Control of active suspension system in the presence of nonlinear spring and damper

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Abstract

Two methods which are computationally simple and easy to apply are developed by using state dependent Riccati equation (SDRE) approach and approximating sequence of Riccati equation (ASRE) approach to control active suspension system in the presence of nonlinear spring and damper. Additionally, effectivenesses of the both control methods developed by utilizing two recently introduced SDRE and ASRE techniques are compared. First, methodologies of both approaches are presented. After that, nonlinear dynamics of the vehicle suspension system is described in terms of conveniently selected state variables for better control performance. Then, a cost function is written by using suspension and tire deflections, sprung mass velocity and acceleration, and unsprung mass velocity variables to improve ride quality, suspension deflection, and tire deflection. Additionally, a convenient representation of this cost function in terms of state variables is obtained to realize better control. A bump expressed as sinusoidal function and roughness of the road expressed as white noise are taken into consideration as the disturbances from the road. Finally, quarter vehicle suspension system equivalent model of Ford Fiesta Mk2 is used as an example and simulations obtained by using the developed control methods are checked against the performance requirements and corresponding passive suspension system.
Key words: Active suspension system, nonlinear spring, nonlinear damper, state-dependent Riccati equation, approximating sequence of Riccati equation

1. Introduction

In the literature there are various studies considering different aspects of the suspension systems [1-6]. Increasing demand for the ride comfort of passenger, handling quality of vehicle, and holding quality of road has led to the usage of active suspension systems (ASSs) increasingly. Recent studies on ASSs control in the presence of nonlinear springs and dampers are given below.


Zhang et al. [27] used an adaptive control method. They used hysteretic leaf spring in suspension system. Ding et al. [28] applied robust event triggered control law. Hysteretic leaf spring is also considered in this study.

Wu et al. [29] developed a fault tolerant control method. Lu et al. [30] proposed active disturbance rejection control technique. Nourisola and Ahmadi [31] designed Hinfinity controller utilizing genetic algorithm.

The aim of this study is to propose computationally simple methods for the control of suspension systems having nonlinear springs and dampers, since the applications of the nonlinear control methods used in the previous studies are computationally intensive. To realize this aim, two methods are developed by using state-dependent Riccati equation (SDRE) and approximating sequence of Riccati equation (ASRE) approaches for the control of nonlinear suspension systems. The proposed methods are computationally simple and easy to apply. Additionally, another aim of this study is to compare the performances of these control methods developed by utilizing two recently introduced SDRE and ASRE techniques.

SDRE approach has attracted the attention of researchers recently. The method is computationally simple, it has design flexibility, and it can be applied in real time. In this approach, nonlinear dynamics is expressed by a state dependent coefficient (SDC) and it is a linear like representation. Because the nonlinear equations are represented by linear equations, the linear control techniques can be used. As a result of these fundamental advantages of the method it has been used for designing filter, observer and controller. Recently, a complete literature review for the theory of SDRE control has been published by Cimen [35]. The method has been applied successfully to different areas having nonlinear dynamics. Examples of the application areas include helicopter, spacecraft, satellite, missile, ship, and robot [36].

ASRE, which is another simple method, has been discovered for systems having nonlinear dynamics. In this method, nonlinear system can be represented by using recursive linear time varying (LTV) equations [37]. As the nonlinear equations are represented by using linear equations, the linear control methods can be used. This technique can be used for systems having nonlinear dynamics that satisfy local Lipschitz requirement which is very mild. In
addition to this, for the solution uniqueness that requirement is standard. The method has also been applied successfully to the application areas with nonlinear dynamics. Examples of these applications are helicopter, spacecraft, satellite, missile, ship, and robot [38].

In this work, the control of ASS in the presence of nonlinear spring and damper is investigated. Two control methods which are computationally simple and easy to apply are developed by utilizing SDRE and ASRE techniques. Additionally, effectivenesses of the both control methods developed by utilizing two recently introduced SDRE and ASRE techniques are compared. First, methodologies of the both approaches are presented. After that, the nonlinear dynamics of the vehicle suspension system (SS) is described by first order differential equations in terms of conveniently selected state variables for better control performance. Then, a cost function is written by using suspension and tire deflections, sprung mass velocity and acceleration, and unsprung mass velocity variables to improve ride quality, suspension deflection, and tire deflection. Additionally, a convenient representation of this cost function in terms of state variables is obtained to realize better control. A bump expressed as sinusoidal function and roughness of the road expressed as white noise are taken into consideration as the disturbances from the road. After that, by using the variables of states and Riccati differential equation control input is formed. Finally, equivalent model of a quarter vehicle SS of Ford Fiesta Mk2 is utilized as an example and simulations obtained by using the developed control methods are checked against the performance requirements and corresponding passive suspension system (PSS) to illustrate the efficacies of the both control methods.

2. SDRE Control of Nonlinear Systems
SDRE technique has been applied increasingly for designing nonlinear filter, observer and controller because it has design flexibility, it is computationally simple, and it can be applied in real time. In this approach, nonlinear dynamics is expressed by a SDC and it is a linear like representation. Because the nonlinear equations are represented by linear equations, the linear control techniques can be used.

A nonlinear system can be described by

\[
\dot{y}(t) = f(y) + B(y)u(t) \quad y(t_0) = y_0 \in \mathbb{R}^n
\]  

(1)

In the above equation \( y(t) \in \mathbb{R}^n \) denotes the vector of states, \( u(t) \in \mathbb{R}^m \) stands for the vector of inputs, \( f(y) \in \mathbb{R}^n \) and \( B(y) \in \mathbb{R}^{m \times n} \) denote nonlinear functions.

**Assumption 1.** Equilibrium point of the system is at the origin \( y = 0 \) when \( u = 0 \) meaning \( f(0) = 0, \quad B(y) \neq 0 \quad \forall y \) and \( f(y) \) is one time continuously differentiable function of \( y \).

**Proposition 1.** Under the above assumptions, the mathematical factorization of \( f(y) \) in the form of state dependent coefficient \( A(y)y(t) \) always exist. The parametrization is expressed by [35]

\[
A(y) = \int_0^{\lambda} \left. \frac{\partial f}{\partial y} \right|_{y=\lambda y} d\lambda
\]

(2)

where \( \lambda \) is used as dummy variable for integration.

The nonlinear differential equation (1) may be rearranged in pseudo linear representation in SDC form having linear structure

\[
\dot{y}(t) = A(y)y(t) + B(y)u(t) \quad y(t_0) = y_0 \in \mathbb{R}^n
\]

(3)
where $A(y) \in \mathbb{R}^{n \times n}$, $A(y)$ and $B(y)$ are called as SDC matrices. Since the above differential equation can be considered as a pointwise linear time invariant (LTI) system, linear control techniques can be used.

**Remark 1.** SDC matrix $A(y)$ is not unique and it can be written infinitely many different forms. Therefore, different $A(y)$ can be used as a design flexibility to improve the performance of the system.

Infinite horizon nonlinear regulator that minimizes the following cost function is considered

$$ J = \frac{1}{2} y^T(t_f)F(y(t_f))y(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \{ y^T(t)Q(y(t))y(t) + u^T(t)P(y(t))u(t) \} \, dt $$

(4)

where $Q(y) \in \mathbb{R}^{n \times n}$ stands for the state weighting matrix, $P(y) \in \mathbb{R}^{m \times m}$ denotes the control weighting matrix, $F(y(t_f)) \in \mathbb{R}^{n \times n}$ stands for the endpoint matrix taken as zero in this case, $t_0 = 0$, $t_f = \infty$.

**Condition 1.** Weighting parameters satisfy $Q(y) = Q^T(y) \succeq 0$ and $P(y) = P^T(y) > 0 \ \forall y$.

**Definition 1.** If $\{A(y),B(y)\}$ is pointwise stabilizable (controllable), SDC representation of the nonlinear system is stabilizable (controllable) [39].

**Definition 2.** If $\{C(y),A(y)\}$ is pointwise detectable (observable), SDC representation of the nonlinear system is detectable (observable). Here $Q(y) = C^T(y)C(y)$ [39].
Remark 2. The stabilizability condition is satisfied \( \forall y \) providing full rank for the \( n \times nm \) state dependent controllability matrix, i.e.,

\[
\text{rank} \left( \begin{bmatrix} B(y) & A(y)B(y) & \cdots & A^{n-1}(y)B(y) \end{bmatrix} \right) = n \quad \forall y
\] (5)

The detectability condition is satisfied \( \forall y \) providing full rank for the \( n \times n^2 \) state dependent observability matrix, i.e.,

\[
\text{rank} \left( \begin{bmatrix} C^T(y) & A^T(y)C^T(y) & \cdots & (A^T(y))^{n-1}C^T(y) \end{bmatrix} \right) = n \quad \forall y
\] (6)

If \( Q(y) \) is selected as \( Q(y) = Q^T(y) > 0 \), the detectability condition is satisfied \( \forall y \).

Definition 3. If \( \text{Re}[\lambda_i(A(y))] < 0 \ \forall y \), SDC representation is pointwise Hurwitz. Here \( \lambda_i(\cdot), i = 1, \cdots, n \) represent eigenvalues of a matrix.

The nonlinear control can be formed by using state dependent coefficients (SDCs) as

\[
u(t) = -P^{-1}(y)B^T(y)R(y)y
\] (7)

Here \( R(y) \) is given by the following algebraic SDRE

\[
R(y)A(y) + A^T(y)R(y) - R(y)B(y)P^{-1}(y)B^T(y)R(y) + Q(y) = 0
\] (8)

to have \( R \geq 0 \).

Remark 3. If the system is scalar, parametrization of the SDCs is unique \( \forall y \neq 0 \) and it can be represented as \( f(y)/y \). However, if the system is not scalar, parametrization of the SDC is infinite.
Proposition 2. If the system is multivariable, for different SDC matrices $A_1(y)$ and $A_2(y)$ assume

$$A(y, \gamma) = \gamma A_1(y) + (1-\gamma)A_2(y)$$

(9)

where $\gamma \in \mathbb{R}$. $A(y, \gamma)$ gives SDC matrices infinite in number for any $\gamma \in \mathbb{R}$ which corresponds to a hyperplane.

Proof. See [35].

Theorem 1. Assuming that the SDC form is selected such that $\text{col}[A(y)]$ is in the neighborhood about the origin and $\{A(y),B(y)\}$ is pointwise stabilizable, $\{C(y),A(y)\}$ is pointwise detectable. Then, SDRE nonlinear control generates locally asymptotically stable solution.

Proof. See [35].

3. ASRE Control of Nonlinear Systems

Most of the nonlinear approaches needs strong requirements and complex numerical methods. ASRE is a simple method for systems having nonlinear dynamics. In this method, nonlinear system can be represented by using recursive LTV equations. As the nonlinear equations are represented by linear equations, the linear control methods can be used. This technique can be used for systems having nonlinear dynamics that satisfy local Lipschitz requirement which is very mild.

A nonlinear system can be described by
\[
\dot{y}(t) = f(y) + B(y)u(t) \quad y(t_0) = y_0 \in \mathbb{R}^n
\]  
(10)

In the above equation, \( y(t) \in \mathbb{R}^n \) denotes the vector of states, \( u(t) \in \mathbb{R}^m \) stands for the vector of inputs, \( f(y) \in \mathbb{R}^n \) and \( B(y) \in \mathbb{R}^{n \times m} \) denote nonlinear functions.

**Assumption 2.** Equilibrium point of the system is at the origin \( y = 0 \) when \( u = 0 \) meaning \( f(0) = 0 \), \( B(y) \neq 0 \) \( \forall y \) and \( f(y) \) is one time continuously differentiable function of \( y \).

**Proposition 3.** Under the above assumptions, the mathematical factorization of \( f(y) \) in the form of state dependent coefficient \( A(y)y(t) \) always exist. The parametrization is expressed by [35]

\[
A(y) = \int_0^1 \frac{\partial f}{\partial y} \bigg|_{y=\Delta y} d\lambda
\]  
(11)

where \( \lambda \) is used as dummy variable for integration.

The nonlinear differential equation (10) may be rearranged in pseudo linear representation in SDC form having linear structure

\[
\dot{y}(t) = A(y)y(t) + B(y)u(t) \quad y(t_0) = y_0 \in \mathbb{R}^n
\]  
(12)

where \( A(y) \in \mathbb{R}^{n \times n} \). \( A(y) \) and \( B(y) \) are called as SDC matrices. Since the above differential equation is in pseudo linear form, linear control techniques can be used.

**Remark 4.** SDC matrix \( A(y) \) is not unique and it can be written infinitely many different forms. Therefore different \( A(y) \) can be used as a design flexibility to improve the performance of the system.
Finite horizon nonlinear regulator that minimizes the following cost function is considered

$$J = \frac{1}{2} y^T(t_f) F(y(t_f)) y(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left[ y^T(t) Q(y(t)) y(t) + u^T(t) P(y(t)) u(t) \right] dt$$  \hspace{1cm} (13)$$

where $Q(y) \in \mathbb{R}^{n \times n}$ stands for the state weighting matrix, $P(y) \in \mathbb{R}^{m \times m}$ denotes the control weighting matrix, $F(y(t_f)) \in \mathbb{R}^{n \times n}$ stands for the endpoint matrix, $t_0 = 0$, $t_f$ is finite.

**Condition 2.** Weighting parameters satisfy $Q(y) = Q^T(y) \geq 0$ and $P(y) = P^T(y) > 0 \ \forall y$.

**Remark 5.** If the system is scalar, parametrization of the SDCs is unique $\forall y \neq 0$ and it can be represented as $f(y)/y$. However, if the system is not scalar, parametrization of the SDC is infinite.

**Proposition 4.** If the system is multivariable, for different SDC matrices $A_1(y)$ and $A_2(y)$ assume

$$A(y, \gamma) = \gamma A_1(y) + (1 - \gamma) A_2(y)$$ \hspace{1cm} (14)$$

where $\gamma \in \mathbb{R}$. $A(y, \gamma)$ gives SDC matrices infinite in number for any $\gamma \in \mathbb{R}$ which corresponds to a hyperplane.

**Proof.** See [35]

The minimization of the nonlinear cost function represented by Eq. (13), subjected to the nonlinear differential equation (12), can be expressed by using successive LTV approximations of both Eq. (13) and Eq. (12) [37].

$$\dot{y}^{[0]}(t) = A(y_0) y^{[0]}(t) + B(y_0) u^{[0]}(t), \quad y^{[0]}(t) = y_0$$  \hspace{1cm} (15)$$
\[ J[0] = \frac{1}{2} y^{T}[0](t_f)F(y(t_f))y^{[0]}(t_f) \]
\[ + \frac{1}{2} \int_{t_0}^{t_f} \left( y^{T}[0](t)Q(y^{[0]}(t)) + u^{T}[0](t)P(y^{[0]}(t))u^{[0]}(t) \right) dt \]

(16)

Here, the first approximation is evaluated by utilizing the state variables initial values resulting LTI system. When the iteration number \( k \geq 1 \), the following approximations can be written where they refer to LTV system

\[ \dot{y}^{[k]}(t) = A(y^{[k-1]}(t))y^{[k]}(t) + B(y^{[k-1]}(t))u^{[k]}(t), \quad k \geq 1, \quad y^{[k]}(0) = y^{[0]} \]

(17)

\[ J[k] = \frac{1}{2} y^{T}[k](t_f)F(y(t_f))y^{[k]}(t_f) \]
\[ + \frac{1}{2} \int_{t_0}^{t_f} \left( y^{T}[k](t)Q(y^{[k-1]}(t))y^{[k]}(t) + u^{T}[k](t)P(y^{[k-1]}(t))u^{[k]}(t) \right) dt, \quad k \geq 1 \]

(18)

LTV approximation of the optimal control input can be formed as

\[ u^{[k]}(t) = -P^{-1}(y^{[k-1]}(t))B^{T}(y^{[k-1]}(t))R^{[k]}(t)u^{[k]}(t) \]

(19)

where \( R^{[k]}(t) \) is given by the following algebraic ASRE

\[ \dot{R}^{[k]}(t) = -A^{T}(y^{[k-1]}(t))R^{[k]}(t) - R^{[k]}(t)A(y^{[k-1]}(t)) - Q(y^{[k-1]}(t)) \]
\[ - R^{[k]}(t)B(y^{[k-1]}(t))P^{-1}(y^{[k-1]}(t))B^{T}(y^{[k-1]}(t))R^{[k]}(t), \quad R^{[k]}(t_f) = F(y(t_f)) \]

(20)

Therefore, LTV approximation of the closed loop system can be represented as

\[ \dot{y}^{[0]}(t) = \dot{A}(y_{0})y^{[0]}(t), \quad y^{[0]}(0) = y_{0} \]

(21)

\[ \dot{y}^{[k]}(t) = \dot{A}(y^{[k-1]}(t))y^{[k]}(t), \quad k \geq 1, \quad y^{[k]}(0) = y^{[0]} \]

(22)

Here

\[ \dot{A}(y^{[k-1]}(t)) = A(y^{[k-1]}(t)) - B(y^{[k-1]}(t))P^{-1}(y^{[k-1]}(t))B^{T}(y^{[k-1]}(t))R^{[k]} \]

(23)

Convergence of LTV system to the original nonlinear system is explained in the theorem given below.
Theorem 2. Assume that the subsequent conditions on $A(y)$ and $B(y)$ hold

(1) $\mu A(y) \leq \mu_0 \quad \forall y \in \mathbb{R}^n$  \hspace{1cm} (24)

(2) $\|A(y) - A(x)\| \leq \alpha \|y - x\| \quad \forall y, x \in \mathbb{R}^n$  \hspace{1cm} (25)

(3) $\|B(y) - B(x)\| \leq \beta \|y - x\| \quad \forall y, x \in \mathbb{R}^n$  \hspace{1cm} (26)

(4) $\|B(y)\| \leq \gamma \quad \forall y \in \mathbb{R}^n$  \hspace{1cm} (27)

where $\mu A(y)$ stands for the logarithmic norm of $A(y)$. $\alpha$, $\beta$, and $\gamma$ are constants. Then, while minimizing approximated cost function $J^{[k]}$, approximations of states $y^{[k]}(t)$ and controls $u^{[k]}(t)$ converge to the nonlinear system states $y(t)$ and controls $u(t)$ minimizing nonlinear cost function $J$.

Proof. Assume that $\Phi^{[k-1]}(t, t_0)$ denotes the transition matrix of $A(y^{[k-1]}(t))$. Thus, $\Phi^{[k-1]}(t, t_0)$ obeys the subsequent Brauer’s well known inequality

$$\|\Phi^{[k-1]}(t, t_0)\| \leq \exp \left[ \int_{t_0}^t \mu A(y^{[k-1]}(\tau)) \, d\tau \right]$$  \hspace{1cm} (28)

Using the assumptions (1) and (2), it can be written as

$$\|\Phi^{[k-1]}(t, t_0) - \Phi^{[k-2]}(t, t_0)\| \leq \alpha \exp(\mu(t - t_0))(t - t_0) \sup_{s \in [t_0, t]} \|y^{[k-1]}(s) - y^{[k-2]}(s)\|$$  \hspace{1cm} (29)

Then, using the above equation and assumptions (1)-(4) the following equation can be obtained as

$$\zeta^{[k]}(t) \leq \alpha \exp(\mu_0 t) \zeta^{[k-1]}(t) \|y_0\| + \int_0^t \exp(\mu_0 (t - s)) \beta \zeta^{[k-1]}(s) \exp((\mu_0 + \gamma) s) \, ds$$

$$+ \|y_0\| \int_0^t \exp(\mu_0 (t - s)) \beta \zeta^{[k-1]}(s) \exp((\mu_0 + \gamma) s) \, ds$$

$$+ \|y_0\| \int_0^t \alpha (t - s) \exp(\mu_0 (t - s)) \zeta^{[k-1]}(t) \exp((\mu_0 + \gamma) s) \, ds$$  \hspace{1cm} (30)
where

$$\zeta^{[k]}(t) = \sup_{t \in [0,t]} \|y^{[k]}(t) - y^{[k-1]}(t)\|$$

(31)

Since

$$\|y^{[k]}(t)\| \leq \exp((\mu_0 + \gamma)t)\|y_0\|$$

(32)

then

$$\zeta^{[k]}(t) \leq \lambda(t)\zeta^{[k-1]}(t)$$

(33)

in the above equation

$$\lambda(t) = \|y_0\left[\alpha \exp(\mu_0 t) + \left(\frac{\beta}{\gamma} + \alpha\right)\exp((\mu_0 + \gamma)t) - \exp(\mu_0 t)\right]\left[1 - \gamma \int_0^t \exp(\mu_0(t-s))ds\right]$$

(34)

Therefore, if $|\lambda(t)| < 1 \forall t \in [0,T]$ then $y^{[k]}(t) \to y(t)$ on $C([t_0,T] \mathbb{R}^n)$.

4. Dynamic Model

Quarter vehicle ASS is considered to show the effectiveness of the control methods on SS. The vertical motion of the quarter vehicle ASS can be represented as a two degree of freedom (DOF) system as given in Figure 1.

Mass of the vehicle body is called as sprung mass, $m_s$, masses of the tire and axles are denoted as unsprung mass, $m_u$. It is assumed that the nonlinearity of the SS is modeled by considering nonlinear spring and nonlinear damper. Nonlinear spring constants are denoted as $k_s$ and $k_{ns}$ and $b_s$ and $b_{ns}$ denote the nonlinear damper constants. $F_a$ stands for the magnitude of the driving force. Stiffness of the tire is denoted by the spring whose coefficient is $k_s$. $p_s$ stands for the position of the sprung mass, $p_u$ is the position of the unsprung mass,
and \( p_r \) stands for the road disturbance at position level. Forces resulting from the nonlinear spring and nonlinear damper of the SS may be taken as given by the following equations [40].

\[
F_s = k_s(p_s - p_u) + k_{ns}(p_s - p_u)^3
\]
\[
F_d = b_s(\dot{p}_s - \dot{p}_u) + b_{ns}(\dot{p}_s - \dot{p}_u)^2 \text{sgn}(\dot{p}_s - \dot{p}_u)
\]

Dynamic equations of the 2 DOF quarter vehicle ASS may be obtained as follows

\[
m_p \ddot{p}_s + b_p(\dot{p}_s - \dot{p}_u) + b_{ns}(\dot{p}_s - \dot{p}_u)^2 \text{sgn}(\dot{p}_s - \dot{p}_u) + k_s(p_s - p_u) + k_{ns}(p_s - p_u)^3 = F_a
\]
\[
m_u \ddot{p}_u - b_u(\dot{p}_u - \dot{p}_s) - b_{ns}(\dot{p}_u - \dot{p}_s)^2 \text{sgn}(\dot{p}_u - \dot{p}_s) - k_s(p_u - p_s) - k_{ns}(p_u - p_s)^3 + k_s(p_u - p_r) = -F_a
\]

Since the selection of state variables is important for better control performance, convenient state variables can be chosen as

\[
y_1 = p_s - p_u
\]
\[
y_2 = \dot{p}_s
\]
\[
y_3 = p_u - p_r
\]
\[
y_4 = \dot{p}_u
\]

where \( y_1 \) represents the suspension deflection (rattle space), \( y_2 \) stands for the sprung mass absolute velocity, \( y_3 \) stands for the tire deflection, and \( y_4 \) denotes the unsprung mass absolute velocity. Thus, the state equations may be represented as follows

\[
\dot{y}(t) = A(y)y(t) + Bu(t) + L\dot{p}_r(t)
\]

Here, \( A(y) \) is given as follows
\[ A(y) = \begin{bmatrix}
-k_s & 0 & \frac{k_{nr}}{m_s} y_1^2 & -\frac{b_s}{m_s} & \frac{1}{m_s} (y_2 - y_4) \text{sgn}(y_2 - y_4) & 0 \\
0 & \frac{k_s}{m_u} y_1^2 & -\frac{b_s}{m_u} & \frac{1}{m_u} (y_2 - y_4) \text{sgn}(y_2 - y_4) & -\frac{k_r}{m_u} \\
b_s & \frac{b_s}{m_s} & \frac{b_{ns}}{m_s} (y_2 - y_4) \text{sgn}(y_2 - y_4) & -1 & 0 \\
-\frac{b_s}{m_u} & \frac{b_{ns}}{m_u} (y_2 - y_4) \text{sgn}(y_2 - y_4) & 1 & 0 \\
\end{bmatrix} \]

(44)

\[ B \] is given as follows

\[ B = \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix}^T \]  

(45)

\[ L \] is denoted by

\[ L = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^T \]  

(46)

\[ y(t) \] is formed as follows

\[ y(t) = [y_1 \ y_2 \ y_3 \ y_4]^T \]  

(47)

\[ u(t) \] is formed as follows

\[ u(t) = F_u \]  

(48)

By using the following cost function, suspension and tire deflections, sprung mass velocity and acceleration, and unsprung mass velocity are minimized to improve ride quality, suspension deflection, and tire deflection.

\[ J = \lim_{T \to \infty} \left[ \int_0^T \left( \dot{y}_2^2 + \rho_1 y_1^2 + \rho_2 y_2^2 + \rho_3 y_3^2 + \rho_4 y_4^2 \right) dt \right] \]  

(49)

To realize better control a convenient representation of the above cost function in terms of state variables can be obtained as
\[ J = \lim_{T \to \infty} \left\{ \int_{0}^{T} [y^T(t)Q(y)y(t) + 2y^T(t)N(y)u(t) + u^T(t)Pu(t)] dt \right\} \]  

(50)

Entries of the symmetric matrix \( Q(y) \) are given below

\[ Q(1,1) = \frac{k_s^2}{m_s^2} + 2 \frac{k_{ns}^2}{m_s^2} y_1^2 \]  

(51)

\[ Q(1,2) = \frac{k_s b_s}{m_s^2} + \frac{k_{ns} b_n}{m_s^2} y_1^2 + \left( \frac{k_s b_n}{m_s^2} + \frac{k_{ns} b_{ns}}{m_s^2} y_1^2 \right) (y_2 - y_4) \]  

(52)

\[ Q(1,3) = 0 \]  

(53)

\[ Q(1,4) = -\frac{k_s b_s}{m_s^2} - \frac{k_{ns} b_n}{m_s^2} y_1^2 + \left( \frac{k_s b_n}{m_s^2} + \frac{k_{ns} b_{ns}}{m_s^2} y_1^2 \right) y_4 \]  

(54)

\[ Q(2,2) = \frac{b_s^2}{m_s^2} + 2 \frac{b_n b_{ns}}{m_s^2} y_2 \]  

(55)

\[ Q(2,3) = 0 \]  

(56)

\[ Q(2,4) = -\frac{b_s^2}{m_s^2} - 3 \frac{b_n b_{ns}}{m_s^2} (y_2 - y_4) \]  

(57)

\[ Q(3,3) = \rho_3 \]  

(58)

\[ Q(3,4) = 0 \]  

(59)

\[ Q(4,4) = \frac{b_s^2}{m_s^2} - 2 \frac{b_n b_{ns}}{m_s^2} y_4 + \frac{b_{ns}^2}{m_s^2} y_4^2 \]  

(60)

Entries of \( N(y) \) are as follows

\[ N(1,1) = -\frac{k_s}{m_s^2} - \frac{k_{ns}}{m_s^2} y_1^2 \]  

(61)
\[ N(2,1) = -\frac{b_1}{m_s} - \frac{b_{ns}}{m_s} (y_2 - y_4) \text{sgn}(y_2 - y_4) \]  
(62)

\[ N(3,1) = 0 \]  
(63)

\[ N(4,1) = \frac{b_s}{m_s} + \frac{b_{ns}}{m_s} (y_2 - y_4) \text{sgn}(y_2 - y_4) \]  
(64)

Scalar \( P \) is as follows

\[ P = \frac{1}{m_s} \]  
(65)

For the minimization of the cost function (50) imposed to the nonlinear differential equation (43), the following nonlinear control law can be formed as

\[ u(t) = -P^{-1} [B^T R(y) + N^T (y)] y(t) \]  
(66)

Here \( R(y) \) is given by the following RDE

\[
\dot{R}(y) = -\left[ A(y) - BP^{-1} N^T (y) \right]^T R(y) - R(y) \left[ A(y) - BP^{-1} N^T (y) \right] \\
- \left[ Q(y) - N(y) P^{-1} N^T (y) \right] + R(y) BP^{-1} B^T R(y) \]  
(67)

Two different road disturbances are taken into consideration. One represents the roughness of road as the disturbance and the other represents the bump on the road as another disturbance.

Roughness of the road is taken as a stochastic process represented by a stationary first-order filtered white noise. Its power spectral density (PSD) function can expressed as [41]

\[ \psi(\omega) = \frac{\sigma^2}{\pi} \frac{av}{\omega^2 + a^2 \nu^2} \]  
(68)

Here \( v \) is the speed of the vehicle, \( \omega \) denotes the circular frequency, \( \sigma^2 \) represents the variance of road roughness, and \( a \) stands for a constant. Values of \( \sigma \) and \( a \) are selected by using the road roughness type. The disturbance \( p_r(t) \) corresponding to the roughness of the
road is expressed by using the PSD function in Eq. (68). It is constructed by using a linear filter whose dynamics expressed below and a white noise process [41]

$$\dot{p}_r(t) = av_p(t) + w(t)$$  \hspace{1cm} (69)

Here \( w(t) \) denotes a Gaussian white noise process and its intensity is \( 2\sigma_{av} \). Solution of the above differential equation gives the following equation

$$p_r(t) = e^{at} p_r(0) + \int_0^t e^{a(t-\tau)} w(\tau) d\tau$$  \hspace{1cm} (70)

When the following condition satisfied

$$a^2 v^2 \ll \omega^2$$  \hspace{1cm} (71)

then Eqs. (68) and (69) take the following forms

$$\psi(\omega) = \frac{\sigma^2}{\pi} \frac{av}{\omega^2}$$  \hspace{1cm} (72)

$$\dot{p}_r(t) = w(t)$$  \hspace{1cm} (73)

It refers to the integrated white noise as the road roughness profile.

A hump represented by a sinusoidal function may be expressed as follows

$$p_r(t) = \begin{cases} 
\frac{h}{2} \left[ 1 - \cos \left( \frac{2\pi v}{D} \right) (t - t_0) \right] & \text{if } t_0 \leq t \leq t_r, \\
0 & \text{otherwise}
\end{cases}$$  \hspace{1cm} (74)

where \( h \) denotes the hump height, \( D \) represents the hump width, \( t_0 \) denotes the time where the hump starts and \( t_r \) stands for the time where the hump finishes for the tire motion. Therefore, \( t_r \) may be represented by using sinusoidal hump parameters \( t_0, D, \) and \( v \) as given below

$$t_r = t_0 + \frac{D}{v}$$  \hspace{1cm} (75)

The sinusoidal hump may be represented as in Figure 2.
5. Case Studies

Equivalent model of a quarter vehicle SS of Ford Fiesta Mk2 is utilized as an example and simulation results of the developed control methods utilizing SDRE and ASRE approaches are checked against the performance requirements and corresponding PSS in which no control to show the effectiveness of the both control methods. Moreover, effectivenesses of the both control methods are compared. The subsequent performance parameters are used to test the effectiveness of the both methods [42]:

1) Suspension deflection steady-state error should be kept as closely as to zero by the control methods.
2) Suspension deflection maximum value should be below 0.1 m.
3) Actuator force maximum value should be below vehicle static weight.
4) Dynamic tire load should be less than or equal to vehicle static weight for good road holding.
5) Sprung mass acceleration maximum value should be below 4.5 m/s$^2$ for good ride comfort [43].

Equivalent quarter vehicle Ford Fiesta Mk2 SS numerical values are as in the followings [40]. Sprung mass, $m_s$, is 216.75 kg and unsprung mass, $m_u$, is 28.85 kg. Coefficients of the nonlinear spring between sprung and unsprung masses, $k_s$ and $k_{ns}$, are 21700 N/m and 2170 N/m, respectively. Coefficients of the nonlinear damper between sprung and unsprung masses, $b_s$ and $b_{ns}$, are 1200 Ns/m and 120 Ns/m, respectively. Coefficient of the tire stiffness in vertical direction, $k_t$, is 184000 N/m.
For the sinusoidal hump as a disturbance, the values parameters are as follows: Height of the hump, $h$, is 0.1 m, width of the hump, $D$, is 0.5 m, time where the hump starts, $t_0$, is 0 s.

The road roughness as disturbance is considered as stationary Wiener process. Its derivative is taken as a white noise having intensity of $2\sigma_{av}$. The values of the parameters for the roughness of the road are as follows: $a$ is taken as 0.15 m$^{-1}$, roughness of the road variance is taken as $9\times10^{-6}$ m$^2$ by considering an asphalt road.

The equilibrium positions of sprung and unsprung masses are used for measurements. Euler’s method is applied for numerical integration. MATLAB® is used for simulations. In simulations, sampling interval of time is selected as 0.001 s. Trial and error methods are used for the selection of the weighting coefficients. After a few trials, the proper weighting coefficients are found as $\rho_1 = 1200$, $\rho_2 = 58$, $\rho_3 = 1200$, and $\rho_4 = 58$. Endpoint matrix is considered to be as $F = 0$.

To test the performances of the both control methods and to compare the performances of the both control methods, three different vehicle speeds are taken into consideration. Performance parameters are fulfilled in both of the control methods for each vehicle speed. Performance parameters of PSS in which no control are also calculated to compare those of ASS performance parameters. Suspension deflection steady-state errors are close to zero, suspension deflection maximum values are below $31\times10^{-3}$ m, tire deflection maximum values are below $5.4\times10^{-3}$ m, and sprung mass acceleration maximum values are below 4.1 m/s$^2$ in both of the control methods for each vehicle speed. In addition to these, actuator force maximum values are below vehicle static weight and dynamic tire loads are below vehicle static weight in both of the control methods for each vehicle speed. Performance parameters
of the SS based on maximum (peak) and root-mean-square (RMS) values of the variables are
given in Table 1 for each vehicle speed.

Here, plots of the simulation results are given for 20 m/s vehicle speed. First, plots of the
values of the variables obtained by applying the control method utilizing SDRE technique are
given. They are compared with the plots of the values of the variables of the equivalent PSS.
Variation of the suspension deflection against time is given in Figure 3. Variation of the tire
deflection against time is depicted in Figure 4. Variation of the sprung mass acceleration
against time is given in Figure 5. Variation of the control force is depicted in Figure 6.
Moreover, as a frequency domain analysis, magnitudes of suspension deflection, tire
deflection, and sprung mass acceleration against frequency are given in Figures 7, 8, and 9,
respectively. In frequency domain analysis, first, power spectral densities of each variable are
calculated. Then, by taking the common logarithm of the power spectral densities of each
variable, magnitudes of each variable are calculated.

Second, plots of the values of the variables obtained by applying the control method utilizing
ASRE technique are given. They are compared with the plots of the values of the variables of
the equivalent PSS. Variation of the suspension deflection against time is given in Figure 10.
Variation of the tire deflection against time is depicted in Figure 11. Variation of the sprung
mass acceleration against time is given in Figure 12. Variation of the control force is depicted
in Figure 13. Moreover, as a frequency domain analysis, magnitudes of suspension
deflection, tire deflection, and sprung mass acceleration against frequency are given in
Figures 14, 15, and 16, respectively.
The case studies reveal that satisfactory performances are obtained for ASS by applying the developed control methods that utilize SDRE and ASRE techniques. Moreover, simulation results of both control methods are close to each other. Simulations obtained by using the developed control methods are checked against the performance requirements and compared with the simulation of corresponding PSS. After crossing the sinusoidal hump, oscillation of ASS suspension deflection and tire deflection are suppressed and they come back to their equilibrium positions in a very short time by using both techniques when compared to the equivalent passive system as seen in Figures 3, 4 and 10, 11. The suspension deflection steady state values are below $1.5 \times 10^{-3}$ m and the tire deflection steady state values are below $1.2 \times 10^{-3}$ m after the ASS settles down to its equilibrium state by means of control. The maximum values of the acceleration of the sprung mass and control force are observed while the vehicle crossing the sinusoidal hump. The maximum values of the acceleration of the sprung mass and control force are in the order of $3 \text{ m/s}^2$ and 900 N, respectively.

As a 2 DOF system, quarter vehicle SS has two undamped natural frequencies. For an LTI quarter vehicle SS, when the suspension spring constant is much less than tire spring constant, the undamped natural frequencies are approximated as $\omega_{n1} = \sqrt{\frac{k_s}{m_s}}$ and $\omega_{n2} = \sqrt{\frac{k_t}{m_u}}$. Here, $\omega_{n1}$ is less than $\omega_{n2}$ and they refer to the sprung and unsprung masses undamped natural frequencies, respectively [44]. For the quarter vehicle SS having nonlinear spring and damper used in this study, the two undamped natural frequencies can be seen in PSS frequency domain analysis simulations given in Figures 7-9 and 14-16.

If the equation of a SS vibration problem is independent of the passive and active suspension forces, it is called as invariant equation. Invariant points refer to the frequencies where ASS transfer function is the same as PSS transfer function irrespective of the actuator forces of
ASS. For an LTI quarter vehicle SS, if the suspension spring constant is much less than tire spring constant, sprung mass acceleration transfer function has an invariant point when the frequency is equal to the unsprung mass natural frequency \( \omega_{inv1} = \sqrt{k_t/m_u} \) and suspension deflection transfer function has an invariant point when the frequency is equal to \( \omega_{inv2} = \sqrt{k_t/(m_s + m_u)} \) [44]. For the nonlinear quarter vehicle SS used in this study, the corresponding invariant points can be seen in Figures 9 and 16 and Figures 7 and 14.

Detectability of the system is satisfied by using a positive definite \( Q \). Stabilizability of the system is guaranteed by inspecting the controllability matrix at each step.

6. Conclusions

In this work, two methods which are computationally simple and easy to apply are developed by utilizing SDRE and ASRE approaches to control ASS in the presence of nonlinear spring and damper. Additionally, effectivenesses of the both control methods developed by utilizing two recently introduced SDRE and ASRE techniques are compared.

The nonlinear dynamics of the vehicle SS is expressed as first order differential equations in terms of conveniently selected state variables for better control performance. Additionally, a convenient representation of a cost function in terms of state variables is obtained to realize better control.

In both of the control methods, nonlinear dynamics is expressed in the form of a SDC and it is a linear like structure. Different SDC forms and weighting constants may be considered as design flexibility. It means that they may be used to improve the performances of the control
techniques. Because the nonlinear equations are represented by linear equations, the linear controllers can be used in both of the methods.

In SDRE method, a nonlinear two point boundary value problem isn’t needed to be solved, therefore, the technique is simple if the time for computation is taken into consideration. SDC structure should be selected in such a way that \( \{C(y),A(y)\} \) must be pointwise detectable and \( \{A(y),B(y)\} \) must be pointwise stabilizable.

In ASRE method, the nonlinear system can be represented by using successive LTV equations. This technique can be used for systems having nonlinear dynamics that satisfy local Lipschitz requirement which is very mild.

SDRE method determines the optimal feedback control online while ASRE method determines the optimal feedback control offline since it uses an iteration procedure. Therefore, when the reference motion is not known in advance, SDRE method is ideal for real time operations. Both of the methods give effective design schemes only solving ARE and then forming nonlinear optimal control input. These schemes are obtained by extending linear quadratic optimal control method to nonlinear system. The methods are computationally simple and overcome the many of the shortcomings and difficulties of linear quadratic optimal control method.

The case studies reveal that satisfactory performances are obtained for ASS by applying the developed control methods that utilize SDRE and ASRE techniques. Moreover, simulation results of both control methods are close to each other. Simulations obtained by using the developed control methods are checked against the performance requirements and compared
with the simulation of corresponding PSS to show the effectivenesses of the both control methods. Performance requirements are fulfilled by using both control methods. After crossing the sinusoidal hump, oscillation of ASS sprung mass is suppressed and it comes back to its equilibrium position in a very short time by using both techniques when compared to the equivalent passive system. The maximum values of the acceleration of the sprung mass and control force are monitored while the vehicle passing the sinusoidal bump.

Nomenclature

\[ a \quad \text{Constant} \]

\[ A(y) \quad \text{SDC matrix} \]

\[ \text{ASRE} \quad \text{Approximating sequence of Riccati equation} \]

\[ \text{ASS} \quad \text{Active suspension system} \]

\[ B(y) \quad \text{SDC matrix} \]

\[ b_x, b_{ns} \quad \text{Nonlinear damper constants} \]

\[ D \quad \text{Hump width} \]

\[ \text{DOF} \quad \text{Degree of freedom} \]

\[ f(y) \quad \text{Nonlinear function} \]

\[ F(\{y(t_j)\}) \quad \text{Endpoint matrix} \]

\[ F_a \quad \text{Magnitude of the driving force} \]

\[ h \quad \text{Hump height} \]

\[ J \quad \text{Cost function} \]

\[ \text{LTI} \quad \text{Linear time invariant} \]

\[ \text{LTV} \quad \text{Linear time varying} \]
\( m_s \)  Mass of the vehicle body (sprung mass)

\( m_u \)  Masses of the tire and axles (unsprung mass)

\( k_s, k_{ns} \)  Nonlinear spring constants

\( k_t \)  Spring coefficient of the stiffness of the tire

\( p_r \)  Road disturbance at position level

\( p_s \)  Position of the sprung mass

\( p_u \)  Position of the unsprung mass

\( P(y) \)  Control weighting matrix

PSD  Power spectral density

PSS  Passive suspension system

\( Q(y) \)  State weighting matrix

RMS  Root-mean-square

SDC  State dependent coefficient

SDRE  State dependent Riccati equation

SS  Suspension system

\( t_0 \)  Time where the hump starts

\( t_r \)  Time where the hump finishes

\( u(t) \)  Vector of inputs

\( v \)  Speed of the vehicle

\( w(t) \)  Gaussian white noise process

\( y(t) \)  Vector of states

\( y_1 \)  Suspension deflection (rattle space)

\( y_2 \)  Sprung mass absolute velocity
$y_3$  Tire deflection
$y_4$  Unsprung mass absolute velocity
$\gamma$  Constant
$\lambda$  Dummy variable for integration
$\lambda_i()$  Eigenvalues of a matrix
$\sigma^2$  Variance of road roughness
$\Phi^{[k-1]}(t, t_0)$  Transition matrix of $A(y^{[k-1]}(t))$
$\psi(\omega)$  PSD function
$\omega$  Circular frequency
$\omega_{inv1}$  Sprung mass acceleration transfer function invariant point frequency
$\omega_{inv2}$  Suspension deflection transfer function invariant frequency
$\omega_{n1}$  Sprung mass undamped natural frequency
$\omega_{n2}$  Unsprung mass undamped natural frequency

References


List of Captions

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Figure 10. $p_s - p_u$.
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Figure 12. $\ddot{p}_r$.
Figure 13. $u$.
Figure 14. Magnitude of $p_s - p_u$.
Figure 15. Magnitude of $p_u - p_r$.
Figure 16. Magnitude of $\ddot{p}_s$.

Table 1. Performance parameters of vehicle SS for three different vehicle speeds.
Figure 1.

Figure 2.
Figure 3.

Figure 4.
Figure 5.

Figure 6.
Figure 7.

Figure 8.
Figure 9.

Figure 10.
Figure 11.

Figure 12.
Figure 13.

Figure 14.
### Table 1.

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