Finite–time synchronization of a new five–dimensional hyper–chaotic system via terminal sliding mode control

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Abstract: This paper constructs a new 5–D nonlinear hyper–chaotic system with attractive and complex behaviors. The standard behaviors of the chaotic system will also be analyzed, including: Equilibrium Point's (EP's), Bifurcation Diagram's (BD's), Poincare Map's (PM's), Lyapunov Exponent's (LE's) and Kaplan–Yorke dimensional. We prove that the introduced new 5–D hyper–chaotic system has complex and nonlinear behaviors. Next, the work describes Fast Terminal Sliding Mode Control (FTSMC) scheme for the control and finite–time fast synchronization of the novel 5–D nonlinear hyper–chaotic system. Proof of stability for both phases has been done for the new controller with the Lyapunov stability theory. For the synchronization, both master–slave subsystems are perturbed by different parameter and model uncertainties. Both steps of the Sliding Mode Controller (SMC) have chaos-based fast convergence properties. Subsequently, it has been shown that the state paths of both master–slave systems can reach each other in a limited–time. One of the features of the novel controller in this paper, is high performance and finite–time stability of the terminal sliding surface due to derivative error and error. Finally, using the MATLAB simulation, the results are confirmed for the new hyper–chaotic system.

Keywords: Hyper–chaotic system; chaos synchronization; terminal sliding mode control; finite–time stability; robustness.

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1. Introduction

With the advent and development of telecommunication, especially wireless communications, encryption and information hiding has become a communication necessity [1]. With the progress of multimedia and communication technologies as well as the limitations of transportation, medical images have played an important role in tele-surgery. At the same time, new communication technologies have enabled medical image sharing and processing. These technologies have also increased security issues such as confidentiality and integrity [2]. Given these advantages, it is risky to send electronic patient records and medical confidential records to common networks such as the Internet. Although sending information to these networks is less expensive, it will have security risks, including the availability of information to everyone. Therefore, one of the necessities of information transfer, especially medical information, is increased security of information transfer [3]. In 1998, Friedrich introduced image encoding using a two-dimensional chaotic function. Using the chaotic adaptive conversion method, he designed a novel idea for chaos–based image encryption based on random encryption [4].

Chaotic systems have a number of intrinsic properties, including high oscillations as well as complex nonlinear dynamical equations [5]. Two important features of chaotic systems are parametric uncertainty and sensitivity to change in their initial conditions. Chaos programs are now highly developed. Due to the unpredictability of these systems, they can be used in many applications, including: nonlinear anti-synchronization [6], chaos-based control [7], encryption [8], robotic [9], biological networks [10], secure communication [11] and neuroscience [12]. Many 3–D nonlinear systems were designed, among them the Chen [13], Lu [14] and Qi [15] systems, which are generic. These systems, despite their good features, have one positive Lyapunov exponents and a simple structure. As a result, these systems have a weak security flaw that makes them easy to break. Therefore, Rossler introduced a hyper–chaotic system with two or more LEs [16]. The hyper-chaotic systems have more nonlinear behaviors and higher fluctuations than the chaotic systems. [6, 17]. Many high–dimensional hyper–chaos systems are designed based on the available low–dimensional nonlinear chaotic systems by two methods as follows:
• By feeding back–the–output of the nonlinear control into the chaos system equation with low–dimensional [18].

• By junction of two low–dimensional nonlinear chaos systems together [19].

One way to create the nonlinear hyper-chaos system is to add the dimensions of a general chaotic system, this may lead to instability. Alternatively, a usual method is to get a novel nonlinear hyper–chaos systems via adding one or more other state variable to a regular nonlinear chaos systems [20], for example, the Chen [13], Lorenz [21], Lu [14], and Qi systems [22]. In a chaos–based secure communication, the master–slave subsystems can be used to transmit secure communication. When these two systems are synchronized, we will have a complete and successful transfer. The chaos–based synchronization is one of the main control approaches [23] has been considered by researchers for many years [24]. For a successful chaos–based synchronization, a suitable control signal is used to move the state trajectories of the two chaotic nonlinear subsystems. In recent years, several linear and nonlinear controllers including: linear and nonlinear feedback control [25], [26, 27], adaptive tracking control [28, 29], backstepping design [30], optimal nonlinear control [31, 32], fuzzy controller [33], PID control [34], stochastic control [35, 36], active control [37], general sliding mode controller [38], linear feedback method [39], passive control [40], finite time stability [41, 42], Sliding Mode Control (SMC) [43], and Terminal Sliding Mode Control TSMC [44] utilized to chaos–based synchronization. In these studies, based on control theories topics, most of them have been investigated asymptotic synchronization.

Among the stated methods, the SMC has special specifications such as: Asymptotic stability, computational simplicity, simple implementation, parametric robustness, reduced order of the system, suitable transient response and less sensitivity to bounded disturbance [45]. In this method, due to the linear sliding surface, the convergence time is unbounded and the system states reach the equilibrium point asymptotically. [46]. Given the importance of time in the transmission of medical information, conditions must be provided to transfer the information as quickly as possible. Compared with the traditional SMC, the TSMC introduces a non-linear term in the sliding surface function to improve the convergence properties of the system to ensure that the system modes converge to a given trajectory in a limited-time. [47].

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Much research has been done over the years on the application and development of SMC design [48, 49]. In [50], two Adaptive Sliding Mode Control (ASMC) for synchronizing of the Genesio–Tesi system with external disturbance and unknown parameters are proposed. In [51], Using a Second–Order Sliding Mode Control (SOSMC), chaos-based synchronizations uncertain with different structure has been investigated. With all the advantages and popularity of the SMC, this method has a major drawback called the chattering phenomenon. This phenomenon is very undesirable in practice and will have effects such as: controller malfunction, mechanical wear in systems and increased energy consumption. Much research has been done by researchers to solve this problem. For example: In [52], a new chatter free SMC strategy with integral operators and both–differential is designed for synchronization and chaos control signal to the nonlinear uncertain chaos systems. In [53], using the chaotic system [54] a new controller for a chaos–based synchronization strategy dedicated to fractional–order nonlinear systems characterized by several dimensions. It is shown that the error goes to zero in the bounded–time. In [55] proposed a new fixed–time chattering–free observer–based SMC scheme for chaos–based synchronization of two–sided teleoperations under unknown time-varying delay. Using a SMC scheme, this system is evaluated for unknown disturbances at a fixed time. The observer-based fixed–time SMC is designed to estimate the unmeasured speed state, while the position state is supposed to be available. In this paper, the fixed time stability method is used for convergence. A new SMC scheme is proposed to synchronize drive system to response system in the presence of time-delay in the communication channel, as well as states and estimating disturbances. Reference [56] investigates a novel robust predefined time chattering free SMC strategy for the nonlinear tracking control problem of a Remotely Operated Vehicle (ROV) with Three-Degree Of Freedom (3-DOF) with uncertainties. Based on defining a novel sliding surfaces, a novel SMC scheme is designed to ease the chatter phenomenon, tracking precision and without damaging the robustness properties. The results shown that the proposed control scheme can solve the design problem of the predefined time tracking controller and also provides robustness to various uncertainties. The Lyapunov stability theory is used to both the sliding phase and the reaching phase. In [57], a novel nine–terms hyper–chaotic system with line equilibrium is first designed. This system has
rich behaviors and attractors. has been developed and all of the attractive features of the system have been analyzed. Finally, synchronization between two new 9-D nonlinear systems using active control is designed. In [58], presents a SMC scheme for nonlinear chaotic systems. The proposed new controller is built on a new SMC reaching law and a nonlinear sliding surface in the bounded–time. In [59], a new robust controller is designed. The new control strategy is proposed for digital secure information between two nonlinear subsystems with unknown parameters and uncertainties, in a finite-time by TSMC and combining adaptive back–stepping approaches. The TSMC provides faster convergence than the general SMC. In [60], two different new controllers are developed by using Non-singular Terminal Sliding Mode Control (NTSMC) and the other Adaptive Non-singular Terminal Sliding Mode Control (ANTSMC) methods with unknown parameters and different uncertainties for synchronization. The concept of TSMC method guarantees controller robustness against various external disturbances and parametric uncertainties and the guarantees system stability in a bounded–time. Based on the above existing results, master–master system is synchronized in an infinite time. However, some engineering problems created in different structures are expected to be synchronized in a bounded–time. The chaos–based fast synchronization has many advantages and features, such as finite-time tracking, optimality of the convergence time, improved robustness and rejection of uncertainties and disturbances.

For an unauthorized receiver, chaos-based decryption is difficult without knowing the dynamics of the system and the initial conditions. One of the security ways is to increase the dynamic of the nonlinear chaotic system. This is due to the fact that it is difficult to recover messages for unauthorized sources using retrieval methods. Another way to increase security is the dynamic complexity of the system, as this will make decoding difficult. For example, in [61], general chaos-based synchronization between two new integer–order and fractional–order hyper–chaos nonlinear systems is studied. The new control signals are constructed using the technique of stability theory and the tracking controller of the fractional–order system. In [62], the chaos–based bounded–time synchronization of the four–dimensional Memristor Chaos Systems (MCS) has been studied. First, an emulator circuit of memristor is created to implement the MCS. Then, based on the presented emulator circuit, the model of the MCS is provided and its
bounded–time chaotic synchronization is achieved under the designed new controller. Finally, sustainability analysis has been performed. In [63], chaos–based synchronization of nonlinear Lu systems with disturbance and uncertainty and minor control scheme with regard to systems dimensions by applying an ASMC is presented. Firstly, an Integral-Type Sliding Mode Control (I-TSMC) is proposed for chaos–based synchronization of nonlinear Lu systems with specify positive parameters. Secondly, a new control signal is used to synchronization of nonlinear master-slave Lu systems; in this case, unspecified positive parameters are estimated using an adaptive control rule. Finally, stability of the designed control scheme is proved using the Lyapunov stability method. In [64], ASMC method is proposed for chaos–based synchronization of 6-D drive-response nonlinear systems in the presence of unknown parameters and external disturbance in the response system. Firstly, two 6-D integer-order drive-response systems in the presence of unknown parameters and external disturbance signal in the response system are designed. Secondly, after identifying chaos in fractional-order dynamic of the foresaid system, chaos–based synchronization of 6-D nonlinear fractional-derivative drive-response systems in the presence of uncertainty, disturbance signal and unknown parameters in the response system is studied, in which fractional-order Riemann-Liouville derivative is used; a novel sliding surface is defined for the new 6-D nonlinear system to specify the proper active control. Finally, controller proofs and numerical simulations for the efficiency of the proposed design are presented in the presence of parametric uncertainty and disturbance.

Asymptotic stability is a weaker concept than finite–time stability. In finite–time stability, system state variables converge to their equilibrium point more rapidly in a finite–time. The term “terminal” refers to the meaning of finite–time stability. Depending on the structure of the systems, there are many applications that need to be stable in a finite–time. The paper makes the following main contributions:

i. Designing and building a 5–D hyper–chaotic system, analyzing and acquiring intrinsic properties.

ii. Designing a novel Fast Terminal Sliding Mode Control (FTSMC) for the chaos–based synchronization of five–dimensional nonlinear master–slave systems.
iii. Designing a new sliding surface and prove the global stability and fast convergence without chattering.

This article is as follows: sec. 2 provides the general dynamical model of the novel 5–D nonlinear system and its benefits and features. Then, the chaos-based fast synchronization problem of hyper–chaos systems in bounded–time is formulated. Sec. 3 presents the proof and design of the TSMC signal for chaos–based synchronization. Sec. 4, numerical simulations are performed to prove the effectiveness of the methods. Sec. 5 presents some conclusions.

2. Problem definition and description

2.1. Model of the novel 5-D hyper–chaos system

The dynamics of the novel 5–D hyper–chaos system is described as:

\[
\begin{align*}
\frac{dx_1(\tau)}{d\tau} &= a_1(x_2 - x_1) - a_2 x_3 \\
\frac{dx_2(\tau)}{d\tau} &= a_3 x_1 - x_2 - a_4 x_4 - x_3 x_3 \\
\frac{dx_3(\tau)}{d\tau} &= -a_5 x_3 + x_2 x_4 + x_4^2 \\
\frac{dx_4(\tau)}{d\tau} &= a_6 x_5 + x_2 \\
\frac{dx_5(\tau)}{d\tau} &= a_7 x_2 + x_1
\end{align*}
\] (1)

where \( a_i, (i = 1, \ldots, 7) \), \( x_i, (i = 1, \ldots, 5) \) and \( \tau \) are the constant positive parameters, state variables and time of the new nonlinear system (1), respectively. The nonlinear system (1) will have hyper–chaotic behaviors by defining the following parameters:

\[ a_1 = 8.83, \ a_2 = 0.75, \ a_3 = 36.36, \ a_4 = 20.779, \ a_5 = 7.79, \ a_6 = 4.1, \ a_7 = 4.286 \]

2.2. Dynamical behaviors and basic properties of the novel 5-D nonlinear system

In this section presents the general properties of the nonlinear system (1) such as: Equilibrium Point's (EP's), chaotic attractors, eigen values, Kaplan–Yorke dimension, Lyapunov Exponent's (LE's), Poincare Map's (PM's), and Bifurcation Diagram's (BD's). By setting the differential equations in new nonlinear system (1) to zero, it is concluded
that the 5-D nonlinear system (1) has equilibrium point at: \( Q = (0, 0, 0, 0, 0) \). The 5-D nonlinear system linearization matrix [65] at the equilibrium point \( Q \) is given by:

\[
J = \frac{\partial F}{\partial x}(x) \bigg|_{Q} = \begin{bmatrix} -a_1 & a_1 & 0 & 0 & -a_2 \\ a_3 & -1 & 0 & -a_4 & 0 \\ 0 & 0 & -a_5 & 0 & 0 \\ 0 & 1 & 0 & 0 & a_6 \\ 1 & a_7 & 0 & 0 & 0 \end{bmatrix}.
\]

(2)

According to linearization matrix (2), the system eigenvalues are found as follows:

\[
(s + a_j)[s^4 + A_3s^3 + A_2s^2 + A_1s + A_0]
\]

\[
A_1 = (a_1 - a_4) + 1 \\
A_2 = (a_1 + a_4a_3 + a_3a_5 + a_5a_7) \\
A_3 = (a_1a_3a_5 + a_3a_5a_7 + a_5a_7a_9 + a_7a_9a_1 + a_9a_1a_3) \\
A_4 = (a_1a_3a_5a_7 + a_3a_5a_7a_9 + a_5a_7a_9a_1 + a_7a_9a_1a_3)
\]

(3)

Using parameter values in (1), the eigenvalues are \( s_1 = -23.2488, s_2 = 8.7260, s_3 = 7.3648, s_4 = -7.79, s_5 = -2.6719 \). Thus, \( Q \) is an unstable saddle. The divergence of nonlinear system (1) is as follows:

\[
\nabla V = \sum_{i=1}^{5} \frac{\partial \tilde{F}}{\partial x_i} = -a_1 - 1 - a_5 = -8.83 - 1 - 7.79 = -17.62 < 0
\]

(4)

So, the convergence speed of system (1) to its attractors is \( e^{-(a_1 - 1 - a_5)x} \). The phase portrait diagrams of the system (1) are depicted in the Fig. 1 and the Fig. 2. The convergence and divergence of the states of the nonlinear systems, are specified by its LE's representation. If LE's are positive, it indicates the chaos and hyper-chaos behaviors of the nonlinear system [66, 67]. The LE's of the novel 5-D nonlinear system (1) with initial conditions \( (x_1(0) = -1.2), (x_2(0) = 3.8), (x_3(0) = 7.7), (x_4(0) = 2.7), (x_5(0) = 1.4) \) are numerically found as \( LE_1 = 2.512, LE_2 = 0.211, LE_3 = -2.155, LE_4 = -4.286, LE_5 = -13.89 \) shown in Fig. 3 and Fig. 4. According to the numerical values obtained for LE's, the Kaplan–Yorke dimensional [68, 69] of the 5-D nonlinear system (1), is defined as: \( D_{KY} = 4.7115 \), which is fractional. The BD's examines the dependence of the parameter values of the chaos nonlinear systems. In Fig. 5 and Fig. 6 BD's of the nonlinear system
The nonlinear system (1) enters into chaotic oscillations with routine period doubling [70, 71]. Another attraction of chaotic nonlinear systems is the use of a PM's to describe the folding properties of the system. This method is one of the most famous topics in the nonlinear dynamical analysis that we can use to prove the behavior and performance of continuous dynamic systems similar to the proposed 5-D nonlinear system (1). Fig. 7 display the PM's of the 5-D nonlinear system (1). According to Fig. 7, the regular set of points, represents the chaotic behavior of the system.

2.3. Problem formulation

In this section, chaos–based fast synchronization are presented between two novel 5–D nonlinear master–slave subsystems with homogeneous parametric uncertainties and unknown disturbances. Next, we use the nonlinear system (1) with changes in parameters and initial conditions to build master–slave subsystems for chaos–based synchronization. Consider the 5–D nonlinear master system as follows:

\[
\frac{dx_{im}(\tau)}{d\tau} = \begin{bmatrix}
-a_i & a_i & 0 & 0 & -a_{2im} \\
-a_{3im} + x_{3im} & -1 & 0 & -a_{4im} & 0 \\
x_{1im} + x_{2im} & 0 & -a_{5im} & 0 & 0 \\
0 & 1 & 0 & 0 & a_{6im} \\
1 & a_{7im} & 0 & 0 & 0
\end{bmatrix} x_{im} \tag{5}
\]

where \(a_{im}, \ldots, a_{7im}\) and \(X_m = x_{1im}, \ldots, x_{5im}\) are the parameters and states of the subsystem (5), respectively. So, for the 5–D nonlinear slave subsystem we define:

\[
\frac{dx_{is}(\tau)}{d\tau} = \begin{bmatrix}
-a_i & a_i & 0 & 0 & -a_{2is} \\
-a_{3is} + x_{3is} & -1 & 0 & -a_{4is} & 0 \\
x_{1is} + x_{2is} & 0 & -a_{5is} & 0 & 0 \\
0 & 1 & 0 & 0 & a_{6is} \\
1 & a_{7is} & 0 & 0 & 0
\end{bmatrix} x_{is} + \lambda \nu(\tau) + d(\tau) \tag{6}
\]

where \(X_s = x_{is}, \ldots, x_{5is}\) are the state variables of subsystem (6) and \(\nu(\tau) = \nu_1, \ldots, \nu_5\) are the nonlinear command signals used to synchronization of master–slave subsystems (5) and (6).

**Assumption 1**: Let the chaos–based synchronization errors of the subsystem (5) and subsystem (6) as follows: \(e_i = x_{is} - x_{im} \quad (i=1, \ldots, 5)\).
**Assumption 2:** Constraints on uncertainties and disturbances are defined as follows:

\[ |f(x(t))| \leq t_1, \ |d(t)| \leq t_2 \]  

where \( t_1 \) and \( t_2 \) denote positive unknown constants.

**Assumption 3:** Suppose \( y_i(t) = x_i(t) \) implies that \( \lim \limits_{t \to \infty} e_i(t) = 0 \).

**Definition 1:** [72] The chaos–based synchronization of the subsystems (5) and (6) is obtained in a bounded-time if \( \lim \limits_{t \to T} \| e(t) \|= 0 \) and \( \| e(t) \|= 0 \) for \( \tau \geq T \), where \( T = T(e(t_0)) > 0 \), \( e(t)=[e_i]^T, (i=1,\ldots,5) \).

**Definition 2:** [73] Hyper–chaotic master and slave subsystems (5) and (6) are fast synchronization, if there is a control signal \( \nu_p(t) \) and a constant \( T > 0 \) such that \( \lim \limits_{\tau \to T} [z^1_p(t) - z^2_p(t)] = 0 \), where \( z^1_p(t) - z^2_p(t) \) for \( \tau > T \), \( z^1(t) \) and \( z^2(t) \) are the solutions of 5-D hyper–chaotic master and slave subsystems (5) and (6).

**Lemma 1:** [11] If the \( \mathcal{H}(\tau) \) is a positive definite performance such that:

\[ \mathcal{H}(\tau) \leq -\delta \mathcal{G}^{\theta}(\tau), \ \forall \tau \geq \tau_0, \ \mathcal{G}(\tau_0) \geq 0 \]  

where \( \delta > 0, 0<\theta<1 \) are constants and known, for any initial time \( \tau_0 \). Then function \( \mathcal{H}(\tau) \) satisfies

\[ \mathcal{G}^{1-\theta}(\tau) \leq \mathcal{G}^{1-\theta}(\tau_0) - \delta(1-\theta)(\tau - \tau_0), \ \tau_0 \leq \tau \leq \tau_1 \]  

and

\[ \mathcal{H}(\tau) = 0, \ \forall \tau \geq \tau_1 \]  

with the settling time \( \tau_1 \) satisfying

\[ \tau_1 \leq \tau_0 + \frac{\mathcal{G}^{1-\theta}(\tau_0)}{\delta(1-\theta)} \]  

**Lemma 2:** Suppose that the function \( \nu(t) \), which is continuous and positive definite, satisfies the following equation [74]:

\[ \dot{\nu}(\tau) \leq -\alpha \nu(\tau) - \beta \nu^\theta(\tau) \ \ \forall \tau \geq \tau_0, \ \nu(\tau_0) \geq 0 \]  

for all times \( \tau_0 \), the function \( \nu(\tau) \) in the finite time \( \tau_0 \), will converge to zero. Thus:
\[
\tau_s = \tau_0 + \frac{1}{\alpha(1-\eta)} \ln \frac{\alpha^\eta (\tau_0) + \beta}{\beta}
\]  
(13)

3. **Main Results**

Consider the dynamical model as:
\[
x(\tau) = \Lambda x(\tau) + B u(\tau) + f(x(\tau)) + d(\tau),
\]
(14)
where \( x(\tau) \) is state variables, \( B \) and \( \Lambda \) are the constant matrices, \( u(\tau) \) is the controller, \( f(x(\tau)) \) is the non-linear functions of the nonlinear first-order system (14) and \( d(\tau) \) sum of the unknown disturbance and uncertainty of the nonlinear system (14). The sliding surface for the system (14) is defined as:
\[
l(\tau) = Gx(\tau)
\]
(15)
where \( G \) is the gain coefficient (row vector) as \( G = [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5] \).

In order to satisfy \( l(\tau) \) converges to origin in the finite–time, the following fast terminal sliding surface is given as:
\[
s(\tau) = \dot{l}(\tau) + \lambda l(\tau) + \mu l^\eta(\tau),
\]
(16)
where \( \lambda, \mu \) and \( \eta \) the positive constant values and ratio of two odd positive integers with \( 1 > \eta > 0 \), respectively.

**Theorem 1:** Let the FTSMC for the nonlinear system (14) is defined as:
\[
\dot{\phi}(\tau) = -(GB)^{-1}\left\{ \left[ \lambda + \mu \eta l(\tau)^{\eta-1} \right] G \left[ \Lambda x(\tau) + f(x(\tau)) + B u(\tau) \right] + G \left[ \lambda^2 x(\tau) + \Lambda f(x(\tau)) + \Lambda B u(\tau) + f'(x(\tau)) \right] + \kappa |s(\tau)|^\eta + \gamma s(\tau) + \chi \text{sgn}(s(\tau)) \right\},
\]
(17)
where \( \kappa \) and \( \gamma \) are optional positive constants and \( \chi \) satisfies
\[
\chi \geq \max \left\{ \left( \lambda + \mu \eta l(\tau)^{\eta-1} \right) G + G \Lambda \right\} d(\tau) + G \dot{d}(\tau).
\]
(18)
With the control scheme (17), the state trajectories of the nonlinear dynamic system (14) move to the sliding surface (15) in finite–time and they stay there.

**Assumption 4:** Function \( f(x(\tau)) \in R^{n_1} \) satisfy the following conditions:
\[
\forall x \in R^{n_1}, |f(x(\tau))| \neq 0.
\]
(19)
**Assumption 5:** $GB$ is a non-singular (invertible) matrix.

**Assumption 6:** The system (14) tracking errors reach to origin in a limited-time using the control scheme (17).

**Proof:** We follow the steps of the finite-time stability of the FTSMC in two phases as follows:

a) Reaching phase: considering the sliding surface (16), the Lyapunov candidate function can be considered as follows:

$$v_1(\tau) = 0.5\zeta s^2(\tau) \quad (20)$$

**Lemma 3:** [75] The following inequalities are established:

$$\left| \sum_{i=1}^{\infty} \frac{\psi(1+\xi)}{\psi(1+k)\psi(1-k+\xi)} D^i s D^{i-k} s \right| \leq \delta |s| \quad (21)$$

where $\delta$ is a positive constant. In Equation (20), $\zeta$ is equal to:

$$\sum_{i=1}^{\infty} \frac{\psi(1+\xi)}{\psi(1+k)\psi(1-k+\xi)} s^i \quad (22)$$

Differentiating the Lyapunov function (22) yields:

$$\dot{v}_1(\tau) = s(\tau)\dot{s}(\tau) + \sum_{i=1}^{\infty} \frac{\psi(1+\xi)}{\psi(1-k+\xi)\psi(1+k)} s(\tau)\dot{s}(\tau)$$

$$\leq s(\tau)\dot{s}(\tau) + \left| \sum_{i=1}^{\infty} \frac{\psi(1+\xi)}{\psi(1-k+\xi)\psi(1+k)} s(\tau)\dot{s}(\tau) \right| \quad (23)$$

According to (21), yields:

$$\dot{v}_1(\tau) \leq s(\tau)\dot{s}(\tau) + \left| \sum_{i=1}^{\infty} \frac{\psi(1+\xi)}{\psi(1+k)\psi(1-k+\xi)} s(\tau)\dot{s}(\tau) \right| \quad (24)$$

By substituting (16) into (24), yields:

$$\dot{v}_1(\tau) \leq s(\tau) \frac{d}{d\tau} (\dot{\hat{r}}(\tau) + \lambda l(\tau) + \mu l''(\tau)) + \delta |s(\tau)| \quad (25)$$

By substituting (14) into (25), yields:
\[ \dot{v}_i(\tau) \leq s(\tau) \frac{d}{d\tau} \left[ (GAx(\tau) + GB\nu(\tau) + Gf(x(\tau)) + Gd(\tau)) + \lambda Gx(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau) \right]^{\gamma} + \delta |s(\tau)| \]

\[ = s(\tau) \left[ (GAx(\tau) + GB\nu(\tau) + Gf(x(\tau)) + Gd(\tau)) + \lambda Gx(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau) \right]^{\gamma} + \delta |s(\tau)| \]

\[ \leq s(\tau) \left[ (GAx(\tau) + Gd(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau))^{\gamma} \right] + \left( s(\tau) + Gf(x(\tau)) + \lambda G\dot{x}(\tau) \right) + \delta |s(\tau)| \]

(26)

**Assumption 7:** The uncertainty disturbances are considered bounded as follows:

\[ |\frac{d^\alpha}{d\tau}(d_i(\tau))| \leq \gamma \]  

(27)

where \( \gamma \) is a positive custom constant.

**Assumption 8:** Assume that the sign function is bounded as:

\[ |\frac{d^\alpha}{d\tau} \rho, \text{sgn}(\mu x_i) | \leq \kappa \]  

(28)

where \( \kappa \) is a positive constant.

Using (17) and (26) we can write:

\[ \dot{v}_i(\tau) \leq s(\tau) \left[ (GAx(\tau) + Gd(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau))^{\gamma} \right] + s(\tau) \left[ (GAx(\tau) + Gd(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau))^{\gamma} \right]^{\gamma} + \left( s(\tau) + Gf(x(\tau)) + \lambda G\dot{x}(\tau) \right) + \delta |s(\tau)| \]

(29)

According to Assumption 7 and Assumption 8, using (18) and (26), we obtain:

\[ \dot{v}_i(\tau) \leq s(\tau) \left[ (GAx(\tau) + Gd(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau))^{\gamma} \right]^{\gamma} - \kappa - \gamma 

(30)

\[ + |s(\tau)| \left[ (GAx(\tau) + Gd(\tau) + \lambda G\dot{x}(\tau) + \mu Gx(\tau))^{\gamma} \right] - \kappa |s(\tau)|^{\gamma} + \delta |s(\tau)| \]

From (27) and (28), one gets:

\[ \dot{v}_i(\tau) \leq - (G + \mu \text{sgn}(s(\tau)) - \delta) |s(\tau)| - \kappa |s(\tau)|^{\gamma} \]

(31)

Therefore:

\[ \dot{v}_i(\tau) \leq - (G - \delta) |s(\tau)| = - \Theta |s(\tau)| \]

(32)
Therefore, the state trajectories of the first-order nonlinear system (14) will converge to the sliding surface \( s(\tau) = 0 \) with \( G > \delta \).

With the designed reaching law (32), switching function \( s \) will reach to the sliding surface in finite-time \( \tau_s \) with proper positive constant \( G \), the stability-time is defined as:

\[
\tau_s = \{ \inf \tau \geq \tau_s : X_s(\tau) = 0 \}
\]

(33)

where \( \tau_s \) is the time to reach the equilibrium point.

By integrating from the equation (32) from 0 to \( \tau_s \), one gets:

\[
v_1(\tau_s) - v_1^{\xi^{-1}}(0) \frac{\tau_s^{\xi^{-1}}}{\Psi(\zeta)} \leq -(G - \delta) \frac{d^{\xi}}{d\tau} |s(\tau)|
\]

(34)

Assuming \( \frac{d^{\xi}}{d\tau} |s(\tau)| \geq \Gamma \) is bounded and \( v_1(\tau_s) = 0 \), we will have:

\[
v_1^{\xi^{-1}}(0) \frac{\tau_s^{\xi^{-1}}}{\Psi(\zeta)} \leq -(G - \delta) \Gamma
\]

(35)

Therefore:

\[
\tau_s \leq \left( \frac{v_1^{\xi^{-1}}(0)}{(G - \delta) \Gamma} \right)^{\frac{1}{\xi - 1}}
\]

(36)

b) Sliding phase: considering the sliding surface (16), the Lyapunov candidate function can be considered as follows:

\[
V(\tau) = 0.5 s(\tau)^2.
\]

(37)

From (16), the time–derivative of the FTSMS surface is found as:

\[
s(\tau) = \dot{l}(\tau) + \left( \lambda + \mu \eta(\tau)^{\eta-1} \right) \ddot{l}(\tau).
\]

(38)

where using (14) and (15), we have:

\[
\dot{s}(\tau) = G\ddot{l}(\tau) + \left( \lambda + \mu \eta(\tau)^{\eta-1} \right) G\dddot{l}(\tau)
\]

\[
= G \left( \lambda \dot{x}(\tau) + B\dot{u}(\tau) + f(x(\tau)) + \dot{d}(\tau) \right) + \left( \lambda + \mu \eta(\tau)^{\eta-1} \right) G\dddot{l}(\tau)
\]

\[
= G \left( \lambda \dot{x}(\tau) + \dot{A}f(x(\tau)) + \lambda B\dot{u}(\tau) + \Lambda \dot{d}(\tau) + \dot{f}(x(\tau)) + \dot{B}\dot{u}(\tau) + \dot{d}(\tau) \right)
\]

\[
+ \left( \lambda + \mu \eta(\tau)^{\eta-1} \right) G \left[ \Lambda x(\tau) + B\dot{u}(\tau) + f(x(\tau)) + \dot{d}(\tau) \right].
\]

(39)
Differentiating the Lyapunov function (37) and using (39) yields:

$$\dot{V}(\tau) = s(\tau) \left[ (\lambda + \mu \eta l(\tau)^{\eta-1})G[Ax(\tau) + f(x(\tau)) + d(\tau) + Bu(\tau)]
+ G \left( A^2 x(\tau) + A f(x(\tau)) + A d(\tau) + ABu(\tau) + B \dot{u}(\tau) + \dot{f}(x(\tau)) + \dot{d}(\tau) \right) \right].$$

(40)

where substituting (17) into (40), yields:

$$\dot{V}(\tau) = -\kappa |s(\tau)|^{\eta+1} - \gamma s(\tau)^2 - \chi s(\tau) \text{sgn}(s(\tau)) + s(\tau) \left[ \left( (\lambda + \mu \eta l(\tau)^{\eta-1})G + GA \right) d(\tau) + G \dot{d}(\tau) \right].$$

(41)

Using (18) and (41), we can write:

$$\dot{V}(\tau) < -\gamma |s(\tau)|^2 - \kappa |s(\tau)|^{\eta+1}
= -\alpha V(\tau) - \beta \dot{V}^{\bar{\eta}}(\tau)$$

(42)

where $\bar{\eta} = (\eta + 1)/2 < 1$, $\alpha = 2\gamma > 0$ and $\beta = 2^{\bar{\eta}}\kappa > 0$. Thus, Lyapunov's function (37) decreases and the sliding surface converges to the origin in a finite–time. Therefore, the proof is complete.

4. Simulation results

This section, we construct a chaos-based fast synchronization between the 5-D nonlinear master-slave subsystems with parametric uncertainty and unknown disturbances. All numerical simulations were performed with Simulink MATLAB software and with a solver of ode45 and step size of 0.001. Here, we used both 5-D nonlinear master-slave subsystems (5) and (6) for the fast synchronization. Fig. 8 display the amazing 5-D nonlinear attractor of the master subsystem (5) with initial condition $x_{1m}(0) = -1.19, x_{2m}(0) = 3.8, x_{3m}(0) = 7.7, x_{4m}(0) = 2.7, x_{5m}(0) = 1.4$ and parameters $a_{1m} = 8.84, a_{2m} = 0.76, a_{3m} = 36.4, a_{4m} = 20.82, a_{5m} = 7.78, a_{6m} = 4.09, a_{7m} = 4.28$.

Similarly, the amazing 5-D nonlinear attractors of the slave subsystem (6) with initial condition $x_{1s}(0) = 1.19, x_{2s}(0) = 4, x_{3s}(0) = -1.5, x_{4s}(0) = 3.8, x_{5s}(0) = -0.75$ and parameters $a_{1s} = 8.83, a_{2s} = 0.75, a_{3s} = 36.36, a_{4s} = 20.779, a_{5s} = 7.79, a_{6s} = 4.1, a_{7s} = 4.286$ shown in Fig. 9. According to Assumption 2, total uncertainties and disturbances are added to the slave subsystem given by equation (6).
We consider the hyper-chaotic systems (5) and (6) with different initial conditions and unequal parameters for fast synchronization. According to Assumption 1, to prove chaos-based fast synchronization, the errors according to subsystems (5) and (6), can be designed as:

\[
\begin{align*}
\dot{e}_1 &= -a_1 e_1 + a_2 e_2 - a_3 e_3 \\
\dot{e}_2 &= a_4 e_1 - e_2 - a_4 e_4 - y_1 y_2 + x_1 x_3 \\
\dot{e}_3 &= a_5 e_5 + e_2 \\
\dot{e}_4 &= a_6 e_6 + e_1 \\
\dot{e}_5 &= a_7 e_5 + e_1 
\end{align*}
\]

where the errors are:

\[
e_i = \sum_{j=1}^{5} y_j - x_j
\]

System (43) in the matrix form is:

\[
\frac{de_i(\tau)}{d\tau} = \Lambda e_i(\tau) + f(e(\tau)) + B\psi(\tau) + D(\tau)
\]  

(44)

where:

\[
e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad f(e(\tau)) = \begin{bmatrix} 0 \\ x_{1s}x_{3s} - x_{1m}x_{3m} \\ -x_{1s}x_{2s} + x_{1m}x_{2m} - x_{1s}^2 + x_{1m}^2 \\ 0 \\ 0 \end{bmatrix}
\]

\[
A = \begin{bmatrix} -8.83 & 8.83 & 0 & 0 & -0.75 \\ 36.36 & -1 & 0 & -20.779 & 0 \\ 0 & 0 & -7.79 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4.1 \\ 1 & 4.286 & 0 & 0 & 0 \end{bmatrix}, \quad D(\tau) = \text{random number}
\]

Then, according to Theorem 1, chaos-based fast synchronization between two subsystems (5) and (6) with error equation (44), is definite in a finite time. Therefore, we use the new control signal (17) for synchronization. Select the positive control gains in the new controller (17) as follows:

\[
G = (85, 5, 0.05, 3795, 2096)
\]

(46)

The \(d(\tau)\) and \(f(\tau)\) functions are specified in (45). The sliding surface (16) parameters are \(\lambda = 10, \mu = 50, \eta = \frac{1}{19}\). Using the control signal (17) with \(\kappa = 20, \gamma = 30\) we are sure we will have chaos-based synchronization in a finite-time. Fig. 10 displays the complete
chaos-based fast synchronization of the master-slave subsystems (5) and (6). According to (45) in the initial conditions, the errors of finite-time fast synchronization without the controller shown in Fig. 11. By applying the control scheme (17), the errors of finite-time fast synchronization obtained are the same as those depicted in Fig. 12. Finally, the control input used for the synchronization is shown in Fig. 13. It is shown that no chattering phenomenon is existed in the control input. According to the simulation results, it is easy to observe that the 5-D nonlinear master–slave subsystems (5) and (6) are synchronized in finite–time. Fig. 14 shows the time series of chaos-based fast synchronization errors designed in this paper with the same controller. As it is known, the designed controller (17) has better results compared to the controller designed in [7]. Moreover, it is obvious from this figure that the designed control method produces a low overshoot and better settling-time.

5. Conclusions

A new five–dimensional hyper–chaos system was reported in this study. The dynamical behaviors of the new system was analyzed using time series trajectories, phase portraits, Poincare Map's (PM's), Lyapunov Exponent's (LE's), Bifurcation Diagram's (BD's) and Kaplan–Yorke dimension. The new 5–D nonlinear system had extremely complicated structure and dynamics. Next, a Fast Terminal Sliding Mode Control (FTSMC) was designed for stabilizing the new nonlinear system with disturbances and uncertainty. The main weakness of FTSMC is that it has encountered the singularity drawback which causes complex-value and a high control effort. A new controller was designed for finite-time synchronization between the two identical proposed 5-D nonlinear master-slave subsystems in the presence of matched disturbances, different initial conditions and unequal parameters. The novel terminal sliding surface can supply a particular convergence characteristic. Finally, the numerical simulations showed the viability of the designed methods. The simulations demonstrated that the analytical results and computational results are similar.

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References


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