Stackelberg models in two-level supply chain with imperfect quality items with allowable shortages

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Abstract

Application of an absolute supply chain model does not invalidate the possibility of few defective items in a supplied lot, therefore it becomes essential to conduct an inspection process for segregating the defective items, subsequently such segregated items are sold at discounted price. Shortages mainly occur with sudden demand or erratic production capacity, and player’s decisions are influenced by it. In this paper, the shortage is considered as a seller’s decision variable and demand is receptive to selling price and marketing expenditure of the buyer. Player’s interaction will be reviewed and determined as non-cooperative Stackelberg game. Further, a supply chain model is endured to substantiate the interaction and democracy among buyer and seller in the supply chain and is pitched by non-cooperative game theoretical approaches. The Stackelberg game approach is used in the non-cooperative method where one player acts as leader and another as follower. Hereafter, unanimous numerical examples along with sensitivity analysis are exhibited to compare amidst two different models with and without shortages to demonstrate the significance of the paper.

Keywords: Game theory; Imperfect quality items; Non-cooperative games; Shortages; Supply chain.

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1. Introduction

In the last few decades, many researchers and academicians have assessed and magnified the supply chain system and fortified methods like seller-buyer coordination, business process management and information sharing (Jüttner et al. [1]). The seller-buyer supply chain is a representation of the relationship between the two players (seller and buyer), where the seller vends goods to the buyer in a lot, who then sells it to the customer (Chen et al. [2], Yang and Zhou [3], and Dai et al. [4]). Concepts such as credit option, quantity discount, buy back and quantity flexibility have been discussed with respect to supply chain management in the past. Chiang et al. [5], Sarmah et al. [6], Weng [7], and Shekarian [8] reviewed the factors affecting closed loop supply chain models. With the assumption of demand as a fixed constant, researchers like Chan and Kingsman [9], Dai and Qi [10], Heuvel et al. [11], and Sucky [12,13] have contributed to supply chain management by determining the ideal lot size and order cycle to optimize profit of the supply chain. Gautam and Khanna [14] presented a sustainable framework under a two level supply chain environment. The seller runs the manufacturing process to assist the marketing demand of the buyer. Researchers like Lee [15], Abad [16], Kim and Lee [17] and Jung and Klein [18] developed supply chain model where end demand fluctuates with the buyer’s price to obtain the optimal policy for both the members. Abad and Jaggi [19] further improved the model proposed by Abad [16] by allowing seller to give a credit period to the buyer in which seller and buyer optimize their strategy under cooperative and non-cooperative game theory structures. Sadjadi et al. [20] and Lee and Kim [21] proposed models for determining optimal policies, where demand is the function of selling price and marketing expenditure. Esmaeili et al. [22] used the demand function to design models of non-cooperative and cooperative game approach following symmetric information pattern i.e. both players work unanimously. Esmaeili and Zeephongsekul [23] proposed a similar model under asymmetric information structure i.e. both the players work ambiguously. Silbermayr [24] discussed the news vendor problem using non-cooperative game approach in the inventory management by describing a single period inventory control model with the focus on horizontal interactions among multiple independent newsvendors.

Aforementioned papers assumed that all the items are of perfect quality, however, it is not uncommon that defective quality items are manufactured during production. These items are then only identified by an inspection process at the buyer’s end. Inventory models on defective items
were investigated by Schwaller [25], and Rosenblatt and Lee [26] and later extended by Salameh and Jaber [27] by allowing defective quality items to be sold at discounted prices post inspection. Maddah and Jaber [28] used Renewal-reward theorem Ross [29] to determine the total expected profit per unit time. Any of the refereed papers did not consider the possibility of shortages, which can occur at any point in the production cycle due to either irregular production or due to an increased in the demand. The buyer may be forced to order large quantities fearing shortages which ultimately increases the holding cost. Essentially, shortage size needs to be optimized. Wee et al. [30] made contributions to the model proposed by Salameh and Jaber [27] by placing, repeated order of shortfalls in each cycle. Eroglu and Ozdemir [31] incorporated the concept of backordering in times of shortages and reviewed the results of defective quality items in lot size and optimal profit. Consequently, numerous related papers for controlling imperfect quality items has been published by Sarkar [32], Sarkar and Moon [33], Roy et al [34], Cheikhrouhou et al. [35] Sarkar et al. [36] reexamine the EPQ model as a single stage manufacturing system with rework process and planed backorders, which allows random defective rates. Three inventory models are developed, and the comparison was shown between modes with three different probability density functions. An inventory model for non-instantaneous deteriorating items proposed by Tiwari et al. [37] where they included acceptable delay in payments and improved the optimal policy with respect to shortages. Jaggi et al. [38] proposed an inventory model with justifiable delays in payments for items of defective quality and shortages are allowed. An inventory model proposed by Khanna et al. [39] with delay in payments for deteriorating imperfect quality items where shortages are allowed and completely backlogged.

Khanna et al. [40] developed integrated vendor-buyer inventory model for imperfect quality items with allowable shortages under the permissible delay in payments. Kishore et al. [41] studied and optimized the production and backordering quantities in order to maximize the total expected profit per unit time. Khanna et al. [42] jointly optimized the number of shipments, the backorder size and the order size in order to minimize the integrated total cost of the seller and the buyer. Jaggi et al. [43] considered two warehouse inventory model for imperfect quality deteriorations items with one level of credit period. The presented inventory model maximized the total profit per unit time by optimizing the ordered quantity. Mittal et al. [44] discussed the effect of inspection on the
retailer's ordering policy under permissible delay in payments for defective deteriorating items, where, price and demand both quantities varies with the time.

Esmaeili [45] presented a new approach to find the lot size by non-cooperative game theoretic approach (Seller-Stackelberg and Buyer-Stackelberg). Yadav et al. [46] presented supply chain models with imperfect quality items with allowable late payments under co-operative and non-cooperative (Seller-Stackelberg) analogue, wherein the market demand of the product depends upon the retail price. Optimal policies of the partners in the supply chain are obtained in each scenario which will enhance the profit of the supply chain. Sarkar et al. [47] developed an integrated inventory model which optimizes the joint cost of a vendor and buyer by the Stackelberg game approach. The buyer used inspection policy to identify the defective items. The fixed number of shipments, variable transportation with the carbon emissions are considered in the model, which makes the model more sustainable. Lu et al. [48] obtained the optimal equilibrium solution between the buyer and the seller by the Stackelberg game theoretic approach under different carbon emission reductions.

Alaei et al. [49] discussed an optimization problem in an advertising environment under non-cooperative (Stackelberg) and cooperative game approach in a supply chain system. The coordination between the partners in the supply chain is discussed through a two-way subsidy strategy under two scenarios (exogenously and endogenously). Jaggi et al. [50] considered a supplier-retailer supply chain in which demand is stock dependent with credit period. The proposed model finds the optimal decisions of the supply chain under three different policies—centralized, Nash equilibrium solution and Supplier-Stackelberg policy. Yadav et al. [51] developed supply chain model to substantiate the interaction and democracy between the members of the supply chain, the buyer and seller, is established by non-cooperative and cooperative game theoretical approaches in which end demand depends upon the retailer price and marketing expenditure cost. Gautam et al. [52] developed two type models in which first model discussed the integrated problem-solving approach and the second model used the Stackelberg policy. The total profit is maximized by jointly optimizing the number of shipments, order quantity and backordering quantity.
Zhang et al. [53] determined a supply chain model based on game theoretical cooperative and non-cooperative approach, in which the decision variable of the seller was taken as shortage and demand are considered to depend on selling price and marketing promotional cost of the buyer. Zhang and Zeephongsekul [54] lengthen the work of Zhang et al. [53] with the same constraints and developed two mechanism design agreement provided to the buyer by the seller as an incentive to increase both the player’s profit.

Game theory is the advanced study of mathematical models of dispute and cooperation between the players or decision makers. Non-cooperative game theory approaches are desired to examine interaction between participants of supply chain and problems related to supply chain. No researcher has yet developed any supply chain models on imperfect quality items by taking shortages as a decision variable with the help of game theoretic approach. In this model, seller-buyer model with imperfect quality items, where shortages are allowed, have been developed using the concept of non-cooperative game. In the present model, the seller delivers items in a lot to the buyer, who then segregates items of degraded quality through an inspection process. These items of sub-par quality are then sold at a discount. The seller’s rate of production is assumed to be linearly dependent on rate of demand. Selling price and marketing expenditure of the leading player are used to determine the end demand, as players with more influence, often gain higher profit than others in a market.

The main contribution of the paper includes the following points:

- The buyer has considered ordering quantity as his decision variable. In the small medium entrepreneurship, based on the market demand, the buyer always raises the requirement for the seller.
- Shortages occur due to uneven production capacity or unpredictable demands and it influences both players’ decisions. Seller regulates it to reduce its effect on supply chain system. Due to shortages, the buyer may be forced to order large quantities to offset the loss of profit, which results in increased holding cost. However, if buyer faces large shortage then he might be inclined towards alternative sources of supply and as a result seller’s profit will be affected. Therefore, optimizing shortage size is important for both the players.
• Order quantity, buyer price and marketing expenditure are taken as the decision variable of the buyer whereas shortages and seller price are the decision variables of the seller. We consider two scenarios, one where the seller acts as a leader (Seller-Stackelberg) and another where buyer acts as a leader (Buyer-Stackelberg).

Authors contribution is given in the tabulated form in Table 1.

< Insert Table 1 >

2. Notations and assumptions

2.1 Notations

Decision variables of the seller

\[ c_b \]  Buyer’s purchasing cost ($ per unit)
\[ S \]  Shortages managed by the seller (units)

Decision variables of the buyer

\[ Q \]  Order quantity (units)
\[ M \]  Cost of marketing expenses ($ per unit)
\[ p_b \]  Retail price of the buyer ($ per unit)

Parameters

\[ C \]  Seller’s Purchasing cost ($ per unit)
\[ A_b \]  Buyer’s Ordering cost ($ per order)
\[ A_s \]  Seller’s Ordering cost ($ per order)
\[ l \]  Percent inventory carrying cost ($ per unit)
\[ H_b \]  Carrying cost of inventory ($ per unit per time)
\[ \alpha \]  Proportion of sub-par quality items delivered to buyer by the seller
\[ T_1 \]  Buyer’s cycle length (years),\[ T_1 = \frac{Q (1 - \alpha)}{D} \]
\[ T_2 \]  Seller’s cycle length (years),\[ T_2 = \frac{Q}{D} \]
\[ T \]  Length of cycle in Stackelberg model (years),\[ T = \text{Max}(T_1, T_2) \]
\(c_s\) Cost of imperfect quality items ($ per unit) \((c_s < c_b)\)

\(\lambda\) Buyer’s rate of screening (unit per year)

\(t\) Screening time to find the defective items, \(t = Q/\lambda\) (years)

\(C_1\) Buyer’s shortages cost ($ per unit)

\(C_2\) Seller’s shortages cost ($ per unit)

\(k\) Scaling constant for Marketing demand \((k > 0)\)

\(r\) Seller’s rate of production (unit per cycle)

\(u\) Scaling constant for production \((u > 1)\)

\(d\) Demand rate (unit per cycle)

\(e\) Price elasticity for marketing demand \((e > 1)\)

\(\beta\) Marketing expenditure elasticity for marketing demand \((0 < \beta < 1, \beta + 1 < e)\)

\(D\) Annual market demand, which is a function of selling price \(p_b\), and marketing expenditure \(M\), such that, \(D = k p_b^{-e} M^\beta\) (Esmaeili et al. [22])

### 2.2 Assumptions

1. The annual market demand is a multiplicative power function of selling price, \(p_b\) and marketing expenditure, \(M\) of the buyer.
2. Infinite planning horizon.
3. Parameters are known in advance and deterministic under symmetric information scenario.
4. The buyer determines the lot size.
5. It is assumed that defective items are distributed uniformly in each lot (Jaggi et al. [38])
6. Shortage occurs due to unanticipated demand or irregular production capacity by the seller so that the effect on the supply chain system is reduced.
7. The production rate is greater than the demand rate. It is assumed that demand rate and production rate is linearly related as \(r = u d\), \(u > 1\), where \(r\) and \(d\) are production rate and demand rate respectively (Zhang et al. [53]).

### 3. Mathematical formulation
In this section, we propose mathematical description of non-cooperative models for both buyer (Buyer-Stackelberg) and seller (Seller-Stackelberg) to optimize supply chain’s expected profits. The supply chain problem is considered to be a two player non-zero-sum game.

3.1 Buyer’s model

The buyer’s objective is to determine his decision variables, \( p_b, M \) and \( Q \) such that total expected profit is maximized. The buyer’s total annual profit, \( TP_b(p_b,M) \) is given by

\[
TP_b(p_b,M,Q) = \text{Sales revenue} - \text{Purchasing cost} - \text{Ordering cost} - \text{Marketing cost} - \text{Holding cost} - \text{Shortages cost}
\]

The holding cost is expressed in terms of the percentage of positive inventory \( l c_b A \) and shortages cost in terms of percentage of negative inventory \( C_1 B \). The values of the constants \( A \) and \( B \) can be found with inventory fluctuation diagram in Figure 1 (Johnson & Montgomery [55])

\[
< \text{Insert Figure 1} >
\]

Both the player have positive and negative inventory during the inventory process. Positive inventory is given by positive area in time \( t_1 \) and \( t_4 \). Negative inventory is given by negative area in time \( t_2 \) and \( t_3 \). The average positive and negative inventory is found as

\[
T_p = t_3 + t_4 = Q/r
\]

\[
l_m = T_p(r - d) - S = Q(1 - u^{-1}) - S, \text{ where } u = r/d
\]

\[
t_1 = \frac{(1-\alpha)l_m}{d}, t_2 = \frac{s}{d}, t_3 = \frac{s}{r-d} \text{ and } t_4 = \frac{l_m}{r-d}
\]

Thus, the positive inventory area is

\[
A = \frac{1}{2}(1 - \alpha)t_1l_m + (\alpha Q)t + \frac{1}{2}t_4l_m = \frac{1}{2d}(1 - \alpha)^2l_m^2 + \frac{\alpha Q^2}{\lambda} + \frac{1}{2(r-d)}l_m^2
\]
The negative inventory area is

\[ B = \frac{1}{2} t_2 S + \frac{1}{2} t_3 S = \frac{S^2}{2d(1 - u^{-1})} \]

Put \( t = \frac{Q}{\lambda}, H_b = Ic_b \) then buyer’s profit becomes

\[ TP_b(p_b, M, Q) = p_b(1 - \alpha)Q + c_s\alpha Q - c_b Q - MQ - A_b - AIC_b - C_1B \]

\[ = p_b(1 - \alpha)Q + c_s\alpha Q - c_b Q - MQ - A_b - \left( \frac{1}{2d} (1 - \alpha)^2 l_m^2 + \frac{\alpha Q^2}{\lambda} + \frac{1}{2(r - d)} l_m^2 \right) Ic_b - C_1 \frac{S^2}{2d(1 - u^{-1})} \]

Let, cycle length, \( T_1 = \frac{(1 - \alpha)Q}{D} \)

Thus, the expected value of buyer’s total profit is given by,

\[ E[TP_b(p_b, M, Q)] = E \left[ \frac{TP_b(p_b, M, Q)}{T_1} \right] \]

Using Renewal theory given by Maddah and Jaber \[28\] to determine expected value of buyer’s total profit per cycle we get,

\[ E[TP^c_b(p_b, M, Q)] = E \left[ \frac{TP_b(p_b, M, Q)}{T_1} \right] \]

\[ E[TP_b(p_b, M, Q)] \]

\[ \frac{D}{Q(1 - E[\alpha])} \left[ p_b(1 - E[\alpha])Q + c_sE[\alpha]Q - c_b Q - MQ - A_b \right. \]

\[ \left. - \left( \frac{l_m^2 E[(1 - \alpha)^2]}{2D} + \frac{E[\alpha]Q^2}{\lambda} + \frac{1}{2(r - D)} l_m^2 \right) Ic_b - C_1 \frac{S^2}{2D(1 - u^{-1})} \right] \]

Demand function is assumed as \( kp_b^{-e} M^\beta \), then
\[
\begin{align*}
\frac{E[TP_b(p_b, M, Q)]}{E[T_1]} &= \frac{kp_b^{-e}M^\beta}{Q(1-E[\alpha])} \left[p_b(1 - E[\alpha])Q + c_s E[\alpha]Q - c_b Q - M Q - A_b - \left(\frac{I_m^2 E[(1-\alpha)^2]}{2D}\right) +
\right] \\
\frac{E[\alpha]Q^2}{\lambda} + \frac{1}{2(r-D)} l_m^2 l_c b - C_1 \frac{s^2}{2D(1-u^{-1})} \right] \\
\end{align*}
\] (1)

To maximize the expected profit, \(E[TP_{b_b}(p_b, M, Q)]\) of the buyer, we have to find optimal values of his decision variables \(p_b, M\) and \(Q\). To determine optimal value of \(p_b\), differentiating equation (1) with respect to \(p_b\) for constant \(M\) and \(Q\) and equating it to zero,

\[
\frac{\partial E[TP_{b_b}(p_b, M, Q)]}{\partial p_b} = 0, \text{ which yields } \]

\[
p_b = \frac{e}{(e-1)(1-E[\alpha])} \left[M + c_b + \frac{A_b}{Q} + \frac{lc_b E[\alpha]Q}{\lambda} - c_s E[\alpha] \right] \] (2)

Refer to Appendix A, where pseudo concavity of expected profit function with respect to \(p_b\) for fixed \(M\) and \(Q\) is proved.

Substituting equation (2) into equation (1), we get

\[
\begin{align*}
E[TP_{b_b}(p_b(M), M, Q(M))] &= \frac{K}{e} \left[\frac{e}{(e-1)(1-E[\alpha])} \left[M + c_b + \frac{A_b}{Q} + \frac{lc_b E[\alpha]Q}{\lambda} - c_s E[\alpha] \right] \right]^{-e+1} M^\beta -
\frac{1}{2Q(1-E[\alpha])} \left[\left(E\left[(1-\alpha)^2\right] + (u-1)^{-1}l_c b (Q(1-u^{-1} - S)^2) - C_1 \frac{s^2}{2(1-u^{-1})(1-E[\alpha])} \right] \right.
\end{align*}
\] (3)

Similarly, to determine the optimal value of \(M\), differentiating equation (1) with respect to \(M\) and equating it to zero we get,

\[
\frac{\partial E[TP_{b_b}(p_b(M), M, Q(M))]}{\partial M} = 0, \text{ gives the value of } M,
\]

\[
M = \frac{\beta}{(e-\beta-1)} \left[c_b + \frac{A_b}{Q} + \frac{lc_b E[\alpha]Q}{\lambda} - c_s E[\alpha] \right] \] (4)

Refer to Appendix B, where concavity of given expected profit function of buyer with respect to \(M\) is proved. Substituting equation (4) into equation (2) which yields
\[ p_b = \frac{e}{(e-\beta-1)(1-E[\alpha])} \left[ c_b + \frac{A_b}{Q} + \frac{Ic_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right] \]  

(5)

By substituting the value of equation (4) and equation (5) in equation (1),

\[ E[T P_c b(Q)] = k \frac{e}{(e-\beta-1)(1-E[\alpha])} \left( c_b + \frac{A_b}{Q} + \frac{Ic_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right)^{-\beta+1} \frac{\beta}{(e-\beta-1)} \left( c_b + \frac{A_b}{Q} + \frac{Ic_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right)^{\beta} \]

\[ + \frac{K(e-\beta-1)(1-E[\alpha])}{(1-E[\alpha])} \left[ c_s E[\alpha] - c_b - M - \frac{A_b}{Q} - E[\alpha]Q \right] - \frac{1}{2Q(1-E[\alpha])} (E[(1-\alpha)^2] + (u-1)^{-1})Ic_b(Q(1-u^{-1}-S)^2 - C_1 \frac{S^2}{2(1-u^{-1})(1-E[\alpha])}) \]

(6)

The first order conditions of equation (6) with respect to \( Q \) are as follows

\[ 2D(A_b - E[\alpha] IC_b \lambda^{-1} Q^2) \]

\[ = ((1-u^{-1})^2Q^2 - S^2) Ic_b(E[(1-\alpha)^2] + (u-1)^{-1}) \]

\[ - \left( \frac{C_1S^2}{(1-u^{-1})(1-E[\alpha])} \right) \]

i.e.

\[ 2ke^{-e} \beta^e \left( c_b + \frac{A_b}{Q} + \frac{Ic_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right)^{\beta-e} (e - \beta - 1)^e (1-E[\alpha])^e (A_b - E[\alpha] Ic_b \lambda^{-1} Q^2) = ((1-u^{-1})^2Q^2 - S^2) \frac{Ic_b(E[(1-\alpha)^2] + (u-1)^{-1})}{(1-E[\alpha])} - \left( \frac{C_1S^2}{(1-u^{-1})(1-E[\alpha])} \right) \]

(7)

It is quite difficult to prove analytically the concavity of equation (6) with respect to \( Q \), thus expected total profit \( E[T P_c^e (p_b(Q), M(Q), Q)] \) defined in equation (6) is concave function with respect to \( Q \) is shown with the help of the graph (Figure 2).
3.2 Seller’s model

To maximize net profit of seller, we have to determine the optimal value of selling price, \( c_b \). The seller’s profit function is expressed as,

\[
TP_s(c_b, Q) = \text{Sales Revenue} - \text{Production cost} - \text{Set up cost} - \text{Holding cost} - \text{Shortages cost}
\]

< Insert Figure 3 >

Figure 3 is a general inventory fluctuation diagram given by Johnson & Montgomery [55]. We used it for the seller to find the positive and negative inventory in the time span of the inventory process.

According to Figure 3, the positive areas on time span \( t_1 \) and \( t_4 \) show the positive inventory. Negative areas \( t_2 \) and \( t_3 \) represent the negative inventory. Positive inventory and negative inventory are calculated below (Zhang et al. [53])

Positive inventory = \( \frac{1}{2} t_1 I_m + \frac{1}{2} t_4 I_m = \frac{(Q(1-u^{-1})-S)^2}{2d(1-u^{-1})} \)

Negative inventory = \( \frac{1}{2} t_2 S + \frac{1}{2} t_3 S = \frac{s^2}{2d(1-u^{-1})} \)

\( T_2 (\text{Cycle length for the seller}) = \frac{Q}{d} \)

The profit of the seller per cycle is,

\[
TP^c_s(c_b, S) = (c_b D - CD - \frac{A_s}{Q} D) - \frac{(Q(1-u^{-1})-S)^2}{2Q(1-u^{-1})}IC - \frac{S^2}{2Q(1-u^{-1})} C_2
\]
\[ TP_c^S(c_b, S) = Kp_b^{-e} M^\beta \left( c_b - C - \frac{A_S}{Q} \right) - \frac{(Q(1-u^{-1}) - S)^2}{2Q(1-u^{-1})} IC - \frac{S^2}{2Q(1-u^{-1})} C_2 \]  

(8)

By differentiating equation (8) with respect to S for fixed \(c_b\) and equating it to zero. By doing so we get the optimal value of S,

\[ S = \frac{ICQ(1-u^{-1})}{C_2 + IC} \]  

(9)

The profit function defined by equation (8) is concave in S, since

\[ \frac{\partial^2}{\partial S^2} TP_c^S(c_b, S) = \frac{-IC + C_2}{(1-u^{-1})Q} < 0 \]  

(10)

Substituting equation (9) into equation (8), we get

\[ TP_c^S(c_b, S) = Kp_b^{-e} M^\beta \left( c_b - C - \frac{A_S}{Q} \right) - \frac{C_2^2 Q(1-u^{-1})}{2(C_2 + IC)^2} IC - \frac{C_2 Q(1-u^{-1})(IC)^2}{2(C_2 + IC)^2} \]  

(11)

Profit function given by equation (11) is linearly increasing with \(c_b\), therefore, to obtain optimal value of \(c_b\), we set it to highest price value obtained through negotiation between seller and buyer. We get,

\[ c_b = Fc_{b0} = F \left( C + \frac{A_S}{Q} + \frac{C_2^2 Q(1-u^{-1})}{2d(C_2 + IC)^2} IC + \frac{C_2 Q(1-u^{-1})(IC)^2}{2d(C_2 + IC)^2} \right) \text{ for some } F > 1 \]  

(12)

4. The non-cooperative Stackelberg games

The Stackelberg game is a strategic game in which a leader moves first and then another player follows by making his best response. Two player’s, i.e. seller and buyer interact with one another. The objective of dominant player is to maximize his gain on the basis of response by follower.

4.1 The Seller-Stackelberg model

In Seller-Stackelberg model, the seller is the leader and buyer acts as a follower. The seller makes his first move by offering values of his decision variables i.e. selling price, \(c_b\), and shortages, \(S\) to the buyer. Based on these values offered by the seller, buyer, as a follower, determines optimal value of his decision variables, i.e. selling price, \(p_b\), marketing expenditure, \(M\) and order quantity,
Q, defined by equation (4), equation (5) and constraint (7) respectively. The seller’s objective is to maximize his profit. Now, the problem can be described mathematically as,

$$\text{Max } \left[ TP^c_s(c_b, S) \right] = (c_b D - CD - \frac{A_s}{Q} D) - ICA_1 - C_2B_1$$

$$= Kp_b \epsilon M^\beta \left( c_b - C - \frac{A_b}{Q} \right) - \frac{(Q(1-u^{-1}) - S)^2}{2Q(1-u^{-1})} IC - \frac{s^2}{2Q(1-u^{-1})} C_2 \tag{13}$$

Subject to

$$p_b = \frac{e}{(e-\beta-1)(1-E[\alpha])} \left[ c_b + \frac{A_b}{Q} + \frac{IC_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right], \beta + 1 < e \tag{14}$$

$$M = \frac{\beta}{(e-\beta-1)} \left[ c_b + \frac{A_b}{Q} + \frac{IC_bE[\alpha]Q}{\lambda} - c_s E[\alpha] \right], \beta + 1 < e \tag{15}$$

and constraint

$$2D(A_b - E[\alpha] IC_b \lambda^{-1} Q^2) = ((1 - u^{-1})^2 Q^2 - S^2) \frac{IC_b(1-\alpha)^2 + (u-1)^{-1}}{(1-E[\alpha])} - \left( \frac{C_1S^2}{(1-u^{-1})(1-E[\alpha])} \right) \tag{16}$$

Cycle length, \(T = \max(T_1, T_2)\)

Substituting the equation (14) and equation (15) into equation (13). The resulting equation with a nonlinear constraint defined by equation (16) gives the optimum solution by any nonlinear programming search tool.

### 4.2 The Buyer-Stackelberg model

In Buyer-Stackelberg model, the initiative of giving the first offer to the Buyer, i.e. he is the leader. Buyer makes his first move by offering selling price, \(p_b\), marketing expenditure, \(M\), and order quantity, \(Q\), to the seller. Based on this offer, seller determines optimal value of his decision variables, i.e. shortages, \(S\), and selling price, \(c_b\), given by equation (9) and equation (12). The buyer’s objective is to optimize his profit. Now, problem is described mathematically as,

$$\text{Max } E[TP^c_b(p_B, M, Q)]$$
\[
E[T P^c_b(p_b, M, Q)] = kp_b^{-e+1}M^\beta + \frac{Kp_b^{-\beta}M^\beta}{(1-E[\alpha])}[c_b E[\alpha] - c_b - M - \frac{A_b}{Q} - \frac{E[\alpha]Q}{\lambda}Ic_b] - \\
\frac{1}{2Q(1-E[\alpha])}((E[(1 - \alpha)^2] + (u - 1)^{-1})Ic_b(Q(1-u^{-1} - S)^2) - C_1 \frac{s^2}{2(1-u^{-1})(1-E[\alpha])})
\]  

Subject to
\[
S = \frac{lCQ(1-u^{-1})}{C_2+Cic}
\]  

\[
c_b = Fc_{b0} = F\left((C + \frac{As}{Q}) + \frac{C_2^2Q(1-u^{-1})}{2d(C_2+Cic)^2}IC + \frac{C_2^2Q(1-u^{-1})(lc)^2}{2d(C_2+Cic)^2}\right)\text{ for some } F > 1
\]  

Using equation (18) and equation (19) on equation (17), we can convert this problem into a non-constrained non-linear function of three variables \((p_b, M, Q)\) which can be solved using any non-linear programming search tool.

5. Numerical illustrations

This section discusses numerical examples for the illustration of some specific features of the given model mentioned above. The first case of numerical examples based on the Seller-Stackelberg and Buyer-Stackelberg models. The second case shows the comparison between the results obtained from models with and without shortage \((S = 0)\). It is explained to reveal the effect of profits of buyer and seller.

5.1 Numerical examples for non-cooperative game with shortages

Example 1

This example shows the effect of items of defective quality in the Seller-Stackelberg game model. Input parameters used are taken from Zhang et al. [53] and Jaggi et al. [38], these are given below,

\[
C = $1.5, C_1 = $1, C_2 = $1, A_b = $40, A_s = $140, C_s = 3.5
\]

\[
l = 0.1, u = 1.1, k = 3500, F = 1.25, e = 1.7, \beta = 0.15, \lambda = 175200 \text{ units/year}
\]

The percentage of items of imperfect quality rate, \(\alpha\), is taken as uniformly distributed on\((a, b)\), \(0 < a < b < 1, i.e., \alpha U(a, b)\). Taking \(a = 0\) and \(b = 0.04\) we get,

\[
E[\alpha] = \frac{a + b}{2} = 0.02
\]
\[ E[(1 - \alpha)^2] = \int_a^b (1 - \alpha)^2 f(\alpha) d\alpha = \frac{a^2 + ab + b^2}{3} + 1 - a - b = 0.960 \]

Equation (13) with equations (14), (15) and constraints defined by (16) gives,

\[ c_b = $4.472, S = 29, Q = 639 \text{ units}, p_b = $14.081, M = $1.218 \]

Using equation (12) to obtain seller’s profit we get, \( [TP^c_s] = $102.32 \) and equation (1) to obtain buyer’s expected profit we get, \( E[TP^c_b] = $322.256 \).

**Example 2**

This example is to show the effect of defective items in Buyer-Stackelberg model. Same input parameters are taken as considered in example 1, with the exception, \( c_s = 1.5 \).

Equation (17) with equations (14) and (15) gives,

\[ p_b = $6.158, M = $0.532, Q = 1436 \text{ units}, c_b = $2.056, S = 17 \]

Using equation (17) to obtain buyer’s expected profit we get, \( E[TP^c_b] = $502.033 \) and equation (18) to obtain seller’s profit, we get, \( [TP^c_s] = $57.864 \).

From the examples, we can infer that when a buyer is the leader as in Buyer-Stackelberg, he is better off as he gains more profit and the selling price which is charged by the buyer to the customer are also less. Shortages are also less in the Buyer-Stackelberg model. Since, production capacity is linearly related to the demand and buyer is better informed about the end demand, the order quantity will be adjusted accordingly. In second example higher order quantity leads to better profit for the buyer. The first example shows that seller’s selling price is high due to which the profit for the seller is also high. We can conclude that both players are better off when they are leader.

**5.2 Numerical example for the models without shortage**

All parameters are considered as in the previous example. In this case, the results obtained in two different models are compared: models without shortage, representing WS. Numerical results are analyzed to know the effect of decision variables and profit functions for both the players, buyer and seller due to the shortages.

**Example 3**
The optimal values obtained in Seller-Stackelberg game model in the WS case are $Q^{WS} = 281$ units, $c_b^{WS} = $4.402, $p_b^{WS} = $14.11 and $M^{WS} = $1.22. With these results, the seller’s expected profit, $E[TP^c_s] = $94.354 and the buyer’s expected profit, $E[TP^c_b] = $326.838.

Comparing the results of two Seller–Stackelberg models of without and with shortages. The marketing expenditure and buyer’s selling price are larger in without shortage case than the shortage case, whereas seller’s price and order quantity are smaller in WS case than with shortage case. Seller’s profit is larger in shortage case than to with shortage case, while buyer’s profit is more in WS case. It is apparent in Seller-Stackelberg game where the seller is advantageous of being a leader and manages the shortages very well, whereas shortages gets a negative impact on buyer’s profit when buyer being follower.

**Example 4**

The optimal values obtained in Buyer-Stackelberg game model in the WS case are, $p_b^{WS} = $6.179 and $M^{WS} = $0.534, $Q^{WS} = 1368$ units and $c_b^{WS} = $2.068. With these results, the seller’s expected profit, $E[TP^c_s] = $57.710 and the buyer’s expected profit, $E[TP^c_b] = $499.867.

Contemplating the outcome with the shortage case, the buyer has larger order quantity and lower selling price in shortage case than to WS case, whereas seller’s price and marketing expenditure are less in shortage case than to WS case. Both the player, seller and buyer have more profit in shortage case than without shortage case. The presence of shortage induces the buyer to order more quantity to fulfill the demand avoiding loss of profit. Thus, it leads to a win-win situation for both the players.

6. Sensitivity analysis

To further analyze the impact of following parameters $a, \beta, e, C_1,$ and $C_2$ on $c_b, p_b, M, S, Q, D, E[TP^c_s],$ and $E[TP^c_b]$ in both Seller-Stackelberg and Buyer-Stackelberg models. We perform sensitivity analysis on the effective parameters. Results are shown through graphs (Figure 4-8).

< Insert Figure 4 >

< Insert Figure 5 >
Further, sensitivity analysis is presented to get the impact of shortages, S, on $c_b$, $p_b$, $M$, $Q$, $E[TP^e_s]$ and $E[TP^e_b]$ in the Seller-Stackelberg and Buyer-Stackelberg game. Results are shown through the bar diagrams (Figure 9-10) and representing the comparison of the results obtained in without and with shortage case in both the non-cooperative game.

**Observations**

1. It is clearly evident from the figure 4, that whenever the fraction of imperfect quantity items increases, the variables $M$, $p_b$ and $Q$ which depend on buyer’s decision are independent of buyer’s leadership position. However, the seller’s decision variables $S$ and $c_b$ are also absolute to seller’s position as a follower or a leader. Both the players, seller and buyer get benefited more when they are leader as compared to when they are follower.

2. From the figure 5, it can be well understood that $S$ and $c_b$, which are the seller’s decision variables, depend on whether the seller is a leader or a follower. For instance, $c_b$ decreases and $S$ increase in the Seller-Stackelberg model, when the value of price elasticity, $e$, is increased. Also, $c_b$ increases and $S$ decrease in the Buyer-Stackelberg model. However, Buyer’s leadership position affects $p_b$ and $Q$, whereas it has no effect on $M$. For example, increase in $e$ causes $M$ to decrease, irrespective of the buyer’s position and $p_b$ and $Q$ increases in the Seller-Stackeberg game and decreases in Buyer-Stackelberg game. Being a leader is more beneficial for both the players in comparison to be a follower.

3. It can be easily seen from the figure 6 that in both the models, as parameter $\beta$ increases, decision variables, $c_b$, $p_b$ and $M$ increases, whereas $Q$ and $S$ decreases, i.e. leadership situation does not influence decision variables of the seller and buyer.
4. Results from figure 7 and 8 indicate that by varying $C_1$ and $C_2$, the profit of seller and buyer is heterogeneously affected in numerous ways for various models. For examples, by increasing $C_1$ in the non-cooperative game results in the buyer’s profit decrease whereas seller’s profit will increase. Both the decision variables of the seller increases before $C_1$ reaches $C_2$ and decreases when it exceeds $C_2$. The decision variables of the buyer $p_b$ and $M$ decreases and $Q$ increases as buyer’s shortage cost increase. In the Buyer-Stackelberg game, as $C_1$ increases, the shortages size, $S$ and order quantity, $Q$ decrease whereas, $c_b$, $p_b$ and $M$ increase.

7. Conclusions

The present study has been conducted on imperfect quality items under supply chain model where shortages are permitted. The shortage is referred as seller’s decision variable which is related to seller-buyer supply chain model wherein the effect of a shortage under symmetric information pattern on supply chain participant’s performance and decision was obtained. It was presumed that the demand is a function of the buyer’s selling price and marketing expenditure. The interaction amidst members were investigated in the non-cooperative situation by applying Stackelberg equilibrium, advantages and disadvantages were also discussed along in the Stackelberg games. It was clear from the study that the both the players, seller and buyer always gained more profit when they are leader. The effect put by shortage on member’s profit was reviewed with numerous examples. The graphs depicted the results of sensitivity analysis.

Result’s indicated that the costs, i.e. $C_1$(buyer’s shortages cost) and $C_2$(seller’s shortages cost) pertaining to the shortage are affecting seller and buyer profit in many ways for both the models. For examples, by increasing $C_1$ in the non-cooperative game decreases buyer’s profit and increases seller’s profit. Both the decision variables of the seller increase before if $C_1$ is ahead $C_2$, seller decision variables increases, whereas it decreases when $C_1$ exceeds $C_2$. Further, we can extend the model by incorporating asymmetric information structure.

References


Table 1. Contribution of several authors in the associated field

Figure 1. Inventory fluctuation diagram with imperfect quality items

Figure 2. Buyer’s expected total profit with respect to $Q$

Figure 3. Inventory fluctuation diagram

Figure 4. The effect of $a$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^b_c]$

Figure 5. The effect of $e$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^b_c]$

Figure 6. The effect of $\beta$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^b_c]$

Figure 7. The effect of $C_1$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^b_c]$

Figure 8. The effect of $C_2$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^b_c]$

Figure 9. The seller-Stackelberg game: without shortages and with shortages

Figure 10. The Buyer-Stackelberg game: without shortages and with shortages
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Figure 6. The effect of $\beta$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^c_b]$. 

- **Seller profit**
  - SP1
  - SP2

- **Marketing expenditure**
  - 0/12 0/13 0/14 0/15 0/16 0/17 0/18

- **Buyer profit**
  - BP1
  - BP2

- **Buyer's shortages cost**
  - $p_b$
  - $Q_b$

- **Marketing expenditure**
  - 0/12 0/13 0/14 0/15 0/16 0/17 0/18

- **Buyer's shortages cost**
  - $M$
  - $C_b$

- **Buyer's shortages cost**
  - $Q$
  - $C_b$

- **Buyer's shortages cost**
  - $C_b$
Figure 7. The effect of $C_1$ parameter on $p_b, M, Q, C_b, S, D, E[TP^c_s], and E[TP^c_b]$
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Appendix A

We have to show that expected profit function $E[T P^c_b(p_b, M, Q)]$ is strictly pseudo concave with respect to $p_b$ for fixed $M$ and $Q$ (refer to Yadav et al., 2018)

For this we will show that for $p_b \neq p_b'$,

$$E[T P^c_b(p_{b1}, M, Q)] \leq E[T P^c_b(p_{b2}, M, Q)] \Rightarrow \nabla(E[T P^c_b(p_{b1}, M, Q)])(p_{b2} - p_{b1}) > 0$$

Expected profit of the buyer from equation (1),

$$E[T P^c_b(p_b, M, Q)] = p_bD + \frac{1}{1 - E[\alpha]} \left[ c_b E[\alpha]D - c_bD - MD - \frac{A_bD}{Q} \left( E[\alpha]Q \right) \right] + \frac{1}{2Q} \left( (E[(1 - \alpha)^2] + (u - 1)^{-1})c_b(Q(1 - u^{-1} - S^2)) - C_1 \frac{S^2}{2(1 - u^{-1})} \right)$$

Suppose the inequality,

$$E[T P^c_b(p_{b1}, M, Q)] \leq E[T P^c_b(p_{b2}, M, Q)]$$

which is equivalent to,

$$D_2 \left[ \frac{1}{1 - E[\alpha]} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} \right) I_c b - c \frac{E[\alpha]}{\lambda} \right] - p_{b2} \leq D_1 \left[ \frac{1}{1 - E[\alpha]} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} I_c b - c \frac{E[\alpha]}{\lambda} \right) - p_{b1} \right]$$

(A.1)

where, $D_i = D(p_{b1}, M) = kp_{b1}^{-e} M^\beta, \ i = 1,2$

To show strictly pseudo concavity of expected profit function $E[T P^c_b(p_b, M, Q)]$ with respect to $p_b$ for fixed $M$ and $Q$, it will be proved that inequality (A.1)

$$\Rightarrow \nabla(E[T P^c_b(p_{b1}, M, Q)])(p_{b2} - p_{b1}) > 0$$

(A.2)

Here,
\[ \nabla (E[T P^c_b(p, M)]) = D_1 (1 - e) p_b + \frac{D_1 e}{(1 - E[\alpha])} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} Ic_b - c_s E[\alpha] \right) \]  \quad (A.3)

Putting the value of equation (A.3) in equation (A.2), we will have

\[ \left( D_1 (e - 1) p_b - \frac{D_1 e}{(1 - E[\alpha])} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} Ic_b - c_s E[\alpha] \right) \right) (p_{b2} - p_{b1}) < 0 \]

To show strictly pseudo concavity of expected profit function \( E[T P^c_b(p, M)] \) with respect to \( p \) for fixed \( M \) and \( Q \), it is sufficient to show that equation (A.1) implies

\[ \left( D_1 (e - 1) p_b - \frac{D_1 e}{(1 - E[\alpha])} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} Ic_b - c_s E[\alpha] \right) \right) (p_{b2} - p_{b1}) < 0 \]  \quad (A.4)

Let, \( P = \frac{1}{(1 - E[\alpha])} \left( c_b + M + \frac{A_b}{Q} + \frac{E[\alpha]Q}{\lambda} Ic_b - c_s E[\alpha] \right) \)

Then equation (A.1) can be rewritten as \( D_2 (P - p_{b2}) \leq D_1 (P - p_{b1}) \)

and equation (A.4) can be rewritten as:

\[ D_1 ((e - 1) p_b - eP)(p_{b2} - p_{b1}) < 0 \]

i.e. \( p_{b2} ((e - 1) p_b - eP) < p_{b1} ((e - 1) p_b - eP) \)  \quad (A.5)

Suppose there are two distinct points \( p_{b2} \) and \( p_{b1} \).

Then either (A) \( p_{b2} > p_{b1} \) or (B) \( p_{b2} < p_{b1} \)  \quad Sadigh et al. [56]

**Case (a):** Let \( p_{b2} > p_{b1} \) that follows \( D_1 > D_2 \). Therefore, to prove equation (A.5), it suffices to show that

\[ \frac{D_2}{D_1} (P - p_{b2}) \leq (P - p_{b1}) \]  \quad (A.6)

Since, \( p_{b2} > p_{b1} \), then we have \( (P - p_{b2}) < (P - p_{b1}) \).

Three possible cases can be considered for the equation (A.6),

(i) \( P - p_{b1} > 0 \) and \( P - p_{b2} > 0 \)

(ii) \( P - p_{b1} > 0 \) and \( P - p_{b2} < 0 \)

(iii) \( P - p_{b1} < 0 \) and \( P - p_{b2} < 0 \)

**Case (i):**

\( P - p_{b1} > 0 \) and \( P - p_{b2} > 0 \)

\[ \Rightarrow P - p_{b1} > P - p_{b2} = \Rightarrow p_{b2} > p_{b1} \] and \( \frac{D_2}{D_1} < 1 = \Rightarrow \) verified equation (A.6).

Here, \( P - p_{b1} > 0 = \Rightarrow P > p_{b1} \)
Since, $0 < (e - 1) < e \Rightarrow (e - 1)p_{b1} < ep_{b1} < eP \Rightarrow (e - 1)p_{b1} - eP < 0$

Hence, equation (A.5 holds.

If $P - p_{b1} < P - p_{b2} \Rightarrow p_{b2} < p_{b1}$, which contradicts the equation (A.6).

**Case (ii):**

$P - p_{b1} > 0$ and $P - p_{b2} < 0$

$\Rightarrow p_{b2} > P$ and $p_{b1} < P \Rightarrow p_{b2} > p_{b1}$

In this case, equation (A.6) holds since, $-p_{b1} > 0$.

Finally, in case (iii):

$P - p_{b1} < 0$ and $P - p_{b2} < 0 \Rightarrow D_2(P - p_{b2}) \leq D_1(P - p_{b1})$

$\Rightarrow p_{b2}^{-e}(P - p_{b2}) \leq p_{b1}^{-e}(P - p_{b1})$

$\Rightarrow \left(\frac{p_{b1}}{p_{b2}}\right)^{e}(P - p_{b2}) \leq (P - p_{b1})$

$\Rightarrow p_{b1}^{e}(P - p_{b2}) \leq p_{b2}^{e}(P - p_{b1})$

$\Rightarrow P(p_{b2}^{e} - p_{b1}^{e}) \geq p_{b1}p_{b2}(p_{b2}^{e-1} - p_{b1}^{e-1})$

$\Rightarrow P \geq \frac{p_{b1}p_{b2}(p_{b2}^{e-1} - p_{b1}^{e-1})}{(p_{b2}^{e} - p_{b1}^{e})}$ \hspace{1cm} (A.7)

Now, the equation (A.7) shows that for a minimum value of

$$P = \frac{p_{b1}p_{b2}(p_{b2}^{e-1} - p_{b1}^{e-1})}{(p_{b2}^{e} - p_{b1}^{e})},$$

Equation (A.6) holds. Now for this

$$(e - 1)p_{b1} - eP = (e - 1)p_{b1} - e\frac{p_{b1}p_{b2}(p_{b2}^{e-1} - p_{b1}^{e-1})}{(p_{b2}^{e} - p_{b1}^{e})} < 0 \Rightarrow \frac{(e - 1)}{e} \frac{p_{b2}(p_{b2}^{e-1} - p_{b1}^{e-1})}{(p_{b2}^{e} - p_{b1}^{e})} < 0$$
\[\frac{(e-1)}{e} < \left( \frac{1 - \left( \frac{p_{b1}}{p_{b2}} \right)^e}{1 - \left( \frac{p_{b1}}{p_{b2}} \right)^1} \right) \Rightarrow \left( \frac{p_{b1}}{p_{b2}} \right)^{e-1} \left[ e + (1 - e) \left( \frac{p_{b1}}{p_{b2}} \right) \right] < 1 \quad \text{(A.8)}\]

Therefore, for each \( e > 1 \) and \( \left( \frac{p_{b1}}{p_{b2}} \right) < 1 \), above equation (A.8) holds. Hence the proof is completed for this case.

**Case (b):** let \( p_{b2} < p_{b1} \) that follows \( D_2 > D_1 \) and it is sufficient to show that

\[\frac{D_2}{D_1} (P - p_{b2}) \leq (P - p_{b1}) \quad \text{(A.9)}\]

In the similar manner as mentioned in CASE (A), three cases can be considered according to (A.9):

(i) \( P - p_{b1} > 0 \) and \( P - p_{b2} > 0 \)

(ii) \( P - p_{b1} > 0 \) and \( P - p_{b2} < 0 \)

(iii) \( P - p_{b1} < 0 \) and \( P - p_{b2} < 0 \)

**Case (i):**

\( P - p_{b1} > 0 \) and \( P - p_{b2} > 0 \) this implies \( P - p_{b1} > P - p_{b2} \Rightarrow p_{b1} < p_{b2} \), contradict with equation (A.29)

**Case (ii):**

\( P - p_{b1} > 0 \) and \( -p_{b2} < 0 \Rightarrow p_{b2} > P \) and \( p_{b1} < P \Rightarrow p_{b1} < p_{b2} \), contradict with equation (A.29).

**Finally, in case (iii):**

\( P - p_{b1} < 0 \) and \( P - p_{b2} < 0 \Rightarrow D_2(P - p_{b2}) \leq D_1(P - p_{b1}) \)

\[\Rightarrow p_{b2}^{-e}(P - p_{b2}) \leq p_{b1}^{-e}(P - p_{b1})\]

\[\Rightarrow \left( \frac{p_{b1}}{p_{b2}} \right)^e (p_{b2} - P) \geq (p_{b1} - P)\]

\[\Rightarrow p_{b1}^e (p_{b2} - P) \geq p_{b2}^e (p_{b1} - P)\]

\[\Rightarrow P (p_{b1}^e - p_{b2}^e) \leq p_{b1} p_{b2} (p_{b1}^{e-1} - p_{b2}^{e-1})\]
=> \[ P \leq \frac{p_{b1} p_{b2} (p_{b1}^{e-1} - p_{b2}^{e-1})}{(p_{b1}^e - p_{b2}^e)} \]  

(A.10)

Similar to the equation (A.7), the equation (A.10) can be proved for the maximum value of \( P = \frac{p_{b1} p_{b2} (p_{b1}^{e-1} - p_{b2}^{e-1})}{(p_{b1}^e - p_{b2}^e)} \). For each \( e > 1 \) and \( \left( \frac{p_{b1}}{p_{b2}} \right) > 1 \). Hence, the proof is completed for \( p_{b2} < p_{b1} \).

Hence, equation (A.5) holds in all possible cases hence the expected profit of the buyer is strictly pseudo concave with respect to \( p_b \) for fixed \( M \) and \( Q \).

Appendix B

\[
E[T P_c^b(p_b(M), M, Q)]
= \frac{K}{e} [p_b]^{-e+1} M^\beta - \frac{1}{2Q} \left( (E[(1 - \alpha)^2] + (u - 1)^{-1}) l c_b (Q(1 - u^{-1} - S)^2) \right)
- C_1 \frac{S^2}{2(1 - u^{-1})}
\]

By equation (2), we have,

\[
p_b = \frac{e}{(e - 1)(1 - E[\alpha])} \left[ M + c_b + \frac{A_p}{Q} + \frac{l c_b E[\alpha] Q}{\lambda} - c_2 E[\alpha] \right]
\]

\[
\frac{\partial p_b}{\partial M} = \frac{e}{(e - 1)((1 - E[\alpha])}
\]

\[
\frac{\partial E[T P_c^b(p_b(M), M, Q)]}{\partial M} = D \left[ - \frac{1}{(1 - E[\alpha])} + \frac{p_b \beta}{eM} \right]
\]
\[
\frac{\partial^2 E[TP^c_b(p_b(M), M, Q)]}{\partial M^2} = D \left[ -\frac{p_b(M)\beta}{eM^2} + \frac{\beta}{M(e - 1)(1 - E[\alpha])} \right] + D \left[ -\frac{1}{(1 - E[\alpha])} + \frac{p_b(M)\beta}{eM} \right] \left[ \frac{\beta}{M} - \frac{e}{p_b(M)} \right]
\]

By equations (4) and (5), we have

\[p_b = \frac{eM}{\beta(1 - E[\alpha])}\] (B.2)

putting the value of equation (B.2) in equation (B.1), we have

\[
\frac{\partial^2 E[TP^c_b(p_b(M), M, Q)]}{\partial M^2} = \frac{D(\beta + 1 - e)}{M(1 - E[\alpha])(e - 1)} < 0, \text{ by assumption } \beta + 1 < e, e > 1
\]

This shows the concavity of the expected profit with respect to \(M\) for fixed \(Q\).

**Biographies**

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and Springer. He is a series editor of Inventory Optimization, Springer Singapore Pvt. Ltd. He has been awarded Best Faculty Award by the Amity School of Engineering and Technology, New Delhi for the year 2016–2017. He guided 4 Ph.D. scholars, and 5 students working with him in the area of Inventory Control and Management. He also served as Dean of Students Activities at Amity School of Engineering and Technology, Delhi, for nine years, and worked as Head, Department of Mathematics in the same institute for one year. He is a member of the editorial boards of Revista Investigacion Operacional, Journal of Control and Systems Engineering, and Journal of Advances in Management Sciences and Information Systems. He actively participated as a core member of organizing committees in the International conferences in India and outside India.