Complex dynamics of a Fitzhugh-Rinzel neuron model considering the effect of electromagnetic induction

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Abstract

Different single computational neuron models have been proposed in the literature. Most of them belong to the Hodgkin–Huxley (H-H) type, in which they can produce complex behavior of the neuron and have efficient computational cost. In this paper, a modified FitzHugh-Rinzel (FH-R) model considering the effect of magnetic induction is proposed. Different features of the model are explored from a complex and nonlinear perspective. For instance, the impact of the magnetic field on the stability of the equilibrium points is studied by stability analysis. Bifurcation analysis reveals that the proposed neuron model has multi-stability. Furthermore, the spatiotemporal behavior of the proposed model is investigated in the complex network consisting of FH-R oscillators. And the effect of external stimuli is explored on wave propagation of the network.

Keyword: FH-R Neuron Model; Stability analysis, Bifurcation diagram, Dynamical Network; Wave Propagation.

1. Introduction

Chaos and fractal geometry are known as the key factors in modeling the brain and nervous system's alpha rhythm. It is also may help to design a more accurate model of neurological disease [1]. It is claimed that the brain's electrical activities are chaotic. Moreover, fractal geometry can be utilized to model the large population of interactive neurons crowded in the brain. [1-3]. A horde of models was developed to understand and grasp the perplexing design of the brain. One of the fundamental and well-known neuron oscillators is the Hodgkin–Huxley (H-H) model [3], in which the axon electrical activities of a squid's nerve cell were studied. This model is comprised of four differential equations with eight auxiliary algebraic functions. The H-H model variables correspond to the membrane potential, the ionic currents, and inactivation of the Na channel. The parameters of the H-H model are biologically meaningful and measurable. There are lots of H-H type models in the literature known as the conductance-based neuronal models [3-5]. The complexity of these models is lesser than the H-H model, which permits the scientist to explore the synaptic and dendritic effects [6], the interplay

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between ionic currents [7], the thermal [8] and photosensitive effect on excitability of the neuron [9, 10] and other topics related to the dynamics of a single cell [11]. These models reasonably can predict an individual neuron's response to external stimulation, but for modeling two- and three-dimensional arrays of connected neurons, it turned out to be too complicated [12].

In Hodgkin-Huxley equations, the variables membrane potential (V) and sodium current (m) undergoes rapid changes compared with the potassium (n) and inactivation of Na current (h). By arbitrarily setting these slowly changing variables constant, In 1961, Fitzhugh proposed a two-dimensional neuron model [13]. Alongside these reductions, using the phase space analysis evolves this model to the practical, intricate neuron model. One year later, in 1962, parallel to Fitzhugh, Nagumo's investigation reaches the same results [14]. Including the externally supplied current (I), the Fitzhugh-Nagumo (FHN) model is presented. FHN model can potentially describe the qualitative nature of the neural activity and impulse propagation similar to the HH model [15]. The homoclinic orbits of the FHN model from the fast-slow perspective is studied in [16]. Synchronization of both directed and undirected electrical couplings of neurons is discussed with the FHN model in [17]. Alongside the FHN neuron model, the synchronization of the network consist of the H-H neuron model is also presented in [18]. The bursting phenomenon in a coupled identical neuron is analyzed using the FHN model in [19]. Emerging the alternative chimeras in the coupled bursting neurons is explored in [20]. The collective behaviour of the hyper neural network [21] and complex neural network consisting the number of sub-networks non-locally coupled with each other [22] are also investigated. Oscillatory chaotic nature especially chaotic spiking to firing death, is identified and studied in FHN oscillators in a network [23]. Most of the studies in the literature investigated the various behaviors of FHN except the bursting nature of neurons. Chattering neurons in a cat neocortex [24] reveal fire periodic bursts of spikes when stimulated, which leads to the brain's oscillations in the gamma frequency [25, 26].

To compensate this drawback of the FHN model, the FitzHugh-Rinzel (FR) model is proposed by adding a super slow variable. In [27, 28], both spiking, bursting, and quiescence behavior of the FR model were studied. Networks of diffusively coupled neurons consist of FR oscillators exploring various complex dynamical behaviors and synchronization effects [29], especially the spatiotemporal pattern identified in coupled systems. A complex pattern with both subgroups of coherent and incoherent oscillators can emerge in the network known as chimeras. In [30-32], spiral waves in a network of phase-locked oscillators are reported as two-dimensional chimeras. Wave propagation in a network of dynamical oscillators has been reported in both excitable [33] and noisy sub-excitable media [34].

2. Mathematical Model

Fitzhugh and Rinzel introduced the FH-R model [35-37] by evolving the FHN neuron model. The traditional FHN model clearly explains the excitations and spike generations of a neuron but cannot replicate cortical neurons' firing patterns [38]. In the case of the FH-R model, an additional slow subsystem is introduced alongside the regular FHN model. This new model can produce the various firing activities for proper choices of the parameters. Here, a modified FH-R neuron model considering the impact of magnetic field is proposed by a fourth-order differential equation as
\[
\begin{align*}
\dot{v} &= v - \frac{v^3}{3} - w + y + I_{\text{ext}} - k_0 v (\alpha + \beta \phi^2) \\
\dot{w} &= \delta (0.7 + v - 0.8 w) \\
\dot{y} &= \mu (c - y - v) \\
\dot{\phi} &= -k_1 v - k_2 \phi
\end{align*}
\]  
\tag{1}

where \( v \) denotes the membrane potential, \( w \) represents the slow current, and \( \phi \) denotes the impact of magnetic field on the membrane. There are six parameters in the model (1) represented by \( I_{\text{ext}} \) denoting the external excitation current, \( \delta \) is the parameter controlling the fast subsystem, \( \mu \) is the parameter controlling the slow subsystem, \( k_0 \) is the electromagnetic induction current’s gain, \( k_1 = \frac{1}{L} \) when \( L \) represents the turn number of the cell or media as a \( L \) turns coil for estimating magnetic field effect, and \( k_2 \) denotes the cell leakage flux’s degree. The parameter \( c \) and the external current \( I_{\text{ext}} \) are taken as the control variables in the paper’s entire discussion. The parameter’s values of the system (1) are set to \( I_{\text{ext}} = 0.73, \ \delta = 0.01, \ \mu = 0.35, \ c = -0.55, \ \alpha = 0.1, \ \beta = 0.03, \ k_0 = 0.1, \ k_1 = 0.01, \ k_2 = 0.5. \)

In the entire paper, two cases are discussed, mentioning case-A for the system (1) without electromagnetic induction and case-B for the system (1) considering electromagnetic induction.

3. Stability Analysis

3.1 Case A

Firstly, the equilibrium states of Case A of the new FH-R model is determined. It is convenient to replace the numerical values for some of the parameters with arbitrary names:

\[
\begin{align*}
\dot{v} &= v - \frac{v^3}{3} - w + y + I_{\text{ext}} = F \\
\dot{w} &= P_4 (P_1 + v + P_2 w) = G \\
\dot{y} &= P_3 (c - y - v) = H \\
\end{align*}
\]  
\tag{2}

Where \( P_1 = 0.7, \ P_2 = 0.8, \ c = -0.55, \ P_4 = \delta = 0.01, \ P_5 = \mu = 0.35, \ I_{\text{ext}} = 0.73. \)

In this study, firstly \( I_{\text{ext}} \) and then \( c \) are chosen as the two bifurcation parameters by keeping the other parameters fixed at their prescribed numerical values. The equilibrium points of Eq (2) are calculated by setting the right-hand side of the equation to zero. Setting \( H=0 \), gives \( y = P_3 - v \), while setting \( G=0 \) gives \( w = \frac{v + P_1}{P_2} \). Substituting into \( F=0 \) leads to a cubic equation for as \( v \) follows:

\[
v^3 + \frac{3}{P_2} v + 3 \left( \frac{P_1}{P_2} - c - I_{\text{ext}} \right) = 0 \]  
\tag{3}

Solving Eq. 3 leads to one real root for \( v \) when the external current is in the range of \([-10,10]\). For \( -2.86552 \leq I_{\text{ext}} \leq 2.53188 \), there is a pair of complex roots with nonzero imaginary parts. The equilibrium point of Eq. 2 for the chosen set of parameters are
\((v^*, w^*, y^*) = (-0.51877, -0.22654, -0.3223E - 10)\). Fig 1 shows the absolute value of the equilibrium point for \(0 \leq I_{ext} \leq 10\).

If the parameter \(c\) is varied in the range of \(-1.5 \leq I_{ext} \leq 1.5\), and the remaining parameters are fixed at their prescribed values, Fig. 2 is obtained.

The third order Jacobian matrix is utilized to analyzed the local stability of the equilibrium point of the model as follows:

\[
J = \begin{pmatrix}
1 - v^2 & -1 & 1 \\
-P_4 & P_3 P_4 & 0 \\
-P_3 & 0 & -P_3
\end{pmatrix}
\]

(4)

The cubic characteristic equation, the determinant of \(J - \lambda I_3\) is

\[
\lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 = 0
\]

(5)

Where

\[
A_2 = P_2 P_4 + P_5 + v_r^2 - 1 \\
A_1 = P_4 + P_5 + P_2 P_4 + (1 - v_r^2)(P_3 + P_2 P_4) \\
A_0 = P_4 P_5 (1 + P_2) + P_2 P_4 (v_r^2 - 1)
\]

(6)

Fig. 3 shows a section of the plots of the three eigenvalues of the characteristic equation (5) as \(I_{ext}\) varies in the range of \([-10, 10]\) in which one of them is real, and the others are complex.

There are two Hopf bifurcations in Fig. 3, where the real part of the complex eigenvalues crosses the imaginary axis, at \(I_{ext} = 0.226\) and \(I_{ext} = 2.666\). The frequencies at the Hopf bifurcations are both equal to \(\omega = 0.49247\) Fig 4 shows the corresponding plots when parameter \(c\) is varied. Now the two Hopf bifurcations occur at \(c = -1.054\) and \(c = 1.344\), with frequencies again given by \(\omega = 0.49247\).

Criteria for detecting the occurrence of a Hopf bifurcation can be concluded by substituting \(\lambda = I \omega\) into Eq. 5. Assuming the same real and imaginary parts of the eigenvalues give:

\[
\omega^2 = \frac{A_0}{A_2} = A_1 > 0
\]

(7)

which concluded the sufficient Hopf bifurcation criteria: \(A_0 = A_1 A_2\). Since both \(c\) and \(I_{ext}\) appear in the expression for \(v_r\), it is not straightforward to obtain the curve of Hopf bifurcations as a function of each of these two parameters analytically. However, the locations of the two Hopf bifurcations, as well as the values of their frequencies, have an underlying symmetry.

Fig. 5 shows the plots of \(\omega^2 = \frac{A_0}{A_2}\) (blue points), \(\omega^2 = A_1\) (red points) and the Hopf bifurcation condition \(HB = A_1 A_2 - A_0\) when \(I_{ext}\) increases. Note the symmetry of the Hopf Bifurcation points about \(I_{ext} = 1.199\), which is at the minimum value of \(\omega^2 = 0.0048\). This symmetry explains why the frequencies at both Hopf bifurcation points are the same and equal to \(\omega = 0.49248\). Moreover, the results of Figs. 3 and 4 are verified by plotting bifurcation transition diagrams of the maxima of \(v\) in each cycle as \(I_{ext}\) increases (Fig. 6 a) and as \(c\)
increases (Fig. 6 b). In each plot, the two Hopf bifurcation points $HB_1$ and $HB_2$ are labeled. In each case, the values for these four points agree with the values from the eigenvalue plots of Figs. 3 and 4.

### 3.2 Case B

By including the effects of electromagnetic induction $\phi(t)$ into the FH-R system Eq (2), the four-dimensional nonlinear equations are obtained:

$$
\begin{align*}
\dot{v} &= v - \frac{v^3}{3} - w + y + I_{ext} - k_0v(\alpha + \beta \phi^2) = F \\
\dot{w} &= P_4(P_1 + v - P_2w) = G \\
\dot{y} &= P_3(c - y - v) = H \\
\dot{\phi} &= -k_1v - k_2\phi = K
\end{align*}
$$

The additional parameters in Eq. (8) and their prescribed values are set to $\alpha = 0.1, \beta = 0.03, k_0 = 0.01, k_2 = 0.5$. The equilibrium state values are:

$$
W_e = \frac{1}{P_2}(P_4 + V_c), \quad Y_c = c - V_e, \quad \phi_c = \frac{k_1}{k_2}V_e
$$

where $V_e$ is the real solution to the modified cubic equation:

$$
(1 + 3\frac{\beta k_0 k_1^2}{k_2^3})V^3 + 3(\frac{1}{P_2} + k_0 \alpha)V + 3(\frac{P_1}{P_2} - c - I_{ext}) = 0
$$

when $k_0 = k_1 = 0$ the Eq (2) of Case A can be recovered. The equilibrium points of Eq. 10 are shown in Fig. 7 a) and Fig. 7 b), by varying the parameters $I_{ext}$ and $c$ in turn. In both cases, the variation of $\phi$ is too tiny: between $-0.572497242204 \times 10^{-10} < \phi_e < 0.505714997709E - 01$ in Fig. 7 a) and between $-0.244182952138E - 01 < \phi_e < 0.222041836952E - 01$ in Fig. 7 b). The range is extended for $c$ in order to capture the Hopf bifurcation values for Case B.

Following the procedure for Case A, the linear stability of the real equilibrium state by computing the fourth-order Jacobian matrix is determined as follows:

$$
J_4 = \begin{pmatrix}
1 - V_e^2 - k_0(\alpha + \beta \phi^2) & -1 & 1 & -2k_0\beta V_e \phi_e \\
-1 & -P_4 & P_4 & 0 \\
0 & P_3 & -P_3 & 0 \\
k_1 & 0 & 0 & -k_2
\end{pmatrix}
$$

The characteristic equation is now quartic $\det(J_4 - \lambda I_4) = 0$:

$$
\Lambda^4 + B_3\Lambda^3 + B_2\Lambda^2 + B_1\Lambda + B_0 = 0
$$

If we define

$$
C = 1 - V_e^2 - k_0(\alpha + \beta \phi^2), D = 2k_0\beta V_e \phi_e
$$

Then,
To find the Hopf bifurcations, we substitute \( \Lambda = 1\Omega \) into Eq. (12) and equate real and imaginary parts. This gives \( \Omega^2 \) as a common positive root of \( \Omega^4 - B_2\Omega^2 + B_0 = 0 \) and \( \Omega^2 = \frac{B_1}{B_3} \). When both conditions hold, we can eliminate \( \Omega \) to obtain:

\[
B_1^2 - B_2B_3 + B_0B_1^2 = 0
\]  

Again, the complexity of the coefficients, with the two control parameters being found in the fixed point expression, renders an analytical solution extremely cumbersome. So, we again rely on numerical integrations to determine the eigenvalues.

Fig. 8 shows a comparison of the real eigenvalues for Case B (blue) with the corresponding real eigenvalues for Case A as \( I_{\text{ext}} \) increases.

Apart from the additional curve due to the \( \phi \) variation, the curves are very nearly identical. The locations of the Hopf bifurcations are shifted slightly to \( HB_1 \) at \( I_{\text{ext}} = 0.23 \) and \( HB_2 \) at \( I_{\text{ext}} = 2.62 \). The frequencies of the bifurcating periodic limit cycles are both equal to \( \Omega = 0.49238 \). Fig. 9 shows the corresponding plots for the imaginary eigenvalues for both Cases.

Fig. 10 shows the analogous plots when \( c \) increases. The two Hopf bifurcation points are located at \( c = -1.054 \) for \( HB_1 \) and \( c = 1.344 \) for \( HB_2 \), with frequencies \( \Omega = 0.492476 \) for the chosen set of parameter values, the linear eigenvalues are \(-0.5, -0.0362, 0.2013 \pm 0.2784i\).

### 4. Bifurcation diagram analysis:

Complete dynamical behavior of the system can be investigated by exploring the impact of the FH-R model’s parameters. The bifurcation diagrams are derived for two cases (A and B) and analyzed.

#### 4.1 Bifurcation of FH-R model for Case-A (without magnetic induction):

Firstly, the bifurcation of the FH-R system without magnetic induction is derived. By considering the parameter \( c \) as the control parameter, the other parameters are the same as Eq. 2. The dynamical behavior in the bifurcation diagram according to the parameter \( c \) is shown in Fig. 11 a). The parameter \( c \) varies from \([-0.6,0]\), and the local maxima of \( v \) are plotted. Different behavior such as limit cycle, period doubling, chaotic oscillation with this parameter range can be observed. Packs of chaotic regions, interestingly, the density of the chaotic region also varies while the parameter increases. Both the period-doubling route to chaos and period halving exit from chaos can be detected

Two different colors of Fig 11 a) correspond to the forward (blue plot) and backward (red plot) continuation. To plot the blue dots, the parameter \( c \) is increased from \(-0.6 \) to 0. While for plotting the bifurcation diagram corresponding to red dots, the parameter \( c \) is decreased from
0 to -0.6. In both cases, the initial conditions are chosen from the end values of the states at each step. The maximum peaks of the variable $v$ are chosen to plot in each step in both red and blue dots. Differences between two bifurcation diagrams show the existence of the multi-stability in the neuron model, which is confirmed by plotting the corresponding Lyapunov exponent's diagrams in Fig. 11 b). The Wolf algorithm is used for calculating the Lyapunov exponent's diagrams [39].

4.2 Bifurcation of FH-R model for Case-B (with magnetic induction):

The second step is dedicated to Case-B. In this case the neuron model is exposed to magnetic induction. The control parameter for bifurcation is excitation current $I_{ext}$ while the other parameters are as the same as Eq. 8. The bifurcation diagram is presented in Fig.12 a). Range of the parameter $I_{ext}$ for the analysis is taken as [0.25, 0.75], and the local maxima of $v$ is plotted. The limit cycle, period doubling, chaotic oscillation with this parameter range can be observed. Packs of chaotic regions, interestingly, the density of the chaotic region also varies while the parameter increases. The property of Antimonotonicity can be clearly seen in the diagram, which such a special property was not observed in earlier studies.

Same as Fig. 11 a), two different colors are chosen to correspond to the forward (blue plot) and backward (red plot) continuation. To plot the blue dots, the parameter $c$ is increased from 0.25 to 0.75. While for plotting the bifurcation diagram corresponding to red dots, the parameter $c$ is decreased from 0.75 to 0.25. in both cases, the initial conditions are chosen from the end values of the states at each step. The maximum peaks of the variable $v$ are selected to plot in each step in both red and blue dots. Differences between two bifurcation diagrams show the existence of the multi-stability in the neuron model, which is confirmed by plotting the corresponding Lyapunov exponent's diagrams in Fig. 12 b).

5. Spatiotemporal dynamics of the FH-R neuron network:

After investigating the local kinetics of the FH-R model, our interest is now to discuss the complex network behavior of the FH-R model. For this, a network constructed by 110×110 FH-R neurons whose local kinetics is governed by Eq. (1) for both Case-A and Case-B with Neumann boundary condition. To explore the propagation of the waves in the network, an external stimulus ($\psi(t)$) is exposed to the center of the network. The variables’ initial states are set to (0,0,0) for Case-A and (0,0,0,0.5) for Case-B. The FH-R neuron network of 110×110 size can be defined as,

$$
\begin{cases}
\dot{v}_j = v_j - \frac{v_j^3}{3} - w_j + y_j + I_{ext} + D(v_{i+1,j} + v_{i-1,j} + v_{j+1} + v_{j-1} - 4v_j) + \chi(t)\psi(t)\psi_{\delta \beta}, \\
\dot{w}_j = \delta(0.7 + v_j - 0.8w_j), \\
y_j = \mu(c - y_j - v_j)
\end{cases}
$$

Case – A: 
(16)
\[\hat{v}_{ij} = v_{ij} - \frac{v_{ij}^3}{3} - w_{ij} + y_{ij} + I_{\text{ext}} - k_0 v(\alpha + \beta \phi^2) + D(v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{ij}) + \chi(t)\psi_{\theta_1} \psi_{\theta_2} \]

**Case - B:**

\[
\begin{align*}
\hat{v}_{ij} & = \delta(0.7 + v_{ij} - 0.8w_{ij}) \\
y_{ij} & = \mu(c - y_{ij} - v_{ij}) \\
\phi_{ij} & = k_1 v_{ij} - k_2 \phi_{ij}
\end{align*}
\]

(17)

Where \( D \) shows the electrical coupling strength. \( \chi(t) = A \sin(\omega t) \) is the external stimuli which are applied to the network when \( \xi_{\theta_1} = 1, \xi_{\theta_2} = 1 \) for \( i = \theta_1 = 55, j = \theta_2 = 55 \), respectively. The parameter values of (16) are \( I_{\text{ext}} = 0.73, \delta = 0.01, \mu = 0.35, c = -0.55 \) and \( \alpha = 0.1, \beta = 0.03, k_0 = 0.1, k_1 = 0.01, k_2 = 0.5 \) are the additional parameters required for (20).

The spatiotemporal behavior of the first variable of the model is explored, and the final patterns at \( t=3000 \) are shown. The entire discussion is first subdivided into Case-A and Case-B, and in each case, we have discussed the impact of the diffusion coefficient and stimuli parameter (frequency/amplitude).

### 5.1 Case-A (without magnetic induction):

The neuron model's spatiotemporal pattern without considering the effect of magnetic induction is investigated in three steps. At first, the effect of coupling strength (D) is explored when the amplitude and frequency of the external stimuli are set to \( A = \omega = 1 \). Fig. 13 exhibits the various wave propagation of the network for five different coupling strengths.

According to the results of Fig. 13, it can be concluded that increasing the coupling strength leads to the emergence of more ordered patterns. For instance, the small value of the coupling strength in Fig. 13 a) ends with irregular patterns. By further increases in the coupling strength (Fig 13 b) c), and d)), some wave's seeds appeared in the network, which is getting strong respectively. Finally, when the coupling strength is set to \( D=1 \), strong regular waves formed in the network.

In the second step, the effect of frequency of the stimuli is investigated on the emergence of the network's spatiotemporal pattern. Accordingly, the frequency of the external stimuli is increased smoothly, and the spatiotemporal pattern of the network is plotted in Fig. 14.

Fig. 14 shows the effect of the external stimuli's frequency on the wave propagation in the network. It can be comprehended that the wave seeds in the network under stimuli with low frequency propagates irregularly. However, if the frequency goes over a specific threshold (\( \omega = 0.1 \)), the wave propagation turns periodic on the network.

The last step is considered the amplitude of the external stimuli as a varying parameter to study the network's spatiotemporal pattern. The collective behavior of the network is examined with six different values of the amplitude in Fig. 15.

Changing the amplitude of the external stimuli can affect the behavior of the network. Fig. 15 shows the final patterns of the network at \( t=3000 \) according to six different amplitude values. The irregular pattern of wave propagation can be seen in the network for amplitude value under the \( A = 1 \). However, the network enters into the regular and periodic mood for a larger
amplitude's value. Comparing the network's spatiotemporal pattern results according to these three different factors revealed that the wave propagation in the network is almost affected by the coupling strength and the frequency of the external stimuli than the amplitude.

5.2 Case-B (with magnetic induction):

Like the last section, the coupling strength and parameters of the external stimuli are considered effective main factors in the network's spatiotemporal behavior. According to the results of Fig. 15, which shows that the amplitude of the network cannot be considered as an effective parameter on wave propagation. To this end, the impact of coupling strength and frequency of the external stimuli are investigated on the emergence and propagation of the seed waves. Accordingly, this section can be divided into two steps. The first one is dedicated to the impact of coupling strength, and the second step investigates the effect of the frequency of the stimuli. Fig. 16 shows the spatiotemporal pattern of six different coupling strengths.

Like Case-A, increasing the coupling strength of the network in Case-B leads to more ordered patterns. Comparing the results of Fig. 16 with Fig. 13, reveals that the coupling strength effect on wave propagation in the network of Case-B is more than Case-A.

In the second step, the spatiotemporal pattern of the network for six different frequency of the external stimuli are shown in Fig. 17.

For $\omega = 0.0001$ in Fig. 17 a) periodic waves propagate in the network, and the spatiotemporal pattern of the network is regular. Increasing the frequency for the range of $0.0001 < \omega < 0.1$ disturbed the network's regularity and led to irregular wave propagation. But the network comes back to the periodic waves propagates for $\omega = 0.1$. Further increases in the frequency of the stimuli retrieved the irregular wave propagation in the network again. Therefore, the overall conclusion is that in contrast with the result of Case-A, increasing the external stimuli's frequency has an inverse effect on propagating regular waves on the network.

6. Conclusion

In this paper, the dynamical behavior of the modified FH-R neuron model exposed to the magnetic field (Case-B) is compared to the FH-R model (Case-A). To this end, the stability of the equilibrium points is studied in both models. And the sufficient criteria for detecting Hopf bifurcation is calculated. By exploring the bifurcation diagram and Lyapunov exponent's diagrams revealed that both Case-A and Case-B categorized the systems with muti-stability. Also, the dynamical behavior of the network consisting of the 110x110 FH-R neuron model is investigated. The effect of coupling strength, frequency, and amplitude of the external stimuli on the emergence and propagation of the waves on the network is reported on both the network consist of Case-A and Case-B neuron models. The results revealed that increasing coupling strength leads to a more regular wave pattern. However, the effect of coupling strength on the emergence of the regular wave patterns in Case-B is more efficient than the Case-A. The frequency of the external stimuli’s effect on the spatiotemporal behavior of the network is different in Case-A and Case-B. in Case-A increasing the frequency of the external stimuli leads to more regular wave propagation in the network. In contrast, the effect of the frequency of the external stimuli in Case-B is producing more irregular waves in the network for larger frequencies. Moreover, the amplitude of the external stimuli does not show any significant effect on the network's wave propagation in both cases.

References
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Figure and Table Captions:

Fig 1: The different values of the fixed points of the neuron model Case-A (Eq. 2) as $I_{ext}$ varies. The blue, black, and red lines correspond to absolute fixed point values of $v, w,$ and $y$. According to the results, increasing the value of the $I_{ext}$ leads to increasing both values of $v, w$ but decrease the value of $y$.

Fig 2: The different values of the fixed points of the neuron model (Eq. 2) as $c$ varies. The blue, black, and red lines correspond to absolute values of $v, w,$ and $y$. Increasing the value of the $I_{ext}$ leads to increasing all values of $v, w$ and $y$.

Fig 3: Eigenvalues of the equilibrium points of variable $v$ by changing the $I_{ext}$. The blue, black, and red curves correspond to the real and imaginary parts of the eigenvalues. The two Hopf bifurcation points, $HB_1$ at $I_{ext} = 0.226$ and $HB_2$ at $I_{ext} = 2.666$ are labeled where the black curve crosses the imaginary axis.

Fig 4: Eigenvalues of the equilibrium points of variable $v$ by changing the $c$. The blue, black, and red curves correspond to the real and imaginary parts of the eigenvalues. The two Hopf bifurcation points, $HB_1$ at $c = -1.054$ and $HB_2$ at $c = 1.344$ are labeled where the black curve crosses the imaginary axis.

Fig 5: The behaviors of $\omega_2$ (blue), $\omega_1$ (red) and the Hopf bifurcation condition $HB$ (black) as $I_{ext}$ increases.

Fig 6: a) The bifurcation transition plots according to changing the $I_{ext}$ as a function of $v_{max}$ b) the bifurcation transition plots according to changing the $c$ as a function of $v_{max}$.
Fig 7: a) The different values of the fixed points of the neuron model (Eq. 10) as $I_{ext}$ varies. The blue, black, red, and green lines correspond to absolute values of $V_e, W_e, Y_e$ and $\Phi_e$.

Fig 8: Comparison of the real eigenvalues for Case B (blue) and Case A (red) as $I_{ext}$ increases.

Fig 9: A comparison of the imaginary parts of the eigenvalues for Cases A (red) and B (blue) as $I_{ext}$ increases.

Fig 10 a) Both the real eigenvalues for Case B (blue) and Case A (red) as $c$ increases. B) both the imaginary parts of the eigenvalues for Cases A and B as $c$ increases.

Fig. 11 a) Bifurcation of the FH-R neuron model with $c$ using forward and backward continuation shown in blue and red plot respectively. B) The corresponding maximum Lyapunov exponents (MLEs) for forward and backward continuation.

Fig. 12 a) Bifurcation of the FH-R neuron model with $I_{ext}$ using forward and backward continuation shown in blue and red plot respectively. B) The corresponding maximum Lyapunov exponents (MLEs) for forward and backward continuation.

Fig. 13 The spatiotemporal behavior of the network for five different coupling strengths a) $D = 0.1$, b) $D = 0.3$, c) $D = 0.5$, d) $D = 0.8$, and e) $D = 1$. Wave propagation in the network with small coupling strength is more irregular than the network with larger coupling strength.

Fig. 14 The spatiotemporal behavior of the network (Case A) for six different frequency a) $\omega = 0.00001$, b) $\omega = 0.0001$, c) $\omega = 0.001$, d) $\omega = 0.01$, e) $\omega = 0.1$, and f) $\omega = 1$. For smaller frequencies, more wave seeds are propagating irregularly. However, increasing the external stimuli's frequencies leads to lower wave seeds with more regular wave propagation in the network.

Fig. 15 The spatiotemporal behavior of the network for six different amplitude a) $A = 0.0001$, b) $A = 0.001$, c) $A = 0.01$, d) $A = 0.1$, e) $A = 1$, and f) $A = 5$. The amplitude of the external stimuli does not have a concrete effect on wave propagation on the network.

Fig. 16 The spatiotemporal behavior of the network (Case-B) for six different coupling strengths a) $D = 0.1$, b) $D = 0.3$, c) $D = 0.5$, d) $D = 0.7$, e) $D = 0.9$ and f) $D = 1$. The larger the coupling strength is, the more regular waves propagate in the network.

Fig. 17 The spatiotemporal behavior of the network (Case-B) for six different frequency a) $\omega = 0.00001$, b) $\omega = 0.001$, c) $\omega = 0.01$, d) $\omega = 0.1$, e) $\omega = 1$, and f) $\omega = 10$. The frequency of the external stimuli in Case-B has an inverse effect on producing regular waves on the network. The larger frequencies lead to irregular wave propagation in the network.
Fig 8

Fig 9

Fig 10
Fig. 11
Fig. 12

Fig. 13