Hybrid integral transform and $p$-version FEM for thermo-mechanical analysis of a functionally graded piezoelectric hollow cylinder subjected to asymmetric loads

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As the first endeavor, a combination of fast Fourier transform (FFT) and $p$-version of finite element method is proposed for electro-thermo-elastic analysis of a thick hollow cylinder under asymmetric thermal loadings. In shells of revolution, the proposed FFT-$p$FE method is accompanied by a significant decrease in the computational costs. Due to the problem periodicity in such structures, the FFT technique is used to discretize the governing equations into a set of harmonics. Each harmonic is then partitioned using higher order finite elements. Hierarchical finite elements based on Legendre polynomial interpolation functions are utilized to discretize 2D governing equations of a functionally graded piezoelectric (FGP) cylinder. 3D governing equations of a FGP hollow cylinder are then discretized by using the higher-order Lagrangian finite elements. The effects of FFT grid-size as well as the order of the interpolation functions are investigated on convergence behavior of the proposed mixed FFT-$p$FE method. The material properties, with the exception of the Poisson’s ratio, are considered to vary along the radius of the cylinder. The governing equations are derived using the principle of virtual displacements. For a 3D FGP hollow cylinder, the influence of axially and circumferentially non-symmetric thermal loadings is presented in contour plots.

**Key words:** FFT; higher-order finite elements; piezoelectric cylinder; asymmetric thermal loads.

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{ij}$</td>
<td>Material properties coefficients</td>
</tr>
<tr>
<td>$D_i$</td>
<td>Electric displacements</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>Piezoelectric constants</td>
</tr>
<tr>
<td>$E_i$</td>
<td>Components of electric field</td>
</tr>
<tr>
<td>$\hat{K}_{ij}$</td>
<td>Components of stiffness matrix</td>
</tr>
<tr>
<td>$N_p(\xi)$</td>
<td>Legendre polynomial of degree $p$</td>
</tr>
<tr>
<td>$\hat{Q}_i$</td>
<td>Components of temperature gradient vector</td>
</tr>
<tr>
<td>$\bar{T}_i$</td>
<td>Temperature variable of $k$th harmonic in Fourier space</td>
</tr>
<tr>
<td>$(\hat{u}_i)^j$</td>
<td>Nodal values of displacement components in $k$th harmonic</td>
</tr>
<tr>
<td>$\mathbf{w}$</td>
<td>Vector of weighting functions</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Coefficients of thermal expansion</td>
</tr>
<tr>
<td>$\lambda_r$</td>
<td>Thermal conductivity coefficient</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>Basis functions</td>
</tr>
<tr>
<td>$\eta_{ij}$</td>
<td>Dielectric constants</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>Components of stress tensor</td>
</tr>
<tr>
<td>$\varepsilon_{ij}$</td>
<td>Components of strain tensor</td>
</tr>
<tr>
<td>$P_i$</td>
<td>Pyroelectric constants</td>
</tr>
</tbody>
</table>
1 Introduction

Over the recent years, the modern industrial productions are concerned with the layered piezoelectric structures produced in the form of thick cylinder. They can be used as smart devices, e.g. in engineering applications with an active control. In this class of piezoelectric structures, material properties suddenly change passing through the cylinder thickness from a layer to the other. So, interface between two dissimilar materials often is accompanied by the stress concentration and huge thermal stresses [1]. Consequently, an advanced class of heterogeneous composite materials has been developed to overcome these issues in which the mechanical properties continuously and gradually vary from one point to the other [1-4]. Therefore, the electro-thermo-elastic analyses of the Functionally Graded Piezoelectric Materials (FGPMs) seems to be attractive. Indeed, there are several numerical and analytical studies concerned with these problems, as briefly reviewed in the sequel.

Ootao et al. [5] has presented an analytical approach for transient thermo-electro-elastic analysis of Functionally Graded Piezoelectric (FGP) hollow cylinder. A three-dimensional exact solution of functionally graded piezoelectric shells under cylindrical bending has been performed by WU and Syu [6] using the perturbation method. The effect of thermo-electro-mechanical loading on a functionally graded shell in the form of thick sphere bonded with piezoelectric layers is carried out by Alashi et al. [7]. Due to their ability in withstanding high-pressure loadings with respect to their weight, shells of revolution have received more attention in numerous applications of structural engineering. Recently, the authors have derived a set of field equations for a functionally graded piezoelectric shell of revolution with arbitrary curvature and variable thickness using tensor analysis in curvilinear coordinate systems [8]. Nejad et al. [9] have developed a general formulation for thermo-elastic analysis of a functionally graded thick shell of revolution with arbitrary curvature and variable thickness by using higher-order shear deformation theory. A semi-analytical method has been frequently used by Santos and his co-workers [10-14] for three-dimensional analysis of axisymmetric shells of revolution. Special emphasis in these articles was given to the coupling between symmetric and anti-symmetric terms in truncated Fourier expansion of the dependent variables and loading. It was also demonstrated that the material properties have significant effect when comparing coupled and uncoupled results. The analysis of shells of revolution is often accompanied by a Fourier representation of unknown variables and the boundary conditions along the circumferential direction. For example, this capability was exploited for the bending and free vibration analyses of layered anisotropic shells of revolution by Noor and Peters [15]. The coupling effect between symmetric and anti-symmetric modes for composite laminated shells of revolution was investigated by using the aforementioned semi-analytical method [16]. Sivadas and Ganesan [17] studied the coupling effects regarding to the symmetric and asymmetric vibration modes of laminated composite shells of revolution using double Fourier series approximation and finite element method. Fourier series is used by Loghman et al. [18] for thermo-elastic analysis of a functionally graded cylinder under asymmetric thermal and mechanical loadings subjected to uniform magnetic field. In the 3D investigation of the FGP cylindrical shells, Chen et al. [19] analyzed dynamic response of a FGP hollow cylinder which encompass a compressible fluid. A semi-analytical method using Laplace transform and DQ methods is introduced by Liang et al. [20] for investigating the dynamic behavior of FGP cylindrical panels in different boundary conditions. Propagating the waves in reinforced FG porous plates was investigated by Gao et al. [21] using CPT, FSDT and HSDT.

Each numerical method may be inherently accompanied by some deficiencies regarding the modeling of geometry, the discretization and the satisfaction of boundary conditions. Semi-analytical and mixed methods are considered to be appropriate to overcome these shortcomings. In this regard, the Fourier spectral method can provide an appropriate base through defining the trial and test functions.
Dehghan et al. [23] and Malekzadeh [24] combined Finite Element Method (FEM) and Differential Quadrature (DQ) technique to gain the ability of mixed method in modeling the complex geometry and boundary conditions as well as the fast convergence. The time-dependency in a nonlinear thermal problem was removed by effective combination of the Laplace transform and finite element method by Lin et al. [25]. Recently, Laplace transform technique and multi-scale FEM was combined to solve coupled partial differential equations of the flow in a dual-permeability system by Liu et al. [26]. Entezari et al. [27] used the ability of Carrera Unified Formulation (1D FE-CUF) to thermo-elastic wave propagation analysis of functionally graded disks. Using this methodology, displacement and temperature field variables are interpolated using one-dimensional finite elements to discretize the governing equations. A layerwise differential quadrature method (LW-DQM) was combined with a non-uniform rational B-spline (NURBS) multi-step time integration scheme by Heydarpour et al. [28]. They investigated thermal shock wave effects in functionally graded panels bonded with piezoelectric layers. An ingenious effort, using the Rayleigh-Ritz method, has been made by Qin et al. [29-31] for analyzing the plates and shells reinforced by carbon Nano-tubes (CNT) and graphene platelets (GPLs). Nowadays, the application of the porous materials and nanocomposites (CNT-filled polyethylene) is to be increased particularly in the offshore industries. Until now, some measures have been taken for investigating these materials [32, 33]. A vast review of literatures reveals that the electro-thermo-elastic analysis of FGP shells of the revolution by mixed numerical FFT-pFE method is scarce. Until now, the transient thermo-elastic analysis of the disk brake has been studied by means of the fast Fourier transform (FFT) and finite element method [34-36]. Currently, a modified Fourier series solution is developed for vibration analysis of the shells of revolution by Jin et al. [37]. Mohazzab and Dozio [38, 39] employed the spectral collocation method for prediction of natural frequencies of laminated curved panels and skew plates, respectively. In this regard, Xie et al. [40] employed the spectral collocation method for free vibration analyzing of the composite shell supported by elastic foundation.

Here, the fast Fourier transform and the $p$-version of the finite element method are combined to gain more advantages in the analysis of shells of revolution. The size and shape of the elements and the approximation properties of the solution space has significant effects on the quality of the finite element solution. The desirable precision in the finite element procedure is obtained by using $h$- and $p$-version techniques. In the $p$-version, this can be achieved by increasing the order of shape functions. Shape functions for a one-dimensional hierarchic element are produced by integrating Legendre polynomials [41]. A major property of these shape functions is favorable orthogonality that leads to sparse and well-conditioned stiffness matrices [42]. Yu et al. [43] used hierarchical finite beam elements to static and dynamic analysis. The rate of convergence in Timoshenko beam improved by Tai and Chan [44] by considering Legendre based hierarchic shape functions. Green’s-function-based finite element formulation (HSP-FEM) is utilized by Wang and Qin [45] for simulating bioheat transfer in the human eye and by Cao et al. [46] for analyzing three-dimensional elastic problems with body forces.

The FFT-pFE method proposed in this paper can be appropriately recognized as a semi-analytical technique since it attains higher precision and has less computational cost in comparison with 3D traditional FE computer programs. Indeed, less computational effort is satisfied by the reduction of three-dimensional to two-dimensional governing equations. In addition, the proposed mixed method benefits from the advantages of the FFT algorithm. The discrete transform can be computed using matrix-vector multiplication with the $2N^2$ operations. The fast Fourier transform technique which is successfully implemented in this paper has an operation count with leading term $(\frac{5}{2})N \log_2 N$. From the numerical aspects, it is accompanied by a great time saving of the computations. The computational routines for FFT and IFFT are available in the software package called ‘FFTW3’. It is a ‘C’ subroutine
library for computing the discrete Fourier transform in one or more dimensions. In general, restrictions of the present mixed method can be explained from two points of view:

1. The grid size of the FFT method should be uniform along the circumferential direction.
2. Periodicity of the geometry (which is automatically satisfied for the periodic domains, such as the cylinder).

In this paper, the following steps are considered. At first, the 2D heat conduction and electro-thermo-elastic partial differential governing equations (PDGEs) of the FGP hollow cylinder with plane strain assumption are discretized using the proposed FFT-pFE method. Discrete Fourier Transform technique is actually used to project the PDGEs from real space into the Fourier space and vice versa. Hierarchical finite elements are introduced here for discretizing the meridian section of the cylinder. In the next step, we deal with 3D thermo-electro-elastic analysis of a FGP hollow cylinder whose meridian section is divided into the subdomains named as second-order 2D finite elements of Lagrange family. According to the existing algorithm, numerical results related to the temperature field, as input data, are imported for electro-thermo-elastic analyzing of a FGP shell. Eventually, the obtained results are inverted into the real space through inverse Fast Fourier Transform (iFFT) technique.

2 Problem formulation in 2D space using hierarchic elements

2.1 The heat conduction problem

In a thermo-elastic analysis, the primary step is to extract the distribution of temperature field within the physical domain of the problem, to which is imposed appropriate boundary conditions. Therefore, the steady-state heat conduction equation is presented here for an FGP hollow cylinder in the absence of heat generation. This equation can be readily derived from the energy conservation law along with the Fourier heat conduction law. For a FG cylinder, this equation in radial and circumferential directions reads [8]

\[
\lambda_\theta \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \lambda_\theta \frac{\partial T}{\partial r} + \lambda_\theta \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{d\lambda_\theta}{dr} \frac{\partial T}{\partial r} = 0, \quad a \leq r \leq b, \quad 0 < \theta \leq 2\pi.
\]

The Dirichlet boundary conditions of the following form are assumed in this work:

\[
T(r = a, \theta) = T_i(\theta), \quad T(r = b, \theta) = T_o(\theta),
\]

in which \( T \) is the temperature field, and \( \lambda_\theta = \lambda_\theta(r) \) is the radially-variable thermal conductivity coefficient. \( T_i \) and \( T_o \) are temperature distributions at inner and outer surfaces of the cylinder whereas \( a \) and \( b \) indicate inner and outer radii of the cylinder, respectively. Here, the thermal conductivity coefficient, \( \lambda_\theta(r) \), is considered to be graded according to a power function of the form

\[
\lambda_\theta = \lambda_{\theta 0} \bar{r}^m, \quad \bar{r} = \frac{r}{r_b},
\]

where \( \lambda_{\theta 0} \) and \( m \) are constants.

2.2 The electro-thermo-elastic problem

In a 2D thermo-electro-elastic analysis, the primary step is to extract the distribution of temperature field within the physical domain of the problem, to which is imposed appropriate boundary conditions. Therefore, the steady-state heat conduction equation is presented here for an FGP hollow cylinder in the absence of heat generation. This equation can be readily derived from the energy conservation law along with the Fourier heat conduction law. For a FG cylinder, this equation in radial and circumferential directions reads [8]
where $\lambda_o$ is some material constant regarding thermal conductivity. $m$ is the power law index of the material. As mentioned previously, the temperature field can be considered as periodic in the circumferential direction, and hence, periodic boundary condition is applied in this direction. At first, Fourier transform is used for solving this partial differential equation. For this purpose the temperature field in Eq. (1) is replaced by the approximation function, including discrete Fourier coefficients. As shown in Figure 1, the temperature field is defined on discrete points $\theta_k$ with $k=0, 2, ..., N-1$. In Fourier space, the integer parameter $k$ is the wavenumber in the circumferential direction. The selected approximation function is defined as

$$T(r, \theta_j) = T_j \approx \sum_{k=\frac{N}{2}}^{N} \hat{T}^k(r)e^{i\theta_j}, \quad j = 0, ..., N - 1$$  \hspace{1cm} (4)

By substituting this ansatz into Eq. (1), the heat equation and its related boundary conditions in radial direction become

$$\lambda_r \frac{d^2 \hat{T}^k}{dr^2} + \lambda_r \frac{\lambda_c}{r} \frac{d\hat{T}^k}{dr} + \lambda_r \frac{d^2 \hat{T}^k}{dr} - \lambda_r \frac{k^2}{r^2} \hat{T}^k = 0, \quad \text{For } k = 0, ..., N - 1,$$  \hspace{1cm} (5)

$$\hat{T}^k(r=a, \theta_q) = \hat{T}_a(\theta_q), \quad \hat{T}^k(r=b, \theta_q) = \hat{T}_b(\theta_q),$$  \hspace{1cm} (6)

where $(...)'$ denotes the first derivative of any arbitrary function with respect to the $r$ variable.

The method of weighted residuals provides a framework for solving partial differential equations approximately. The general form of the partial differential equation to be solved is typically written as

$$\hat{f}(\hat{T}^k(r)) - f = R.$$  \hspace{1cm} (7)

In general, an approximate solution, say $\hat{T}^k$, does not exactly satisfy Eq. (5). Thus, the weighted residual method is utilized to find a solution such that the residual $R$ is to be minimized in the weighted integral sense

$$\int_{\Omega} \mathbf{w} R d\Gamma = 0,$$  \hspace{1cm} (8)

where, $\mathbf{w}$ is a vector of weighting functions and

$$d\Gamma = rdr.$$

Substituting Eq. (7) into Eq. (8) yields

$$\int_{\Omega} \mathbf{w} \left[ \frac{1}{r} \frac{d}{dr} \left( \frac{r \frac{d\hat{T}^k}{dr}}{\lambda_c} \right) + \frac{\lambda_c'}{\lambda_r} \frac{d\hat{T}^k}{dr} - \frac{k^2}{r^2} \hat{T}^k \right] rdr = 0,$$

For $k = 0, ..., N - 1, \quad q = 1, 2, ..., N_q.$
By assuming \( w_q \) to be sufficiently differentiable weight functions, the original equation can be easily transformed into the weak form using the integration by parts technique. In the finite element context and based on the Bubnov-Galerkin method, the weight functions \( w_i \) are the same as the interpolant polynomials. In each hierarchical element, it is presumed that the temperature variable \( \hat{T}^k(r) \) is approximated as a series function containing the product of basis functions \( \phi_j^e \) and nodal values:

\[
\hat{T}^k(r) = \sum_{j=1}^{N} \psi_j^e(r) \hat{T}_j^k, 
\]

(10)

where \( \hat{T}_j^k \) are the nodal values regarding the temperature field in complex domain. Higher-order Legendre polynomials are considered as the basis functions \( \psi_j \) in a hierarchical element. These basis functions and further details on hierarchical elements will be presented in the next section. Now, the stiffness matrix can be obtained using the Gauss-Legendre quadrature technique and is arranged in matrix form as

\[
\begin{bmatrix} \hat{K} \end{bmatrix} \{ \hat{T} \} = \{ \hat{Q} \},
\]

(11)

where \( \begin{bmatrix} \hat{K} \end{bmatrix} \) and \( \{ \hat{Q} \} \) are stiffness matrix and temperature gradient vector which can be written as followings

\[
\hat{K}_0 = \int_{\Omega} \left( r \frac{d\psi'}{dr} \frac{d\psi'}{dr} + \frac{k}{r} \psi' \psi' - r \frac{\lambda}{\lambda} \psi' \frac{d\psi'}{dr} \right) dr,
\]

\[
\hat{Q} = \int_{\Gamma} \psi' q_n ds,
\]

\[
q_n = \left\{ \frac{\lambda}{\lambda} \frac{d\hat{T}}{dr} n_r \right\}.
\]

Solving this set of equations yields the nodal values of the temperature field for each wavenumber.

2.2 The electro-thermo-elastic problem

2.2.1 FFT-pFE analysis

In this section, the mixed FFT-pFE method is implemented for the electro-thermo-elastic analysis of a hollow FGP cylinder. For this purpose, a hollow cylinder, which its stress is negligible at the reference temperature, operates in a thermal environment. Thermal loading together with mechanical and electrical constraints causes thermal stresses in the shell. In order to derive the PDGEs of the FGP hollow cylinder with assumed boundary conditions, the Hamilton’s principle is employed here. It should be noted that the Fourier spectral method acts upon strong form and the Finite Element Method needs to be executed on the weak form of the PDGEs. So, the Hamilton’s principle can appropriately provide these conditions, simultaneously. A special case of the Hamilton’s principle that deals with
elastic mediums is known as the principle of minimum total potential energy. For an FGP cylinder, it takes form [8]

$$\delta \Pi = \delta \left( V + H \right) = 0,$$

(12)

where $\delta$ is the variation symbol and $\Pi$ is named the total potential energy of the elastic body. The electric enthalpy $H$ and the energy of the applied loads $V$ can be expressed as

$$H = \frac{1}{2} \iiint \left[ \sigma^T \varepsilon - \{D\}^T \{E\} \right] \text{dv},$$

$$V = -\iint u_r \{r \sigma_r n_r + \sigma_{\theta \phi} n_\theta \} \text{ds} - \iint u_\theta \{r \sigma_{\theta r} n_r + \sigma_{\theta \phi} n_\theta \} \text{ds}.$$  

(13)

The kinematic relationships between strain components and displacement field are defined as

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} \right), \quad \varepsilon_{zz} = \varepsilon_{rr} = \varepsilon_{\theta\theta} = 0,$$

$$E_r = -\frac{\partial \varphi}{\partial r}, \quad E_\theta = -\frac{1}{r} \frac{\partial \varphi}{\partial \theta}, \quad E_z = 0$$

(14)

where $\varepsilon_{ij} (i, j = r, \theta, z)$ are the strain components in the cylindrical coordinate system. $u_r$ and $u_\theta$ are displacement components in radial and circumferential directions, respectively. Moreover, $E_i (i = r, \theta, z)$ represents the electric field corresponding to the electric potential $\varphi$. The plane-strain electro-thermo-elastic constitutive relations for an isotropic FGP hollow cylinder can be written as

$$\begin{bmatrix} \sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{zz} \\
\sigma_{r\theta} \\
D_r \\
D_\theta \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\
C_{21} & C_{22} & C_{23} & 0 \\
C_{31} & C_{32} & C_{33} & 0 \\
0 & 0 & 0 & C_{44} \\
e_{11} & e_{21} & 0 & \varepsilon_{rr} \\
0 & 0 & 0 & 2\varepsilon_{\theta\theta} \end{bmatrix} + \begin{bmatrix} \eta_{11} & 0 & 0 & 0 \\
0 & \eta_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} E_r \\
E_\theta \end{bmatrix} + \begin{bmatrix} P_1 \\
P_2 \end{bmatrix} T$$

(15)

where $\sigma_{ij} (i, j = r, \theta, z)$ and $D_i (i = r, \theta)$ represent stress components and electric displacements. $a$ is the coefficient of thermal expansion. $e_{ij}, \eta_{ij}$ and $P_i$ denote the piezoelectric, dielectric and pyroelectric constants, respectively. In this work, the material properties are considered to be graded along the $r$-direction as
\[
\begin{align*}
C_{ij} &= C_{ij}^{0}r^{l}, & e_{ij} &= e_{ij}^{0}r^{l}, & \eta_{ij} &= \eta_{ij}^{0}r^{l} \\
\alpha_{ij} &= \alpha_{ij}^{0}r^{b}, & P_{i} &= P_{i}^{0}r^{b} + P_{i}^{1}r^{i},
\end{align*}
\]

where \(l\) is a power-law index of the elastic constants, the piezoelectric coefficient and the dielectric constants. \(b\) indicates a power-law index for the coefficient of thermal expansion. Both \(l\) and \(b\) inhomogeneous parameters are considered to be applied in the pyroelectric constants. Substituting Eqs. (13)-(15) into Eq. (12) yields the following integral form

\[
\int \left[ \begin{array}{c}
C_{11} \left( \frac{\partial u_{r}}{\partial r} \right) + C_{12} \left( \frac{u_{r}}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} \right) - \epsilon_{11} \left( \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) - \beta_{1}T \left( \frac{\partial w_{1}}{\partial r} \right) \\
C_{12} \left( \frac{\partial u_{r}}{\partial r} \right) + C_{22} \left( \frac{u_{r}}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} \right) - \epsilon_{21} \left( \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) - \beta_{2}T \left( \frac{\partial w_{2}}{\partial r} \right) \\
C_{44} \left( \frac{1}{r} \frac{\partial u_{r}}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} \right) - \epsilon_{41} \left( \frac{1}{r} \frac{\partial \varphi}{\partial r} \right) + \eta_{11} \left( \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right) + P_{r}T \left( \frac{\partial w_{3}}{\partial r} \right) \\
\end{array} \right] \frac{1}{r} \frac{\partial w_{i}}{\partial r} dr \theta = 0
\]

where

\[
\begin{align*}
w_{1} &= \delta u_{r}, & w_{2} &= \delta u_{\theta}, & w_{3} &= \delta \varphi \\
\beta_{1} &= C_{11} \alpha_{r} + C_{12} \alpha_{\theta} + C_{13} \alpha_{z} \\
\beta_{2} &= C_{12} \alpha_{r} + C_{22} \alpha_{\theta} + C_{23} \alpha_{z}
\end{align*}
\]

Now, the Fourier transform method can be used to discretize governing equations of an FGP cylinder into a set of harmonics in the circumferential direction. For this purpose, approximation functions according to the FFT approach are considered for displacement components and electric potential of the cylinder:

\[
\begin{align*}
u_{r}(r, \theta) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{j}(r) (\hat{u}_{r})^{k} e^{ik\theta}, \\
u_{\theta}(r, \theta) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{j}(r) (\hat{u}_{\theta})^{k} e^{ik\theta}, \\
\varphi(r, \theta) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \psi_{j}(r) (\hat{\varphi})^{k} e^{ik\theta},
\end{align*}
\]

where \((\hat{u}_{r})^{k}\), \((\hat{u}_{\theta})^{k}\), and \((\hat{\varphi})^{k}\) are nodal values of displacement components and electric potential in the Fourier space. Using the integration by parts technique with respect to the circumferential variable \(\theta\), we can transform a part of the governing equations into the strong form. By substituting approximation functions of Eq. (19) into Eq. (17), for each harmonic \(k\) with \(k=1, \ldots, N\), we have

\[(\delta \hat{u}_{r})^{k}:\]
\[ C_{11} \sum_{j=1}^{N} D_{j} \left( \hat{u}_{j} \right) + C_{12} \left( \sum_{j=1}^{N} A_{j} \left( \hat{u}_{j} \right) + \sum_{j=1}^{N} G_{j} \left( \left( \hat{u}_{j} \right) + ik \left( \hat{u}_{\theta} \right) \right) \right) + C_{22} \sum_{j=1}^{N} B_{j} \left( \left( \hat{u}_{j} \right) + ik \left( \hat{u}_{\theta} \right) \right) \]
\[ + C_{44} \left( \sum_{j=1}^{N} B_{j} \left( k^2 \left( \hat{u}_{j} \right) + ik \left( \hat{u}_{\theta} \right) \right) - ik \sum_{j=1}^{N} A_{j} \left( \hat{u}_{\theta} \right) \right) - e_{2k^2} \sum_{j=1}^{N} B_{j} \hat{\phi}_{j} \]
\[ + e_{41} \sum_{j=1}^{N} D_{j} \hat{\phi}_{j} + e_{42} \sum_{j=1}^{N} A_{j} \hat{\phi}_{j} - \left( \beta_{1} \sum_{j=1}^{N} F_{j} \bar{T}_{j}^{k} - \beta_{2} \sum_{j=1}^{N} D_{j} \bar{T}_{j}^{k} \right) = 0 \]

(\delta \hat{u}_{\theta})_{j}:

\[-ik C_{12} \sum_{j=1}^{N} A_{j} \left( \hat{u}_{j} \right) - C_{22} \sum_{j=1}^{N} B_{j} \left( ik \left( \hat{u}_{j} \right) - k^2 \left( \hat{u}_{\theta} \right) \right) \]
\[-C_{44} \left( \sum_{j=1}^{N} B_{j} \left( ik \left( \hat{u}_{j} \right) - \left( \hat{u}_{\theta} \right) \right) + \sum_{j=1}^{N} A_{j} \left( \hat{u}_{\theta} \right) - \sum_{j=1}^{N} G_{j} \left( ik \left( \hat{u}_{j} \right) - \left( \hat{u}_{\theta} \right) \right) - \sum_{j=1}^{N} D_{j} \left( \hat{u}_{\theta} \right) \right) \]
\[-ike_{21} \sum_{j=1}^{N} A_{j} \hat{\phi}_{j} - ike_{42} \sum_{j=1}^{N} \left( B_{j} - G_{j} \right) \hat{\phi}_{j} + ik \beta \sum_{j=1}^{N} E_{j} \bar{T}_{j}^{k} = 0 \]

(\delta \hat{\phi}_{j}):

\[ e_{41} \sum_{j=1}^{N} D_{j} \left( \hat{u}_{j} \right) + e_{42} \sum_{j=1}^{N} A_{j} \left( \hat{u}_{j} \right) + ik \sum_{j=1}^{N} G_{j} \left( \left( \hat{u}_{j} \right) + ik \left( \hat{u}_{\theta} \right) \right) - \eta_{1} \sum_{j=1}^{N} D_{j} \hat{\phi}_{j} - \eta_{2} k^2 \sum_{j=1}^{N} B_{j} \hat{\phi}_{j} \]
\[ + e_{42} \left( k^2 \sum_{j=1}^{N} B_{j} \left( \hat{u}_{j} \right) + ik \sum_{j=1}^{N} B_{j} \left( \hat{u}_{\theta} \right) + ik \sum_{j=1}^{N} A_{j} \left( \hat{u}_{j} \right) \right) + P_{1} \sum_{j=1}^{N} F_{j} \bar{T}_{j}^{k} - ik P_{2} \sum_{j=1}^{N} E_{j} \bar{T}_{j}^{k} = 0 \]

where

\[ A_{j} = \int_{0}^{1} \left( \frac{dv_{j}}{dr} \right) dr, \quad B_{j} = \int \left( \frac{1}{r} \right) \psi_{j} \psi_{j} dr. \]
\[ D_{j} = \int \left( \frac{dv_{j}}{dr} \right) \left( \frac{dv_{j}}{dr} \right) dr, \quad E_{j} = \int \psi_{j} \psi_{j} dr. \]
\[ F_{j} = \int \left( \frac{dv_{j}}{dr} \right) \psi_{j} dr, \quad G_{j} = \int \left( \frac{dv_{j}}{dr} \right) \psi_{j} dr. \]

As mentioned previously, the electro-thermo-elastic governing equations of the FGP cylinder, for each harmonic \( k \), can now be discretized using FEM. This procedure is shown in the flowchart depicted in Figure 2, and it is explained in the sequel.

### 2.2.2 Hierarchical discretization

The main idea of this section is to introduce a set of hierarchic \( p \)-order shape functions so that they can be used in a hierarchic finite element. In an auxiliary index space \( \mathcal{N}(\Omega) \) with \( \Omega := \{ \xi | -1 \leq \xi \leq 1 \} \), two shape functions associated to element sides at \( \xi = \pm 1 \) are defined as [47]
and the other shape functions associated with element region at interval \(-1 < \xi < 1\) are

\[
\hat{\psi}_{p+1} = \int N_p(\xi) d\xi, \quad \xi \in \Omega, \quad p > 1
\]  

(25)

where

\[
N_p(\xi) = \frac{1}{(p-1)!} \frac{1}{2^{p-1}} \frac{d^p}{d\xi^p} \left[ (\xi^2 - 1)^p \right]
\]  

(26)

in which \(N_p(\xi)\) shows the Legendre polynomial in which subscript \(p\) indicates degree of the related polynomials. Now, unknown variables of the problem are approximated using the aforementioned shape functions:

\[
\begin{bmatrix}
\hat{u}_1^r \\
\hat{u}_2^r \\
\hat{\phi}_r
\end{bmatrix} =
\begin{bmatrix}
[\hat{\psi}_1 \ldots \hat{\psi}_{p+1}]
& 0 & 0 \\
0 & [\hat{\psi}_1 \ldots \hat{\psi}_{p+1}] & 0 \\
0 & 0 & [\hat{\psi}_1 \ldots \hat{\psi}_{p+1}]
\end{bmatrix}
\begin{bmatrix}
\{\hat{u}_1\} \\
\{\hat{u}_2\} \\
\{\hat{\phi}\}
\end{bmatrix}
\]  

(27)

where

\[
\begin{align*}
\{\hat{u}_1\}^r &= \begin{bmatrix} \hat{u}_1^r \end{bmatrix} \\
\{\hat{u}_2\}^r &= \begin{bmatrix} \hat{u}_2^r \end{bmatrix} \\
\{\hat{\phi}\}^r &= \begin{bmatrix} \hat{\phi}_1 \ \hat{\phi}_2 \ \ldots \ \hat{\phi}_{p+1} \end{bmatrix}
\end{align*}
\]  

\[
\begin{align*}
\{a_1\}^r &= \begin{bmatrix} a_1^r \ \ldots \ a_{p+1}^r \end{bmatrix} \\
\{b_1\}^r &= \begin{bmatrix} b_1^r \ \ldots \ b_{p+1}^r \end{bmatrix} \\
\{d_1\}^r &= \begin{bmatrix} d_1^r \ \ldots \ d_{p+1}^r \end{bmatrix}
\end{align*}
\]  

(28)

The coefficients \((a_i)^r\), \((b_i)^r\) and \((d_i)^r\), \(i = 1, \ldots, p-1\) are auxiliary parameters of a hierarchical finite element which can be eliminated using the Element Condensation (EC) technique. Substituting approximation functions of Eq. (27) into the governing Eqs (20)-(22), results the following set of equations in the matrix form:

\[
\begin{bmatrix}
[K_{11}^r] & [K_{12}^r] & \{U\}^r \\
[K_{21}^r] & [K_{22}^r] & \{D\}^r
\end{bmatrix} =
\begin{bmatrix}
\{\hat{F}\}^r \\
\{\hat{F}_2\}^r
\end{bmatrix}
\]  

(29)

where

\[
\begin{align*}
\{\hat{F}_1\}^r &= \begin{bmatrix} \hat{F}_1^r \end{bmatrix} \\
\{\hat{F}_2\}^r &= \begin{bmatrix} \hat{F}_2^r \end{bmatrix}
\end{align*}
\]  

\[
\begin{align*}
\{\hat{G}_1\} = \begin{bmatrix} \hat{G}_{11} \ \hat{G}_{12} \ \{\hat{T}\} \end{bmatrix} \\
\{\hat{G}_2\} = \begin{bmatrix} \hat{G}_{21} \ \hat{G}_{22} \ \{\hat{T}\} \end{bmatrix}
\end{align*}
\]  

(30)
Using EC technique in the absence of auxiliary parameters, the assembly procedure is performed to obtain the imaginary unknown variables in the frequency space. So, we have

$$\{ \hat{U} \} = \left[ \begin{array}{c} \hat{U}_1 \\hat{U}_2 \\hat{U}_3 \\hat{U}_4 \\hat{U}_5 \\hat{U}_6 \end{array} \right]$$

Using EC technique in the absence of auxiliary parameters, the assembly procedure is performed to obtain the imaginary unknown variables in the frequency space. So, we have

$$\{ \hat{K}^* \} \{ \hat{U} \} = \{ \hat{F}^* \}$$

where

$$\{ \hat{K}^* \} = \left[ \begin{array}{c} \hat{K}_{11} \\hat{K}_{12} \\hat{K}_{21} \end{array} \right]$$

$$\{ \hat{F}^\prime \} = \left[ \begin{array}{c} \hat{F}_1 \\hat{F}_2 \end{array} \right]$$

Since the obtained results from this section are in frequency space, it is necessary to invert them into the physical space. The inverse FFT technique can be effectively used for this aim. In the present investigation, the FFTW3 library is used for the required FFT and inverse FFT.

3 3D discretization of the FGP hollow cylinder using higher-order elements

3.1 3D heat conduction problem

In this section, the ability of the proposed FFT-FE method is investigated for three dimensional heat conduction analyzing a thick hollow cylinder (Figure 3) under asymmetric thermal excitations. Review of the literature in the past decades indicates that the finite element solution of heat conduction problem in a cylindrical or spherical coordinate system is scarce [48]. In this study, both the geometry and the governing equations are discretized in the cylindrical coordinate system using FFT-pFE method. For this purpose, three-dimensional heat equation in cylindrical coordinate and in the absence of heat generation is considered as followings [8],

$$\lambda_r \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \lambda_r \frac{\partial T}{\partial r} + \lambda_r \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + a \frac{\partial T}{\partial r} + \lambda_r \frac{\partial^2 T}{\partial z^2} = 0, \quad a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq h. \quad (34)$$

where $\lambda_r$ is defined as Eq. (3). As shown in Figure 4, related thermal boundary conditions at inner and outer surfaces of the cylinder are considered to be in the following form:

$$T(r = a, z, \theta) = T_i(z, \theta),$$

$$T(r = b, z, \theta) = 0,$$

$$q_n(r, z = 0, \theta) = 0,$$

$$q_n(r, z = h, \theta) = 0,$$

where $q_n$ is heat input (or output) into (or from) the two ends of the cylinder, $T$ is the temperature field respected to the reference temperature and $T_i$ is the distribution of the temperature across the inner surface of the cylinder.

Now, using weighted residual method, the weak form of the above heat equation can be represented as:
\[
\int_\Omega \left[ \frac{\partial w}{\partial r} \left( \lambda_r \frac{\partial T}{\partial r} \right) + \frac{\partial w}{\partial z} \left( \lambda_z \frac{\partial T}{\partial z} \right) - \frac{w}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] r d\theta d\theta - \int_{\Gamma} w q_n ds = 0
\]

where

\[
q_n = \left( \lambda_r \frac{\partial T}{\partial r} n_r + \lambda_z \frac{\partial T}{\partial z} n_z + \lambda_\theta \frac{1}{r} \frac{\partial T}{\partial \theta} n_\theta \right)
\]

In which, \( w \) is a vector of weight functions from the Lagrange family used for interpolating the field variable, \( T \), and heat flux, \( q_n \). Mutually, the discrete Fourier transform technique is well used to discretize heat equation in the circumferential direction. A set of approximate functions, including Fourier coefficients of the field variable are substituted in the above equation. As mentioned previously, the field variable is defined at discrete points \( \theta_k, k = 1,2,...,N \) in which \( k \) is referred to the wave numbers. The following relation is then introduced for interpolating the temperature variable:

\[
T(r, \theta, z) = \sum_{k=1}^{N} \hat{T}_k(r, z)e^{ik\theta}
\]

By substituting this relation into the Eq. (36), the residual heat equation in terms of the radial and axial variables becomes:

\[
\int_\Omega \left[ \frac{\partial w}{\partial r} \left( \lambda_r \frac{\partial \hat{T}_k}{\partial r} \right) + \frac{\partial w}{\partial z} \left( \lambda_z \frac{\partial \hat{T}_k}{\partial z} \right) + \frac{wk^2}{r^2} \hat{T}_k \right] r d\theta d\theta = 0 \quad \text{For } k = 1,...,N
\]

Related boundary conditions are to be

\[
\begin{align*}
\hat{T}(r = a, z, \theta_k) &= \hat{T}_k(z, \theta_k), \\
\hat{T}(r = b, z, \theta_k) &= 0, \\
\hat{q}(r, z = 0, \theta_k) &= 0, \\
\hat{q}(r, z = h, \theta_k) &= 0.
\end{align*}
\]

It should be noted that, in the aforementioned equation and with respect to the circumferential variable spatial derivatives should be retained at strong form. Now, the meridian cross-section of the cylinder can be discretized using finite elements, including \( N_e \) nodes in each harmonic. 2D rectangular linear and higher order elements from the Lagrange family are used to discretize the geometry and heat equation, respectively.

In each element, \( \hat{T}_k(r, z) \) is approximated by the interpolation functions, \( \chi_j^e \), so we have

\[
\hat{T}_k^e(r, z) = \sum_{j=1}^{N_e} \chi_j^e(r, z)(\hat{T}_k^e)
\]

where \( (\hat{T}_k^e) \) are nodal values corresponding to the temperature field in the Fourier space. According to the Galerkin method, the weight functions, \( w_e \), are considered to be the same as the interpolation functions. In computation of the weighted integrals, a local coordinate system \( (\xi, \eta) \) is defined for the sake of simplicity. We need to have an appropriate mapping between parametric space and physical
space for this achievement. \( r, z \) variables in terms of the local coordinate parameters \((\xi, \eta)\) as well as the transformation Jacobian matrix can be defined as:

\[
\begin{bmatrix}
  \mathbf{r} \\
  \mathbf{z}
\end{bmatrix} = \begin{bmatrix}
  \sum_{j=1}^{n} r_j' X_j'(\xi) \\
  \sum_{j=1}^{n} z_j' X_j'(\xi)
\end{bmatrix} = \begin{bmatrix}
  r_1 & r_2 & \ldots & r_m \\
  z_1 & z_2 & \ldots & z_m
\end{bmatrix} \begin{bmatrix}
  X_1 \\
  X_2 \\
  \ldots \\
  X_m
\end{bmatrix}^T
\]  
\( \text{(42)} \)

\[
\begin{bmatrix}
  \frac{\partial T}{\partial \xi} \\
  \frac{\partial T}{\partial \eta}
\end{bmatrix} = \begin{bmatrix}
  \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
  \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta}
\end{bmatrix} \begin{bmatrix}
  \frac{\partial T}{\partial r} \\
  \frac{\partial T}{\partial \xi}
\end{bmatrix}
\]  
\( \text{(43)} \)

Using this geometrical approximation, we can solve the governing equation of the problem. As we considered in the previous section, the Gauss-Legendre method is used for numerical computing of the stiffness matrix. Discretized model of the thick hollow cylinder is shown in Figure 5. Discretized heat conduction equation in the matrix form yields,

\[
\begin{bmatrix}
  \mathbf{K}
\end{bmatrix} \{ \hat{\mathbf{T}} \} = \{ \hat{\mathbf{Q}} \}
\]  
\( \text{(44)} \)

### 3.2 Thermo-electro-elastic demonstration

In this section the previous FFT-pFE methodology is considered for three-dimensional thermo-electro-elastic analysis of a FGP hollow cylinder. In spite of what was assumed in the previous section for the finite elements family, 3D discretization technique now rises from the classical second-order finite elements. Accordingly, we deal with a functionally graded hollow cylinder (Figure 3) whose kinematic relations between strain and displacement components (in cylindrical coordinate) are as followings:

\[
e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right), \quad e_{zz} = \frac{\partial u_z}{\partial z},
\]

\[
e_{r\theta} = \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} - \frac{u_\theta}{r} \right), \quad e_{\theta\phi} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial r} + \frac{\partial u_\phi}{\partial \phi} \right), \quad e_{rr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial r} + \frac{\partial u_r}{\partial z} \right),
\]  
\( \text{(45)} \)

\[
E_r = -\frac{\partial \phi}{\partial r}, \quad E_\theta = -\frac{\partial \phi}{\partial \phi}, \quad E_z = -\frac{\partial \phi}{\partial z}.
\]

The generalized constitutive relations of orthotropic piezoelectric materials polarized in the axial direction are considered as below [19]

\[
\begin{align*}
\sigma_{rr} &= C_{11} e_{rr} + C_{12} e_{\theta\theta} + C_{13} e_{zz} - \beta T - e_{31} E_z, \quad \sigma_{\theta\theta} = 2C_{44} e_{\theta\theta} - e_{32} E_z, \\
\sigma_{z\phi} &= C_{15} e_{rr} + C_{25} e_{\theta\theta} + C_{35} e_{zz} - \beta T - e_{33} E_z, \quad \sigma_{\phi\phi} = 2C_{66} e_{\phi\phi}, \\
\sigma_z &= C_{13} e_{rr} + C_{23} e_{\theta\theta} + C_{33} e_{zz} - \beta T - e_{33} E_z, \quad \sigma_{\phi} = 2C_{66} e_{\phi\phi}, \\
D_r &= e_{13} E_r + \eta_1 E_z + P_{1} T, \quad D_\phi = 2e_{35} e_{\phi\phi} + \eta_2 E_\phi + P_{2} T, \\
D_z &= 2(e_{31} e_{rr} + e_{32} e_{\theta\theta} + e_{33} e_{zz}) + \eta_3 E_z + P_{3} T.
\end{align*}
\]  
\( \text{(46)} \)

where
\[
\begin{align*}
\beta_1 &= C_1\alpha_r + C_1\alpha_\theta + C_1\alpha_z, \\
\beta_2 &= C_2\alpha_r + C_2\alpha_\theta + C_2\alpha_z, \\
\beta_3 &= C_3\alpha_r + C_3\alpha_\theta + C_3\alpha_z,
\end{align*}
\]

(47)

Similarly, the Hamilton’s principle of Eq. (12) is used to derive the governing equations of the FGP hollow cylinder. Considering these approximation functions for the field variables of the thermo-electro-elastic problem

\[
\begin{align*}
\text{u}_r(r, \theta, z) &= \sum_{k=1}^{N} \sum_{j=1}^{N} \chi_j (r, z) (\hat{u}_r)^j e^{ik\theta}, \\
\text{u}_\theta(r, \theta, z) &= \sum_{k=1}^{N} \sum_{j=1}^{N} \chi_j (r, z) (\hat{u}_\theta)^j e^{ik\theta}, \\
\text{u}_z(r, \theta, z) &= \sum_{k=1}^{N} \sum_{j=1}^{N} \chi_j (r, z) (\hat{u}_z)^j e^{ik\theta}, \\
\varphi(r, \theta, z) &= \sum_{k=1}^{N} \sum_{j=1}^{N} \chi_j (r, z) (\hat{\varphi})^j e^{ik\theta},
\end{align*}
\]

(48)

and substituting kinematic and constitutive relations of Eq.s (45), (46) into the prescribed energy method, the following four sets of discretized governing equations at each harmonic \( k \) with \( k=1, \ldots, N \) yields

\[
\left( \delta \hat{u}_r \right)^j = \sum_{j=1}^{N} D_{11}^j (\hat{u}_r)^j + \sum_{j=1}^{N} A_{12}^j (\hat{u}_r)^j + \sum_{j=1}^{N} G_{12}^j \left( (\hat{u}_r)^j + ik (\hat{u}_\theta)^j \right) + \sum_{j=1}^{N} B_{22}^j \left( (\hat{u}_r)^j + ik (\hat{u}_\theta)^j \right) \\
+ \sum_{j=1}^{N} H_{55}^j (\hat{u}_r)^j + \sum_{j=1}^{N} P_{55}^j (\hat{u}_r)^j + k^2 \sum_{j=1}^{N} \left( \hat{u}_r \right)^j - ik \left( \sum_{j=1}^{N} \hat{A}_{60}^j (\hat{u}_r)^j - \sum_{j=1}^{N} \hat{B}_{60}^j (\hat{u}_r)^j \right) \\
+ \sum_{j=1}^{N} T_{13}^j (\hat{u}_r)^j + \sum_{j=1}^{N} \left( T_{23}^j (\hat{u}_r)^j + \sum_{j=1}^{N} \hat{P}_{13}^j (\hat{\varphi})^j + \sum_{j=1}^{N} \hat{T}_{13}^j (\hat{\varphi})^j + \sum_{j=1}^{N} \hat{T}_{23}^j (\hat{\varphi})^j \right) \\
- \sum_{j=1}^{N} F \hat{T}^j + \sum_{j=1}^{N} E \hat{T}^j = 0
\]

(49)

\[
\left( \delta \hat{u}_\theta \right)^j = \\
-ik \sum_{j=1}^{N} \hat{A}_{12}^j (\hat{u}_r)^j - \sum_{j=1}^{N} \hat{B}_{22}^j (\hat{u}_r)^j - k^2 (\hat{u}_\theta)^j \\
- \sum_{j=1}^{N} H_{55}^j (\hat{u}_\theta)^j + ik \sum_{j=1}^{N} A_{55}^j (\hat{u}_r)^j - ik \sum_{j=1}^{N} B_{55}^j (\hat{u}_r)^j \\
+ \sum_{j=1}^{N} G_{60}^j (\hat{u}_r)^j = \sum_{j=1}^{N} B_{60}^j (\hat{u}_r)^j + \sum_{j=1}^{N} \hat{A}_{60}^j (\hat{u}_r)^j \\
-ik \sum_{j=1}^{N} \hat{T}_{13}^j (\hat{\varphi})^j + ik \sum_{j=1}^{N} \hat{M}_{13}^j (\hat{\varphi})^j + ik \sum_{j=1}^{N} \hat{E} \hat{T}^j = 0
\]

(50)
\[
\left(\delta \tilde{u}_r\right)^k = : \\
\sum_{j=1}^{N_{rr}} \tilde{E}_{33}^j \left(\tilde{u}_r\right)^j_i + \sum_{j=1}^{N_{rr}} \tilde{M}_{33}^j \left(\tilde{u}_r\right)^j_i + ik \left(\tilde{u}_o\right)^j_i + \sum_{j=1}^{N_{rr}} \left(\tilde{T}_{33}^n \left(\tilde{u}_r\right)^n_j + D_{33}^n \left(\tilde{u}_o\right)^n_j\right) \\
+ \sum_{j=1}^{N_{rr}} \left(-ik \tilde{I}_{3j}^n \left(\tilde{u}_o\right)^n_j + k^2 \tilde{B}_{3j}^n \left(\tilde{u}_r\right)^n_j\right) + \sum_{j=1}^{N_{rr}} \tilde{H}_{33}^n \left(\tilde{u}_o\right)^n_j \\
+ \sum_{j=1}^{N_{rr}} \tilde{H}_{33}^n \delta^n_j + k^2 \sum_{j=1}^{N_{rr}} \tilde{B}_{33}^n \delta^n_j + \sum_{j=1}^{N_{rr}} \tilde{D}_{33}^n \delta^n_j - \sum_{j=1}^{N_{rr}} \tilde{N}_3^n \delta^n_j = 0.
\] 

(51)

\[
\delta \tilde{\phi}_j^k = : \\
\sum_{j=1}^{N_{rr}} \left(\tilde{E}_{33}^j \left(\tilde{u}_r\right)^j_i + \tilde{D}_{33}^j \left(\tilde{u}_o\right)^j_i + \sum_{j=1}^{N_{rr}} \tilde{M}_{33}^j \left(\tilde{u}_r\right)^j_i + ik \left(\tilde{u}_o\right)^j_i + \sum_{j=1}^{N_{rr}} \tilde{T}_{33}^n \left(\tilde{u}_r\right)^n_j + D_{33}^n \left(\tilde{u}_o\right)^n_j\right) \\
+ \sum_{j=1}^{N_{rr}} \left(k^2 \tilde{B}_{3j}^n \left(\tilde{u}_r\right)^n_j - ik \tilde{I}_{3j}^n \left(\tilde{u}_o\right)^n_j\right) + \sum_{j=1}^{N_{rr}} \tilde{H}_{33}^n \left(\tilde{u}_o\right)^n_j \\
- \sum_{j=1}^{N_{rr}} \tilde{D}_{33}^n \delta^n_j - k^2 \sum_{j=1}^{N_{rr}} \tilde{B}_{33}^n \delta^n_j - \sum_{j=1}^{N_{rr}} \tilde{H}_{33}^n \delta^n_j \\
+ \sum_{j=1}^{N_{rr}} \tilde{F}^n_j \delta^n_j - \sum_{j=1}^{N_{rr}} \tilde{E}_{22}^n \delta^n_j + \sum_{j=1}^{N_{rr}} \tilde{N}_3^n \delta^n_j = 0.
\] 

(52)

where

\[
\tilde{A}_{mn} = \int_{\Omega} C_{mn} \left(\frac{\partial X_j}{\partial r}\right) drdz \\
\tilde{B}_{mn} = \int_{\Omega} C_{mn} \left(1 \right) X_j drdz \\
\tilde{D}_{mn} = \int_{\Omega} C_{mn} \left(\frac{\partial X_j}{\partial r}\right) \frac{\partial X_j}{\partial r} drdz \\
\tilde{G}_{mn} = \int_{\Omega} C_{mn} \left(\frac{\partial X_j}{\partial r}\right) X_j drdz \\
\tilde{H}_{mn} = \int_{\Omega} C_{mn} \left(\frac{\partial X_j}{\partial z}\right) \frac{\partial X_j}{\partial r} drdz \\
\tilde{T}_{mn} = \int_{\Omega} C_{mn} \left(\frac{\partial X_j}{\partial r}\right) \left(\frac{\partial X_j}{\partial z}\right) drdz \\
\tilde{E}_{mn} = \int_{\Omega} P_{mn} X_j drdz \\
\tilde{F}_{mn} = \int_{\Omega} P_{mn} \left(\frac{\partial X_j}{\partial r}\right) X_j drdz \\
\tilde{N}_{mn} = \int_{\Omega} P_{mn} \left(\frac{\partial X_j}{\partial z}\right) X_j drdz \\
\tilde{D}_{mn} = \int_{\Omega} \eta_{mn} \left(\frac{\partial X_j}{\partial r}\right) \left(\frac{\partial X_j}{\partial r}\right) r drdz \\
\tilde{B}_{mn} = \int_{\Omega} \eta_{mn} \left(\frac{1}{r}\right) X_j X_j drdz \\
\tilde{H}_{mn} = \int_{\Omega} \eta_{mn} \left(\frac{\partial X_j}{\partial r}\right) \left(\frac{\partial X_j}{\partial z}\right) r drdz \\
\tilde{L}_{mn} = \int_{\Omega} \left(\frac{\partial X_j}{\partial z}\right) \left(\frac{\partial X_j}{\partial r}\right) r drdz \\
\tilde{E}_{mn} = \int_{\Omega} \left(\frac{\partial X_j}{\partial r}\right) \left(\frac{\partial X_j}{\partial z}\right) r drdz
\] 

(53)
\[
\tilde{M}_{mn}^{ij} = \int_{\Omega} e_{mn} \left( \frac{\partial \chi_i}{\partial z} \right) \chi_j d\Omega,
\]
\[
\tilde{L}_{mn}^{ij} = \int_{\Omega} e_{mn} \left( \frac{\partial \chi_i}{\partial r} \right) \chi_j d\Omega,
\]
\[
\tilde{B}_{mn}^{ij} = \int_{\Omega} e_{mn} \left( \frac{1}{r} \right) \chi_i \chi_j d\Omega,
\]
\[
\tilde{J}_{mn}^{ij} = \int_{\Omega} e_{mn} \chi_i d\Omega,
\]
\[
\tilde{H}_{mn}^{ij} = \int_{\Omega} e_{mn} \left( \frac{\partial \chi_i}{\partial z} \right) \frac{\partial \chi_j}{\partial z} d\Omega,
\]
\[
\tilde{N}_{mn}^{ij} = \int_{\Omega} \beta_{mn} \left( \frac{\partial \chi_i}{\partial r} \right) \chi_j d\Omega,
\]
\[
\tilde{F}_{mn}^{ij} = \int_{\Omega} \beta_{mn} \chi_i d\Omega.
\]

Rewriting these governing equations in the matrix form for \( k = 1, \ldots, N \) yields:

\[
\begin{bmatrix}
\hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} & \hat{k}_{14} & \hat{u}_{1}^r & \vdots & 0 \\
\hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} & \hat{k}_{24} & \hat{u}_{2}^r & \vdots & \vdots \\
\hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} & \hat{k}_{34} & \hat{u}_{3}^r & \vdots & \vdots \\
\hat{k}_{41} & \hat{k}_{42} & \hat{k}_{43} & \hat{k}_{44} & \hat{u}_{4}^r & \vdots & \vdots \\
\end{bmatrix}
+ 
\begin{bmatrix}
G_{11}^r & 0 & \cdots & 0 \\
G_{22}^r & \ddots & \vdots & \vdots \\
G_{33}^r & \ddots & 0 \\
G_{44}^r & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
\hat{f}_{1}^r \\
\hat{f}_{2}^r \\
\hat{f}_{3}^r \\
\hat{f}_{4}^r \\
\end{bmatrix}
= 
\begin{bmatrix}
\hat{T}_{1} \\
\hat{T}_{2} \\
\hat{T}_{3} \\
\hat{T}_{4} \\
\end{bmatrix}.
\]

In the followings, we can solve this linear system of equations, formed in the harmonic space, using FE approach to obtain the results in compliance with the nodal degrees of freedom within the hollow cylinder. Mutually, using ifft technique, the harmonic numerical results can be inverted into the real space.

4 Numerical results

4.1 2D FFT-pFE results (By hierarchical elements)

4.1.1 Convergence study

The aim of this section is to illustrate the convergence, accuracy and effectiveness of the suggested method for the electro-thermo-elastic analysis of an FGP hollow cylinder. Indeed, the solution of the heat conduction equation under uniform boundary temperature at inner surface \( (T_i) \) and outer surface \( (T_o) \) of the cylinder are considered here. In this regard, The following formula can be used for computing the relative error of the obtained results:

\[
\text{Relative Error} = \frac{\| \text{Present solution - Exact solution} \|}{\| \text{Exact solution} \|},
\]

with \( \| \| \) being an appropriate norm.

Results extracted from the proposed hybrid method are compared with those of reference [49]. The convergence rate obtained by linear elements is shown in Figure 6. It is observed that increasing the number of elements in the radial direction is accompanied by a linear convergence behavior of the results. This is known as \( h \)-version finite element approach in which the density of linear elements is raised until the desired precision is achieved. On the other hand, the convergence is gained in the \( p \)-
version approach using higher-order shape functions. Figure 7 shows a super algebraic convergence behavior of the FFT-pFE method, while, the order of the hierarchical shape functions is to be increased. Furthermore, the influence of the element density is presented for various element distributions.

In Figure 8, the convergence behavior of the Fourier spectral method is shown for some periodic functions. These functions are assumed to be as boundary conditions of the inner surface of the cylinder. As shown, the rate of convergence varies by varying the complexity of the periodic functions. As expected, for \( T(\theta) = \sin \theta \), the desired method can appropriately attain machine precision in finite harmonics. It is obvious from the figure that FFT technique has an exponentially varying convergence behavior. Using a thorough review, it is concluded that FFT is more vigorous and accurate than FEM for analyzing the problems with periodic domain.

4.1.2 Asymmetric demonstration

In this section, at first the results of the FGP cylinder under thermal loading are verified using existing literature [5]. For this purpose, axisymmetric exact solution of an FGP cylinder is extracted. To obtain the results, the properties of the cadmium selenide as an FGP material are used, as given below [5]:

\[
\begin{align*}
\alpha_0^0 &= 2.458 \times 10^{-6} \text{ } 1/\text{K}, \\
\alpha_0^0 &= 4.396 \times 10^{-6} \text{ } 1/\text{K}, \\
C_{11}^0 &= 83.6 \text{ GPa}, \\
C_{12}^0 &= 39.3 \text{ GPa}, \\
C_{22}^0 &= 74.1 \text{ GPa}, \\
\epsilon_1^0 &= 0.347 \text{ } \text{C}/\text{m}^2, \\
\epsilon_2^0 &= -0.16 \text{ } \text{C}/\text{m}^2, \\
\eta_1^0 &= 9.03 \times 10^{-11} \text{ C}/\text{Nm}, \\
R_1^0 &= -2.94 \times 10^{-6} \text{ C}/\text{m}^2 \text{K}, \\
\lambda_{10} &= 12.9 \text{ W/mK}. 
\end{align*}
\]

(56)

For convenience, some dimensionless values are defined as below:

\[
\begin{align*}
\rho &= \frac{r}{r_b}, \quad \bar{T} = \frac{T}{T_0}, \quad \bar{\sigma}_y = \frac{\sigma_y}{\alpha_0 Y_0}, \\
\bar{\bar{\epsilon}}_y = \frac{\epsilon_y}{\alpha_0 Y_0 T_0}, \quad \bar{u}_r = \frac{u_r}{\alpha_0 T_0 r_b}, \\
\bar{\bar{C}}_{ij}^0 &= \frac{C_{ij}^0}{Y_0}, \quad \bar{\alpha}_y = \frac{\alpha_y}{\alpha_0},
\end{align*}
\]

(57)

where the typical parameters of material properties used to normalize the numerical results are considered:

\[ \alpha_0 = \alpha_0^0, \quad Y_0 = 42.8 \text{ GPa}, \quad d_0 = -3.92 \times 10^{-12} \text{ C/N} \]

(58)

Here, two sets of thermo-electro-elastic governing equations of a hollow thick cylinder in dimensionless form are presented [5]

\[
\begin{align*}
\bar{C}_{11} r^m \frac{\partial^2 \bar{u}}{\partial r^2} + (m + 1) \bar{C}_{11} r^{m-1} \frac{\partial \bar{u}}{\partial r} &+ r^{m-2} \left( m \bar{C}_{12} - \bar{C}_{22} \right) \bar{u} + \bar{e}_1 i r^n \frac{\partial^2 \bar{\phi}}{\partial r^2} \\
+ r^{m-1} \left[ (1 + m) - \bar{e}_1 i \right] \frac{\partial \bar{\phi}}{\partial r} &- \left[ (1 + m + n) \beta_1^0 \bar{u} - \beta_2^0 \right] r^{m+n-1} - \bar{\beta}_1^0 r^{m+n} \frac{\partial \bar{T}}{\partial r} = 0, \quad \text{ (59)} \\
\bar{e}_1 i r^n \frac{\partial \bar{u}}{\partial r} &+ \left[ (1 + m) \bar{e}_1 i + \bar{e}_2^0 \right] r^{n-1} \frac{\partial \bar{u}}{\partial r} + m \bar{e}_1 i r^{n-2} \bar{u} - \bar{\eta}_1^0 r^n \frac{\partial^2 \bar{\phi}}{\partial r^2} - (1 + m) \bar{\eta}_1^0 r^{m+n} \frac{\partial \bar{\phi}}{\partial r} \\
&+ (1 + m + n) \bar{\eta}_1^0 r^{m+n-1} - \bar{\eta}_1^0 r^{m+n} \frac{\partial \bar{T}}{\partial r} = 0, \quad \text{ (60)}
\end{align*}
\]
where

\[
\begin{align*}
\bar{\beta}^0 & = \bar{C}_{11} \alpha^0_2 + \bar{C}_{12} \alpha^0_1 \\
\bar{\beta}^\infty & = \bar{C}_{22} \alpha^0_2 + \bar{C}_{21} \alpha^0_1
\end{align*}
\]  

(61)

In the sequel, the solution is assumed to be in the following form

\[
\begin{align*}
\bar{u}_r & = \bar{u}_{rc} + \bar{u}_p \\
\bar{\phi} & = \bar{\phi}_c + \bar{\phi}_p
\end{align*}
\]  

(62)

The first term on the right-hand side of the above relation with subscript \(c\) indicates the homogenous solution and the other one gives the particular solution. In the first step, the particular solution can be written as

\[
\begin{align*}
\bar{u}_p (r) & = X_1 \bar{r}^{\sigma_{k+1}} + X_2 \bar{r}^{\sigma_{k+1}} \\
\bar{\phi}_p (r) & = X_3 \bar{r}^{\sigma_{k+1}} + X_4 \bar{r}^{\sigma_{k+1}}
\end{align*}
\]  

(63)

By substituting Eq. (63) into Eqs. (59) and (60), the unknown coefficients \(x_i, (i = 1, 2, 3, 4)\) can be extracted. Utilizing the change of variable \(\bar{r} = e^\alpha\), the homogeneous expression of governing equations can be shown as [5]

\[
\begin{align*}
\left( D^2 + m\bar{D} - \alpha \right) \bar{u}_{rc} + \left[ D^2 + (m - \beta) \bar{D} \right] \bar{\phi}_c & = 0 \\
\left[ D^2 + (m + \beta) \bar{D} + \beta m \right] \bar{u}_{rc} - \gamma \left( D^2 + m\bar{D} \right) \bar{\phi}_c & = 0
\end{align*}
\]  

(64)\hspace{1cm} (65)

where \(\bar{D} = d/d\bar{s}\),

\[
\begin{align*}
\alpha & = \frac{\bar{C}_{12}^0 - \bar{C}_{11}^0}{\bar{C}_{22}^0}, \quad \beta = \frac{\bar{e}_3^0}{\bar{e}_1^0}, \quad \gamma = \frac{\bar{C}_{11}^0 \bar{m}^0_1}{\left( \bar{e}_1^0 \right)^3}, \quad \bar{\Phi}_c = \frac{\bar{e}_1^0}{\bar{C}_{11}^0} \bar{\phi}_c
\end{align*}
\]  

(66)

From homogeneous Eqs. (64) and (65), the following equation can be obtained

\[
\left( \gamma + 1 \right) \left( \bar{D}^3 + 2m\bar{D}^2 \right) - \left[ \beta^2 + \alpha\gamma - m^2 \left( \gamma + 1 \right) - m\beta \right] \bar{D} - m \left( \beta^2 - \beta m + \alpha\gamma \right) \bar{u}_{rc} = 0
\]  

(67)

Considering the general solution \(\bar{u}_{rc} = e^{(i\lambda)}\) leads to the characteristic equation

\[
\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0
\]  

(68)

where

\[
\begin{align*}
a_1 & = 2m, \quad a_2 = -\frac{1}{1 + \gamma} \left[ \beta^2 + \alpha\gamma - m^2 \left( 1 + \gamma \right) - m\beta \right] \\
a_3 & = -\frac{m}{1 + \gamma} \left( \beta^2 - \beta m + \alpha\gamma \right)
\end{align*}
\]  

(69)

According to the material properties selected in this example, we will obtain three distinct real roots and the general solutions of depending variables are to be [5]
\[ \bar{u}_{rc} = \sum_{j=1}^{3} U_j \bar{r}^{\lambda_j} \]  
\[ \bar{\phi}_e = \sum_{j=1}^{3} M_j U_j \bar{r}^{\lambda_j} + M_4 U_4 \]  

(70)  
(71)  

where

\[ M_j = \frac{C_0^{c+1}}{e_j^c \beta_j^{c+y}} \left[ (y+1)\lambda_j + m\gamma + m + \frac{m\beta - \alpha\gamma}{\lambda_j} \right], \quad j=1,2,3 \]  
\[ M_4 = \frac{C_0^{c+1}}{e_4^c} \]  

(72)  

Assuming that both the inner and outer surfaces of the cylinder are traction free, the electro-elastic boundary conditions are presented as follows

\[ \begin{cases} \sigma_{rr}(r=0.7) = 0, & r=1 \\ \sigma_{rr}(r=1) = 0 \\ \phi(r=0.7) = 0, & r=1 \\ \phi(r=1) = 0 \end{cases} \]  

(73)  

Moreover, the boundary conditions corresponding to heat conduction equation are considered to be

\[ \begin{cases} T(r=0.7) = 0 \\ T(r=1.0) = T_0 \end{cases} \]  

(74)  

The inhomogeneous parameters \( t=0.01 \) and \( b=0.01 \) are considered for the numerical calculations. The variations of temperature, radial displacement and electric potential of an FGP cylinder along the thickness direction are depicted in Figures 9-11 in terms of several inhomogeneous parameters \( m \). It can be observed that the obtained results from the proposed combined FFT-\( p \)FE method have a good agreement with exact solution.

As the verification process has completed, the results of electro-thermo-elastic analysis of an FGP hollow cylinder under asymmetric thermal loadings are presented. The power index of material properties and the temperature boundary condition in the outer radii of the hollow cylinder are considered to be as follows

\[ m = 2, \quad d = 0.01, \quad l = 0.01, \quad \bar{T}(r=1,\theta) = \sin 2\theta \]  

(75)  

Figure 12 indicates an asymmetric two-dimensional distribution of the temperature field on the cylinder section. In this analysis, \( N_e=60 \) hierarchic elements along the thickness direction and \( N=80 \) grid points along the circumferential direction are considered to obtain the results. Harmonic variation of the unknown variables, including the radial and circumferential displacements and electric potential are presented in Figures 13-15. Post-processing is then accomplished and the distribution of radial and hoop stresses are depicted in Figures 16 and 17.
4.2 3D FFT-pFE results (By quadratic Lagrange elements)

4.2.1 Heat conduction of a FGP cylinder

As mentioned previously, the main idea in the present mixed method is to reduce the computational efforts through eliminating one of the spatial dimensions (i.e. circumferential). This feature is considered to be prominent in 3D analysis in which there exists a large number of computations. In order to verify the proposed mixed FFT-pFE method for 3D thermal analyzing, a thick hollow cylinder under thermal excitations is modeled in the ABAQUS software. Thermal boundary condition relevant to the inner radii of the hollow cylinder is \( T(r=0.6, z, \theta) = T_0 + \left(z^2\right) e^{i2\theta} \). The boundary temperature of the outer surface of the cylinder is considered to be unchanged at \( T=0 \). Mutually, heat fluxes through the two ends of the cylinder (\( z=0.4 \)) are assumed to vanish. For the case of simplicity, we used ABAQUS ability in modeling asymmetric loading by means of AnalyticalField subroutine. Considering the above boundary conditions, the temperature field variable can be extracted for all nodal points across the cylinder section. For better demonstration, paths including nodal points (e.g. \( r=0.8 \) path of the \( z=4 \) cross section) are selected and the corresponding temperature nodal values are compared to those extracted from the FFT-pFE method (Figure 18).

4.2.2 Thermo-electro-elastic responses

After solving heat conduction equation of a FGP hollow cylinder, the desired temperature field is used in the thermo-electro-elastic analysis. To obtain the results, the properties of the PZT-4 as a well-known piezoelectric material [50] are considered here. As an arbitrary boundary condition, in this study, we constrain axial component of the displacement field regarding two ends of the FGP hollow cylinder. Electrical conditions at the inner and outer cylindrical surfaces are considered to be open circuit. So, we have

\[
\sigma_n = \sigma_{nc} = D_r = 0, \quad \text{at} \quad r = a, b
\]

In order to illustrate the convergence behavior and computational efficiency of FFT-pFE method in the 3D thermo-electro-elastic analysis, Figure 19 is presented. Radial displacement at middle surface of the FGP hollow cylinder is considered to estimate the relative error. \( n_m \times n_c \), labeled at horizontal axis of the Figure 19, indicates mesh density in the meridian section of the FGP cylinder. As shown in Figure 20 and Figure 21 the contour plot of the non-dimensional components of displacement field in the radial and circumferential directions are depicted. Consequently, the ability of the proposed FFT-pFE method considering asymmetric boundary excitations is demonstrated as in the Figures 22-25.

5 Conclusions

A combination of the Fast Fourier Transform technique and \( p \)-version of the finite element method was performed in this paper for the electro-thermo-elastic analysis of a hollow FGP cylinder. By considering the geometric feature of the shells of revolution, integral transform technique is suitably used in the circumferential direction. So, the governing equations can be divided into a set of harmonics. Afterwards, pertinent individual equations are discretized using higher order finite element method. In fact, this procedure decreases one dimension of the related governing equations. So, a noticeable decrease in computational cost can be achieved. In this regard, the influence of the FFT grid-size as well as the higher order interpolation functions are indicated. As expected, the trigonometric Fourier
functions have the upmost rate of the convergence as FFT grid-size increases. It can be concluded from the convergence study that the FFT technique has an exponentially varying convergence behavior whereas the hierarchic finite element method shows an algebraic behavior. Nevertheless, it was apparent from the results that the rate of convergence of the hierarchic elements is much more than that of the linear finite elements. Since, the FFT technique needs to be implemented on the strong form and finite element method deals with the weak formulation, Hamilton’s principle can be used to gather both the strong and weak form of the governing equations, simultaneously. The FFT technique helps us to employ the asymmetric thermal loadings. The results indicated that the proposed method is in good agreement with the exact analytical solutions from the literature. The proposed FFT-\(p\)FE method enjoys both the ability of FEM in modeling the complicated geometry and the simplicity and accuracy of FFT. As a future extension of this work, the proposed method can be extended to three-dimensional electro-thermo-elastic analysis of homogeneous, composite and functionally graded thick shells of revolution.

6 References


**Figure 1.** Discretized geometry of an infinitely long FGP cylinder in the presence of finite element grids and FFT harmonics.

**Figure 2.** FFT-pFE analysis flowchart.

**Figure 3.** Variation of the temperature field along the cylinder axis.

**Figure 4.** Circumferential distribution of the temperature field.

**Figure 5.** Discretized model of the cylinder.
Figure 6. Logarithmic representation of relative error versus the number of finite elements in the $h$-version FEM.

Figure 7. Variation of the relative error versus Legendre polynomial degree.

Figure 8. Logarithmic representation of the relative error versus the number of wavenumbers in the FFT technique.

Figure 9. Temperature distribution along the thickness of cylinder ($p=7$, $Nel=80$).

Figure 10. Distribution of radial displacement along the thickness of cylinder ($p=7$, $Nel=80$).

Figure 11. Distribution of electric potential along the thickness of cylinder ($p=7$, $Nel=80$).

Figure 12. Two-dimensional asymmetric distribution of temperature.

Figure 13. Two-dimensional asymmetric distribution of radial displacement.

Figure 14. Two-dimensional asymmetric distribution of circumferential displacement.

Figure 15. Two-dimensional asymmetric distribution of electric potential.

Figure 16. Two-dimensional asymmetric distribution of radial stresses.

Figure 17. Two-dimensional asymmetric distribution of hoop stresses.

Figure 18. Distribution of the temperature field in the middle surface of the cylinder ($z=4$, $r=0.8$).

Figure 19. Convergence study of the FFT-$p$FE method for thermo-electro-elastic analyzing.

Figure 20. Contour plot of the radial component of displacement field across the $N=1$ surface.

Figure 21. Contour plot of the circumferential component of displacement field across the $N=1$ surface.

Figure 22. 2D distribution of the radial displacement across the middle surface of the FGP cylinder.

Figure 23. 2D distribution of the circumferential displacement across the middle surface of the FGP cylinder.

Figure 24. 2D distribution of the axial displacement across the middle surface of the FGP cylinder.

Figure 25. 2D distribution of the electric potential across the middle surface of the FGP cylinder.
Figure 26.

Start

Transforming of governing equations into complex domain by FFT approximation

Fourier transform of physical properties and boundary conditions

Loop on each frequency:

Thermo-elastic analysis in transformed domain

Loop on Finite Elements:

Obtaining the stiffness matrix and load vector using Heuristic FEM method

Solving FE equations to calculate nodal values of temperature and displacement variables

Return to real domain by Inverse Fast Fourier Transform

Obtain temperature and displacement nodal values in real domain

Post processing: Calculate nodal stress and strain

Figure 27
Figure 28.

\[ T(r = a, \theta, z) = T_i(\theta, z) \]

Figure 29.

\[ T(r = a, \theta, z) = T_i(\theta, z) \]
Figure 30.

Figure 31.
Figure 32.

Figure 33.
Figure 34.

Figure 35.
Figure 36.

Figure 37.
Figure 38.

Figure 39.
Figure 40.

Figure 41.
Figure 42.

Figure 43.
Figure 44.

![Graph showing Relative Error vs. 2D FE-Mesh size]

Figure 45.

![Heatmap showing Radial Displacement, $\bar{u}_r$, distribution across $r$ and $z$]

34
Figure 46.

Figure 47.
Figure 48.

Figure 49.
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