A new approximate analytical method and its convergence for nonlinear time-fractional partial differential equations

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ABSTRACT. The main goal of this paper is to present a new approximate analytical method called modified generalized Taylor fractional series method (MGTFSM) for solving general nonlinear time-fractional partial differential equations. The fractional derivative is considered in the Caputo sense. We establish the convergence results of the proposed method. The basic idea of the MGTFSM is to construct the solution in the form of infinite series which converges rapidly to the exact solution of the given problem. The main advantage of the proposed method compare to current methods is that method solves the nonlinear problems without using linearization, discretization, perturbation or any other restriction. The efficiency and accuracy of the MGTFSM is tested by means of different numerical examples. The results prove that the proposed method is very effective and simple for solving the nonlinear time-fractional partial differential equations problems.

Keywords: Fractional model; Riemann-Liouville integral; Caputo derivative; Numerical method; Approximate analytical solution.

1. INTRODUCTION

In recent years, many mathematicians and physicists have been interested in the subject of nonlinear fractional partial differential equations (NFPDEs) because of its many applications in various fields where they can be used for
modelling a wide range of physical phenomena such as statistical mechanics, mathematical physics, theoretical neuroscience, fluid dynamics, electrochemistry, viscoelasticity, population dynamics, cancer modelling and mathematical finance. So, the theory of NFPDEs is a powerful theory to give a solution to engineering problems. See for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Obtaining approximate analytical solutions of the NFPDEs is one of the very important subjects in mathematics, science and technology. For this purpose, many different numerical and analytical methods have been constructed and developed, among them are: modification of Adomian decomposition method (MADM) [16], variational iteration method (VIM) [17], homotopy analysis method (HAM) [18], homotopy perturbation transform method (HPTM) [19], new iterative method (NIM) [20], fractional Elzaki projected differential transform method (FEPDTM) [21], generalized differential transform method (GDTM) [22], modification of the reduced differential transform method (MRDTM) [23], fractional Taylor operational matrix method (FTOMM) [24], fractional residual power series method (FRPSM) [25].

The MGTFSM was first proposed to obtain numerical solution for a certain class of NFPDEs, see [26]. The MGTFSM consists of an iterative algorithm. This method is effective and easy to obtain a power series solution for linear and nonlinear fractional differential equations without resorting to linearization, perturbation, or discretization. Unlike other series methods, the MGTFSM does not want to match the coefficients of the similar conditions and a repeated connection isn’t needed. The present method computes the coefficients of the power series by a bond of algebraic equations of some variables. In addition, the MGTFSM does not need any transformation during the change from the low order to the higher order, thus it is possible to work with the present method directly on the given example by choosing an suitable initial estimate approximation. This method has been tested to be powerful, effective, and can easily deal with a wide range of linear and nonlinear fractional differential equations. The main objective of this paper is to extend this method and establish its convergence to solve general nonlinear time-fractional partial differential equations with arbitrary nonlinear terms.

The rest of the paper is organized as follows: In Section 2, we introduce some basic definitions and properties of fractional calculus. In Section 3, we present our obtained results in the form of a new theorems to solve the general nonlinear time-fractional partial differential equations by using the MGTFSM. In Section 4, we present some numerical examples to exhibit the efficiency and effectiveness of the proposed method, In Section 5, we discuss our obtained results represented by Figures and Tables. Section 6, summarizes the conclusions of this work.
2. Basic definitions of fractional calculus

In this section, we introduce some basic definitions and properties of fractional calculus which are used further in this paper. For more details see, [27, 28].

Definition 1. A real function $f(t), t > 0$, is considered to be in the space $C_{\mu} ([0, \infty]), \mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $f(t) = t^p h(t)$, where $h \in C ([0, \infty])$, and it is said to be in the space $C_{\mu}^n$ if $f^{(n)} \in C_{\mu} ([0, \infty]), n \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ of a function $f \in C_{\mu} ([0, \infty]), \mu \geq -1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} f(\xi)d\xi, \alpha > 0, t > 0,$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 3. The Caputo fractional derivative of order $n - 1 < \alpha \leq n$ of a function $f \in C_{n-1}^\alpha ([0, \infty]), n \in \mathbb{N}$, is defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \xi)^{n-\alpha-1} f^{(n)}(\xi)d\xi.$$

For this definition we have the following properties

1) $D^\alpha (C) = 0$, where $C$ is a constant.

2) $D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta > n - 1, \\ 0, & \text{if } \beta \leq n - 1. \end{cases}$

3) $D^\alpha [\lambda f(t) + \nu g(t)] = \lambda D^\alpha [f(t)] + \nu D^\alpha [g(t)], (\lambda, \nu) \in \mathbb{R}^2.$

4) $D^\alpha [f(t)g(t)] = g(t)D^\alpha [f(t)] + f(t)D^\alpha [g(t)].$

Definition 4. The Caputo time-fractional derivative of order $n - 1 < \alpha \leq n$ of a function $u \in C_{n-1}^\alpha (\mathbb{R} \times [0, \infty]), n \in \mathbb{N}$, is defined as

$$D^\alpha_t u(x, t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \xi)^{n-\alpha-1} \frac{\partial^n u(x, \xi)}{\partial \xi^n} d\xi.$$

Definition 5. The Mittag-Leffler function is defined as follows

\[ E_\alpha (z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \alpha > 0, z \in \mathbb{C}. \quad (8) \]

For \( \alpha = 1 \), \( E_\alpha (z) \) reduces to \( e^z \).

3. Analytical study of MFTSM

Theorem 1. Consider a general nonlinear time-fractional partial differential equation

\[ D_t^\alpha u(x, t) = \mathcal{L}(u(x, t)) + \mathcal{N}(u(x, t)), x \in I \subset \mathbb{R}, t \geq 0, \quad (9) \]

with the initial conditions

\[ u(x, 0) = u_0(x), \quad (10) \]

where \( D_t^\alpha \) is the Caputo fractional derivative of order \( \alpha, 0 < \alpha \leq 1 \), \( \mathcal{L}(u(x, t)) \) is a linear differential operator and \( \mathcal{N}(u(x, t)) \) represents a general nonlinear differential operator. Then, by MGTFSM the solution of equations (9)-(10) is given in the form of infinite series as follows

\[ u(x, t) = \sum_{j=0}^{\infty} u_j(x) t^{j\alpha} \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}, x \in I \subset \mathbb{R}, 0 \leq t < R, \quad (11) \]

where \( u_j(x) \) are the coefficients of the series (11) and \( R \) is the radius of convergence.

Proof. In order to achieve our goal, we consider the following generalized nonlinear fractional partial differential equation (9) with the initial conditions (10).

We assume the solution as an infinite series given by

\[ u(x, t) = \sum_{j=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}. \quad (12) \]

Consequently, the approximate solution of equations (9)-(10), can be written as

\[ u_n(x, t) = \sum_{j=0}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = u_0(x) + \sum_{j=1}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}. \quad (13) \]

By applying the operator \( D_t^\alpha \) on equation (13), and using the properties (1) and (2), we obtain the formula

\[ D_t^\alpha u_n(x, t) = \sum_{j=0}^{n-1} u_{j+1}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}. \quad (14) \]
We substitute (13) and (14) in the equation (9), and the following recurrence relation is obtained

\[
0 = \sum_{j=0}^{n-1} u_{j+1}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} - \mathcal{L} \left( \sum_{j=0}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} \right) \\
- \mathcal{N} \left( \sum_{j=0}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} \right).
\]

To find the functions \( u_n(x) \), \( n = 1, 2, 3, \ldots \), we follow the same methodology to obtain the coefficients of the Taylor series, so we need to solve the following equation

\[
D_t^{(n-1)\alpha} \{ G(x, t, \alpha, n) \} \downarrow_{t=0} = 0,
\]

where

\[
G(x, t, \alpha, n) = \sum_{j=0}^{n-1} u_{j+1}(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} - \mathcal{L} \left( \sum_{j=0}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} \right) \\
- \mathcal{N} \left( \sum_{j=0}^{n} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} \right).
\]

Now, we calculate the first terms of the sequence \( \{ u_n(x) \}_{1}^{N} \).

For \( n = 1 \) we have

\[
G(x, t, \alpha, 1) = u_1(x) - \mathcal{L} \left( \frac{u_0(x) + u_1(x) t^{\alpha}}{\Gamma(\alpha + 1)} \right) \\
- \mathcal{N} \left( \frac{u_0(x) + u_1(x) t^{\alpha}}{\Gamma(\alpha + 1)} \right).
\]

Solve the equation \( G(x, t, \alpha, 1) = 0 \), gives

\[
u_1(x) = \mathcal{L}(u_0(x)) + \mathcal{N}(u_0(x)).
\]

To find \( u_2(x) \), we consider \( G(x, t, \alpha, 2) \) and we solve

\[
D_t^\alpha \{ G(x, t, \alpha, 2) \} \downarrow_{t=0} = 0.
\]

To find \( u_3(x) \), we consider \( G(x, t, \alpha, 3) \) and we solve

\[
D_t^{2\alpha} \{ G(x, t, \alpha, 3) \} \downarrow_{t=0} = 0.
\]

To find \( u_4(x) \), we consider \( G(x, t, \alpha, 4) \) and we solve

\[
D_t^{3\alpha} \{ G(x, t, \alpha, 4) \} \downarrow_{t=0} = 0.
\]

and so on.
In general, to obtain the coefficient function \( u_k(x) \) we solve
\[
D_t^{(k-2)\alpha} \{ G(x, t, \alpha, k) \} \big|_{t=0} = 0,
\]
where
\[
G(x, t, \alpha, k) = \sum_{j=0}^{k-1} u_{j+1}(x) \frac{t^j}{\Gamma(j+1)} - \mathcal{L} \left( \sum_{j=0}^{k} u_j(x) \frac{t^j}{\Gamma(j+1)} \right) - \mathcal{N} \left( \sum_{j=0}^{k} u_j(x) \frac{t^j}{\Gamma(j+1)} \right).
\]
Finally, the solution of equations (9)-(10), can be expressed as
\[
u(x, t) = \lim_{n \to \infty} u_n(x, t)
= \lim_{n \to \infty} \sum_{j=0}^{n} u_j(x) \frac{t^j}{\Gamma(j+1)}
= \sum_{j=0}^{\infty} u_j(x) \frac{t^j}{\Gamma(j+1)}.
\]

The proof is complete. \qed

**Theorem 2.** If there exists a constant \( 0 < \gamma < 1 \) such that
\[
\|u_{n+1}(x, t)\| \leq \gamma \|u_n(x, t)\|, \quad n \in \mathbb{N}, \quad x \in I \subset \mathbb{R}, \quad 0 \leq t < R,
\]
then, the sequence of approximate solution (11) converges to the exact solution.

**Proof.** For all \( x \in I \subset \mathbb{R}, \ 0 \leq t < R \), we have
\[
\|u(x, t) - u_n(x, t)\| = \left\| \sum_{j=n+1}^{\infty} u_j(x, t) \right\| 
\leq \sum_{j=n+1}^{\infty} \|u_j(x, t)\|
\leq \sum_{j=n+1}^{\infty} \gamma \|u_{j-1}(x, t)\|
\leq \sum_{j=n+1}^{\infty} \gamma^2 \|u_{j-2}(x, t)\|
\leq ... \leq \|u_0(x)\| \sum_{j=n+1}^{\infty} \gamma^j
= \frac{\gamma^{n+1}}{1-\gamma} \|u_0(x)\|.
\]
Since \( 0 < \gamma < 1 \) and \( u_0(x) \) is bounded, then
\[
\lim_{n \to \infty} \|u(x, t) - u_n(x, t)\| = 0.
\]
This completes the proof. \qed
4. Applications of MGTFSM to Nonlinear Time-Fractional Partial Differential Equations

In this section, some numerical examples are given to exhibit the efficiency and effectiveness of the MGTFSM for solving nonlinear time-fractional partial differential equations

Example 1. We consider the cubic nonlinear time-fractional Schrödinger equation

\[
i D_t^\alpha u = -u_{xx} + 2|u|^2 u, \tag{29}
\]

with the initial condition

\[
u(x, 0) = e^{ix}, \tag{30}
\]

where \( D_t^\alpha \) is the Caputo time-fractional derivative of order \( \alpha, 0 < \alpha \leq 1 \), \( u \) is a complex function of \( (x, t) \in \mathbb{R} \times \mathbb{R}^+ \) and \( i^2 = -1 \).

For \( \alpha = 1 \), the exact solution of equations (29)-(30) is given by

\[
u(x, t) = e^{i(x-3t)}. \tag{31}
\]

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of equations (29)-(30) in the form

\[
u(x, t) = \sum_{i=0}^{\infty} u_j(x) t^{j\alpha} \Gamma(j\alpha + 1), \tag{32}
\]

and the coefficients of the series (32), are given as

\[
u_j(x) = (-1)^j (3it^\alpha)^j e^{ix} \text{ for } j = 0, 1, 2, 3, ... \tag{33}
\]

Therefore, the solution of equations (29)-(30), can be expressed as

\[
u(x, t) = e^{ix} \left( 1 - \frac{(3it^\alpha)}{\Gamma(\alpha + 1)} + \frac{(3it^\alpha)^2}{\Gamma(2\alpha + 1)} - \frac{(3it^\alpha)^3}{\Gamma(3\alpha + 1)} + \cdots \right)
\]

\[
= e^{ix} \sum_{j=0}^{\infty} \frac{(-1)^j (3it^\alpha)^j}{\Gamma(j\alpha + 1)} = e^{ix} E_\alpha (-3it^\alpha), \tag{34}
\]

where \( E_\alpha (-3it^\alpha) \) is the Mittag-Leffler function, defined by equation (8).

Taking \( \alpha = 1 \) in equation (34), will give the following result

\[
u(x, t) = e^{ix} \left( 1 - 3it + \frac{(3it)^2}{2!} - \frac{(3it)^3}{3!} + \cdots \right)
\]

\[
= e^{i(x-3t)}, \tag{35}
\]

which is the exact solution of classical Schrödinger equation.
Example 2. We consider the nonlinear time-fractional gas dynamic equation [30]

\[ D_t^\alpha u = -uu_x + u(1 - u), \quad (36) \]

with the initial condition

\[ u(x, 0) = e^{-x}, \quad (37) \]

where \( D_t^\alpha \) is the Caputo time-fractional derivative of order \( \alpha, 0 < \alpha \leq 1 \) and \( u \) is a function of \( (x, t) \in \mathbb{R} \times \mathbb{R}^+ \).

For \( \alpha = 1 \), the exact solution of equations (36)-(37) is given by

\[ u(x, t) = e^{t-x}. \quad (38) \]

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of equations (36)-(37) in the form

\[ u(x, t) = \sum_{j=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}, \quad (39) \]

and the coefficients of the series (39), are given as

\[ u_j(x) = e^{-x} \text{ for } j = 0, 1, 2, 3, \ldots \quad (40) \]

Therefore, the solution of equations (36)-(37), can be expressed as

\[ u(x, t) = e^{-x} \left( 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots \right) \]

\[ = e^{-x} \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = e^{-x} E_\alpha(t^\alpha), \quad (41) \]

where \( E_\alpha(t^\alpha) \) is the Mittag-Leffler function, defined by equation (8).

Taking \( \alpha = 1 \) in equation (41), will give the following result

\[ u(x, t) = \left( 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) e^{-x} \]

\[ = e^{t-x}, \quad (42) \]

which is the exact solution of classical gas dynamic equation.

Example 3. We consider the nonlinear time-fractional reaction-diffusion-convection equation [31]

\[ D_t^\alpha u = u_{xx} - u_x + uu_x - u^2 + u, \quad (43) \]

with the initial condition

\[ u(x, 0) = e^x, \quad (44) \]

where \( D_t^\alpha \) is the Caputo time-fractional derivative of order \( \alpha, 0 < \alpha \leq 1 \) and \( u \) is a function of \( (x, t) \in \mathbb{R} \times \mathbb{R}^+ \).
For $\alpha = 1$, the exact solution of equations (43)-(44) is given by

$$u(x, t) = e^{x+t}. \quad (45)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of equations (43)-(44) in the form

$$u(x, t) = \sum_{j=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}, \quad (46)$$

and the coefficients of the series (46), are given as

$$u_j(x) = e^x \text{ for } j = 0, 1, 2, 3, ... \quad (47)$$

Therefore, the solution of equations (43)-(44), can be expressed as

$$u(x, t) = e^x \left( 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots \right)$$

$$= e^x \sum_{i=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = e^x E_\alpha (t^\alpha), \quad (48)$$

where $E_\alpha (t^\alpha)$ is the Mittag-Leffler function, defined by equation (8).

Taking $\alpha = 1$ in equation (48), will give the following result

$$u(x, t) = e^x \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) = e^{x+t}, \quad (49)$$

which is the exact solution of classical reaction-diffusion-convection equation.

**Example 4.** Consider the nonlinear time-fractional Fokker-Planck equation [29]

$$D_t^\alpha u = \left( \frac{x}{3} u \right)_x - \left( \frac{4}{x} u^2 \right)_x + (u^2)_{xx}, \quad (50)$$

with the initial condition

$$u(x, 0) = x^2, \quad (51)$$

where $D_t^\alpha$ is the Caputo time-fractional derivative of order $\alpha$, $0 < \alpha \leq 1$ and $u$ is a function of $(x, t) \in \mathbb{R} \times \mathbb{R}^+$. For $\alpha = 1$, the exact solution of equations (50)-(51) is given by

$$u(x, t) = x^2 e^t. \quad (52)$$

By applying the steps involved in the MGTFSM as presented in Section 3, we have the solution of equations (50)-(51) in the form

$$u(x, t) = \sum_{i=0}^{\infty} u_j(x) \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)}, \quad (53)$$
and the coefficients of the series (53), are given as
\[ u_j(x) = x^2 \text{ for } j = 0, 1, 2, 3, \ldots \] (54)

Therefore, the solution of equations (50)-(51), can be expressed as
\[
\begin{align*}
  u(x, t) &= x^2 \left( 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots \right) \\
  &= x^2 \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(j\alpha + 1)} = x^2 E_\alpha (t^\alpha),
\end{align*}
\] (55)

where \( E_\alpha (t^\alpha) \) is the Mittag-Leffler function, defined by equation (8).

Taking \( \alpha = 1 \) in equation (55), will give the following result
\[
\begin{align*}
  u(x, t) &= x^2 \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) = x^2 e^t,
\end{align*}
\] (56)

which is the exact solution of classical Fokker-Planck equation.

**Example 5.** Consider the 2-dimensional nonlinear time-fractional biological population equation
\[ D_t^\alpha u = (u^2)_{xx} + (u^2)_{yy} - u(1 + \frac{8}{9}u), \] (57)

with the initial condition
\[ u(x, y, 0) = e^{(x+y)/3}, \] (58)

where \( D_t^\alpha \) is the Caputo time-fractional derivative of order \( \alpha, 0 < \alpha \leq 1 \) and \( u \) is a function of \( (x, y, t) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \).

For \( \alpha = 1 \), the exact solution of equations (57)-(58) is given by
\[ u(x, y, t) = e^{(1/3)(x+y) - t}. \] (59)

By applying the steps involved in the MGTTFSM as presented in Section 3, we have the solution of equations (57)-(58) in the form
\[
\begin{align*}
  u(x, y, t) &= \sum_{i=0}^{\infty} u_i(x, y) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)},
\end{align*}
\] (60)

and the coefficients of the series (60), are given as
\[ u_i(x, y) = (-1)^i e^{(x+y)/3} \text{ for } i = 0, 1, 2, 3, \ldots \] (61)

Therefore, the solution of equations (57)-(58), can be expressed as
\[
\begin{align*}
  u(x, y, t) &= e^{(x+y)/3} \left( 1 - \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \cdots \right) \\
  &= e^{(x+y)/3} \sum_{i=0}^{\infty} \frac{(-1)^i t^{i\alpha}}{\Gamma(i\alpha + 1)} = e^{(x+y)/3} E_\alpha (-t^\alpha),
\end{align*}
\] (62)
where $E_\alpha (-t^\alpha)$ is the Mittag-Leffler function, defined by equation (8).

Taking $\alpha = 1$ in equation (62), will give the following result

$$u(x, y, t) = e^{(x+y)/3} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^{(1/3)(x+y)-t}, \quad (63)$$

which is the exact solution of classical biological population equation.

5. Numerical results and discussion

To show the accuracy of the MGTFSM algorithm in handling examples provided, the two-dimensional plots of the 4th MGTFSM approximate solutions and exact solutions when $t \in [0, 1]$ for different levels of fractional order $\alpha$ such that $\alpha = \{0.7, 0.8, 0.9, 1\}$, are given in Figures 1–4, respectively. Tables 1–4 show the numerical values of the 4th MGTFSM approximate solutions, the exact solutions and the absolute values of the errors for Examples 2–5, respectively, when $\alpha = \{0.7, 0.8, 0.9, 1\}$ and some selected grid points $t$ with step size 0.1.

From these Figures and Tables, it is clearly observed that when $\alpha \to 1$ the approximate solution obtained by the MGTFSM is very close to the exact solution, which indicates accuracy and efficiency of the proposed method.

6. Conclusion

In this paper, a new approximate analytical method called the MGTFSM has been successfully applied to obtain the numerical solutions of general nonlinear time-fractional partial differential equations with arbitrary nonlinear terms. The method was applied to many numerical examples. The results show that the MGTFSM is an efficient and easy to use method for finding numerical solutions for these problems. The obtained approximate solutions using the proposed method is in excellent agreement with it corresponding exact solutions. Finally, we conclude the MGTFSM is powerful and efficient in finding approximate and analytical solutions for wider classes of nonlinear fractional problems.

In future works, we will extend this approach to include another set of NFPDEs with high-order fractional derivatives $n\alpha$, where $n - 1 < n\alpha \leq n$ and $n \in \mathbb{N}^*$.

Acknowledgments

The authors thank the anonymous referee for his/her careful reading of the paper and his/her valuable remarks that improved the final version of the paper.
Nomenclature:
MGTFSM : modified generalized Taylor fractional series method.
\( u(1,t) \) : exact solution \( u(x,t) \) at \( x = 1 \).
\( u(1,1,t) \) : exact solution \( u(x,y,t) \) at \( x = y = 1 \).
\( u_{exact} \) : exact solution.
\( u_{MGTFSM} \) : 4th MGTFSM approximate solution.
\( |u_{exact} - u_{MGTFSM}| \) : absolute error between \( u_{exact} \) and \( u_{MGTFSM} \).

References


**Biographies**

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Figure 1. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1,t)$ for Example 2.

Figure 2. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1,t)$ for Example 3.

Figure 3. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1,t)$ for Example 4.

Figure 4. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1,1,t)$ for Example 5.

**List of Table captions :**

Table 1. The numerical values of the 4th MGTFSM approximate solution, exact solution $u(1,t)$ and the absolute value of the error for Example 2.

Table 2. The numerical values of the 4th MGTFSM approximate solution, exact solution $u(1,t)$ and the absolute value of the error for Example 3.

Table 3. The numerical values of the 4th MGTFSM approximate solution, exact solution $u(1,t)$ and the absolute value of the error for Example 4.
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**Figure 1.** The behavior of the $4^{th}$ MGTFSM approximate solution and exact solution $u(1,t)$ for Example 2.

**Figure 2.** The behavior of the $4^{th}$ MGTFSM approximate solution and exact solution $u(1,t)$ for Example 3.
Figure 3. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1, t)$ for Example 4.

Figure 4. The behavior of the 4th MGTFSM approximate solution and exact solution $u(1, 1, t)$ for Example 5.
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Table 1. The numerical values of the 4th MGFTSM approximate solution, exact solution $u(1, t)$ and the absolute value of the error for Example 2.

| $t$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 1$ | exact solution | $|u_{exact} - u_{MGFTSM}|$ |
|-----|----------------|----------------|----------------|-------------|----------------|----------------------------|
| 0   | 0.46191        | 0.43746        | 0.41969        | 0.40657     | 0.40657        | $3.1175 \times 10^{-8}$   |
| 0.3 | 0.61312        | 0.56399        | 0.52622        | 0.49658     | 0.49659        | $7.8386 \times 10^{-6}$   |
| 0.5 | 0.77963        | 0.70816        | 0.65172        | 0.60643     | 0.60653        | $1.0439 \times 10^{-4}$   |
| 0.7 | 0.97106        | 0.87781        | 0.80211        | 0.74024     | 0.74082        | $1.9680 \times 10^{-6}$   |
| 0.9 | 1.19200        | 1.07800        | 0.98249        | 0.90272     | 0.90484        | $2.1210 \times 10^{-3}$   |

Table 2. The numerical values of the 4th MGFTSM approximate solution, exact solution $u(1, t)$ and the absolute value of the error for Example 3.

| $t$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 1$ | exact solution | $|u_{exact} - u_{MGFTSM}|$ |
|-----|----------------|----------------|----------------|-------------|----------------|----------------------------|
| 0   | 3.4130         | 3.2325         | 3.1011         | 3.0042      | 3.0042         | $2.3035 \times 10^{-7}$   |
| 0.3 | 4.5304         | 4.1674         | 3.8882         | 3.6692      | 3.6693         | $5.7920 \times 10^{-5}$   |
| 0.5 | 5.7607         | 5.2327         | 4.8156         | 4.4809      | 4.4817         | $7.7137 \times 10^{-4}$   |
| 0.7 | 7.1752         | 6.4862         | 5.9269         | 5.4696      | 5.4739         | $4.3000 \times 10^{-3}$   |
| 0.9 | 8.8079         | 7.9656         | 7.2597         | 6.6702      | 6.6859         | $1.5673 \times 10^{-2}$   |

Table 3. The numerical values of the 4th MGFTSM approximate solution, exact solution $u(1, t)$ and the absolute value of the error for Example 4.

| $t$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 1$ | exact solution | $|u_{exact} - u_{MGFTSM}|$ |
|-----|----------------|----------------|----------------|-------------|----------------|----------------------------|
| 0   | 1.2556         | 1.1892         | 1.1408         | 1.1052      | 1.1052         | $8.4742 \times 10^{-8}$   |
| 0.3 | 1.6666         | 1.5331         | 1.4304         | 1.3498      | 1.3499         | $2.1308 \times 10^{-5}$   |
| 0.5 | 2.1193         | 1.9250         | 1.7715         | 1.6484      | 1.6487         | $2.8377 \times 10^{-4}$   |
| 0.7 | 2.6396         | 2.3861         | 2.1804         | 2.0122      | 2.0138         | $1.5819 \times 10^{-3}$   |
| 0.9 | 3.2403         | 2.9304         | 2.6707         | 2.4538      | 2.4596         | $5.7656 \times 10^{-3}$   |
Table 4. The numerical values of the 4\textsuperscript{th} MGTFSM approximate solution, exact solution $u(1, 1, t)$ and the absolute value of the error for Example 5.

| $t$ | $\alpha = 0.7$ | $\alpha = 0.8$ | $\alpha = 0.9$ | $\alpha = 1$ | exact solution | $|u_{\text{exact}} - u_{\text{MGTFSM}}|$ |
|-----|---------------|---------------|---------------|------------|---------------|-----------------|
| 0.1 | 1.2556        | 1.1892        | 1.1408        | 1.1052     | 1.1052        | $8.4742 \times 10^{-8}$ |
| 0.3 | 1.6666        | 1.5331        | 1.4304        | 1.3498     | 1.3499        | $2.1308 \times 10^{-5}$ |
| 0.5 | 2.1193        | 1.9250        | 1.7715        | 1.6484     | 1.6487        | $2.8377 \times 10^{-4}$ |
| 0.7 | 2.6396        | 2.3861        | 2.1804        | 2.0122     | 2.0138        | $1.5819 \times 10^{-3}$ |
| 0.9 | 3.2403        | 2.9304        | 2.6707        | 2.4538     | 2.4596        | $5.7656 \times 10^{-3}$ |