	236.504	197.062	177.342	136.384	118.179
Q2			4276.026	1068.747	854.928	712.382	569.836	474.806	427.291	328.605	284.745
Q3			8552.051	2137.493	1709.856	1424.764	1139.673	949.612	854.582	657.211	569.490
ARL	**2**		4628.375	1157.094	925.675	771.396	617.117	514.264	462.838	356.029	308.559
CV			1	1	0.999	0.999	0.999	0.999	0.999	0.998	0.998
Q1			1331.357	332.731	266.156	221.773	177.389	147.801	133.006	102.279	88.623
Q2			3207.798	801.689	641.283	534.344	427.406	356.114	320.468	246.434	213.529
Q3			6415.597	1603.38	1282.565	1068.689	854.812	712.228	640.936	492.868	427.059
ARL	**2.5**		3703.699	925.925	740.740	617.284	493.827	411.523	370.370	284.901	246.914
CV			1	1	1	1	1	0.999	0.999	0.998	0.998
Q1			1065.344	266.228	212.954	177.438	141.921	118.244	106.405	81.817	70.889
Q2			2566.862	641.456	513.095	427.522	341.948	284.899	256.375	197.131	170.801
Q3			5133.724	1282.911	1026.191	855.043	683.896	569.798	512.749	394.262	341.602
ARL	**2.7**		3429.721	857.431	685.945	571.621	457.297	381.081	342.973	263.825	228.649
CV			1	0.999	0.999	0.999	0.999	0.999	0.999	0.998	0.998
Q1			986.526	246.524	197.190	164.301	131.412	109.486	98.523	75.754	65.634
Q2			2376.955	593.979	475.114	395.871	316.627	263.798	237.384	182.523	158.140
Q3			4753.91	1187.958	950.228	791.741	633.254	527.596	474.766	365.046	316.281
ARL	**3**		3087.249	771.812	617.45	514.542	411.634	343.028	308.725	237.481	205.817
CV			1	0.999	0.999	0.999	0.999	0.999	0.998	0.998	0.998
Q1			888.002	221.893	177.486	147.881	118.276	98.539	88.671	68.175	59.066
Q2			2139.571	534.633	427.637	356.306	284.976	237.422	213.645	164.263	142.315
Q3			4279.142	1069.266	855.274	712.613	569.952	474.844	427.291	328.526	284.629

Table A.3. ARL, Cv and quartiles of the run length distribution assuming 0.0027 as the false alarm probability of the two-sided chart with α_0=1.00, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.024	1.103	1.133	1.163	1.212	1.265	1.302	1.428	1.524
CV			0.153	0.306	0.342	0.374	0.418	0.458	0.482	0.548	0.586
Q1			0.077	0.121	0.134	0.146	0.165	0.184	0.197	0.239	0.269
Q2			0.185	0.293	0.323	0.353	0.398	0.443	0.475	0.575	0.649
Q3			0.369	0.585	0.646	0.706	0.795	0.887	0.949	1.151	1.299
ARL	**0.4**		1.078	1.397	1.544	1.718	2.049	2.493	2.877	4.725	7.017
CV			0.268	0.533	0.593	0.647	0.716	0.774	0.808	0.888	0.926
Q1			0.109	0.229	0.276	0.329	0.429	0.561	0.674	1.209	1.871
Q2			0.264	0.551	0.664	0.795	1.035	1.352	1.623	2.915	4.509
Q3			0.527	1.102	1.328	1.589	2.071	2.704	3.245	5.830	9.017
ARL	**0.5**		1.092	1.492	1.688	1.929	2.412	3.106	3.745	7.289	12.579
CV			0.289	0.574	0.638	0.694	0.765	0.823	0.856	0.929	0.959
Q1			0.116	0.259	0.320	0.394	0.537	0.740	0.926	1.949	3.473
Q2			0.279	0.625	0.772	0.949	1.294	1.784	2.232	4.697	8.368
Q3			0.559	1.250	1.544	1.898	2.588	3.568	4.463	9.394	16.737
ARL	**0.6**		1.105	1.587	1.836	2.155	2.825	3.859	4.876	11.402	23.261
CV			0.308	0.608	0.675	0.732	0.804	0.861	0.892	0.955	0.978
Q1			0.122	0.289	0.366	0.461	0.658	0.959	1.253	3.134	6.547
Q2			0.294	0.697	0.881	1.112	1.586	2.311	3.019	7.551	15.774
Q3			0.588	1.394	1.763	2.223	3.173	4.622	6.039	15.103	31.549
ARL	**0.9**		1.138	1.872	2.311	2.931	4.439	7.303	10.735	46.459	161.613
CV			0.348	0.683	0.753	0.812	0.880	0.929	0.952	0.989	0.997
Q1			0.136	0.377	0.507	0.689	1.127	1.953	2.942	13.221	46.349
Q2			0.329	0.907	1.223	1.661	2.716	4.707	7.089	31.855	111.675
Q3			0.657	1.815	2.445	3.322	5.432	9.414	14.178	63.709	223.349
ARL	**1**		1.148	1.968	2.479	3.225	5.129	8.997	13.950	74.827	293.494
CV			0.359	0.701	0.772	0.831	0.897	0.943	0.963	0.993	0.998
Q1			0.140	0.405	0.557	0.775	1.326	2.442	3.868	21.382	84.289
Q2			0.338	0.977	1.342	1.868	3.196	5.883	9.319	51.519	203.088
Q3			0.677	1.953	2.685	3.735	6.392	11.765	18.637	103.038	406.175
ARL	**1.3**		1.174	2.257	3.019	4.232	7.787	16.634	30.399	281.180	774.348
CV			0.385	0.746	0.818	0.874	0.934	0.969	0.983	0.998	0.999
Q1			0.151	0.491	0.715	1.067	2.093	4.639	8.601	80.747	222.622
Q2			0.363	1.184	1.723	2.571	5.043	11.179	20.722	194.553	536.390
Q3			0.726	2.368	3.447	5.143	10.087	22.359	41.445	389.105	1072.780
ARL	**1.5**		1.189	2.453	3.410	5.017	10.173	24.811	50.541	515.646	794.908
CV			0.399	0.769	0.841	0.895	0.949	0.979	0.990	0.999	0.999
Q1			0.157	0.549	0.829	1.294	2.780	6.992	14.395	148.198	228.537
Q2			0.378	1.328	1.997	3.118	6.699	16.849	34.684	357.072	550.641
Q3			0.755	2.648	3.994	6.237	13.398	33.698	69.369	714.144	1101.283
ARL	**2**		1.223	2.959	4.503	7.451	19.188	64.445	162.827	687.865	617.053
CV			0.427	0.814	0.882	0.930	0.974	0.992	0.997	0.999	0.999
Q1			0.169	0.698	1.146	1.996	5.375	18.396	46.698	197.743	177.371
Q2			0.408	1.681	2.760	4.809	12.951	44.322	112.516	476.445	427.362
Q3			0.815	3.362	5.521	9.619	25.901	88.645	225.032	952.889	854.723
ARL	**2.5**		1.252	3.489	5.779	10.697	34.642	149.835	370.370	569.056	493.827
CV			0.449	0.845	0.909	0.952	0.985	0.997	0.999	0.999	0.999
Q1			0.179	0.852	1.514	2.931	9.821	42.961	106.405	163.563	141.921
Q2			0.432	2.053	3.648	7.063	23.664	103.511	256.374	394.093	341.948
Q3			0.865	4.105	7.297	14.125	47.327	207.021	512.749	788.186	683.896
ARL	**2.7**		1.262	3.709	6.346	12.267	43.333	199.239	443.726	527.443	457.272
CV			0.456	0.855	0.918	0.958	0.988	0.998	0.999	0.999	0.999
Q1			0.183	0.916	1.678	3.383	12.322	57.174	127.508	151.592	131.405
Q2			0.442	2.206	4.042	8.151	29.689	137.756	307.221	365.249	316.610
Q3			0.883	4.413	8.0841	16.302	59.377	275.511	614.442	730.498	633.221
ARL	**3**		1.277	4.047	7.259	14.952	59.746	282.346	505.086	474.878	411.578
CV			0.466	0.868	0.929	0.966	0.992	0.998	0.999	0.999	0.999
Q1			0.188	1.014	1.941	4.156	17.044	81.082	145.160	136.469	118.259
Q2			0.454	2.442	4.677	10.014	41.065	195.361	349.752	328.814	284.938
Q3			0.907	4.885	9.353	20.027	82.131	390.721	699.504	657.627	569.875

Table A.4. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the lower-sided chart with α_0=1.50, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.010	1.042	1.053	1.064	1.081	1.099	1.111	1.149	1.176
CV			0.1	0.199	0.224	0.245	0.274	0.299	0.316	0.360	0.387
Q1			0.062	0.089	0.096	0.102	0.111	0.119	0.125	0.141	0.151
Q2			0.151	0.215	0.231	0.246	0.268	0.288	0.301	0.339	0.365
Q3			0.301	0.431	0.463	0.493	0.535	0.575	0.602	0.679	0.729
ARL	**0.4**		1.036	1.163	1.212	1.264	1.352	1.450	1.523	1.784	2.001
CV			0.187	0.374	0.418	0.457	0.510	0.557	0.586	0.663	0.707
Q1			0.086	0.146	0.165	0.184	0.214	0.246	0.269	0.349	0.415
Q2			0.207	0.353	0.397	0.443	0.515	0.593	0.649	0.843	1.000
Q3			0.414	0.705	0.794	0.886	1.029	1.185	1.297	1.685	2.001
ARL	**0.5**		1.044	1.202	1.265	1.335	1.452	1.588	1.692	2.079	2.423
CV			0.206	0.410	0.458	0.501	0.558	0.609	0.639	0.721	0.766
Q1			0.091	0.161	0.184	0.208	0.246	0.289	0.322	0.439	0.540
Q2			0.219	0.389	0.444	0.501	0.594	0.698	0.775	1.057	1.302
Q3			0.438	0.778	0.887	1.002	1.188	1.396	1.550	2.114	2.604
ARL	**0.6**		1.051	1.241	1.319	1.407	1.558	1.739	1.879	2.433	2.954
CV			0.221	0.441	0.492	0.538	0.599	0.652	0.684	0.768	0.813
Q1			0.095	0.176	0.203	0.232	0.280	0.336	0.379	0.543	0.696
Q2			0.229	0.423	0.489	0.558	0.675	0.809	0.913	1.309	1.677
Q3			0.459	0.846	0.977	1.117	1.349	1.619	1.826	2.618	3.354
ARL	**0.9**		1.071	1.357	1.485	1.634	1.912	2.273	2.577	3.964	5.562
CV			0.258	0.513	0.571	0.623	0.691	0.748	0.782	0.865	0.906
Q1			0.106	0.215	0.257	0.304	0.389	0.496	0.586	0.989	1.451
Q2			0.256	0.519	0.619	0.732	0.936	1.196	1.411	2.384	3.497
Q3			0.512	1.038	1.238	1.465	1.872	2.392	2.823	4.768	6.994
ARL	**1**		1.077	1.395	1.541	1.715	2.043	2.483	2.863	4.690	6.952
CV			0.268	0.532	0.592	0.646	0.714	0.773	0.807	0.887	0.925
Q1			0.109	0.228	0.275	0.329	0.428	0.558	0.669	1.199	1.852
Q2			0.263	0.549	0.662	0.792	1.031	1.345	1.613	2.891	4.463
Q3			0.526	1.099	1.324	1.584	2.062	2.689	3.226	5.781	8.926
ARL	**1.3**		1.094	1.509	1.714	1.968	2.481	3.228	3.925	7.891	14.046
CV			0.293	0.581	0.645	0.701	0.773	0.831	0.863	0.934	0.964
Q1			0.117	0.265	0.328	0.405	0.557	0.776	0.978	2.123	3.895
Q2			0.283	0.639	0.791	0.977	1.343	1.869	2.357	5.115	9.385
Q3			0.565	1.276	1.583	1.954	2.686	3.739	4.714	10.230	18.771
ARL	**1.5**		1.104	1.585	1.832	2.149	2.813	3.837	4.844	11.296	23.069
CV			0.307	0.607	0.674	0.731	0.803	0.859	0.891	0.955	0.978
Q1			0.122	0.289	0.365	0.459	0.655	0.953	1.244	3.104	6.492
Q2			0.294	0.695	0.878	1.107	1.578	2.296	2.997	7.478	15.641
Q3			0.588	1.391	1.757	2.214	3.156	4.592	5.995	14.956	31.282
ARL	**2**		1.127	1.774	2.142	2.646	3.812	5.878	8.195	28.769	87.139
CV			0.336	0.660	0.730	0.789	0.859	0.911	0.937	0.982	0.994
Q1			0.132	0.347	0.457	0.606	0.946	1.543	2.211	8.132	24.924
Q2			0.318	0.835	1.102	1.459	2.278	3.717	5.326	19.593	60.054
Q3			0.636	1.671	2.204	2.919	4.557	7.434	10.653	39.186	120.107
ARL	**2.5**		1.148	1.963	2.472	3.213	5.102	8.939	13.866	76.945	370.370
CV			0.359	0.700	0.772	0.829	0.897	0.942	0.963	0.993	0.999
Q1			0.140	0.404	0.555	0.772	1.319	2.425	3.843	21.992	106.405
Q2			0.338	0.974	1.337	1.859	3.177	5.843	9.260	52.987	256.374
Q3			0.676	1.947	2.674	3.718	6.354	11.686	18.521	105.974	512.749
ARL	**2.7**		1.155	2.039	2.611	3.462	5.715	10.553	17.112	115.483	681.474
CV			0.366	0.714	0.785	0.843	0.908	0.951	0.970	0.996	0.999
Q1			0.143	0.427	0.596	0.844	1.496	2.889	4.778	33.078	195.904
Q2			0.345	1.029	1.435	2.033	3.603	6.963	11.511	79.699	472.015
Q3			0.690	2.057	2.870	4.066	7.207	13.925	23.023	159.399	944.030
ARL	**3**		1.165	2.155	2.825	3.859	6.756	13.513	23.461	215.017	1755.504
CV			0.377	0.732	0.804	0.861	0.923	0.962	0.978	0.998	0.999
Q1			0.147	0.461	0.658	0.959	1.796	3.742	6.604	61.712	504.883
Q2			0.355	1.111	1.586	2.311	4.327	9.016	15.913	148.692	1216.476
Q3			0.710	2.223	3.173	4.623	8.654	18.031	31.826	297.383	2432.951

Table A.5. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the upper-sided chart with α_0=1.50, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		138768.8	34692.19	27753.75	23128.13	18502.5	15418.75	13876.88	10674.52	9251.251
CV			1	1	1	1	1	1	1	1	1
Q1			39921.14	9980.177	7984.113	6653.403	5322.694	4435.554	3991.985	3070.724	2661.275
Q2			96186.83	24046.45	19237.09	16030.85	12824.61	10687.12	9618.371	7398.667	6412.132
Q3			192373.7	48092.89	38474.18	32061.7	25649.22	21374.23	19236.74	14797.33	12824.26
ARL	**0.4**		34695.94	8673.985	6939.188	5782.657	4626.125	3855.104	3469.594	2668.918	2313.063
CV			1	1	1	1	1	1	1	1	1
Q1			9981.256	2495.206	1996.136	1663.423	1330.709	1108.9	997.996	767.656	665.283
Q2			24049.05	6012.002	4809.532	4007.886	3206.239	2671.808	2404.593	1849.606	1602.946
Q3			48098.09	12024	9619.064	8015.772	6412.478	5343.616	4809.185	3699.213	3205.893
ARL	**0.5**		27757.75	6939.438	5551.55	4626.292	3701.034	3084.195	2775.775	2135.212	1850.517
CV			1	1	1	1	1	1	1	1	1
Q1			7985.263	1996.208	1596.938	1330.757	1064.577	887.124	798.397	614.118	532.217
Q2			19239.86	4809.705	3847.695	3206.355	2565.015	2137.454	1923.674	1479.67	1282.334
Q3			38479.72	9619.411	7695.389	6412.709	5130.029	4274.909	3847.348	2959.339	2564.668
ARL	**0.6**		23132.29	5783.073	4626.459	3855.382	3084.306	2570.255	2313.229	1779.407	1542.153
CV			1	1	1	1	1	1	1	1	1
Q1			6654.602	1663.543	1330.805	1108.98	887.156	739.272	665.331	511.759	443.506
Q2			16033.74	4008.174	3206.47	2672.001	2137.531	1781.218	1603.062	1233.044	1068.592
Q3			32067.47	8016.348	6412.941	5344.001	4275.063	3562.437	3206.123	2466.089	2137.185
ARL	**0.9**		15423.2	3855.799	3084.639	2570.533	2056.426	1713.688	1542.32	1186.4	1028.213
CV			1	1	1	1	1	1	1	1	1
Q1			4436.833	1109.1	887.252	739.352	591.453	492.854	443.554	341.162	295.655
Q2			10690.2	2672.29	2137.762	1781.411	1425.059	1187.491	1068.708	822.003	712.356
Q3			21380.4	5344.579	4275.524	3562.822	2850.119	2374.983	2137.416	1644.006	1424.713
ARL	**1**		13881.38	3470.344	2776.275	2313.563	1850.85	1542.375	1388.138	1067.798	925.425
CV			1	1	1	1	1	1	1	1	1
Q1			3993.279	998.212	798.541	665.427	532.313	443.569	399.198	307.043	266.084
Q2			9621.49	2405.112	1924.021	1603.293	1282.565	1068.746	961.837	739.795	641.109
Q3			19242.98	4810.225	3848.041	3206.586	2565.13	2137.493	1923.674	1479.589	1282.218
ARL	**1.3**		10679.13	2669.784	2135.827	1779.856	1423.885	1186.571	1067.914	821.472	711.943
CV			1	1	1	1	1	0.999	0.999	0.999	0.999
Q1			3072.052	767.905	614.295	511.889	409.482	341.211	307.076	236.179	204.669
Q2			7401.865	1850.206	1480.096	1233.355	986.615	822.121	739.875	569.054	493.134
Q3			14803.73	3700.413	2960.192	2466.711	1973.231	1644.243	1479.749	1138.109	986.269
ARL	**1.5**		9255.916	2313.979	1851.183	1542.653	1234.122	1028.435	925.592	711.994	617.061
CV			1	1	1	1	1	1	0.999	0.999	0.999
Q1			2662.617	665.547	532.408	443.649	354.891	295.718	266.132	204.684	177.374
Q2			6415.366	1603.581	1282.796	1068.939	855.082	712.510	641.225	493.169	427.368
Q3			12830.73	3207.163	2565.592	2137.878	1710.163	1425.021	1282.449	986.339	854.735
ARL	**2**		6943.187	1735.797	1388.638	1157.198	925.758	771.465	694.319	534.092	462.879
CV			1	1	1	1	0.999	0.999	0.999	0.999	0.999
Q1			1997.286	499.214	399.342	332.761	266.180	221.793	199.599	153.505	133.018
Q2			4812.304	1202.816	962.184	801.762	641.340	534.392	480.919	369.857	320.497
Q3			9624.607	2405.632	1924.367	1603.524	1282.68	1068.785	961.837	739.715	640.994
ARL	**2.5**		5555.549	1388.887	1111.11	925.925	740.740	617.284	555.555	427.350	370.370
CV			1	1	1	0.999	0.999	0.999	0.999	0.999	0.999
Q1			1598.088	399.414	319.503	266.228	212.954	177.438	159.679	122.797	106.405
Q2			3850.466	962.357	769.816	641.456	513.095	427.522	384.735	295.869	256.375
Q3			7700.933	1924.713	1539.632	1282.911	1026.191	855.043	769.469	591.739	512.749
ARL	**2.7**		5144.397	1286.099	1028.88	857.399	685.919	571.599	514.44	395.723	342.960
CV			1	1	1	1	0.999	0.999	0.999	0.999	0.998
Q1			1479.807	369.844	295.846	246.515	197.183	164.295	147.851	113.699	98.519
Q2			3565.478	891.109	712.818	593.958	475.097	395.856	356.236	273.948	237.375
Q3			7130.955	1782.219	1425.637	1187.915	950.193	791.712	712.472	547.896	474.750
ARL	**3**		4630.457	1157.614	926.092	771.743	617.395	514.495	463.046	356.189	308.698
CV			1	1	0.999	0.999	0.999	0.999	0.999	0.998	0.998
Q1			1331.956	332.881	266.276	221.873	177.469	147.867	133.066	102.325	88.663
Q2			3209.241	802.051	641.571	534.585	427.599	356.274	320.612	246.545	213.626
Q3			6418.483	1604.101	1283.142	1069.17	855.197	712.549	641.225	493.09	427.252

Table A.6. ARL, CV and quartiles of the run length distribution assuming 0.0027 as the false alarm probability for the two-sided chart with α_0=1.50, λ_0=2.5, $\lambda_1 \in$ (0.1,0.4,0.5,0.6,0.9,1,1.3,1.5,2,2.5,2.7,3) and $\alpha_1 \in$ (0.1,0.4,0.5,0.6,0.75,0.9,1,1.3,1.5)

	λ	α	**0.1**	**0.4**	**0.5**	**0.6**	**0.75**	**0.9**	**1**	**1.3**	**1.5**
ARL	**0.1**		1.011	1.045	1.057	1.069	1.089	1.108	1.122	1.164	1.193
CV			0.104	0.208	0.233	0.255	0.285	0.312	0.329	0.375	0.403
Q1			0.064	0.092	0.099	0.105	0.115	0.124	0.129	0.147	0.158
Q2			0.153	0.221	0.238	0.254	0.276	0.298	0.312	0.353	0.381
Q3			0.307	0.442	0.476	0.507	0.555	0.596	0.624	0.707	0.762
ARL	**0.4**		1.039	1.177	1.231	1.289	1.387	1.499	1.582	1.884	2.142
CV			0.194	0.388	0.433	0.474	0.528	0.577	0.607	0.685	0.730
Q1			0.088	0.152	0.172	0.193	0.225	0.261	0.288	0.380	0.457
Q2			0.212	0.366	0.414	0.464	0.543	0.629	0.693	0.916	1.102
Q3			0.423	0.732	0.828	0.928	1.086	1.259	1.386	1.833	2.204
ARL	**0.5**		1.047	1.219	1.289	1.367	1.499	1.655	1.774	2.231	2.647
CV			0.213	0.425	0.474	0.518	0.577	0.629	0.661	0.743	0.789
Q1			0.093	0.168	0.193	0.219	0.262	0.310	0.347	0.484	0.606
Q2			0.224	0.405	0.464	0.527	0.630	0.748	0.836	1.166	1.461
Q3			0.448	0.809	0.928	1.054	1.260	1.495	1.672	2.331	2.921
ARL	**0.6**		1.055	1.262	1.349	1.446	1.617	1.825	1.989	2.651	3.296
CV			0.229	0.456	0.509	0.556	0.618	0.673	0.705	0.789	0.835
Q1			0.098	0.183	0.213	0.245	0.299	0.362	0.412	0.607	0.796
Q2			0.235	0.441	0.512	0.589	0.719	0.873	0.992	1.464	1.917
ARL	**0.9**		1.076	1.389	1.530	1.699	2.017	2.442	2.806	4.538	6.649
CV			0.266	0.529	0.589	0.641	0.710	0.768	0.802	0.883	0.922
Q1			0.109	0.226	0.271	0.324	0.420	0.546	0.653	1.156	1.765
Q2			0.262	0.544	0.654	0.781	1.012	1.316	1.573	2.785	4.253
Q3			0.524	1.088	1.308	1.561	2.025	2.631	3.145	5.569	8.505
ARL	**1**		1.083	1.429	1.592	1.788	2.167	2.687	3.146	5.463	8.515
CV			0.276	0.548	0.609	0.664	0.734	0.792	0.826	0.904	0.939
Q1			0.112	0.239	0.291	0.351	0.465	0.618	0.752	1.423	2.303
Q2			0.269	0.576	0.701	0.846	1.119	1.489	1.812	3.428	5.549
Q3			0.539	1.153	1.402	1.693	2.239	2.978	3.624	6.857	11.097
ARL	**1.3**		1.100	1.553	1.782	2.072	2.670	3.569	4.434	9.683	18.531
CV			0.302	0.597	0.663	0.719	0.791	0.848	0.880	0.947	0.973
Q1			0.120	0.279	0.349	0.437	0.613	0.875	1.126	2.639	5.186
Q2			0.289	0.672	0.842	1.052	1.477	2.109	2.712	6.359	12.495
Q3			0.579	1.343	1.684	2.104	2.954	4.218	5.424	12.718	24.989
ARL	**1.5**		1.111	1.636	1.914	2.276	3.057	4.304	5.573	14.349	31.879
CV			0.316	0.623	0.691	0.749	0.820	0.876	0.906	0.965	0.984
Q1			0.125	0.304	0.389	0.497	0.726	1.0885	1.455	3.983	9.026
Q2			0.301	0.733	0.938	1.198	1.749	2.622	3.505	9.596	21.748
Q3			0.602	1.467	1.875	2.396	3.498	5.243	7.009	19.191	43.496
ARL	**2**		1.135	1.841	2.258	2.839	4.234	6.819	9.849	39.392	125.695
CV			0.345	0.676	0.746	0.805	0.874	0.924	0.948	0.987	0.996
Q1			0.135	0.367	0.492	0.663	1.068	1.814	2.687	11.188	36.016
Q2			0.325	0.885	1.185	1.597	2.572	4.371	6.474	26.957	86.778
Q3			0.651	1.770	2.369	3.193	5.145	8.742	12.948	53.913	173.556
ARL	**2.5**		1.156	2.049	2.627	3.489	5.779	10.697	17.329	106.788	370.370
CV			0.367	0.715	0.787	0.845	0.909	0.952	0.971	0.995	0.999
Q1			0.144	0.429	0.600	0.852	1.514	2.931	4.840	30.577	106.405
Q2			0.346	1.035	1.446	2.053	3.649	7.063	11.662	73.673	256.374
Q3			0.692	2.070	2.893	4.105	7.297	14.125	23.324	147.345	512.749
ARL	**2.7**		1.163	2.132	2.781	3.776	6.522	12.773	21.678	154.307	467.899
CV			0.375	0.727	0.800	0.857	0.920	0.960	0.977	0.997	0.999
Q1			0.147	0.454	0.646	0.935	1.729	3.529	6.091	44.247	134.462
Q2			0.353	1.095	1.556	2.253	4.165	8.502	14.676	106.611	323.976
Q3			0.706	2.190	3.111	4.506	8.329	17.004	29.353	213.221	647.953
ARL	**3**		1.174	2.258	3.022	4.236	7.793	16.613	30.222	248.968	539.485
CV			0.385	0.746	0.818	0.874	0.934	0.969	0.983	0.998	0.999
Q1			0.151	0.492	0.716	1.068	2.095	4.634	8.549	71.479	155.056
Q2			0.363	1.185	1.725	2.574	5.047	11.165	20.599	172.224	373.596
Q3			0.727	2.370	3.449	5.148	10.095	22.330	41.199	344.449	747.191