Scientia Iranica D (2023) 30(1), 228–236

SCIENTIA IRANICA

Sharif University of Technology
Scientia Iranica
Transactions D: Computer Science & Engineering and Electrical Engineering
http://scientiairanica.sharif.edu

Reducing the number of elements of linear antenna arrays using Fourier's coefficients equating method

M. Khalaj-Amirhosseini*

School of Electrical Engineering, Iran University of Science and Technology, Tehran, Iran.

Received 1 March 2021; received in revised form 30 May 2021; accepted 23 August 2021

KEYWORDS
Nonuniformly spaced arrays;
Uniformly spaced arrays;
Fourier's coefficients equating method;
Reduced number of elements arrays;
Synthesis of antenna arrays.

Abstract. An analytic method is proposed in this study to reduce the number of elements of Uniformly Spaced Antenna Arrays (USAAs). To this end, both excitations and positions of a Nonuniformly Spaced Antenna Array (NSAA) are obtained by equating Fourier's coefficients of array factors of NSAA to those of predesigned USAA in two steps. In the first step, the array is considered a USAA and excitations of elements are determined. In the second step, the position of elements is determined for the excitations obtained in the first step. These two steps are repeated several times to increase the accuracy. In fact, this method is the extension of Fourier's Coefficients Equating (FCE) method previously introduced to design NSAAs. The effectiveness of the presented method for both pencil beam and shaped beam patterns is verified by some comprehensive examples.

1. Introduction

Reducing the number of elements of antenna arrays with a desired beam pattern is important in some applications such as satellite communication to decrease the profile, weight, and cost and to promote the simplicity of antenna systems.

Usually, antenna arrays are designed so that the spaces between all elements are the same. These ordinary types of arrays are called Uniformly Spaced Antenna Arrays (USAAs) [1–2]. To reduce the number of elements while keeping the array factor unchanged, the spaces between the elements should be different. The special types of arrays created in this way are called Nonuniformly Spaced Antenna Arrays (NSAAs) [3–22].

Finding the positions of NSAA elements is an important matter. Several methods have been proposed for the synthesis of NSAAs so far. These methods can be divided into two main groups. The first group includes various optimization procedures such as Genetic Algorithm (GA), Dynamic Programming (DP), Particle Swarm Optimization (PSO), and Differential Evolution Algorithm (DEA) [3–11]. The second group includes various analytical procedures such as perturbation, integral techniques, probabilistic approaches, Matrix Pencil Method (MPM), and iterations methods [12–29]. In [20], the zeros of array factors of NSAAs were equated to the zeros of the exact-Chebyshev pattern to reach a near-Chebyshev pattern, called Zeros Matching Method (ZMM). In [21], ZMM was generalized to design arbitrary patterns. In [22], the periodicity of array factors was used and the Fourier's coefficients of NSAA was equated to those of USAA, called Fourier's Coefficients Equating (FCE) method.

NSAAs can be divided into two types. The first kind of NSAAs is designed so that their elements are excited uniformly [18–23]. This kind of NSAAs has the advantage that does not need complex circuits, providing nonuniform amplitude distribution. The second

*. *E-mail address: khalaja@iust.ac.ir*

doi: 10.24200/sci.2021.57801.5423

kind of NSAAs is designed so that their elements can be excited nonuniformly [24–29]. This kind of NSAAs enjoys the capability to reduce the number of elements.

This paper proposes the previously presented FCE method to design NSAAs in order to reduce the number of elements of predesigned USAAs. After designing a USAA with the desired array factor, an NSAA with a fewer elements is synthesized by FCE method. This method is used to find not only the positions of elements but also the excitations of them. In fact, the application of FCE method is extended from designing the first type of NSAAs (uniformly excited) to the second type of NSAAs (nonuniformly excited). In this method, both excitations and positions of the elements are obtained using FCE method [22], which are separately considered in two steps. In the first step, the array is considered to be USAA and excitations of elements are determined. In the second step, the positions of elements are determined for the excitations obtained in the first step. The presented method uses an analytic procedure different from optimization methods. Finally, some thorough examples are presented to demonstrate the effectiveness of the presented method to reduce the number of elements of previously design USAAs.

The paper is organized as follows. In Section 2, the array factors of USAAs and NSAAs are reviewed. In Section 3, the FCE method is applied to design NSAAs with the reduced number of elements. In Section 4, matrix equations are obtained for FCE method. In Section 5, some examples are provided to indicate the effectiveness of the presented method.

2. USAA and NSAA arrays

Without loss of generality, here, we discuss only arrays containing elements in odd numbers. Figure 1(a) shows the USAAs having $L_0 = 2N_0 + 1$ elements of equal distance d_0. Figure 1(b) shows nonuniformly spaced antenna arrays (NSAA) having $L = 2N + 1$ elements. The number of elements of NSAAs is presumed to be less than that of USSAs, i.e., $L < L_0$.

The position of the nth element ($n = -N, ..., -2, -1, 0, 1, 2, ..., N$) is as follows:

$$x_n = (n + e_n)d, \tag{1}$$

where e_n is defined as the nth deviation, assuming $e_0 = 0$. Also, d is the nearly average distance in NSAAs which is obtained in the following based on having the same length as USAAs.

$$d = \frac{L_0 - 1}{L - 1} d_0 < \lambda. \tag{2}$$

The average distance d should be less than a wavelength to prevent the presence of grating lobes.

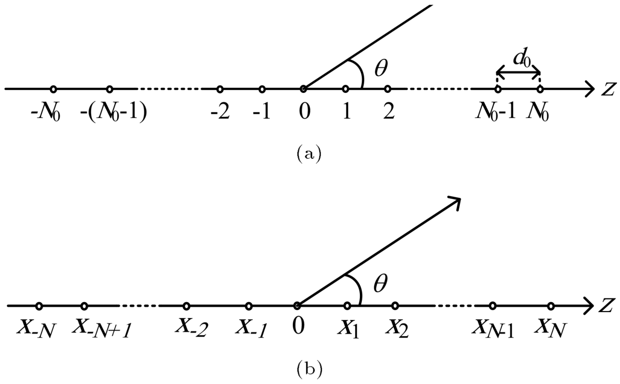

Figure 1. (a) Typical configuration of USAAs of $L_0 = 2N_0 + 1$ antennas of distance d_0. (b) Typical configuration of NSAAs of $L = 2N + 1$ antennas of unequal distances.

The array factors of USAAs and NSAAs are respectively written as follows:

$$F_0(\theta) = \sum_{n=-N_0}^{N_0} I_n \exp(jknd_0 \cos\theta), \tag{3}$$

$$F(\theta) = \sum_{n=-N}^{N} C_n \exp(jkx_n \cos\theta), \tag{4}$$

where $k = 2\pi/\lambda$ is the wavenumber and I_n and C_n are the excitation currents of USAA and NSAA, respectively.

3. Application of FCE method

Both functions $F_0(\theta)$ and $F(\theta)$ in Eqs. (3) and (4) are periodic and even with respect to the angle θ. Thus, they can be represented in the form of Fourier's series as follows [30]:

$$F(\theta) = \sum_{m=0}^{\infty} F_m \cos(m\theta), \tag{5}$$

where $F_m s$ are the Fourier's coefficients given by:

$$F_m = \frac{\varepsilon_m}{2\pi} \int_0^{2\pi} F(\theta) \cos(m\theta) d\theta, \tag{6}$$

in which ε_m is defined as one or two for $m = 0$ and $m \neq 0$, respectively.

Substituting functions $F_0(\theta)$ and $F(\theta)$ into Eq. (6) and considering the following well-known identity [30], we have:

$$\int_0^{2\pi} \exp(jx\cos(\theta)) \cos(m\theta) d\theta = 2\pi j^m j_m(x). \tag{7}$$

The Fourier's coefficients of USAA and NSAA are respectively determined as follows:

$$F_m^{USAA} = \varepsilon_m j^m \sum_{n=-N_0}^{N_0} I_n J_m(knd_0), \tag{8}$$

$$\begin{aligned} F_m^{NSAA} &= \varepsilon_m j^m \sum_{n=-N}^{N} C_n J_m(kx_n) \\ &\cong \varepsilon_m j^m \sum_{n=-N}^{N} C_n J_m(knd) \\ &+ \varepsilon_m j^m \sum_{\substack{n=-N \\ n\neq 0}} C_n kd J_m'(knd) e_n \\ &= \varepsilon_m j^m \sum_{n=-N}^{N} C_n J_m(knd) \\ &+ \varepsilon_m j^m (kd) \sum_{\substack{n=-N \\ n\neq 0}}^{N} C_n \Bigg(\frac{m J_m(knd)}{knd} \\ &- J_{m+1}(knd) \Bigg) e_n. \end{aligned} \tag{9}$$

In Eq. (9), $J_m'(x)$ is the derivative Bessel function and can be represented below [30]:

$$J_\nu'(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x). \tag{10}$$

In Fourier's Coefficients Equating (FCE) method, the Fourier's coefficients of NSAA are equated to those of USAA [22]. Therefore, Eqs. (8) and (9) must be equal to each other. This equating gives us the following equations:

$$\begin{aligned} &k_0 d \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(\frac{m J_m(knd)}{knd} - J_{m+1}(knd) \right) e_n \\ &\quad \cong \sum_{n=-N_0}^{N_0} I_n J_m(knd_0) - \sum_{n=-N}^{N} C_n J_m(knd). \end{aligned} \tag{11}$$

Eq. (11) holds for $m = 0$ to M which is the upper bound of m in truncated series in Eq. (5). The parameter M should be as large as possible. The minimum value of this parameter must be specified in the view of Eq. (8) so that the largest term in the Mth coefficient can be less than the smallest term in the zero coefficient. It means that $|J_M(kN_0d_0)| < |J_0(kN_0d_0)|$. One can see that this condition holds only when $M > 1.3kNd_0$.

The solution obtained by Eq. (11) minimizes the following errors:

$$error1 = \sqrt{\frac{1}{M+1} \sum_{m=0}^{M} \left| F_m^{(NSAA)} - F_m^{(USAA)} \right|^2}, \tag{12}$$

$$error2 = \sqrt{\frac{1}{\pi} \int_0^\pi \left| F(\theta) - F_{desired}(\theta) \right|^2 d\theta}. \tag{13}$$

The error2 evaluates the similarity of synthesized patterns to the desired one and its integral can be calculated through discretization.

4. Finding the deviations of NSAAS

Eq. (11) makes a linear set of equations consisting of $M+1$ linear equations and $2N$ unknown variables C_n and e_n; therefore, it would be represented as a matrix equation as follows:

$$\begin{aligned} [Q]_{(M+1)\times 2N} [e]_{2N\times 1} &= [P_0]_{(M+1)\times L_0} [I]_{L_0\times 1} \\ &- [P]_{(M+1)\times L} [C]_{L\times 1}. \end{aligned} \tag{14}$$

The matrix equation (Eq. (14)) can be solved in two steps. In the first step, deviations are considered zero, i.e., $[e] = 0$, and excitations C_n are determined as follows.

$$[C] = [P]^{-1} [P_0][I], \tag{15}$$

where $[P]^{-1}$ is the pseudo inverse of matrix $[P]$. In the second step, deviations are determined as follows:

$$[e] = [Q]^{-1} ([P_0][I] - [P][C]), \tag{16}$$

where $[Q]^{-1}$ is the pseudo inverse of matrix $[Q]$.

It is worthy to note that since the matrices $[P_0]$, $[P]$, and $[Q]$ in Eqs. (15) and (16) are purely real, the deviations would be real if and only if the current excitations of USAA and $I_n s$ are being real. This condition is equivalent to that in which the pattern of USAA is being conjugately symmetric about $\theta = 90°$.

To increase the accuracy of approximations existing in Eq. (11), it is better to obtain unknown variables C_n and e_n in several iterations rather than in one iteration. For this purpose, one can use two above-mentioned steps several times alternately. Hence, the matrices $[P]$ and $[Q]$ must be changed in each iteration, accordingly. In the iteration number it, the parameter n existing in the first and third terms of Eq. (11) must be replaced with $\left(n + \sum_{s=1}^{it-1} e_n^{(s)}\right)$. Here, $e_n^{(s)}$ is the obtained deviation in the iteration number s. The final deviations would be the sum of all deviations obtained in all it iterations as follows:

$$e_n = \sum_{s=1}^{IT} e_n^{(s)}. \tag{17}$$

5. Verifying FCE method

To verify the proposed FCE method to design NSAAs, some examples are presented and discussed. They are Chebyshev, Taylor-Kaiser, raised linear, flat-top, and cosecant excitations.

1. *Some Chebyshev Excitations.* A reference USAA with $L_0 = 21$ elements and $d_0 = 0.5\lambda$ having the Chebyshev pattern of SLL $= -30$ dB is available. Its array pattern is reconstructed through an NSAA with the reduced number of elements by the proposed FCE method. The reconstructed patterns are obtained in three steps. First, another USAA with $L = 13$ elements and $d = 0.833\lambda$ is designed and its amplitudes are obtained. Second, the positions of the elements are changed. Then, these two steps are repeated until $it = 30$ iterations. Figure 2 compares these three obtained patterns with the reference one. It is seen that when both amplitudes and positions are changed alternately many times, the reconstructed pattern tends to the reference one.

Figure 3 depicts the magnitude of Fourier's coefficients of reference pattern of USAA and reconstructed pattern of NSAA. There is good agreement between the two groups of Fourier's coefficients. Since the pattern of USAA is symmetric about $\theta = 90°$, it repeats equally twice within the range $\theta = [0° - 180°]$ and, therefore, the odd harmonics are zero.

Figure 4 shows the reconstructed pattern of NSAA with three different elements $L = 11$, 13, and 15. It is seen that only the pattern of $L = 11$ is far from the reference one. It is because the average distance between the elements is $d = \lambda$, which is not less than λ, contrary to Eq. (2).

Figure 5 illustrates the defined error2 versus the number of iterations, it, for some SLLs and $L = 13$ and 15 (four first curves). It is seen that as the number of iterations, it, and the number of elements, L, increase, the error2 decreases. One can see that the variation type of the defined error1 is similar to the defined error2. Therefore, one may find the same optimum it from both defined error1 and error2.

Figure 6 shows the reconstructed patterns of NSAA with $L = 13$ elements for four SLLs equal to -20, -30, -40, and -50 dB. It is seen that as SLL decreases, the difference of the reconstructed pattern from the reference one increases slightly.

Figure 7 shows the excitation currents of reference USAA and reconstructed NSAAs with $L = 11$, 13, and 15 elements for SLL $= -30$ dB. In addition, Figure 8 shows the position deviations of reconstructed NSAAs with $L = 11$, 13, and 15 elements for SLL $= -30$, -40, and -50 dB. All deviations are real and their imaginary parts are exactly zero.

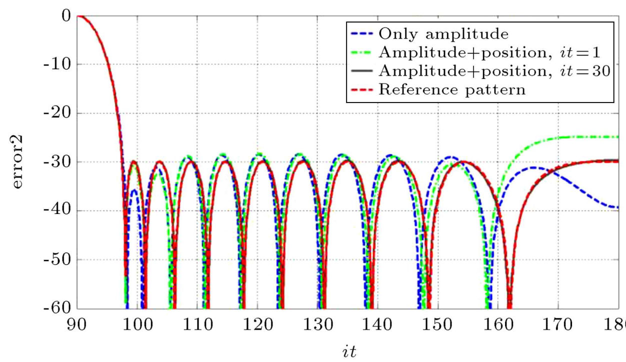

Figure 2. Reference pattern and reconstructed patterns of NSAAs with $L = 13$ elements for Chebyshev excitation of SLL $= -30$ dB.

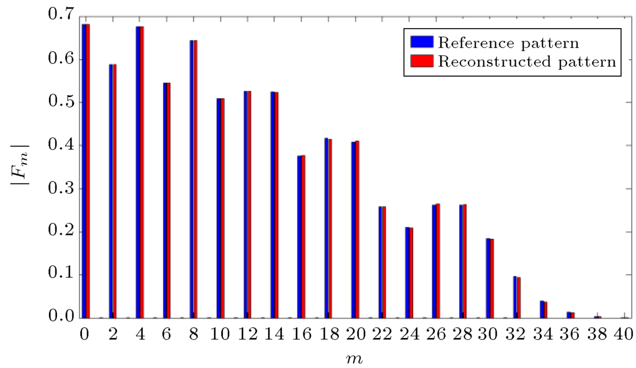

Figure 3. The magnitude of Fourier's coefficients of patterns of USAA and NSAA with $L = 13$ for Chebyshev pattern of SLL $= -30$ dB.

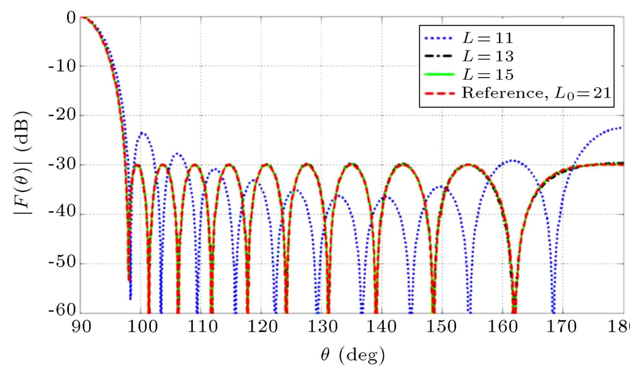

Figure 4. Reference pattern and reconstructed patterns of NSAAs with $L = 11$, 13 and 15 elements for Chebyshev excitation of SLL $= -30$ dB.

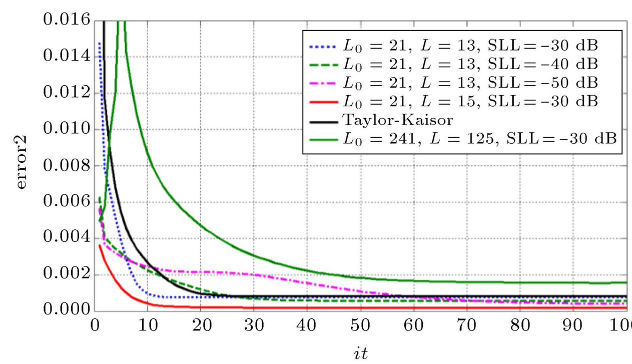

Figure 5. Defined error2 versus the number of iteration, it, for examples 1 and 2.

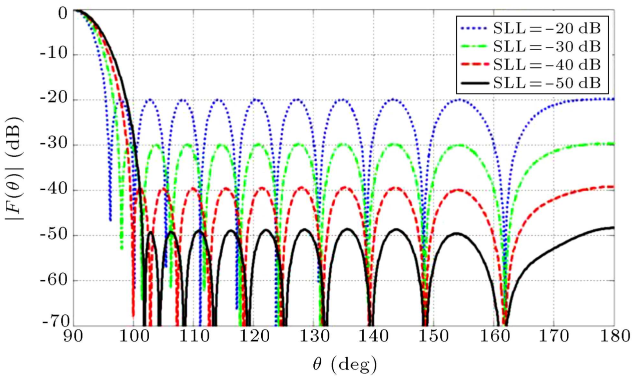

Figure 6. Patterns of reconstructed NSAAs with $L = 13$ elements for Chebyshev excitation of SLL $= -20$, -30, -40 and -50 dB.

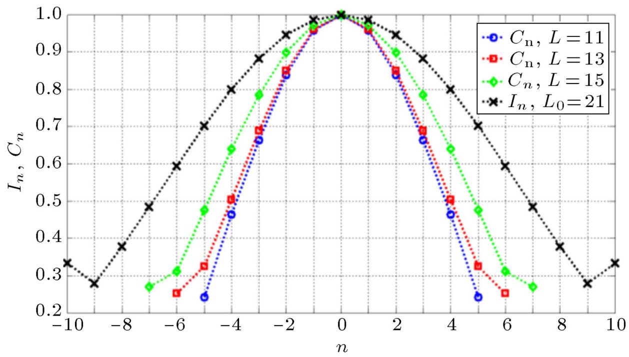

Figure 7. Excitation currents of reference USAA and reconstructed NSAAs for SLL $= -30$ dB.

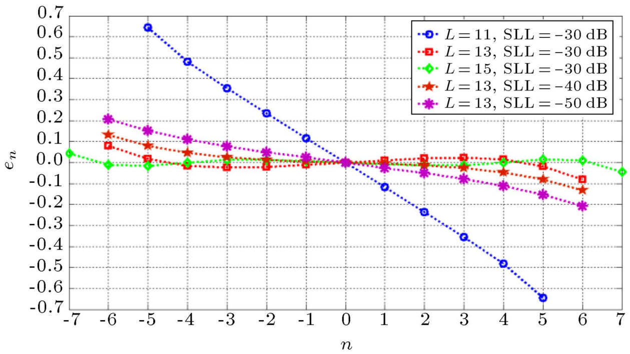

Figure 8. Deviations of reconstructed NSAAs for Chebyshev patterns.

To verify the efficiency of FCE method for electrically large arrays, a USAA with $L_0 = 241$ elements and $d_0 = 0.5\lambda$ having Chebyshev pattern of SLL $= -30$ dB and HPBW $= 0.5°$ is considered. The pattern of this array is reconstructed using an NSAA with the minimum number of elements, i.e., $L = 125$ and $d = 0.97\lambda$, by the proposed FCE method. Figure 5 depicts the defined error2 versus *it*. Figure 9 compares the reconstructed pattern obtained by $it = 100$ iteration with the reference one. There is good agreement between the two patterns, in particular their SLL and HPBW.

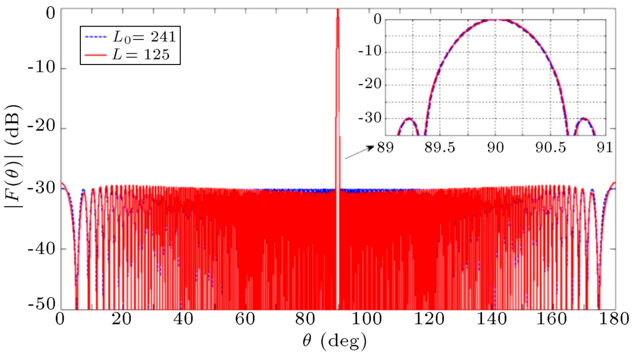

Figure 9. Patterns of electrically large USAA and NSAA for Chebyshev excitation of SLL $= -30$ dB and HPBW $= 0.5°$.

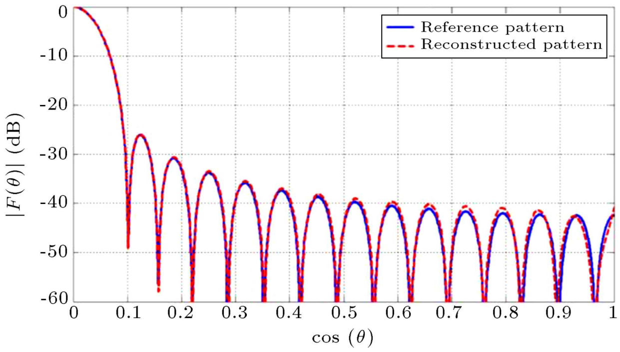

Figure 10. Reference pattern and reconstructed patterns for Taylor-Kaisor excitation of SLL $= -25$ dB.

Table 1. Comparison of elements excitations and positions.

C_n [24]	x_n/λ [24]	C_n [FCE]	x_n/λ [FCE]
1.0000	0	1.0000	0
0.9786	0.8831	0.9786	0.8822
0.9163	1.7652	0.9163	1.7634
0.8190	2.6451	0.8190	2.6422
0.6955	3.5211	0.6956	3.5169
0.5565	4.3905	0.5568	4.3850
0.4137	5.2485	0.4144	5.2417
0.2778	6.0842	0.2791	6.0768
0.1570	6.8661	0.1587	6.8603

2. *Taylor-Kaisor excitation.* A USAA with $L_0 = 29$ elements and $d_0 = 0.5\lambda$ having Taylor-Kaiser pattern with SLL $= -25$ dB was reported in [21]. This array is reconstructed with $L = 17$ elements by both matrix pencil [24,25] and the proposed FCE methods. Figure 5 depicts defined error2 versus *it*. Figure 10 compares the reconstructed pattern obtained by $it = 30$ iteration with the reference one. Table 1 indicates both excitation currents and positions of the elements obtained by FCE and matrix pencil method [24,25]. It confirms that the excitations and positions of elements found from

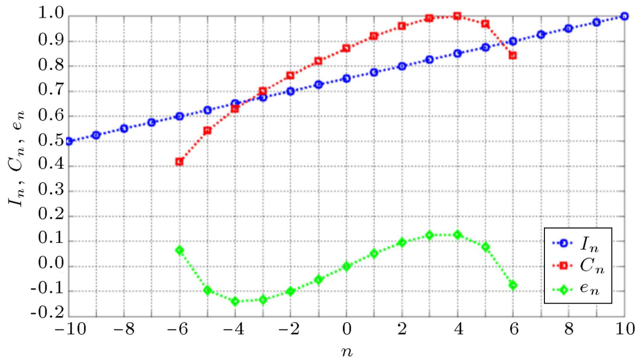

Figure 11. Excitation currents of reference USAA and excitation currents and deviations of reconstructed NSAA for raised linear.

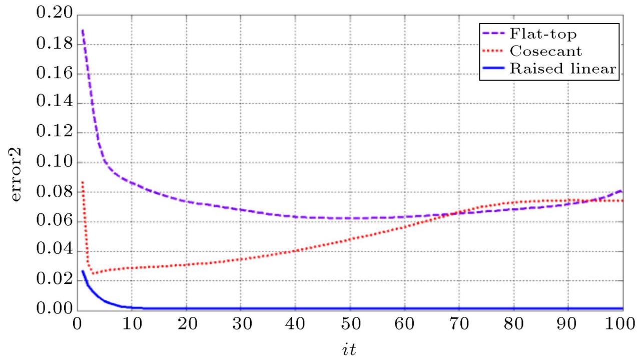

Figure 12. Defined error2 versus the number of iteration, it, for examples 3, 4 and 5.

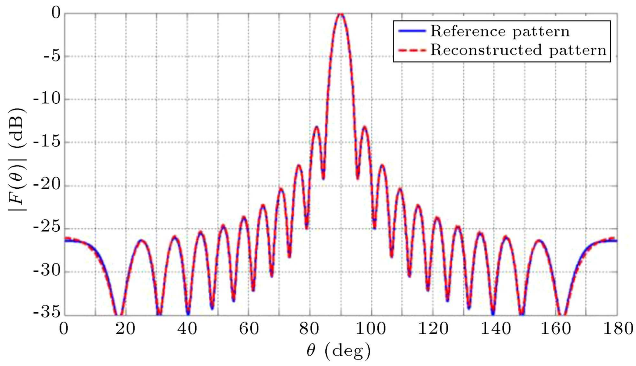

Figure 13. Amplitude of reference pattern and reconstructed patterns for raised linear excitation.

FCE method are close to those found through other methods.

3. *Raised linear excitation.* As a classic example, consider a reference USAA with $L_0 = 21$ elements and $d_0 = 0.5\lambda$ having a raised linear excitation with max/min $= 2$, as shown in Figure 11. This excitation is asymmetric, but real; therefore, the resulting pattern would be conjugately symmetric. This array is reconstructed with $L = 13$ elements by FCE method. Figure 12 depicts defined error2 versus it. Figures 13 and 14 compare the amplitude

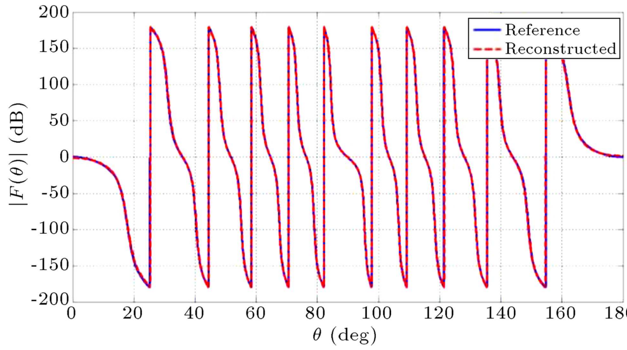

Figure 14. Phases of reference pattern and reconstructed patterns for raised linear excitation.

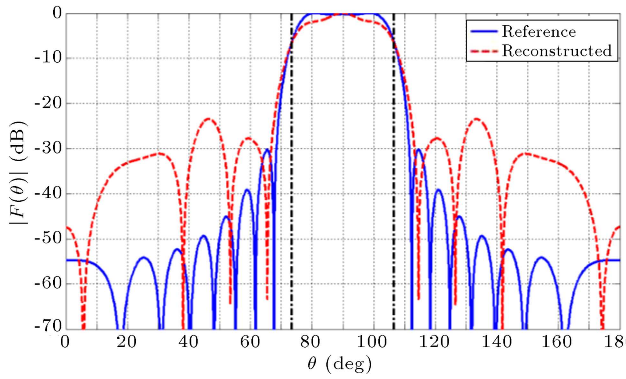

Figure 15. Reference pattern and reconstructed patterns for flat-top pattern.

and phase of the reconstructed pattern obtained by $it = 30$ iteration with those of the reference one. The agreement between the two patterns is quite clear. Figure 11 shows the excitation currents of reference USAA and reconstructed NSAA as well as the position deviations of the reconstructed NSAA.

4. *Flat-top pattern.* Consider a reference USAA with $L_0 = 21$ elements and $d_0 = 0.5\lambda$ having a shaped beam, which is a symmetric flat-top beam characterized by $\theta = 73.4°$ to $\theta = 106.6°$. The reference pattern was synthesized using sampling method [1] considering $F(73.4°) = F(106.6°) = 0.5F(90°)$. The excitations of this array are real, but are both positive and negative. This array is reconstructed with $L = 13$ elements by the FCE method. Figure 12 depicts defined error2 versus it. It is seen that the best number of iterations is $it = 50$. Figure 15 compares the reconstructed pattern obtained by $it = 50$ iterations with the reference one. There is some difference out of the flat-top region. Figure 16 shows the excitation currents of reference USAA and reconstructed NSAA as well as the position deviations of reconstructed NSAA.

5. *Cosecant pattern.* Consider a reference USAA with $L_0 = 21$ elements and $d_0 = 0.5\lambda$ having a shaped beam, which is cosecant beam, i.e., proportional

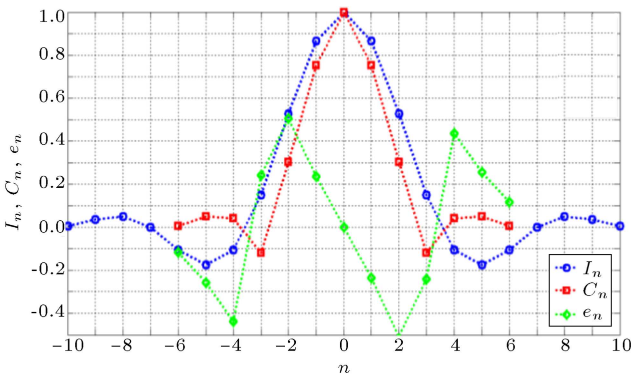

Figure 16. Excitation currents of reference USAA and excitation currents and deviations of reconstructed NSAA for flat-top pattern.

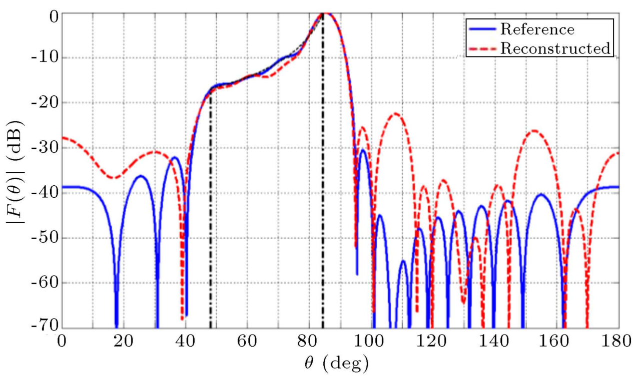

Figure 17. Reference pattern and reconstructed patterns for cosecant pattern.

to $1/sin(\pi/2-\theta)$, characterized by $\theta = 48.2°$ to $\theta = 84.5°$. The reference pattern was synthesized using sampling method [1] considering $F(90°) = 0.6F(84.5°)$.

Unlike the previous examples, the excitations of this array are not real, but are complex because their corresponding pattern is asymmetric about $\theta = 90°$. Hence, the deviations obtained from Eq. (16) could be complex and we have to keep its real part and ignore its imaginary ones. This somewhat attenuates the agreement between the reconstructed and reference patterns.

The pattern of this array is reconstructed with $L = 13$ elements by FCE method. Figure 12 depicts defined error2 versus it. It is seen that the best number of iterations is $it = 3$. Figure 17 compares the reconstructed pattern obtained by $it = 3$ iterations with the reference one. There is some difference out of the cosecant region. Figures 18 and 19 show the amplitude and phase of excitation currents of reference USAA and reconstructed NSAA, respectively. Moreover, Figure 18 shows the real part of the position deviations of reconstructed NSAA.

Figure 20 depicts the magnitude of Fourier's

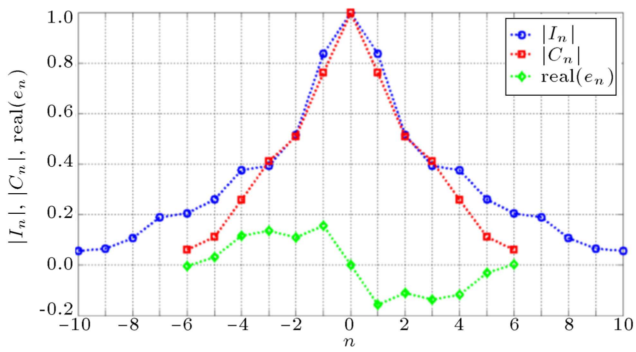

Figure 18. Amplitude of the excitation currents of reference USAA and reconstructed NSAA for cosecant pattern, and also, the real part of deviations of reconstructed NSAA.

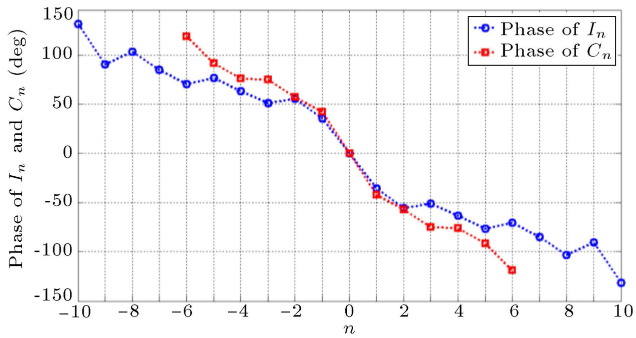

Figure 19. Phase of the excitation currents of reference USAA and reconstructed NSAA for cosecant pattern.

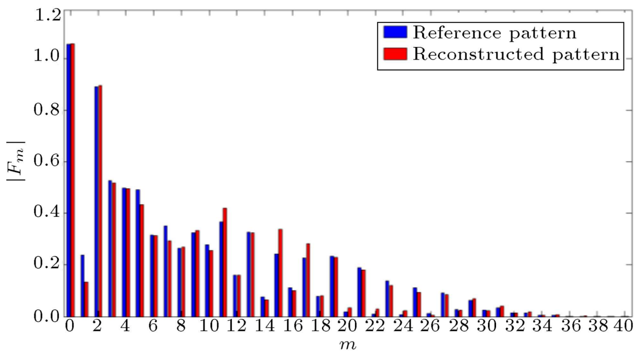

Figure 20. The magnitude of Fourier's coefficients of patterns of USAA and NSAA with $L = 13$ for cosecant pattern.

coefficients of the reference pattern of USAA and reconstructed pattern of NSAA. There is relatively good agreement between the two groups of Fourier's coefficients, despite ignoring the imaginary part of deviations. Since the pattern of USAA is asymmetric about $\theta = 90°$, both odd and even harmonics are non-zero.

6. Conclusion

Fourier's Coefficients Equating (FCE) method was employed to design NSAAs so that their elements could be fewer than those of USAAs. The Fourier's coefficients of array factors of NSAAs were equated to those of USAAs to obtain both the excitations and positions of their elements. The FCE method and an iteration approach were linked to each other so that the accuracy could be raised. Five types of array factors were given to demonstrate the effectiveness of the presented method. They were Chebyshev, Taylor-Kaisor, raised linear, flat-top, and cosecant. As a whole, the array factors of designed NSAA have good agreement with the desired ones as long as the average distance between the elements is less than a wavelength.

References

1. Balanis, C. A., *Antenna Theory: Analysis and Design*, New York: John Wiley & Sons (2005).
2. Khalaj-Amirhosseini, M. "Synthesis of linear and planar arrays with sidelobes of individually arbitrary levels", *Int. J. of RF and Microwave Computer Aided Design*, **29**(3), mmce. 21637, pp. 1–9 (2018).
3. Kurup, D.G., Himdi, M., and Rydberg, A. "Synthesis of uniform amplitude unequally spaced arrays using the differential evolution algorithm", *IEEE Trans. Antenna Propag.*, **51**(9), pp. 2210–2217 (2003).
4. Lin, C., Qing, A., and Feng, Q. "Synthesis of unequally spaced antenna arrays by using differential evolution", *IEEE Trans. Antennas Propag.*, **58**(8), pp. 2553–2558 (2010).
5. Mahmoud, K.R. "Synthesis of unequally-spaced linear array using modified central force optimization algorithm", *IET Microwaves, Antennas & Propagation*, **10**(10), pp. 1011–1021 (2016).
6. Schjaer-Jacobsen, H. and Madsen, K. "Synthesis of nonuniformly spaced arrays using a general nonlinear minimax optimization method", *IEEE Trans. Antennas Propag.*, **24**(4), pp. 501–506 (1976).
7. Oraizi, H. and Fallahpour, M. "Nonuniformly spaced linear array design for the specified beamwidth/ sidelobe level or specified irectivity/sidelobe level with coupling considerations", *Progress in Electromagnetic Research M*, **4**, pp. 185–209 (2008).
8. Asif Zaman, M. and Abdul-Matin, Md. "Nonuniformly spaced linear antenna array design using firefly algorithm", *Int. Journal of Microwave Science and Technology*, Article ID 256759, **2012**, pp. 1–8 (2012).
9. Bae, J.-H., Kim, K.-T., Lee, J.-H., et al. "Design of steerable non-uniform linear array geometry for sidelobe reduction" *Microwave and Optical Technology Lett.*, **36**(5), pp. 363–367 (2003).
10. Yuan, P., Guo, C., Ding, J., et al. "Synthesis of nonuniform sparse linear array antenna using whale optimization algorithm", *Asia-Pacific Conference on Antennas and Propagation* (APCAP), China (2017).
11. Gangwar, V.S., Singh, A.K., and Singh, S.P. "An effective approach for the synthesis of unequally spaced antenna array by estimating optimum elements density on the aperture", *IEEE Antennas and Wireless Propagation Letters*, **16**, pp. 2278–2282 (2017).
12. Harrington, R.F. "Sidelobe reduction by nonuniform element spacing", *IRE Trans. Antennas Propagation*, **9**(2), pp. 187–192 (1961).
13. Ishimaru, A. "Theory of unequally spaced arrays", *IEEE Trans. Antennas Propagat.*, **10**(6), pp. 691–702 (1962).
14. Redlich, R.W. "Iterative least squares synthesis of nonuniformly spaced linear array", *IEEE Trans. Antennas Propagation*, **21**(1), pp. 106–108 (1973).
15. Hodjat F. and Hovanessian, S.A. "Nonuniformly spaced linear and planar array antennas for sidelobe reduction", *IEEE Trans. Antennas Propagation*, **26**(2), pp. 198–204 (1978).
16. Yu, C.-C. "Sidelobe reduction of asymmetric linear array by spacing perturbation", *Electronics Letters*, **33**(9), pp. 730–732 (1997).
17. Ren, X.F., Azevedo, J.A., and Casimiro, A.M. "Synthesis of non-uniformly spaced arrays using the Fourier transform and window techniques", *IET Microwaves, Antennas & Propagation*, **3**(8), pp. 1245–1253 (2009).
18. Ishimaru, A. "Unequally spaced arrays based on the Poisson sum formula", *IEEE Trans. Antennas Propagation*, **62**(4), pp. 1549–1554 (2014).
19. You, B.Q., Cai, L.R., Zhou, J.H., et al. "Hybrid approach for the synthesis of unequally spaced array antennas with sidelobes reduction", *IEEE Antennas and Wireless Propagation Letters*, **14**, pp. 1569–1572 (2015).
20. Khalaj-Amirhosseini, M., Vecchi, G., and Pirinoli, P. "Near-Chebyshev pattern for nonuniformly spaced arrays using zeros matching method", *IEEE Trans. Antennas and Propagation*, **65**(10), pp. 5155–5161 (2017).
21. Khalaj-Amirhosseini, M. "Design of nonuniformly spaced arrays using zeros matching method", *Int. Journal of RF and Microwave Computer-Aided Engineering*, **28**(9), mmce.21490, pp. 1–5 (2018).
22. Khalaj-Amirhosseini, M. "Design of nonuniformly spaced antenna arrays using Fourier's coefficients equating method", *IEEE Trans. Antennas and Propagation*, **66**(10), pp. 5326–5332 (2018).
23. Yang, K., Zhao, Z.Q., and Liu, Q. H. "Fast pencil beam pattern synthesis of large unequally spaced antenna arrays", *IEEE Trans. Antennas and Propagation*, **61**(2), pp. 627–634 (2013).
24. Liu, Y., Nie, Z., and Liu, Q.H. "Reducing the number of elements in a linear antenna array by the matrix pencil method", *IEEE Trans. Antennas Propag.*, **56**(9), pp. 2955–2962 (2008).
25. Liu, Y., Liu, Q.H., and Nie, Z. "Reducing the number of elements in the synthesis of shaped-beam patterns by the forward-backward matrix pencil method", *IEEE Trans. Antennas Propag.*, **58**(2), pp. 604–608 (2010).

26. Alijani, M. and Neshati, M.H. "A new noniterative method for pattern synthesis of unequally spaced linear arrays", *Int. Journal of RF and Microwave Computer-Aided Engineering*, **29**(11), e21921 (2019).

27. Goudos, S.K., Boursianis, A., Siakavara, K., et al. "Novel methodology for designing antenna arrays with reduced number of elements based on differential evolution with ranking-based mutation operators", *Int. Applied Computational Electromagnetics Society Symposium* (ACES), China (2018).

28. You, P., Liu, Y., Chen, S.-L., et al. "Synthesis of unequally spaced linear antenna arrays with minimum element spacing constraint by alternating convex optimization", *IEEE Antennas and Wireless Propagation Letters*, **16**, pp. 3126–3130 (2017).

29. Shen, H. and Wang, B. "An effective method for synthesizing multiple-pattern linear arrays with a reduced number of antenna elements", *IEEE Trans. Antennas Propagation*, **65**(5), pp. 2358–2366 (2017).

30. Sokolnikoff, I.S. and Sokolnikoff, E.S. *Higher Mathematics for Engineers and Physicists*, McGraw-Hill (1941).

Biography

Mohammad Khalaj-Amirhosseini was born in Tehran, Iran in 1969. He received the BSc, MSc, and PhD degrees in Electrical Engineering from the Iran University of Science and Technology (IUST), Tehran in 1992, 1994, and 1998, respectively. He is currently a Professor with the School of Electrical Engineering, IUST. His current research interests include electromagnetics, microwaves, antennas, radio wave propagation, and electromagnetic compatibility.