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An improved MULTIMOORA method for multi-valued neutrosophic multi-criteria group decision-making based on prospect theory

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Abstract. At present, there are many subways being constructed in many cities. Constructing subways requires an appropriate scheme that can help to minimize costs while ensuring the quality of the project. This paper places great importance on introducing a Multi-Criteria Group Decision-Making (MCGDM) method for selecting an appropriate construction scheme for subways. The process of selecting the mentioned scheme is subject to high complexity due to a great deal of fuzzy and uncertain information that can be presented by Multi-Valued Neutrosophic Numbers (MVNNs). In addition, in order to handle the interaction of inputs, an Improved Generalized Multi-Valued Neutrosophic Weighted Heronian Mean (IGMVNWHM) operator is introduced. Subsequently, a new distance measure between two MVNNs is defined for deriving the objective criteria weights. Considering that decision-makers are not completely rational, we develop an improved multi-valued neutrosophic MULTIMOORA method based on prospect theory. The paper concludes by providing an example of applying the proposed method for selecting an appropriate construction scheme for a subway, and analyzing the impact of various parameters. Furthermore, a comparative analysis is conducted to demonstrate the validity and advantages of the proposed method.

1. Introduction

As quality of life has improved, more families now own cars their own cars, which has led to a significant increase in traffic congestion. Subways enjoy a number of advantages such as convenience, speediness, and punctuality. Furthermore, an appropriate construction scheme for subways can be selected using Multi-Criteria Group Decision-Making (MCGDM). There are several criteria that should be considered in this regard including technology level, environmental condition, public intervention risk, and force of supervision. Previous studies on selecting appropriate construction schemes have often used numerical values to represent criteria [1–3]. However, the uncertainty and intricacy of real decision problems require a more sophisticated approach to expressing evaluation information. Relying

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solely on crisp numerical values is no longer sufficient to convey decision information accurately.

Zadeh introduced fuzzy sets to tackle uncertainty and vagueness [4]. Since then, the field of fuzzy sets has experienced many advances [5–9]. Although fuzzy sets can handle vagueness and uncertainty through membership functions, they may not be sufficient to address more complex problems. Therefore, Atanassov presented Intuitionistic Fuzzy Sets (AIFSs) [10–12], which incorporated both membership and non-membership degrees. AIFSs are utilized in neural networks [13,14] and medical diagnoses [15]. Thereafter, they have been employed in Atanassov Interval-Valued Intuitionistic Fuzzy Sets (AIVIFSs) [16]. Nevertheless, AIFSs and AIVIFS are only capable of handling fuzzy that has a single specific number for its membership and non-membership degrees. For instance, the degree of truth of a statement may vary between decision-makers, such that one may assign a degree of 0.5 while another may assign 0.6. To resolve this problem, Hesitant Fuzzy Sets (HFS) were introduced [17–19]. Then, the generalized HFS and Dual Hesitant Fuzzy Sets (DHFSs) were produced [20,21].

Although AIFSs have undergone several improvements since their inception, they are not able to handle all types of uncertainty in real decision-making. Inconsistent and incomplete information cannot be approached by AIFSs. In some cases, decision-makers may hold a belief that a statement is true with a degree of (0.4) or false (0.3), while the expert may be unsure (0.3) [22]. To solve this problem, Neutrosophic Sets (NSs) were employed [23,24]. At first, each section of NSs lies in [0,1] [25,26]. However, it is difficult to utilize NSs in practice. Therefore, Sahin and Kucuk [27] introduced single-valued neutrosophic sets (SVNSs). Moreover, many achievements have been made in SVNSs [28–30].

However, decision-makers may be hesitant to provide every attribute value for each membership on SVNSs. For example, an expert may think that the statement is true (0.5 or 0.6), false (0.2 or 0.3), or unsure (0.3 or 0.4). Multi-Valued Neutrosophic Sets (MVNSs) can be used to handle this problem. Liu et al. [31] initially defined MVNSs and, at the same time, the Weighted Average (WA) and Weighted Geometric (WG) operators were expressed. Then, Peng et al. [32] proposed a multi-valued neutrosophic qualitative flexible (QUALIFLEX) approach for Multiple-Criteria Decision Making (MCDM). The expert group's diverse professional backgrounds can lead to varying opinions on the construction scheme, making them hesitant to provide evaluation information for each alternative. Therefore, this study utilizes MVNSs to depict the evaluation information.

The aggregation operators play an notable role in dealing with the MCDM problem and many research findings have been produced in this field. The most commonly used aggregation operators are WA operator and WG operator [33,34]. Furthermore, the interrelations of evaluation values should be taken into consideration. Several operators that are able to deal with this problem have been presented, namely Power Average (PA) operator [35] and Bonferroni Mean (BM) operator [36]. Similar to these operators, Heronian Mean (HM) operator has the same function. Numerous researchers have explored HM operators [37,38]. Liu et al. [39] brought HM operators into NSs to expand the scope of its use. Then, based on the lack of idempotency in existing HM operators, Peng et al. [40] introduced the Improved Generalized Weighted HM (IGWHM) operator and Improved Generalized Weighted Geometric HM (IGWGHM) operator. Furthermore, HM operators were extended to neutrosophic HFS [41]. To deal with some unreasonable evaluation values, Liu [42] combined power operators with HM operators. Considering the interactions between experts when selecting a construction scheme for subways, it is necessary to introduce HM operator into this MCGDM problem.

In the process of practical application, the theoretical methods related to MCGDM often encounter the same problem, and the evaluation results obtained from different evaluation methods are different. This type of problem about the robustness of decision-making analysis has attracted the attention of many scholars [43]. In order to gather the advantages of different evaluation methods, MULTIMOORA method was proposed [44]. Because the MULTIMOORA method includes three different decision-making methods, MULTIMOORA method was found more robust than MOORA method [45]. Currently, the MULTIMOORA method has been extended to cover AIFSs [46], HFS [47], and NSs [48]. It also has been applied into personnel selection [49], supplier selection [50], and quality management [51], among other areas. However, existing studies on the MULTIMOORA method rarely consider the bounded rationality of decision-makers and it has not been extended to MVNSs.

On the basis of the above analysis, the contributions of our research are listed below:

1. In order to express the assessments of decision-makers, MVNSs are often utilized. To handle the interactions of inputs, an Improved Generalized Multi-Valued Neutrosophic Weighted Heronian Mean (IGMVNWHM) operator has been developed for aggregating the evaluation matrix;
2. A new distance measure between two Multi-Valued Neutrosophic Numbers (MVNNs) is defined. Then, a distance-based method for deriving the objective criteria weights is developed;
3. This paper extends the MULTIMOORA method

to MVNSs. In addition, an improved multi-valued neutrosophic MULTIMOORA method is presented based on prospect theory (IMVN-PT-MULTIMOORA). By taking into account the fact that decision-makers are not always completely rational, the method can solve practical decision-making problems effectively.

The rest of the paper is organized below. In Section 2, some basic theories are stated. In Section 3, IGMVNWHM operator, distance measure, and IMVN-PT-MULTIMOORA method are presented. Subsequently, a solution framework for Multi-Valued Neutrosophic MCGDM (MVN-MCGDM) problem is presented in Section 4. In Section 5, illustrative example, influence of the parameter analysis, and comparative analysis are given. In Section 6, some conclusions are drawn.

2. Preliminaries

2.1. MVNSs

Definition 1 [52,53]. Let X be a space of points (objects), with a generic element in X denoted by x. MVNS A in X is:

$$A = \{x\,\langle T_A(x),\ I_A(x),\ F_A(x)\rangle\,|x \in X\,)\},$$

where T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. γ, η, and ξ represent any real values in T_A, I_A, and F_A, satisfying $0 \le \gamma, \eta, \xi \le 1$. $\#T_A$, $\#I_A$, and $\#F_A$ are the number of all elements in T_A, I_A, and F_A.

If there is just one element in X, then A can be called an MVNN, and A is represented by $\langle T_A, I_A, F_A\rangle$; if T_A, I_A, and F_A only have one value, then the MVNN will be reduced to an SVNN.

Definition 2 [54,55]. Let $A = \{x\langle T_A(x), I_A(x), F_A(x)\rangle|x \in X)\}$, and $B = \{x\langle T_B(x), I_B(x), F_B(x)\rangle\,|x \in X\,)\}$ be two MVNNs. Moreover, let $\forall x \in X$ and all values of $T_A(x)$, $I_A(x)$ and $F_A(x)$ be ranked in ascending order. γ^i, η^i, and ξ^i are the ith values in $T(x)$, $I(x)$, and $F(x)$. Then, we have: $A \le B$ if ${\gamma_A}^k \le {\gamma_B}^k$, ${\gamma_A}^{\#\mathrm{T}} \le {\gamma_B}^{\#T}$, ${\eta_A}^l \ge {\eta_B}^l$, ${\eta_A}^{\#\mathrm{I}} \ge {\eta_B}^{\#I}$, ${\xi_A}^m \ge {\xi_B}^m$, and ${\xi_A}^{\#\mathrm{F}} \ge {\xi_B}^{\#F}$, where $k = 1, 2, \cdots, \#T$, $l = 1, 2, \cdots, \#I$, $m = 1, 2, \cdots, \#F$, $\#T = \min(\#T_A(x), \#T_B(x))$, $\#I = \min(\#I_A(x), \#I_B(x))$, and $\#F = \min(\#F_A(x), \#F_B(x))$.

Definition 3. Let $A = \langle T_A, I_A, F_A\rangle$ and $B = \langle T_B, I_B, F_B\rangle$ be two MVNNs and $\lambda > 0$. The algebraic operations can be defined as follows:

1. $$A \oplus B = \left\langle \begin{array}{c} \bigcup_{\gamma_A \in T_A, \gamma_B \in T_B} \{\gamma_A + \gamma_B - \gamma_A\gamma_B\}, \\ \bigcup_{\eta_A \in I_A, \eta_B \in I_B} \{\eta_A\eta_B\}, \\ \bigcup_{\xi_A \in F_A, \xi_B \in F_B} \{\xi_A\xi_B\} \end{array} \right\rangle;$$

2. $$A \otimes B = \left\langle \begin{array}{c} \bigcup_{\gamma_A \in T_A, \gamma_B \in T_B} \{\gamma_A\gamma_B\}, \\ \bigcup_{\eta_A \in I_A, \eta_B \in I_B} \{\eta_A + \eta_B - \eta_A\eta_B\}, \\ \bigcup_{\xi_A \in F_A, \xi_B \in F_B} \{\xi_A + \xi_B - \xi_A\xi_B\} \end{array} \right\rangle$$

3. $$\lambda A = \left\langle \begin{array}{c} \bigcup_{\gamma_A \in T_A} 1 - (1 - \gamma_A)^\lambda, \\ \bigcup_{\eta_A \in I_A} {\eta_A}^\lambda, \\ \bigcup_{\xi_A \in F_A} {\xi_A}^\lambda \end{array} \right\rangle;$$

4. $$A^\lambda = \left\langle \begin{array}{c} \bigcup_{\gamma_A \in T_A} \{(\gamma_A)^\lambda\}, \\ \bigcup_{\eta_A \in I_A} \{1 - (1 - \eta_A)^\lambda\}, \\ \bigcup_{\xi_A \in F_A} \{1 - (1 - \xi_A)^\lambda\} \end{array} \right\rangle$$

2.2. HM operators

HM operators can tackle interrelationships among the aggregated arguments. In this subsection, two definitions of extended HM operators are introduced below.

Definition 4 [56,57]. Let $\theta = [0, 1]$, $s, t \ge 0$, $P^{s,t} : \theta^n \to \theta$ and then, the generalized HM operator is defined as follows:

$$GHM(z_1, z_2, \cdots, z_m) = \left(\frac{2}{m(m+1)} \sum_{h=1}^{m} \sum_{k=h}^{m} {z_h}^s {z_k}^t\right)^{\frac{1}{s+t}}. \tag{1}$$

Definition 5. Let $s, t \ge 0$, and $z_h (h = 1, 2, \cdots, m)$ be a set of nonnegative numbers. $Q = (q_1, q_2, \cdots, q_m)^T$ is the weight vector of $z_h (h = 1, 2, \cdots, m)$ and satisfies $q_h > 0$ and $\sum_{h=1}^{m} q_h = 1$. Then, IGWHM operator is defined below:

$$IGWHM^{s,t}(z_1, z_2, \cdots, z_n) = \frac{\left(\sum_{h=1}^{m} \sum_{k=h}^{m} q_h q_k {x_h}^s {x_k}^t\right)^{\frac{1}{s+t}}}{\left(\sum_{h=1}^{m} \sum_{k=h}^{m} q_h q_k\right)^{\frac{1}{s+t}}}. \tag{2}$$

2.3. The MULTIMOORA method

Let $A = \{A_1, A_2, \cdots, A_m\}$ be a collection of schemes; $C = \{C_1, C_2, \cdots, C_n\}$ be a set of criteria. $V = [v_{ij}]_{m\times n}$ represents an original evaluation matrix, where v_{ij} denotes attribute information for scheme A_i under attribute C_j. In order to facilitate comparison, it is necessary to normalize V and obtain normalized evaluation matrix $V^* = [{v_{ij}}^*]_{m\times n}$.

$${v_{ij}}^* = \frac{v_{ij}}{\sqrt{\sum_{i=1}^{m} {v_{ij}}^2}}. \tag{3}$$

2.3.1. The ratio system method

The comprehensive evaluation value of each scheme is derived from the following equation:

$$y^* = \sum_{j=1}^{g} {v_{ij}}^* - \sum_{j=g+1}^{n} {v_{ij}}^*, \tag{4}$$

where g represents the number of benefit criteria.

The best alternative can be obtained by the following formula:

$$A^* = \left\{ A_i \left| \max_i y_i{}^* \right. \right\}. \quad (5)$$

2.3.2. The reference point method

First, each optimal reference point of criterion is obtained below:

$$r_j{}^* = \begin{cases} \max\limits_i v_{ij}{}^*, & j \le g \\ \min\limits_i v_{ij}{}^*, & j > g \end{cases} \quad (6)$$

Then, the comprehensive evaluation value of each scheme is derived from the following formula:

$$z_i{}^* = \max_j \left| r_j{}^* - v_{ij}{}^* \right|, \quad (7)$$

where $|r_j{}^* - v_{ij}{}^*|$ represents deviation of each attribute value from the reference point.

Finally, the best alternative can be obtained as follows:

$$A^* = \left\{ A_i \left| \min_i z_i{}^* \right. \right\}. \quad (8)$$

2.3.3. The full multiplicative form method

The comprehensive evaluation value of each scheme is derived from the following equation:

$$u_i{}^* = \frac{\prod_{j=1}^{g} v_{ij}{}^*}{\prod_{j=g+1}^{n} v_{ij}{}^*}, \quad (9)$$

where $\prod_{j=1}^{g} v_{ij}{}^*$ and $\prod_{j=g+1}^{n} v_{ij}{}^*$ represent the products of benefit criterion and cost criterion, respectively.

Then, the best alternative can be obtained as follows:

$$A^* = \left\{ A_i \left| \max_i u_i{}^* \right. \right\}. \quad (10)$$

2.4. Prospect theory

The prospect theory was developed by modifying the theory of maximum subjective expected utility [58]. The first phase of this theory involves the processing and reference point selection. The next phase involves judging and calculating information by value function and weight function. Such a decision-making process can reflect the limited rationality of the decision-maker.

The core of prospect theory is prospect value. It is expressed below:

$$V = \sum_{i=1}^{n} \pi(p_i) v(\Delta x_i), \quad (11)$$

while $\pi(p_i)$ represents the probability weight function considering risk attitude and $v(\Delta x_i)$ indicates the value function formed by the decision-maker's subjective feelings. The probability weight function $\pi(p)$ and value function $v(\Delta x_i)$ are expressed as follows:

$$\pi(p) = \begin{cases} \dfrac{p^\gamma}{(p^\gamma + (1-p^\gamma))^{1/\gamma}}, & \Delta x \ge 0 \\ \dfrac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}, & \Delta x \le 0 \end{cases} \quad (12)$$

$$v(x) = \begin{cases} (\Delta x)^\alpha, & \Delta x \ge 0 \\ -\lambda(\Delta x)^\beta, & \Delta x \le 0 \end{cases} \quad (13)$$

while Δx represents the difference between the decision criterion value and the reference point; α and β represents risk attitude coefficients. The greater the value of $0 \le \alpha$, $\beta \le 1$, the more risk-taking decision-makers are. λ represents the loss avoidance coefficient.

3. Methodology

In this section, we propose the IGMVNWHM based on the IGWHM operator. Then, some properties about aggregation operator are presented. Secondly, a new distance measure between two MVNNs is defined. Finally, according to the prospect theory, an IMVN-PT-MULTIIMOORA method is presented for dealing with the MCGDM problem.

3.1. GMVNWHM operator

Definition 6. Let $s, t \ge 0$ and $\partial_h = \langle T_h, I_h, F_h \rangle$ $(h = 1, 2, \cdots, m)$ be a set of MVNNs with the weights $Q = (q_1, q_2, \cdots, q_m)^T$, satisfying $q_h \ge 0$ and $\sum q_h = 1$, and then the IGMVNWHM operator is defined as follows:

$$IGMVNWHM^{s,t}(\partial_1, \partial_2, \cdots, \partial_m) = \left(\frac{\underset{h=1}{\overset{m}{\oplus}} \underset{k=h}{\overset{m}{\oplus}} (q_h q_k \partial_h{}^s \partial_k{}^t)}{\sum_{h=1}^{m} \sum_{k=h}^{m} q_h q_k} \right)^{\frac{1}{s+t}}. \quad (14)$$

According to the algebraic operations in Section 2.2, the following theorems can be obtained.

Theorem 1. Let $s, t \ge 0$ and $\partial_h = \langle T_h, I_h, F_h \rangle$ $(h = 1, 2, \cdots, m)$ be a set of m elements. $Q = (q_1, q_2, \cdots, q_m)^T$ are the weights of all elements, satisfying $q_h \ge 0$ and $\sum q_h = 1$, γ_{A_h}, η_{A_h}, and ξ_{A_h}, respectively, representing all elements in T_{A_h}, I_{A_h} and F_{A_h}; and with γ_{B_h}, η_{B_h}, and ξ_{B_h} represent all elements in T_{B_h}, I_{B_h}, and F_{B_h}, respectively. Then, the value aggregated by Eq. (14) is still an MVNN, and expanded form of Eq. (15) is shown in Box I.

Proof.

$$\partial_h{}^s = \left\langle \bigcup_{\gamma_h \in T_h} \{\gamma_h{}^s\}, \bigcup_{\eta_h \in I_h} \{1 - (1 - \eta_h)^s\}, \bigcup_{\xi_h \in F_h} \{1 - (1 - \xi_h)^s\} \right\rangle,$$

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_m)=$$

$$\left\langle \begin{array}{l} \bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}\left\{\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}\left(1-{\gamma_h}^s{\gamma_k}^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}\left(1-(1-\eta_h)^s(1-\eta_k)^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}\left(1-(1-\xi_h)^s(1-\xi_k)^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\}. \end{array} \right\rangle, \tag{15}$$

Box I

$$q_1q_1{\partial_1}^s{\partial_1}^t\oplus q_1q_2{\partial_1}^s{\partial_2}^t\oplus q_2q_2{\partial_2}^s{\partial_2}^t=$$

$$\left\langle \begin{array}{l} \bigcup_{\gamma_1\in T_1,\gamma_2\in T_2}1-\left(1-{\gamma_1}^s{\gamma_1}^t\right)^{q_1q_1}\left(1-{\gamma_1}^s{\gamma_2}^t\right)^{q_1q_2}\left(1-{\gamma_2}^s{\gamma_2}^t\right)^{q_2q_2}, \\ \bigcup_{\eta_1\in I_1,\eta_2\in I_2}\left(1-(1-\eta_1)^s(1-\eta_1)^t\right)^{q_1q_1}\left(1-(1-\eta_1)^s(1-\eta_2)^t\right)^{q_1q_2}\left(1-(1-\eta_2)^s(1-\eta_2)^t\right)^{q_2q_2}, \\ \bigcup_{\xi_1\in F_1,\xi_2\in F_2}\left(1-(1-\xi_1)^s(1-\xi_1)^t\right)^{q_1q_1}\left(1-(1-\xi_1)^s(1-\xi_2)^t\right)^{q_1q_2}\left(1-(1-\xi_2)^s(1-\xi_2)^t\right)^{q_2q_2} \end{array} \right\rangle.$$

Then, we have:

$$\left(\frac{q_1q_1{\partial_1}^s{\partial_1}^t\oplus q_1q_2{\partial_1}^s{\partial_2}^t\oplus q_2q_2{\partial_2}^s{\partial_2}^t}{\sum_{h=1}^2\sum_{k=h}^2 q_hq_k}\right)^{\frac{1}{s+t}}=$$

$$\left\langle \begin{array}{l} \bigcup_{\gamma_1\in T_1,\gamma_2\in T_2}\left\{\left(1-\left(\prod_{h=1}^{2}\prod_{k=h}^{2}\left(1-{\gamma_h}^s{\gamma_k}^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\eta_1\in I_1,\eta_2\in I_2}\left\{1-\left(1-\left(\prod_{h=1}^{2}\prod_{k=h}^{2}\left(1-(1-\eta_h)^s(1-\eta_k)^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\xi_1\in F_1,\xi_2\in F_2}\left\{1-\left(1-\left(\prod_{h=1}^{2}\prod_{k=h}^{2}\left(1-(1-\xi_h)^s(1-\xi_k)^t\right)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}}\right)^{\frac{1}{s+t}}\right\} \end{array} \right\rangle.$$

Box II

$${\partial_k}^t=\left\langle\bigcup_{\gamma_k\in T_k}\{{\gamma_k}^t\},\bigcup_{\eta_k\in I_k}\{1-(1-\eta_k)^t\},\bigcup_{\xi_k\in F_k}\{1-(1-\xi_k)^t\}\right\rangle.$$

Then, we have:

$$q_hq_k{\partial_h}^s\otimes{\partial_k}^t=$$

$$\left\langle \begin{array}{l} \bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}\{1-\left(1-{\gamma_h}^s{\gamma_k}^t\right)^{q_hq_k}\}, \\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\{\left(1-(1-\eta_h)^s(1-\eta_k)^t\right)^{q_hq_k}\}, \\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\{\left(1-(1-\xi_h)^s(1-\xi_k)^t\right)^{q_hq_h}\} \end{array} \right\rangle. \tag{16}$$

Based on the above equations, the following properties could be obtained easily:

1. If $m=2$, based on Eqs. (14) and (16), we can determine that:

$$IGMVNWHM^{s,t}(\partial_1,\partial_2)$$

$$=\left(\frac{\overset{2}{\underset{h=1}{\oplus}}\overset{2}{\underset{k=h}{\oplus}}\left(q_hq_k{\partial_h}^s{\partial_k}^t\right)}{\sum_{h=1}^m\sum_{k=h}^m q_hq_k}\right)^{\frac{1}{s+t}}$$

$$=\left(\frac{q_1q_1{\partial_1}^s{\partial_1}^t\oplus q_1q_2{\partial_1}^s{\partial_2}^t\oplus q_2q_2{\partial_2}^s{\partial_2}^t}{\sum_{h=1}^2\sum_{k=h}^2 q_hq_k}\right)^{\frac{1}{s+t}},$$

and its expanded form is shown in Box II.

2. Assuming that Eq. (15) holds for $m=g$, we

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_g)=\left(\frac{1}{\sum_{h=1}^{g}\sum_{k=h}^{g}q_hq_k}\overset{g}{\underset{h=1}{\oplus}}\overset{g}{\underset{k=h}{\oplus}}(q_hq_k{\partial_h}^s\otimes{\partial_k}^t)\right)^{\frac{1}{s+t}}$$

$$\left\langle\begin{array}{c}\bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}\left\{\left(1-\left(\prod_{h=1}^{g}\prod_{k=h}^{g}(1-{\gamma_h}^s{\gamma_k}^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g}\sum_{k=h}^{g}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\left\{1-\left(1-\left(\prod_{h=1}^{g}\prod_{k=h}^{g}(1-(1-\eta_h)^s(1-\eta_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g}\sum_{k=h}^{g}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\left\{1-\left(1-\left(\prod_{h=1}^{g}\prod_{k=h}^{g}(1-(1-\xi_h)^s(1-\xi_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g}\sum_{k=h}^{g}q_hq_k}}\right)^{\frac{1}{s+t}}\right\}\end{array}\right\rangle.$$

Box III

$$\overset{g}{\underset{h=1}{\oplus}}\overset{g}{\underset{k=h}{\oplus}}(q_hq_k{\partial_h}^s\otimes{\partial_k}^t)\oplus\overset{g+1}{\underset{h=1}{\oplus}}(q_hq_{g+1}{\partial_h}^s\otimes{\partial_{g+1}}^t)$$

$$=\left\langle\begin{array}{c}\bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}(1-\prod_{h=1}^{g}\prod_{k=h}^{g}(1-{\gamma_h}^s{\gamma_k}^t)^{q_hq_k})\prod_{h=1}^{g+1}(1-{\gamma_h}^s{\gamma_{g+1}}^t)^{q_hq_{g+1}},\\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\prod_{h=1}^{g}\prod_{k=1}^{g}(1-(1-\eta_h)^s(1-\eta_k)^t)^{q_hq_k}\prod_{h}^{g+1}(1-(1-\eta_h)^s(1-\eta_{g+1})^t)^{q_hq_{g+1}},\\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\prod_{h=1}^{g}\prod_{k=1}^{g}(1-(1-\xi_h)^s(1-\xi_k)^t)^{q_hq_k}\prod_{h}^{g+1}(1-(1-\xi_h)^s(1-\xi_{g+1})^t)^{q_hq_{g+1}}\end{array}\right\rangle.$$

Further:

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_{g+1})=$$

$$\left\langle\begin{array}{c}\bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}\left\{\left(1-\left(\prod_{h=1}^{g+1}\prod_{k=i}^{g+1}(1-{\gamma_h}^s{\gamma_k}^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g+1}\sum_{k=h}^{g+1}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\left\{1-\left(1-\left(\prod_{h=1}^{g+1}\prod_{k=h}^{g+1}(1-(1-\eta_h)^s(1-\eta_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g+1}\sum_{k=h}^{g+1}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\left\{1-\left(1-\left(\prod_{h=1}^{g+1}\prod_{k=h}^{g+1}(1-(1-\xi_h)^s(1-\xi_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{g+1}\sum_{k=h}^{g+1}q_hq_k}}\right)^{\frac{1}{s+t}}\right\}\end{array}\right\rangle.$$

Box IV

can obtain the IGMVNWHM operator as shown in Box III.

When $m = g + 1$, according to Definition 3, let $A = \langle T_A, I_A, F_A\rangle$ and $B = \langle T_B, I_B, F_B\rangle$ be two MVNNs and $\lambda > 0$. The algebraic operations can be defined as follows:

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_g,\partial_{g+1})$$

$$=\left(\frac{1}{\sum_{h=1}^{g+1}\sum_{k=h}^{g+1}q_hq_k}\overset{g+1}{\underset{h=1}{\oplus}}\overset{g+1}{\underset{k=h}{\oplus}}(q_hq_k{\partial_h}^s\otimes{\partial_k}^t)\right)^{\frac{1}{s+t}}$$

$$=\left(\frac{1}{\sum_{h=1}^{g+1}\sum_{k=h}^{g+1}q_hq_k}\left(\overset{g}{\underset{h=1}{\oplus}}\overset{g}{\underset{k=h}{\oplus}}\left(q_hq_k{\partial_h}^s\otimes{\partial_k}^t\right)\oplus\overset{g+1}{\underset{h=1}{\oplus}}q_hq_{g+1}{\partial_h}^s\otimes{\partial_{g+1}}^t\right)\right)^{\frac{1}{s+t}}.$$

Then, we have the expanded formulas as shown in Box IV. Since Eq. (15) holds for $m = g + 1$, it can hold for all m's. Therefore, we can obtain the IGMVNWHM operator as shown in Box V. In addition, there are some properties of IGMVNWHM operators.

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_m)=\left(\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}\overset{m}{\underset{h=1}{\oplus}}\overset{m}{\underset{k=h}{\oplus}}(q_hq_k{\partial_h}^s\otimes{\partial_k}^t)\right)^{\frac{1}{s+t}}$$

$$\left\langle\begin{array}{c}\bigcup_{\gamma_h\in T_h,\gamma_k\in T_k}\left\{\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_h}^s{\gamma_k}^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\eta_h\in I_h,\eta_k\in I_k}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_h)^s(1-\eta_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\},\\ \bigcup_{\xi_h\in F_h,\xi_k\in F_k}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi_h)^s(1-\xi_k)^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\}\end{array}\right\rangle.$$

Box V

Theorem 2 (Monotonicity). Let $A_h=\langle T_{A_h},I_{A_h},F_{A_h}\rangle$ and $B_h=\langle T_{B_h},I_{B_h},F_{B_h}\rangle\,(h=1,2,\cdots,m)$ be two sets of MVNNs. If $A_h\le B_h$ for all h, $\gamma_{A_h}\le\gamma_{B_h}$, $\eta_{A_h}\ge\eta_{B_h}$, and $\xi_{A_h}\ge\xi_{B_h}$, then:

$$IGMVNWHM^{s,t}(A_1,A_2,\cdots,A_m)\le IGMVNWHM^{s,t}(B_1,B_2,\cdots,B_m), \quad (17)$$

where γ_{A_h}, η_{A_h}, and ξ_{A_h} represent all elements in T_{A_h}, I_{A_h}, and F_{A_h}, respectively; γ_{B_h}, η_{B_h}, and ξ_{B_h} represent all elements in T_{B_h}, I_{B_h}, and F_{B_h}, respectively.

Proof. Since $\gamma_{A_h}\le\gamma_{B_h}$ for all h and $s,t\ge 0$, we have:

$${\gamma_{A_h}}^s\cdot{\gamma_{A_k}}^t\le{\gamma_{B_h}}^s\cdot{\gamma_{B_k}}^t,$$

$$1-{\gamma_{A_h}}^s\cdot{\gamma_{A_k}}^t\ge 1-{\gamma_{B_h}}^s\cdot{\gamma_{B_k}}^t;$$

$$\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{A_h}}^s\cdot{\gamma_{A_k}}^t)^{q_hq_k}\ge\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{B_h}}^s\cdot{\gamma_{B_k}}^t)^{q_hq_k}.$$

Therefore, we have:

$$\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{A_h}}^s\cdot{\gamma_{A_k}}^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\le\left(1-(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{B_h}}^s\cdot{\gamma_{B_k}}^t)^{q_hq_k})^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}.$$

In the same way, we can proceed the proof as follows:

(i) Since $\eta_{A_h}\ge\eta_{B_h}$ for all h's and $s,t\ge 0$, we can derive the following:

$$(1-\eta_h)^s\ge(1-\eta_{B_h})^s\ ,\ (1-\eta_{A_k})^t\ge(1-\eta_{B_k})^t.$$

In addition,

$$1-(1-\eta_{A_h})^s(1-\eta_{A_k})^t\le 1-(1-\eta_{B_h})^s(1-\eta_{B_k})^t;$$

$$\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{A_h})^s(1-\eta_{A_k})^t)^{q_hq_k}\le\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{B_h})^s(1-\eta_{B_k})^t)^{q_hq_k}.$$

Therefore, we have:

$$1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{A_h})^s(1-\eta_{A_k})^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\le 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{B_h})^s(1-\eta_{B_k})^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}.$$

(ii) According to above formulas, it is not difficult to prove:

$$1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi_{A_h})^s(1-\xi_{A_k})^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\le 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi_{B_h})^s(1-\xi_{B_k})^t)^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}.$$

$$\left\langle \begin{array}{c} \bigcup_{\gamma_{A_h}\in T_{A_h},\gamma_{A_k}\in T_{A_k}} \left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{A_h}}^{s}\cdot{\gamma_{A_k}}^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \\ \bigcup_{\eta_{A_h}\in I_{A_h},\eta_{A_k}\in I_{A_k}} 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{A_h})^{s}(1-\eta_{A_k})^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \\ \bigcup_{\xi_{A_h}\in F_{A_h},\xi_{A_k}\in F_{A_k}} 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi_{A_h})^{s}(1-\xi_{A_k})^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \end{array} \right\rangle$$

$$\leq \left\langle \begin{array}{c} \bigcup_{\gamma_{B_h}\in T_{B_h},\gamma_{B_k}\in T_{B_k}} \left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-{\gamma_{B_h}}^{s}\cdot{\gamma_{B_k}}^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \\ \bigcup_{\eta_{B_h}\in I_{B_h},\eta_{B_k}\in I_{B_k}} 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta_{B_h})^{s}(1-\eta_{B_k})^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \\ \bigcup_{\xi_{B_h}\in F_{B_h},\xi_{B_k}\in F_{B_k}} 1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi_{B_h})^{s}(1-\xi_{B_k})^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}} \end{array} \right\rangle.$$

Box VI

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_m)=$$

$$\left\langle \begin{array}{c} \bigcup_{\gamma\in T}\left\{\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-\gamma^{s}\gamma^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\eta\in I}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\eta)^{s}(1-\eta)^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k=h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\}, \\ \bigcup_{\xi\in F}\left\{1-\left(1-\left(\prod_{h=1}^{m}\prod_{k=h}^{m}(1-(1-\xi)^{s}(1-\xi)^{t})^{q_hq_k}\right)^{\frac{1}{\sum_{h=1}^{m}\sum_{k-h}^{m}q_hq_k}}\right)^{\frac{1}{s+t}}\right\} \end{array} \right\rangle.$$

Box VII

Based on Item (i) and (ii), we can obtain the expanded formulas as shown in Box VI, i.e., $IGMVNWHM^{s,t}(A_1,A_2,\cdots,A_m) \leq IGMVNWHM^{s,t}(B_1,B_2,\cdots,B_m)$.

Theorem 3 (Idempotency). Let $\partial_h=\langle T_h,I_h,F_h\rangle$ $(h=1,2,\cdots,m)$ be a set of MVNNs and $\partial=\langle T,I,F\rangle$. If $\gamma_h=\gamma$, $\eta_h=\eta$, and $\xi_h=\xi$ for all h's, then:

$$IGMVNWHM^{s,t}(\partial_1,\partial_2,\cdots,\partial_m)=\partial, \tag{18}$$

where $\gamma_h,\ \eta_h,\ \xi_h$ and $\gamma,\ \eta,\ \xi$ represent all the elements in $T_h,\ I_h,\ F_h$ and $T,\ I,\ F$, respectively.

Proof. Since $\partial_h=(T,I,F)$ $(h=1,2,\cdots,m)$ and based on Eq. (15), we can derive IGMVNWHM as shown in Box VII. Then, we have:

$$IGMVNWHM^{s,t}(\sigma_1,\sigma_2,...,\sigma_m)$$

$$=\left\langle \begin{array}{c} \bigcup_{\gamma\in T}(1-(1-\gamma^{s+t}))^{\frac{1}{s+t}}, \\ \bigcup_{\eta\in I}1-(1-(1-(1-\eta)^{s+t}))^{\frac{1}{s+t}}, \\ \bigcup_{\xi\in F}1-(1-(1-(1-\xi)^{s+t}))^{\frac{1}{s+t}} \end{array} \right\rangle$$

$$=\left\langle \begin{array}{c} \bigcup_{\gamma\in T}\gamma \\ \bigcup_{\eta\in I}\eta \\ \bigcup_{\xi\in F}\xi \end{array} \right\rangle=\langle T,I,F\rangle.$$

3.2. Distance measure between two MVNNs

Definition 7. Let $C=\langle T_C,\ I_C,\ F_C\rangle$ and $E=(T_E,I_E,F_E)$ be two MVNNs; then, the distance between C and E can be obtained by the following formula as shown in Box VIII.

Theorem 4. Let $C=\langle T_C,\ I_C,\ F_C\rangle$, $D=(T_D,I_D,F_D)$, and $E=(T_E,I_E,F_E)$ be three MVNNs. The distance measure in Definition 7 satisfies the following properties:

$$d_{gd}(C,\ E) = \left(\frac{1}{6}\left[\begin{array}{l}\frac{1}{\#T_C}\sum\limits_{\gamma_C\in T_C}\min\limits_{\gamma_E\in T_E}|\gamma_C-\gamma_E|^{\mu}+\frac{1}{\#T_E}\sum\limits_{\gamma_E\in T_E}\min\limits_{\gamma_C\in T_C}|\gamma_E-\gamma_C|^{\mu}+\frac{1}{\#I_C}\sum\limits_{\eta_C\in I_C}\min\limits_{\eta_E\in I_E}|\eta_C-\eta_E|^{\mu}\\ +\frac{1}{\#I_E}\sum\limits_{\eta_E\in I_E}\min\limits_{\eta_C\in I_C}|\eta_E-\eta_C|^{\mu}+\frac{1}{\#F_C}\sum\limits_{\xi_C\in F_C}\min\limits_{\xi_E\in F_E}|\xi_C-\xi_E|^{\mu}+\frac{1}{\#F_E}\sum\limits_{\xi_E\in F_E}\min\limits_{\xi_C\in F_C}|\xi_E-\xi_C|^{\mu}\end{array}\right]\right)^{\frac{1}{\mu}}. \tag{19}$$

Box VIII

1. $d_{gd}(C,\ C) = 0$
2. $d_{gd}(C,\ D) = d_{gd}(D,\ C)$
3. If $C \leq D \leq E$, then $d_{gd}(C,\ D) \leq d_{gd}(C,\ E)$ and $d_{gd}(D,\ E) \leq d_{gd}(C,\ E)$.

Proof. Clearly, the distance measure satisfies Properties 1 and 2. The proof of Property 3 is shown below. Since $C \leq D \leq E$ and from Definition 2, we can obtain that ${\gamma_C}^k \leq {\gamma_D}^k \leq {\gamma_E}^k$, ${\gamma_C}^{\#\mathrm{T}} \leq {\gamma_D}^{\#T} \leq {\gamma_E}^{\#T}$, ${\eta_C}^l \geq {\eta_D}^l \geq {\eta_E}^l$, ${\eta_C}^{\#\mathrm{I}} \geq {\eta_D}^{\#I} \geq {\eta_E}^{\#I}$, ${\xi_C}^m \geq {\xi_D}^m \geq {\xi_E}^m$, and ${\xi_C}^{\#\mathrm{F}} \geq {\xi_D}^{\#F} \geq {\xi_E}^{\#F}$, where $k = 1,\ 2,\ \cdots,\ \#T$, $l = 1,\ 2,\ \cdots,\ \#I$, $m = 1,\ 2,\ \cdots,\ \#F$, $\#T = \min(\#T_C(x),\ \#T_D(x),\ \#T_E(x))$, $\#I = \min(\#\ I_C(x),\ \#I_D(x),\ \#I_E(x))$, and $\#F = \min(\#F_C(x),\ \#F_D(x),\ \#F_E(x))$. Subsequently, the following inequalities can be obtained:

$$\left|{\gamma_C}^k - {\gamma_D}^k\right| \leq \left|{\gamma_C}^k - {\gamma_F}^k\right|,$$

$$\left|{\eta_C}^l - {\eta_D}^l\right| \leq \left|{\eta_C}^l - {\eta_F}^l\right|,$$

$$\left|{\xi_C}^m - {\xi_D}^m\right| \leq \left|{\xi_C}^m - {\xi_F}^m\right|,$$

and:

$$\left|{\gamma_D}^k - {\gamma_C}^k\right| \leq \left|{\gamma_E}^k - {\gamma_C}^k\right|,$$

$$\left|{\eta_D}^l - {\eta_C}^l\right| \leq \left|{\eta_E}^l - {\eta_C}^l\right|,$$

$$\left|{\xi_D}^m - {\xi_C}^m\right| \leq \left|{\xi_E}^m - {\xi_C}^m\right|,$$

then,

$$\frac{1}{\#T_C}\sum_{\gamma_C\in T_C}\min_{\gamma_D\in T_D}|\gamma_C-\gamma_D|^{\mu} \leq \frac{1}{\#T_C}\sum_{\gamma_C\in T_C}\min_{\gamma_E\in T_E}|\gamma_C-\gamma_E|^{\mu},$$

$$\frac{1}{\#I_C}\sum_{\eta_C\in I_C}\min_{\eta_D\in I_D}|\eta_C-\eta_D|^{\mu} \leq \frac{1}{\#I_C}\sum_{\eta_C\in I_C}\min_{\eta_E\in I_E}|\eta_C-\eta_E|^{\mu},$$

$$\frac{1}{\#F_C}\sum_{\xi_C\in F_C}\min_{\xi_D\in F_D}|\xi_C-\xi_D|^{\mu} \leq \frac{1}{\#F_C}\sum_{\xi_C\in F_C}\min_{\xi_E\in F_E}|\xi_C-\xi_E|^{\mu},$$

$$\frac{1}{\#T_D}\sum_{\gamma_D\in T_D}\min_{\gamma_C\in T_C}|\gamma_D-\gamma_C|^{\mu} \leq \frac{1}{\#T_E}\sum_{\gamma_E\in T_E}\min_{\gamma_C\in T_C}|\gamma_E-\gamma_C|^{\mu},$$

$$\frac{1}{\#I_D}\sum_{\eta_D\in I_D}\min_{\eta_C\in I_C}|\eta_D-\eta_C|^{\mu} \leq \frac{1}{\#I_E}\sum_{\eta_E\in I_E}\min_{\eta_C\in I_C}|\eta_E-\eta_C|^{\mu},$$

$$\frac{1}{\#F_D}\sum_{\xi_D\in F_D}\min_{\xi_C\in F_C}|\xi_D-\xi_C|^{\mu} \leq \frac{1}{\#F_E}\sum_{\xi_E\in F_E}\min_{\xi_C\in F_C}|\xi_E-\xi_C|^{\mu}.$$

Thus, $d_{gd}(C,\ D) \leq d_{gd}(C,\ E)$. Similarly, we can also get $d_{gd}(D,\ E) \leq d_{gd}(C,\ E)$.

3.3. IMVN-PT-MULTIMOORA method

The traditional MULTIMOORA method assumes that decision-makers are entirely rational, which is inconsistent with the actual condition. Therefore, it is necessary to use relevant theories to solve this problem. In this section, we connect the prospect theory to MULTIMOORA method. In addition, a new method for determining weights has been applied to the multi-valued neutrosophic MULTIMOORA method.

Let $A = \{A_1,\ A_2,\ \cdots,\ A_m\}$ and $C = \{C_1, C_2, \cdots, C_n\}$ be a collection of alternatives and a set of criteria, respectively. Assume that $Q = \{q_1,\ q_2,\ \cdots,\ q_n\}$ ($q_k \in [0,\ 1]$ and $\sum_{k=1}^{n} q_k = 1$) is weight. The decision matrix is $B = [b_{hk}]_{m\times n}$, where b_{hk} is an MVNN that represents the assessment of alternative A_h ($h = 1, 2, ..., m$) with the criterion C_k

$(k = 1, 2, ..., n)$. The standardization is calculated below:

$$\tilde{b_{hk}} = \begin{cases} \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} (d(b_{hk},\ \tilde{b_k}))^\alpha, & b_{hk} > \tilde{b_k} \\ 0, & b_{hk} = \tilde{b_k} \\ -\lambda \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}} (d(b_{hk},\ \tilde{b_k}))^\beta, & b_{hk} < \tilde{b_k} \end{cases} \tag{20}$$

$$b_{hk}{}^* = \frac{\tilde{b_{hk}}}{\sqrt{\sum_{h=1}^{m} (\tilde{b_{hk}})^2}}, \tag{21}$$

where $\tilde{b_{hk}}$ denotes the prospect value, $\tilde{b_k}$ the reference point value of criterion C_k, $d(b_{hk},\ \tilde{b_k})$ the distance between evaluation value and reference point, b^*_{hk} the standardized form of $\tilde{b_{hk}}$, and $B^* = [b_{hk}{}^*]$ the standardized decision matrix.

Before we conduct the MULTIMOORA method, the weights of criteria should be obtained. Firstly, for each criterion, the optimistic and pessimistic evaluation values are represented as follows: Optimistic and pessimistic values include $B^+ = (b_1{}^+,\ b_2{}^+,\ \cdots,\ b_n{}^+)$ and $B^- = (b_1{}^-,\ b_2{}^-,\ \cdots,\ b_n{}^-)$, respectively. Then, according to the distance measure represented in Definition 7, we can get the distances between evaluation value of each criterion and optimistic/pessimistic values.

$$d_k{}^+ = \sum\nolimits_{h=1}^{m} d(b_{hk},\ b_k{}^+),$$

$$d_j{}^- = \sum\nolimits_{h=1}^{m} d(b_{hk},\ b_k{}^-). \tag{22}$$

Based on the TOPSIS method, the dispersion measure of criterion C_k can be obtained below:

$$e_k = \frac{d_k{}^+}{d_k{}^+ + d_k{}^-}. \tag{23}$$

Finally, the criterion weight can be obtained according to dispersion measure.

$$q_k = \frac{e_k}{\sum_{k=1}^{n} e_k}. \tag{24}$$

3.3.1. The IMVN-PT-ratio system method

The comprehensive evaluation value of each scheme is derived from the following equation:

$$y_h{}^* = \sum_{k=1}^{g} q_k b_{hk}{}^* - \sum_{k=g+1}^{n} q_k b_{hk}{}^*, \tag{25}$$

where g and $n-g$ represent the number of benefit and cost criterion, respectively.

The best alternative can be obtained by the following formula:

$$A_{RS}{}^* = \left\{ A_h \middle| \max_h y_h{}^* \right\}. \tag{26}$$

3.3.2. The IMVN-PT-reference point method

Firstly, each optimal reference point of criterion is obtained below:

$$r_k{}^* = \begin{cases} \max_h b_{hk}{}^*, & k \le g \\ \min_h b_{hk}{}^*, & k > g \end{cases} \tag{27}$$

Then, the comprehensive evaluation value of each scheme is derived from the following formula:

$$z_h{}^* = \max_k q_k \left| r_k{}^* - b_{hk}{}^* \right|, \tag{28}$$

Finally, the best alternative can be obtained as follows:

$$A_{RP}{}^* = \left\{ A_h \middle| \min_h z_h{}^* \right\}. \tag{29}$$

3.3.3. The IMVN-PT-full multiplicative form method

The comprehensive evaluation value of A_h is obtained as follows:

$$u_h{}^* = \frac{\prod_{k=1}^{g} (b_{hk}{}^*)^{q_k}}{\prod_{k=g+1}^{n} (b_{hk}{}^*)^{q_k}}. \tag{30}$$

The best alternative is obtained as follows:

$$A_{FMF}{}^* = \left\{ A_h \middle| \max_h u_h{}^* \right\}. \tag{31}$$

Based on the dominance theory, the final ranks can be collected from the above three parts of MULTIMOORA method.

4. Solution framework for MVN-MCGDM problem

Considering that the evaluation information is described by MVNNs, let $A = \{A_1, A_2, \cdots, A_m\}$ be the set of schemes, $D = \{D_1,\ D_2,\ \cdots,\ D_l\}$ be a set of decision-makers, and the weights of decision-makers be expressed by $\phi = (\phi_1,\ \phi_2,\ \cdots,\ \phi_l)$, $\phi_k \in [0,1]$ and $\sum_{k=1}^{l} \phi_k = 1$. The criteria are expressed by $C = \{C_1, C_2, \cdots, C_n\}$. $Q = (q_1, q_2, \cdots, q_n)$ represents the importance of the criteria, satisfying $q_k \in [0,1]$ $(k = 1, 2, \cdots, n)$ and $\sum q_k = 1$. We assume that the decision matrix is $B^r = [b_{hk}{}^r]_{m \times n}$, where $b_{hk}{}^r = \langle T_{hk}, I_{hk}, F_{hk} \rangle$ is an MVNN that represents the assessment of scheme A_h $(h = 1, 2, ..., m)$ under the criterion C_j $(k = 1, 2, ..., n)$ obtained from decision-maker D_r.

The solution framework is shown in Figure 1 and the detailed steps are stated below:

Step 1: Tidy up original data.

Gather and transform the evaluation information into MVNNs; then, normalize the decision matrix of each decision-maker based on the following equation.

$$b_{hk} = \begin{cases} \partial_{hk}, & \text{if } C_k \text{ is a benefit criteria} \\ \partial_{hk}{}^c, & \text{else} \end{cases} \tag{32}$$

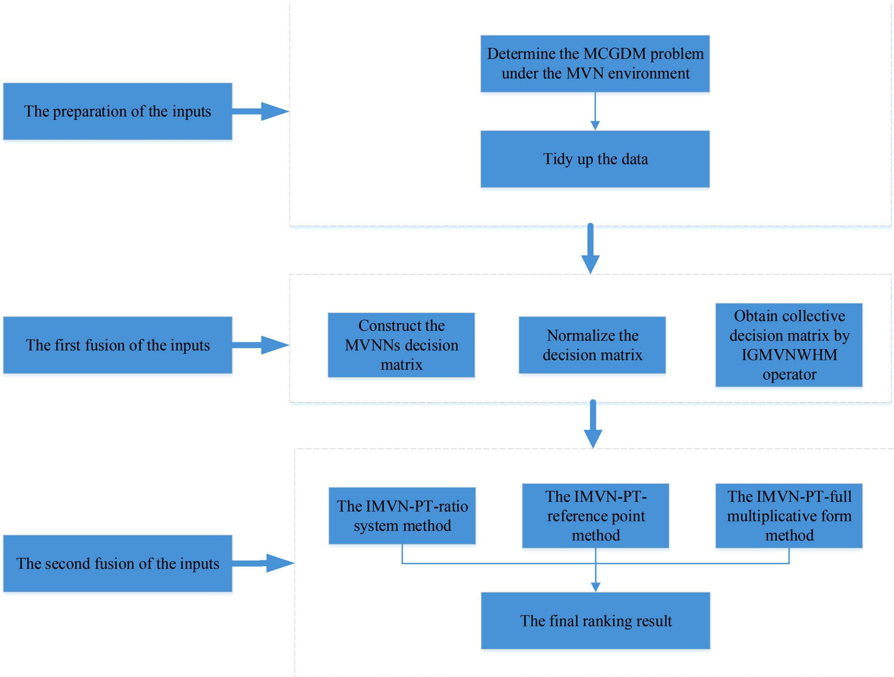

Figure 1. Solution framework for MVN-MCGDM problem.

where ∂_{hk}^c is the complement of ∂_{hk}, satisfying ${\partial_{hk}}^c = \langle F_{hk}, 1 - I_{hk}, T_{hk} \rangle$.

Step 2: Obtain the collective decision matrix by IGMVNWHM operator.

According to Eq. (14), we can aggregate decision matrix B^r of each decision-maker into a collective decision matrix $CB = [b_{hk}]_{m \times n}$.

Step 3: Measure the ranking result using IMVN-PT-MULTIMOORA method.

Step 3.1: Conduct the IMVN-PT-ratio system method.

Based on the distance measure and Eqs. (22)–(24), we can obtain the weights of criterion Q. Then, utilizing Eqs. (20) and (21), we can derive the prospect value and the reference point. Finally, the ranking result can be obtained using Eqs. (25) and (26).

Step 3.2: Conduct the IMVN-PT-reference point method.

According to Eqs. (27)–(29), the ranking result can be obtained.

Step 3.3: Conduct the IMVN-PT-full multiplicative form method.

According to Eqs. (30) and (31), the ranking result can be obtained.

Step 4: Calculate the final ranking result.

The dominance theory is employed to collect three ranking results of the MULTIMOORA method.

5. Case study

The following case applies the IMVN-PT-MULTIMOORA method to deal with the MCGDM problem of selecting an appropriate scheme for subway construction. This case demonstrates the validity and advantages of the proposed method utilizing sensitive analysis and comparative analysis.

Due to the improvements in the quality of life, coupled with the increasing number of families owning their own cars, the issue of traffic congestion is becoming more prominent. The subway experiences some advantages, such as convenience, speediness, and punctuality. When a subway system needs to be constructed, the government invites a group of experts to select one from four alternatives denoted A_1, A_2, A_3, and A_4. Considering the limited knowledge of each expert, we choose several experts to form an expert group $D = \{D_1, D_2, D_3, D_4, D_5\}$. Moreover, the chosen experts should possess not only good professional knowledge but also extensive practical experience. The information of experts is shown in Table 1. Based on the literature review [59,60] and consulting some experts in this field, the following factors need to be considered: technology level (C_1), environmental conditions (C_2), the risk of public intervention (C_3), and force of supervision (C_4).

5.1. *Steps of the proposed method*

Based on the solution framework for MVN-MCGDM problem in Section 4, we can obtain the detailed results of each step:

Table 1. Information of experts.

Experts	Education	Positional titles	Employment position	Working years
Expert 1	MSc	Engineer	Project manager	12
Expert 2	MSc	Engineer	Technical manager	15
Expert 3	PhD	Senior engineer	Economic manager	15
Expert 4	PhD	Senior engineer	Risk manager	18
Expert 5	PhD	Senior engineer	General manager	20

Table 2. Evaluation information collected from Expert 1.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8, 0.9\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.9\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.4\}, \{0.8\}, \{0.7\}\rangle$	$\langle\{0.8\}, \{0.2\}, \{0.1\}\rangle$
A_2	$\langle\{0.6, 0.7\}, \{0.3\}, \{0.3\}\rangle$	$\langle\{0.6\}, \{0.3\}, \{0.1\}\rangle$	$\langle\{0.2\}, \{0.6\}, \{0.7\}\rangle$	$\langle\{0.6\}, \{0.1\}, \{0.2\}\rangle$
A_3	$\langle\{0.7\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.8\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.2, 0.3\}, \{0.7\}, \{0.9\}\rangle$	$\langle\{0.7\}, \{0.2\}, \{0.2\}\rangle$
A_4	$\langle\{0.5\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.4, 0.5\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.2\}, \{0.7\}, \{0.6\}\rangle$	$\langle\{0.5\}, \{0.1\}, \{0.3\}\rangle$

Table 3. Evaluation information collected from Expert 2.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.8, 0.9\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.4\}, \{0.8\}, \{0.7\}\rangle$	$\langle\{0.9\}, \{0.2\}, \{0.1\}\rangle$
A_2	$\langle\{0.7\}, \{0.3\}, \{0.3\}\rangle$	$\langle\{0.6\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.2\}, \{0.6\}, \{0.6\}\rangle$	$\langle\{0.6\}, \{0.1, 0.2\}, \{0.2\}\rangle$
A_3	$\langle\{0.8\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.8\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.3\}, \{0.6\}, \{0.8\}\rangle$	$\langle\{0.7\}, \{0.2\}, \{0.1\}\rangle$
A_4	$\langle\{0.5\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.4\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.3\}, \{0.7\}, \{0.6\}\rangle$	$\langle\{0.5\}, \{0.1\}, \{0.3\}\rangle$

Table 4. Evaluation information collected from Expert 3.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.9\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.9\}, \{0.1, 0.2\}, \{0.1\}\rangle$	$\langle\{0.2\}, \{0.8\}, \{0.6, 0.7\}\rangle$	$\langle\{0.9\}, \{0.2\}, \{0.1\}\rangle$
A_2	$\langle\{0.7\}, \{0.3\}, \{0.3\}\rangle$	$\langle\{0.6, 0.7\}, \{0.3\}, \{0.1\}\rangle$	$\langle\{0.2\}, \{0.6\}, \{0.7\}\rangle$	$\langle\{0.6\}, \{0.1\}, \{0.2\}\rangle$
A_3	$\langle\{0.6, 0.7\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.7\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.3\}, \{0.7\}, \{0.9\}\rangle$	$\langle\{0.7\}, \{0.2\}, \{0.1\}\rangle$
A_4	$\langle\{0.4, 0.5\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.4\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.2\}, \{0.7\}, \{0.6\}\rangle$	$\langle\{0.5\}, \{0.1\}, \{0.2\}\rangle$

Table 5. Evaluation information collected from Expert 4.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.9\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.8\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.2\}, \{0.5\}, \{0.6\}\rangle$	$\langle\{0.8\}, \{0.2\}, \{0.1\}\rangle$
A_2	$\langle\{0.6\}, \{0.3\}, \{0.3\}\rangle$	$\langle\{0.6\}, \{0.3\}, \{0.1\}\rangle$	$\langle\{0.2, 0.3\}, \{0.6\}, \{0.7\}\rangle$	$\langle\{0.6\}, \{0.2\}, \{0.3\}\rangle$
A_3	$\langle\{0.7\}, \{0.2\}, \{0.1\}\rangle$	$\langle\{0.8\} \{0.1\} \{0.2\}\rangle$	$\langle\{0.2\}, \{0.7\}, \{0.9\}\rangle$	$\langle\{0.7, 0.8\}, \{0.2\}, \{0.2\}\rangle$
A_4	$\langle\{0.5\}, \{0.1\}, \{0.1\}\rangle$	$\langle\{0.5\}, \{0.1\}, \{0.2\}\rangle$	$\langle\{0.2\}, \{0.7\}, \{0.6\}\rangle$	$\langle\{0.4\}, \{0.1\}, \{0.3\}\rangle$

Step 1: Tidy up original data.

To prevent the personal opinions of experts from being influenced by other experts during the scheme evaluation process, all construction plans will be sent to the experts via email. At the same time, in order to make the evaluation results as accurate as possible, some background information should be provided to the expert group. Then, evaluation information is obtained from experts. Next, the evaluation information about each scheme under criterion is transformed into MVNNs, as shown in Tables 2–6. Then, each evaluation matrix provided by the expert is normalized according to Eq. (32). The normalized evaluation matrix is represented as $B^r = [b_{hk}]_{4\times 4}(h, k = 1, 2, 3, 4)$.

Step 2: Obtain the collective decision matrix by IGMVNWHM operator.

According to Eq. (14), we can aggregate evaluation matrix B^r of each decision-maker to obtain a collective evaluation matrix $CB = [b_{hk}]_{m\times n}$, where $s = t = 1$. The weight vector of experts is subjectively determined, which is represented as $\phi = (0.1,\ 0.2,\ 0.2,\ 0.25,\ 0.25)$.

Table 6. Evaluation information collected from Expert 5.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.8\},\{0.2\},\{0.1\}\rangle$	$\langle\{0.9\},\{0.1\},\{0.1\}\rangle$	$\langle\{0.2\},\{0.4\},\{0.6\}\rangle$	$\langle\{0.8,0.9\},\{0.2\},\{0.1\}\rangle$
A_2	$\langle\{0.6\},\{0.3\},\{0.3\}\rangle$	$\langle\{0.6\},\{0.3\},\{0.1\}\rangle$	$\langle\{0.2\},\{0.6\},\{0.7\}\rangle$	$\langle\{0.6\},\{0.1\},\{0.2\}\rangle$
A_3	$\langle\{0.8\},\{0.2\},\{0.1\}\rangle$	$\langle\{0.8\},\{0.2\},\{0.2\}\rangle$	$\langle\{0.2\},\{0.7\},\{0.9\}\rangle$	$\langle\{0.8\},\{0.2\},\{0.2\}\rangle$
A_4	$\langle\{0.4\},\{0.1\},\{0.1\}\rangle$	$\langle\{0.4\},\{0.1\},\{0.2\}\rangle$	$\langle\{0.2\},\{0.7\},\{0.6\}\rangle$	$\langle\{0.4,0.5\},\{0.1\},\{0.3\}\rangle$

Table 7. Collective evaluation matrix.

	C_1	C_2	C_3	C_4
A_1	$\langle\{0.850,0.860\},\{0.150\},\{0.100\}\rangle$	$\langle\{0.860,0.878\},\{0.100,0.110\},\{0.100\}\rangle$	$\langle\{0.630,0.651\},\{0.401\},\{0.287\}\rangle$	$\langle\{0.844,0.870\},\{0.200\},\{0.100\}\rangle$
A_2	$\langle\{0.641,0.651\},\{0.300\},\{0.300\}\rangle$	$\langle\{0.600,0.621\},\{0.279\},\{0.123\}\rangle$	$\langle\{0.682\},\{0.400\},\{0.2,0.224\}\rangle$	$\langle\{0.600\},\{0.122,0.141\},\{0.224\}\rangle$
A_3	$\langle\{0.732,0.748\},\{0.200\},\{0.100\}\rangle$	$\langle\{0.782\},\{0.123\},\{0.177\}\rangle$	$\langle\{0.884\},\{0.319\},\{0.237,0.246\}\rangle$	$\langle\{0.728,0.754\},\{0.200\},\{0.156\}\rangle$
A_4	$\langle\{0.456,0.475\},\{0.100\},\{0.100\}\rangle$	$\langle\{0.423,0.436\},\{0.100\},\{0.200\}\rangle$	$\langle\{0.600\},\{0.300\},\{0.218\}\rangle$	$\langle\{0.450,0.475\},\{0.100\},\{0.279\}\rangle$

The final collective evaluation matrix is presented in Table 7.

Step 3: Calculate the ranking result using IMVN-PT-MULTIMOORA method.

Step 3.1: Conduct the IMVN-PT-ratio system method.

Firstly, the optimistic values of each criterion are determined.

$$B^+ = (\langle\{0.850,0.860\},\{0.150\},\{0.100\}\rangle,$$
$$\langle\{0.860,0.878\},\{0.100,0.110\},\{0.100\}\rangle,$$
$$\langle\{0.884\},\{0.319\},\{0.237,0.246\}\rangle,$$
$$\langle\{0.844,0.870\},\{0.200\},\{0.100\}\rangle),$$

and:

$$B^- = (\langle\{0.456,0.475\},\{0.100\},\{0.100\}\rangle,$$
$$\langle\{0.423,0.436\},\{0.100\},\{0.200\}\rangle,$$
$$\langle\{0.630,0.651\},\{0.401\},\{0.287\}\rangle,$$
$$\langle\{0.450,0.475\},\{0.100\},\{0.279\}\rangle).$$

Then, according to the distance measure represented in Definition 7, we can get the distances between the evaluation value of each criterion and its optimistic/pessimistic values. In this paper, we assume $\mu = 1$.

$$d_1^+ = 0.382, \quad d_2^+ = 0.384, \quad d_3^+ = 0.148 \quad \text{and} \quad d_4^+ = 0.419.$$

$$d_1^- = 0.457, \quad d_2^- = 0.453, \quad d_3^- = 0.234, \quad \text{and} \quad d_4^- = 0.454.$$

Thus, the weights of criteria are obtained below:

$$Q = (0.255,\ 0.258,\ 0.217,\ 0.270).$$

According to the investigation and experiment of Kahneman and Tversky [61], we set $\alpha = \beta = 0.88$, $\lambda = 2.25$, $\gamma = 0.61$, $\delta = 0.69$. The ranking result is shown in Table 8.

Step 3.2: Conduct the IMVN-PT-reference point method.

According to Eqs. (27)–(29), the ranking result is shown in Table 8.

Step 3.3: Apply the IMVN-PT-full multiplicative form method.

According to Eqs. (30) and (31), the ranking result is presented below.

Step 4: Measure the final ranking result.

Based on the dominance theory, the final ranking result is $A_1 \succ A_3 \succ A_2 \succ A_4$.

Table 8. Ranking results obtained by the proposed method.

	IMVN-PT-ratio system method		IMVN-PT-reference point method		IMVN-PT-full multiplicative form method	
	Calculating result	Rank	Calculating result	Rank	Calculating result	Rank
A_1	0.125	1	0	1	0.426	1
A_2	–0.374	3	0.186	3	–0.653	4
A_3	0.164	2	0.047	2	0.241	2
A_3	–0.805	4	0.324	4	–0.162	3

Table 9. Results obtained by different values of λ.

Parameters	Methods	A_1	A_2	A_3	A_4
$\lambda = 1$	y_i^*	0.361	–0.292	0.319	–0.672
	z_i^*	0	0.244	0.086	0.354
	u_i^*	0.716	–0.471	0.352	–0.103
	Ranking	1	3	2	4
$\lambda = 1.5$	y_i^*	0.230	–0.343	0.235	–0.759
	z_i^*	0	0.216	0.066	0.344
	u_i^*	0.563	–0.568	0.295	–0.139
	Ranking	1	3	2	4
$\lambda = 2$	y_i^*	0.152	–0.367	0.182	–0.795
	z_i^*	0	0.195	0.052	0.330
	u_i^*	0.463	–0.629	0.256	–0.156
	Ranking	1	3	2	4
$\lambda = 2.5$	y_i^*	0.102	–0.379	0.148	–0.812
	z_i^*	0	0.179	0.043	0.318
	u_i^*	0.395	–0.673	0.228	–0.166
	Ranking	1	3	2	4
$\lambda = 3$	y_i^*	0.068	–0.387	0.124	–0.822
	z_i^*	0	0.168	0.036	0.309
	u_i^*	0.345	–0.708	0.207	–0.171
	Ranking	1	3	2	4

5.2. Influence of the parameter

To obtain the effect of distinct parameter λ on the result of this decision-making process, we conduct this analysis.

First, we let λ vary from 1 to 3. Then, the results of IMVN-PT-ratio system, IMVN-PT-reference point, and IMVN-PT-full multiplicative form method are shown in Table 9. Finally, we graph the results in Figures 2 and 4.

From Table 9 and Figures 2–4, we can observe that the results obtained by the three components of the IMVN-PT-MULTIMOORA method decrease as the parameter increases. The ranking results do not undergo any changes due to the change of parameter λ. The optimal alternative is always A_1. In Figure 2, with the change of parameter λ, the numerical value of A_1 is larger than that of A_3 initially and, then, smaller than that of A_3. In addition, the gap of numerical results between A_1 and A_3 is narrowed further following an increase in the value of the parameter λ, as shown in Figures 3 and 4. Based on this, we conducted a test on the parameter λ. We found that when the value of the

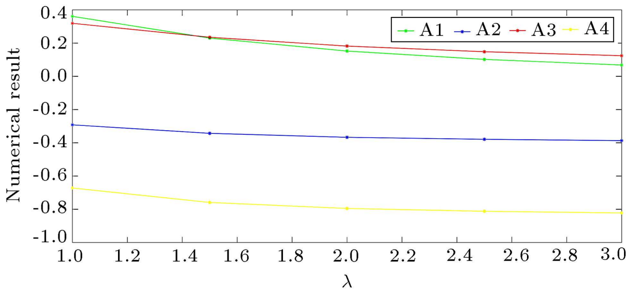

Figure 2. Results obtained by IMVN-PT-ratio system method.

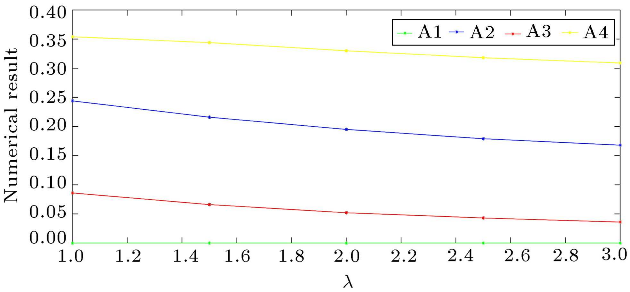

Figure 3. Results obtained by IMVN-PT-reference point method.

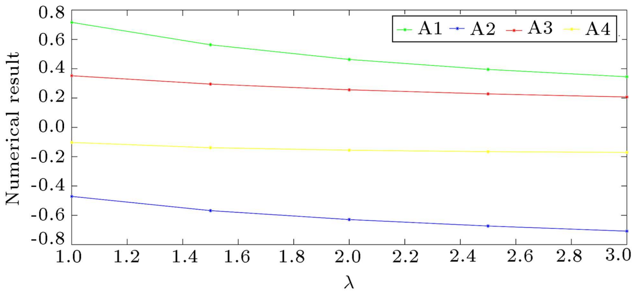

Figure 4. Results obtained by IMVN-PT-full multiplicative form method.

parameter λ is close to 34, the optimal alternative is selected. This implies that as decision-makers become increasingly sensitive to loss, the optimal alternative is more likely to be option A_3. It also proves that it is necessary to introduce prospect theory into multi-criteria decision-making.

5.3. Comparative analysis

5.3.1. Validity of the proposed method

Given that the proposed method connects HM aggregation operator, prospect theory, and MULTIMOORA method, we can select some existing methods based on these theories and methods.

In the method presented by Li et al. [62], some single-valued neutrosophic number HM operators, including the NNIGWHM and NNIGWGHM operators, are presented to integrate criterion values. Then, schemes will be ranked based on the values of the score and accuracy functions.

Tian et al. [48] presented an improved MULTIMOORA method for MCDM problem based on the closeness coefficient of TOPSIS and variation coefficient method. In addition, they defined distance measure for neutrosophic linguistic sets.

Table 10. Ranking results derived from different methods.

Method	Ranking
NNIGWHM operator (presented by Li et al. [62])	$A_1 \succ A_3 \succ A_4 \succ A_2$
NNIGWGHM operator (presented by Li et al. [62])	$A_1 \succ A_3 \succ A_4 \succ A_2$
MULTIMOORA method (presented by Tian et al. [48])	$A_1 \succ A_3 \succ A_4 \succ A_2$
The proposed method	$A_1 \succ A_3 \succ A_4 \succ A_2$

In order to apply the method proposed by the above researchers, MVNNs should be transformed into SVNNs. We can derive SVNNs by calculating the average values of all possible truth-membership, indeterminacy-membership, and falsity-membership degrees in MVNNs. For example, $a = \langle \{0.2, 0.3\}, \{0.3, 0.5\}, \{0.4, 0.5\} \rangle$ can be transformed into $a_1 = \langle 0.25, 0.4, 0.45 \rangle$.

The ranking results obtained by the methods by Li et al. [62], Tian et al. [48], and the proposed method are shown in Table 10.

5.3.2. Advantages of the proposed method

Ji et al. [52] defined the operations of MVNSs and comparison methods. At the same time, some aggregation operators are presented to solve MCDM problems. To show the advantages of the proposed method, we select WA and WG operators in this paper for instance. The results of this comparison are shown in Table 11.

Biswas et al. [63] presented a novel TOPSIS-based approach to the MCGDM problem in single-valued neutrosophic environments. In this paper, the authors obtain the evaluation information by linguistic terms. Then, an objective method is used to obtain the weight vector of each decision-maker.

Ji et al. [64] defined the projection measure of MVNNs, presented a projection-based TODIM method, and applied it to personnel selection.

Based on the above, we conduct a comparison, the results of which are presented in Table 11.

Table 11 indicates that A_1 is always the best construction scheme for subways, while the worst scheme is A_4 or A_2. The proposed method produced the same ranking as the WA operator and TOPSIS method. Furthermore, the rankings obtained from the proposed method exhibit minimal differences when compared to those obtained through the use of the WG operator and projection-based TODIM method. From Table 11, we can conclude that our method is more reasonable than

Table 11. Ranking results obtained by different methods.

Method	Final ranking
WA operator (presented by Ji et al. [52])	$A_1 \succ A_3 \succ A_2 \succ A_4$
WG operator (presented by Ji et al. [52])	$A_1 \succ A_3 \succ A_4 \succ A_2$
TOPSIS method (presented by Biswas et al. [63])	$A_1 \succ A_3 \succ A_2 \succ A_4$
Projection-based TODIM method (presented by Ji et al. [64])	$A_1 \succ A_3 \succ A_4 \succ A_2$
The proposed method	$A_1 \succ A_3 \succ A_2 \succ A_4$

the WA, WG operators, TOPSIS, and projection-based TODIM method for the following reasons.

Although the WA operator produced the same ranking as the proposed method, the proposed method considered the interactions of the inputs. The WG operator also did not consider the interactions of the inputs. The TOPSIS and projection-based TODIM method ignore the bounded rationality of decision-makers. The proposed method utilizes IGMVNWHM operator to aggregate the assessments of all experts. Then, the collective decision matrix is input into the IMVN-PT-MULTIMOORA method to deal with practical problems better.

6. Conclusion

To tackle the challenge of handling fuzzy and uncertain information while selecting construction schemes for subways, the decision-making problem was addressed using a combination of Multi-Valued Neutrosophic Numbers (MVNNs), Heronian Mean (HM) operator, prospect theory, and MULTIMOORA method. This study initially introduced some basic concepts and theorems about Multi-Valued Neutrosophic Sets (MVNSs) and HM operator. Then, the IGMVNWHM operator based on HM operator was presented. It considered the interactions among inputs. Subsequently, the new distance measure was defined between two MVNNs. Then, motived by TOPSIS and the variation coefficient method, an improved MULTIMOORA method was proposed. Considering that decision-makers were not completely rational, we introduced the prospect theory to this method. Finally, a solution framework to select construction schemes for subways in multi-valued neutrosophic environments was developed. In addition, an application example was introduced to prove the validity and advantages of the proposed method. At the same time, the rankings were analyzed as the parameters changed.

The contributions and innovations of this paper are described as follows. First, MVNNs were used to present assessments of construction schemes for subways. Second, IGMVNWHM operator was introduced, which considered the interactions of inputs. Third, a new distance measure was defined between two MVNNs. Fourth, an IMVN-PT-MULTIMOORA method was presented.

In the future, the IGMVNWHM operator could be employed in other neutrosophic environments, such as probabilistic multi-valued neutrosophic sets. At the same time, we can explore other methods to solve multi-valued neutrosophic evaluation information problems.

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