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A new approximate analytical method and its convergence for nonlinear time-fractional partial differential equations

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Abstract. The main objective of this paper is to present a new approximate analytical method called Modified Generalized Taylor Fractional Series Method (MGTFSM) for solving general nonlinear time-fractional partial differential equations. The fractional derivative is considered in the Caputo sense. The convergence results of the proposed method are established here. The basic idea of the MGTFSM is to construct the solution in the form of infinite series that converges rapidly to the exact solution of the given problem. The main advantage of the proposed method, compared to current methods, is that the method solves the nonlinear problems without using linearization, discretization, perturbation, or any other restriction. The efficiency and accuracy of the MGTFSM are tested by means of different numerical examples. The results prove that the proposed method is very effective and simple for solving the nonlinear time-fractional partial differential equations problems.

1. Introduction

In recent years, many mathematicians and physicists have been interested in the subject of Nonlinear Fractional Partial Differential Equations (NFPDEs) because of their many applications in various fields where they can be used for modeling a wide range of physical phenomena such as statistical mechanics, mathematical physics, theoretical neuroscience, fluid dynamics, electrochemistry, viscoelasticity, population dynamics, cancer modeling, and mathematical finance. So, the theory of NFPDEs is a powerful theory to give a solution to engineering problems (see, e.g., [1–15]).

Obtaining approximate analytical solutions to the NFPDEs is one of the very important subjects in mathematics, science, and technology. For this purpose, many different numerical and analytical methods have been constructed and developed, among which Modification of Adomian Decomposition Method (MADM) [16], Variational Iteration Method (VIM) [17], Homotopy Analysis Method (HAM) [18], Homotopy Perturbation Transform Method (HPTM) [19], New Iterative Method (NIM) [20], Fractional Elzaki Projected Differential Transform Method (FEPDTM) [21], Generalized Differential Transform Method (GDTM) [22], Modification of the Reduced Differential Transform Method (MRDTM) [23], Fractional Taylor Operational Matrix Method (FTOMM) [24], and Fractional Residual Power Series Method (FRPSM) [25].

The Modified Generalized Taylor Fractional Series Method (MGTFSM) was first proposed to obtain

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a numerical solution to a certain class of NFPDEs, see [26]. The MGTFSM consists of an iterative algorithm. This method is effective and easy to obtain a power series solution to linear and nonlinear fractional differential equations without resorting to linearization, perturbation, or discretization. Unlike other series methods, the MGTFSM does not match the coefficients of the similar conditions and a repeated connection is not required. The present method computes the coefficients of the power series by a bond of algebraic equations of some variables. In addition, the MGTFSM does not need any transformation during the change from low to higher order; thus, it is possible to work with the present method directly on the given example by choosing a suitable initial estimate approximation. This method has been tested to be powerful, effective, and easily capable to deal with a wide range of linear and nonlinear fractional differential equations. The main objective of this paper is to extend this method and establish its convergence to solve general nonlinear time-fractional partial differential equations with arbitrary nonlinear terms.

The rest of the paper is organized as follows. Section 2 presents some basic definitions and properties of fractional calculus. Section 3 introduces our obtained results in the form of a new theorems to solve the general nonlinear time-fractional partial differential equations by using the MGTFSM. Section 4 gives some numerical examples to exhibit the efficiency and effectiveness of the proposed method. Section 5 discusses our obtained results represented by figures and tables. Section 6 summarizes the conclusions of this work.

2. Basic definitions of fractional calculus

In this section, some basic definitions and properties of fractional calculus, which are used further in this paper, are introduced. For more details, see [27,28].

Definition 1. *A real function $f(t), t > 0$, is considered to be in the space $\mathrm{C}_\mu([0,\infty])$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ so that $f(t) = t^p h(t)$, where $h \in C([0,\infty])$, and it is said to be in the space C_μ^n if $f^{(n)} \in \mathrm{C}_\mu([0,\infty])$, $n \in \mathbb{N}$.*

Definition 2. *The Riemann-Liouville fractional integral of order $\alpha \geq 0$ of a function $f \in \mathrm{C}_\mu([0,\infty]), \mu \geq -1$, is defined as follows:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad t > 0, \tag{1}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 3. *The Caputo fractional derivative of order $n-1 < \alpha \leq n$ of a function $f \in \mathrm{C}_{-1}^n([0,\infty]), n \in \mathbb{N}$, is defined as*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} f^{(n)}(\xi) d\xi. \tag{2}$$

For this definition, we have the following properties:

$$D^\alpha(C) = 0, \tag{3}$$

where C is a constant.

$$D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta > n-1, \\ 0, & \text{if } \beta \leq n-1. \end{cases} \tag{4}$$

$$D^\alpha [\lambda f(t) + \nu g(t)] = \lambda D^\alpha [f(t)] + \nu D^\alpha [g(t)], \quad (\lambda, \nu) \in \mathbb{R}^2, \tag{5}$$

$$D^\alpha [f(t) g(t)] = g(t) D^\alpha [f(t)] + f(t) D^\alpha [g(t)]. \tag{6}$$

Definition 4. *The Caputo time-fractional derivative of order $n-1 < \alpha \leq n$ of a function $u \in \mathrm{C}_{-1}^n(\mathbb{R} \times [0,\infty]), n \in \mathbb{N}$, is defined as:*

$$D_t^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial \xi^n} d\xi. \tag{7}$$

Definition 5. *The Mittag-Leffler function is defined as follows:*

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \tag{8}$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z.

3. Analytical study of MFTSM

Theorem 1. *Consider a general nonlinear time-fractional partial differential equation:*

$$D_t^\alpha u(x,t) = \mathcal{L}(u(x,t)) + \mathcal{N}(u(x,t)), \quad x \in I \subset \mathbb{R}, t \geq 0, \tag{9}$$

with the initial conditions:

$$u(x,0) = u_0(x), \tag{10}$$

where D_t^α is the Caputo fractional derivative of order $\alpha, 0 < \alpha \leq 1$, $\mathcal{L}(u(x,t))$ is a linear differential operator, and $\mathcal{N}(u(x,t))$ represents a general nonlinear differential operator. Then, by MGTFSM, the solution to Eqs. (9)–(10) is given in the form of infinite series as follows:

$$u(x,t) = \sum_{j=0}^\infty u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \quad x \in I \subset \mathbb{R}, \quad 0 \leq t < R, \tag{11}$$

where $u_j(x)$ is the coefficient of Series (11) and R is the radius of convergence.

Proof. In order to achieve our goal, we consider the following generalized nonlinear time-fractional partial differential equation (9) with the initial conditions (10).

We assume the solution as an infinite series given by:

$$u(x,t)=\sum_{j=0}^{\infty}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{12}$$

Consequently, the approximate solution of Eqs. (9) and (10) can be written as follows:

$$u_n(x,t)=\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}$$
$$=u_0(x)+\sum_{j=1}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{13}$$

By applying the operator D_t^{α} to Eq. (13) and using Properties (1) and (2), the following formula is obtained:

$$D_t^{\alpha}u_n(x,t)=\sum_{j=0}^{n-1}u_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{14}$$

We substitute Eqs. (13) and (14) into Eq. (9) and obtain the following recurrence relation:

$$0=\sum_{j=0}^{n-1}u_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}-\mathcal{L}\left(\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)$$
$$-\mathcal{N}\left(\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right). \tag{15}$$

To find the functions $u_n(x), n = 1,2,3,..$ we follow the same methodology to obtain the coefficients of the Taylor series; therefore, we need to solve the following equation:

$$D_t^{(n-1)\alpha}\left\{G(x,t,\alpha,n)\right\}\downarrow_{t=0}=0, \tag{16}$$

where:

$$G(x,t,\alpha,n)=\sum_{j=0}^{n-1}u_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}$$
$$-\mathcal{L}\left(\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)$$
$$-\mathcal{N}\left(\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right). \tag{17}$$

Now, we calculate the first terms of the sequence $\{u_n(x)\}_1^N$. For $n=1$, we have:

$$G(x,t,\alpha,1)=u_1(x)-\mathcal{L}\left(u_0(x)+u_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)$$
$$-\mathcal{N}\left(u_0(x)+u_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right). \tag{18}$$

Solution to the equation $G(x,t,\alpha,1)=0$ gives:

$$u_1(x)=\mathcal{L}(u_0(x))+\mathcal{N}(u_0(x)). \tag{19}$$

To find $u_2(x)$, we consider $G(x,t,\alpha,2)$ and solve:

$$D_t^{\alpha}\left\{G(x,t,\alpha,2)\right\}\downarrow_{t=0}=0. \tag{20}$$

To find $u_3(x)$, we consider $G(x,t,\alpha,3)$ and solve:

$$D_t^{2\alpha}\left\{G(x,t,\alpha,3)\right\}\downarrow_{t=0}=0. \tag{21}$$

To find $u_4(x)$, we consider $G(x,t,\alpha,4)$ and solve:

$$D_t^{3\alpha}\left\{G(x,t,\alpha,4)\right\}\downarrow_{t=0}=0, \tag{22}$$

and so on.

In general, to obtain the coefficient function $u_k(x)$, we solve:

$$D_t^{(k-2)\alpha}\left\{G(x,t,\alpha,k)\right\}\downarrow_{t=0}=0, \tag{23}$$

where:

$$G(x,t,\alpha,k)=\sum_{j=0}^{k-1}u_{j+1}(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}$$
$$-\mathcal{L}\left(\sum_{j=0}^{k}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right)$$
$$-\mathcal{N}\left(\sum_{j=0}^{k}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}\right). \tag{24}$$

Finally, the solution of Eqs. (9) and (10) can be expressed as follows:

$$u(x,t)=\lim_{n\to\infty}u_n(x,t)=\lim_{n\to\infty}\sum_{j=0}^{n}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}$$
$$=\sum_{j=0}^{\infty}u_j(x)\frac{t^{j\alpha}}{\Gamma(j\alpha+1)}. \tag{25}$$

The proof is complete.□

Theorem 2. *If there exists a constant* $0<\gamma<1$ *such that:*

$$\|u_{n+1}(x,t)\|\leq\gamma\|u_n(x,t)\|,$$
$$n\in\mathbb{N},\quad x\in I\subset\mathbb{R},\quad 0\leq t<R, \tag{26}$$

then the sequence of approximate solution (11) converges to the exact solution.

Proof. For all $x \in I \subset \mathbb{R}, 0 \leq t < R$, we have:

$$\|u(x,t) - u_n(x,t)\| = \left\| \sum_{j=n+1}^{\infty} u_j(x,t) \right\|$$
$$\leq \sum_{j=n+1}^{\infty} \|u_j(x,t)\| \leq \sum_{j=n+1}^{\infty} \gamma \|u_{j-1}(x,t)\|$$
$$\leq \sum_{j=n+1}^{\infty} \gamma^2 \|u_{j-2}(x,t)\| \leq ...$$
$$\leq \|u_0(x)\| \sum_{j=n+1}^{\infty} \gamma^j = \frac{\gamma^{n+1}}{1-\gamma} \|u_0(x)\|. \quad (27)$$

Since $0 < \gamma < 1$ and $u_0(x)$ is bounded, we have:

$$\lim_{n \longrightarrow \infty} \|u(x,t) - u_n(x,t)\| = 0. \quad (28)$$

This completes the proof.□

4. Applications of MGTFSM to nonlinear time-fractional partial differential equations

In this section, some numerical examples are given to exhibit the efficiency and effectiveness of the MGTFSM in solving nonlinear time-fractional partial differential equations.

Example 1. *We consider the cubic nonlinear time-fractional Schrödinger equation [29]:*

$$iD_t^\alpha u = -u_{xx} + 2|u|^2 u, \quad (29)$$

with the initial condition:

$$u(x,0) = e^{ix}, \quad (30)$$

where D_t^α is the Caputo time-fractional derivative of order $\alpha, 0 < \alpha \leq 1$, u is a complex function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, and $i^2 = -1$.

For $\alpha = 1$, the exact solution of Eqs. (29) and (30) is given by:

$$u(x,t) = e^{i(x-3t)}. \quad (31)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of Eqs. (29) and (30) in the form:

$$u(x,t) = \sum_{i=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \quad (32)$$

and the coefficients of Series (32) are given as follows:

$$u_j(x) = (-1)^j (3i)^j e^{ix} \quad \text{for} \quad j = 0, 1, 2, 3, ... \quad (33)$$

Therefore, the solution of Eqs. (29) and (30) can be expressed as follows:

$$u(x,t) = e^{ix}$$
$$\left(1 - \frac{(3it^\alpha)}{\Gamma(\alpha+1)} + \frac{(3it^\alpha)^2}{\Gamma(2\alpha+1)} - \frac{(3it^\alpha)^3}{\Gamma(3\alpha+1)} + \cdots\right)$$
$$= e^{ix} \sum_{j=0}^{\infty} \frac{(-1)^j (3it^\alpha)^j}{\Gamma(j\alpha+1)}$$
$$= e^{ix} E_\alpha(-3it^\alpha), \quad (34)$$

where $E_\alpha(-3it^\alpha)$ is the Mittag-Leffler function defined by Eq. (8). Taking $\alpha = 1$ to Eq. (34) will bear the following result:

$$u(x,t) = e^{ix} \left(1 - 3it + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \cdots\right)$$
$$= e^{i(x-3t)}, \quad (35)$$

which is the exact solution to the classical Schrödinger equation.

Example 2. *We consider the nonlinear time-fractional gas dynamic equation [30]:*

$$D_t^\alpha u = -uu_x + u(1-u), \quad (36)$$

with the initial condition

$$u(x,0) = e^{-x}, \quad (37)$$

where D_t^α is the Caputo time-fractional derivative of order $\alpha, 0 < \alpha \leq 1$, and u is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Eqs. (36) and (37) is given by:

$$u(x,t) = e^{t-x}. \quad (38)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of Eqs. (36) and (37) in the form:

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \quad (39)$$

and the coefficients of Series (39) are given as follows:

$$u_j(x) = e^{-x} \quad \text{for} \quad j = 0, 1, 2, 3, ... \quad (40)$$

Therefore, the solution of Eqs. (36) and (37) can be expressed as follows:

$$u(x,t) = e^{-x}$$
$$\left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$
$$= e^{-x} \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = e^{-x} E_\alpha(t^\alpha), \quad (41)$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function defined by Eq. (8). Taking $\alpha = 1$ in Eq. (41) will give the following result:

$$u(x,t) = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) e^{-x} = e^{t-x}, \quad (42)$$

which is the exact solution of classical gas dynamic equation.

Example 3. *We consider the nonlinear time-fractional reaction-diffusion-convection equation [31]:*

$$D_t^\alpha u = u_{xx} - u_x + uu_x - u^2 + u, \quad (43)$$

with the initial condition:

$$u(x,0) = e^x, \quad (44)$$

where D_t^α is the Caputo time-fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and u is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Eqs. (43) and (44) is given by:

$$u(x,t) = e^{x+t}. \quad (45)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of Eqs. (43) and (44) in the form:

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \quad (46)$$

and the coefficients of Series (46) are given as follows:

$$u_j(x) = e^x \quad \text{for} \quad j = 0,1,2,3,... \quad (47)$$

Therefore, the solution of Eqs. (43) and (44), can be expressed as follows:

$$u(x,t) = e^x \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) = e^x \sum_{i=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = e^x E_\alpha(t^\alpha), \quad (48)$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function, defined by Eq. (8). Taking $\alpha = 1$ in Eq. (48) will give the following result:

$$u(x,t) = e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{x+t}, \quad (49)$$

which is the exact solution of classical reaction-diffusion-convection equation.

Example 4. *Consider the nonlinear time-fractional Fokker-Planck equation [29]:*

$$D_t^\alpha u = \left(\frac{x}{3}u\right)_x - \left(\frac{4}{x}u^2\right)_x + \left(u^2\right)_{xx}, \quad (50)$$

with the initial condition:

$$u(x,0) = x^2, \quad (51)$$

where D_t^α is the Caputo time-fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and u is a function of $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Eqs. (50) and (51) is given by:

$$u(x,t) = x^2 e^t. \quad (52)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of Eqs. (50) and (51) in the form:

$$u(x,t) = \sum_{i=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha+1)}, \quad (53)$$

and the coefficients of Series (53) are given as follows:

$$u_j(x) = x^2 \quad \text{for} \quad j = 0,1,2,3,... \quad (54)$$

Therefore, the solution of Eqs. (50) and (51) can be expressed as follows:

$$u(x,t) = x^2 \left(1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) = x^2 \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} = x^2 E_\alpha(t^\alpha), \quad (55)$$

where $E_\alpha(t^\alpha)$ is the Mittag-Leffler function, defined by Eq. (8). Taking $\alpha = 1$ in Eq. (55) will give the following result:

$$u(x,t) = x^2 \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = x^2 e^t, \quad (56)$$

which is the exact solution to the classical Fokker-Planck equation.

Example 5. *Consider the 2-dimensional nonlinear time-fractional biological population equation [32]:*

$$D_t^\alpha u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} - u\left(1 + \frac{8}{9}u\right), \quad (57)$$

with the initial condition:

$$u(x,y,0) = e^{(x+y)/3}, \quad (58)$$

where D_t^α is the Caputo time-fractional derivative of order $\alpha, 0 < \alpha \leq 1$ and u is a function of $(x,y,t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$.

For $\alpha = 1$, the exact solution of Eqs. (57) and (58) is given by:

$$u(x,y,t) = e^{(1/3)(x+y)-t}. \tag{59}$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of Eqs. (57) and (58) in the form:

$$u(x,y,t) = \sum_{i=0}^{\infty} u_i(x,y)\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \tag{60}$$

and the coefficients of Series (60) are given as follows:

$$u_i(x,y) = (-1)^i e^{(x+y)/3} \quad \text{for} \quad i = 0,1,2,3,\cdots \tag{61}$$

Therefore, the solution of Eqs. (57) and (58) can be expressed as follows:

$$u(x,y,t) = e^{(x+y)/3}$$

$$\left(1 - \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$

$$= e^{(x+y)/3} \sum_{i=0}^{\infty} \frac{(-1)^j t^{j\alpha}}{\Gamma(j\alpha+1)} = e^{(x+y)/3} E_\alpha(-t^\alpha), \tag{62}$$

where $E_\alpha(-t^\alpha)$ is the Mittag-Leffler function defined by Eq. (8). Taking $\alpha = 1$ in Eq. (62) will yield the following result:

$$u(x,y,t) = e^{(x+y)/3}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)$$

$$= e^{(1/3)(x+y)-t}, \tag{63}$$

which is the exact solution of classical biological population equation.

5. Numerical results and discussion

To show the accuracy of the MGTFSM algorithm in handling examples provided, the two-dimensional plots of the 4th MGTFSM approximate solutions and exact solutions when $t \in [0,1]$ at different levels of fractional order α such that $\alpha = \{0.7, 0.8, 0.9, 1\}$ are given in Figures 1–4, respectively. Tables 1–4 show the numerical values of the 4th MGTFSM approximate solutions the exact solutions and the absolute values of the errors for Examples 2–5, respectively, when $\alpha = \{0.7, 0.8, 0.9, 1\}$ with some selected grid points t with the step size 0.1. From these figures and tables, it is clearly observed that when $\alpha \to 1$, the approximate solution obtained by the MGTFSM is very close to the exact solution, which indicates the accuracy and efficiency of the proposed method.

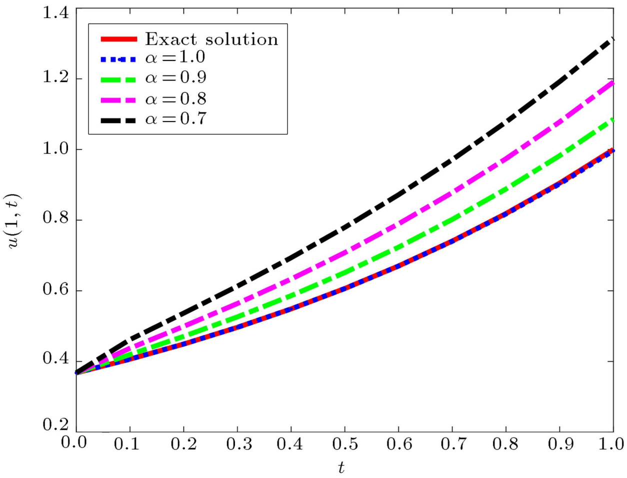

Figure 1. The behavior of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution and exact solution $u(1,t)$ for Example 2.

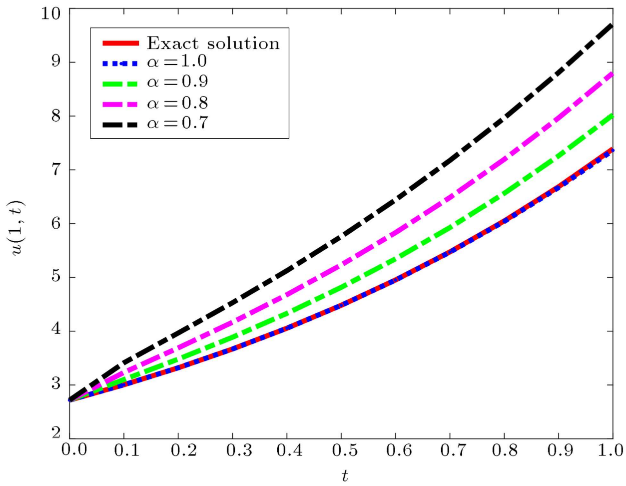

Figure 2. The behavior of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution and exact solution $u(1,t)$ for Example 3.

6. Conclusion

In this paper, a new approximate analytical method called the Modified Generalized Taylor Fractional Series Method (MGTFSM) was successfully applied to obtain the numerical solutions of general nonlinear time-fractional partial differential equations with arbitrary nonlinear terms. The method was applied to many numerical examples. The results showed that the MGTFSM was an efficient and convenient method for finding numerical solutions to these problems. The obtained approximate solutions using the proposed method were in excellent agreement with the corresponding exact solutions. Finally, it can be concluded that the MGTFSM was powerful and efficient in finding approximate and analytical solutions to wider classes of nonlinear fractional problems.

In future works, we will extend this approach

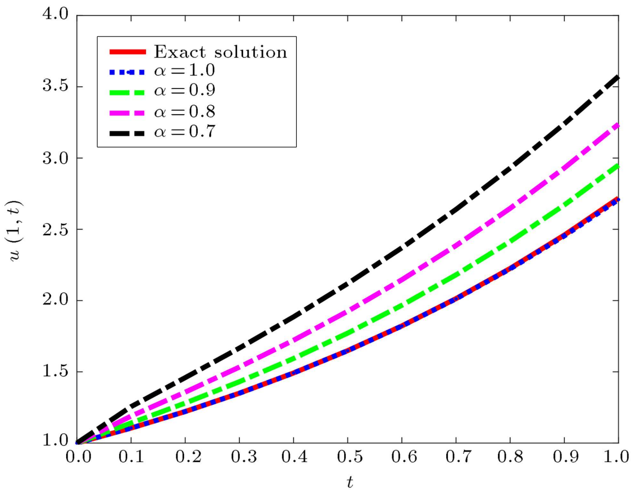

Figure 3. The behavior of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution and exact solution $u(1,t)$ for Example 4.

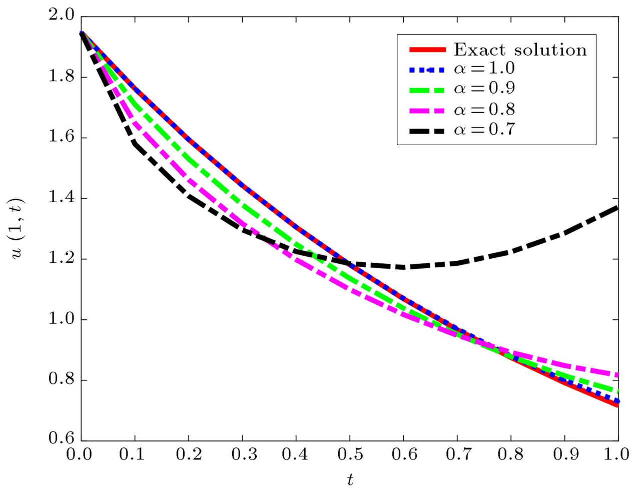

Figure 4. The behavior of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution and exact solution $u(1,1,t)$ for Example 5.

Table 1. The numerical values of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution, exact solution $u(1,t)$, and the absolute value of the error for Example 2.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{MGTFSM} \rvert$
0.1	0.46191	0.43746	0.41969	0.40657	0.40657	3.1175×10^{-8}
0.3	0.61312	0.56399	0.52622	0.49658	0.49659	7.8386×10^{-6}
0.5	0.77963	0.70816	0.65172	0.60643	0.60653	1.0439×10^{-4}
0.7	0.97106	0.87781	0.80211	0.74024	0.74082	1.9680×10^{-6}
0.9	1.19200	1.07800	0.98249	0.90272	0.90484	2.1210×10^{-3}

Table 2. The numerical values of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution, exact solution $u(1,t)$, and the absolute value of the error for Example 3.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{MGTFSM} \rvert$
0.1	3.4130	3.2325	3.1011	3.0042	3.0042	2.3035×10^{-7}
0.3	4.5304	4.1674	3.8882	3.6692	3.6693	5.7920×10^{-5}
0.5	5.7607	5.2327	4.8156	4.4809	4.4817	7.7137×10^{-4}
0.7	7.1752	6.4862	5.9269	5.4696	5.4739	4.3000×10^{-3}
0.9	8.8079	7.9656	7.2597	6.6702	6.6859	1.5673×10^{-2}

Table 3. The numerical values of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution, exact solution $u(1,t)$, and the absolute value of the error for Example 4.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{MGTFSM} \rvert$
0.1	1.2556	1.1892	1.1408	1.1052	1.1052	8.4742×10^{-8}
0.3	1.6666	1.5331	1.4304	1.3498	1.3499	2.1308×10^{-5}
0.5	2.1193	1.9250	1.7715	1.6484	1.6487	2.8377×10^{-4}
0.7	2.6396	2.3861	2.1804	2.0122	2.0138	1.5819×10^{-3}
0.9	3.2403	2.9304	2.6707	2.4538	2.4596	5.7656×10^{-3}

Table 4. The numerical values of the 4th Modified Generalized Taylor Fractional Series Method (MGTFSM) approximate solution, exact solution $u(1,1,t)$, and the absolute value of the error for Example 5.

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{MGTFSM} \rvert$
0.1	1.2556	1.1892	1.1408	1.1052	1.1052	8.4742×10^{-8}
0.3	1.6666	1.5331	1.4304	1.3498	1.3499	2.1308×10^{-5}
0.5	2.1193	1.9250	1.7715	1.6484	1.6487	2.8377×10^{-4}
0.7	2.6396	2.3861	2.1804	2.0122	2.0138	1.5819×10^{-3}
0.9	3.2403	2.9304	2.6707	2.4538	2.4596	5.7656×10^{-3}

to include another set of Nonlinear Fractional Partial Differential Equations (NFPDEs) with high-order fractional derivatives $n\alpha$, where $n-1 < n\alpha \leq n$ and $n \in N^*$.

Nomenclature

MGTFSM	Modified Generalized Taylor Fractional Series Method
$u(1,t)$	Exact solution $u(x,t)$ at $x=1$
$u(1,1,t)$	Exact solution $u(x,y,t)$ at $x=y=1$
u_{exact}	Exact solution
u_{MGTFSM}	4th MGTFSM approximate solution
$\lvert u_{exact}-u_{MGTFSM}\rvert$	Absolute error between u_{exact} and u_{MGTFSM}

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