Cattaneo-Christov heat and mass flux models on time-dependent swirling flow through oscillatory rotating disk

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Abstract

This analysis emphasis on the time invariant impressions of Cattaneo-Christov heat and mass flux theories are implemented to overcome the initial instant disturbances throughout whole medium. The motion of three-dimensional, incompressible, magnetized viscous fluid flow induced by the oscillatory disk. Porous media is used to saturate the rotating disk. Similarity transformations are accomplished to normalize the flow problem. Successive over Relaxation (SOR) technique is implemented to discuss the new findings of normalized non-linear resulting system. It is perceived that increase in porosity parameter results in decrease of oscillatory velocity profiles. The characterization of porous media is useful in geothermal and petroleum reservoirs. Time varying oscillatory curves for concentration and temperature decay for varying concentration and thermal relaxation times parameters, respectively. Moreover, an interesting nature of phase-log shift is also observed in temperature and concentration profiles. Three-dimensional flow features are also labeled for velocity, temperature and concentration fields.
Keywords: Unsteady flow; Porous medium; Magnetohydrodynamics; Cattaneo-Christov theory; Numerical solution

1. Introduction

The combined heat-mass transportation emerges frequently in widespread manufacturing and industrial wise applications like nuclear processes, nuclear reactor, cooling processes, hot wiring, distillation columns, power generation, marine engineering, solar technology, transpiration processes, packed bed processes, production of glass-fiber, furnace design and energy production. Heat conduction Fourier’s law is the most famous model in field of continuum mechanics. Differential equation of parabolic type for temperature is obtained in Fourier’s law and has major short comings of having instant initial disturbances throughout whole medium. Such unrealistic feature termed as paradox of heat conduction. To overcome these various techniques have been implemented over time through which not all have been utilized successfully [1]. Maxwell-Cattaneo law [2] is one of those techniques in which time partial derivative is added in constitutive equation by Cattaneo. The law overcome the shortcoming of Fourier’s law by acquiring damped hyperbolic equation and describe relationship among temperature flourished and heat flux. Khan et al. [3] communicated the series solutions by implementing HAM of the problem of three-dimensional Burgers fluid flow through stretching surface with characterization of Cattaneo-Christov model of heat and mass flux theories. Liu et al. [4] obtained numerical solutions to feature the heat conduction influence by characterizing Cattaneo-Christov theory using space fractional derivative. Upadhay et al. [5] computed numerical solutions of dusty Eyring-Powell nanofluid flow over stretching sheet subject to Cattaneo-Christov model. The implementation of HAM is utilized to study the salient features of Cattaneo-Christov heat-mass flux theories on squeezing fluid flow by Farooq et al. [6]. Shehzad

Rotating flows have fascinating phenomenon because of the richness involved in scientific and engineering applications. Understanding of such phenomenon providing modeling capabilities to design turbomachines and various other products for power plants, turbo machinery, oil industry and medical engineering such as jet engines, vacuum cleaners pumps and swirl flow through diffuser. Especially fluid flow subjected to the oscillatory motion of rotating surfaces has innumerable practical applications worldwide. The analysis of such type incorporates the thermal stability analysis of chemical reactors, thermal radiators, condensers, separation systems, centrifuges, radiators, centrifugal pumps, heat exchangers, jet engines and designing process of the thermal subsystems as implication in boilers. Chawla et al. [11] examined incompressible flow of viscous fluid subjected to porous rotating disk. Mahmood et al. [12] obtained numerical solutions for unsteady viscous fluid flow through rotating disc, lubricated by thin coating of power-law fluid. Yin et al. [13] executed analytical technique to discuss heat transfer attributes of nanofluid flow across rotating disk. Mustafa [14] accomplished numerical technique to analyze the distinctions of nanofluid due to rotating disk. Turkyilmazoglu [15] investigated heat transfer phenomenon of viscous fluid flow near vertically moving and rotating disk. Lok et al. [16] established numerical solutions of stagnation-point flow through permeable rotating disk of

The topic of porous media with applications in industry and engineering is an active area of research. The pioneer work in fluid transport area and heat transport through porous media started from the last century. The applications of such fascinating area found in thermal-insulation engineering, chemical waste spreading, geothermal reservoirs, petroleum reservoirs, grain storage, catalytic reactors, coal combustors and microsphere insulation of packed cryogenic [19-25]. On basis of such specific applications of porous media, the flow may be external or internal through porous medium. In this regard most of the research work has done by implementing Darcy law [26]. The model based on the creeping flow assumption subject to infinitely enlarged uniform medium such that average volume velocity and pressure gradient keeps direct relationship [27]. Khan et al. [28] solved the problem of second grade fluid flow through oscillatory moving surface submerged into porous medium. Ali et al. [29] performed simulations using HAM to discuss the flow characteristics of couple stress fluid via oscillatory sheet subjected to porous medium. Ali et al. [30] discussed the flow problem of viscoelastic fluid through oscillatory moving sheet via porous medium. Hasnain and Abbas [31] examined flow of two immiscible liquids through inclined annulus of concentric cylinders in porous medium. Sheikholeslami and Shehzad [32] performed numerical simulation using CVFEM to analyze heat transfer features of nanoparticles in enclosure merged in porous medium. Sheikholeslami et al. [33] discussed the nanofluid flow behavior through oscillatory vertical plate, which takes place in porous medium.
The main goal of our study is to study the numerical findings of incompressible, three-dimensional MHD flow of viscous fluid. The motion of fluid is generated due to oscillatory rotating disk saturated by the porous space. One of premier imperfections of Fourier’s theory is that temperature field is parabolic in nature due to which early disturbance is detected all over the medium. Such behavior contradicts with causality principle. To rectify such unrealistic attribute, Christov [2] include time derivative term in constitutive expression between temperature and heat flux which extends the heat equation into damped hyperbolic one and solves the major short comings in Fourier’s model. Therefore the utilization of Cattaneo-Christov heat and mass flux theories is made for the present flow problem. The magnetized flow problems of viscous material through rotation of oscillatory disk under double diffusive Cattaneo-Christov theory still not studied in the literature. The present work is the extension of [34] with addition Cattaneo-Christov theories of double diffusion. Mathematical model is developed in section 2. Section 3 mentioned the numerical solutions. Discussions of results are elaborated in section 4. Conclusions are discussed in section 5.

2. Mathematical modeling

A three-dimensional, unsteady, incompressible flow of hydromagnetic viscous fluid is assessed. Constant rotation and periodic oscillation of disk generates fluid motion and is equipped in porous space. Angular speed of disk is $\gamma$ and amplitude of oscillation is denoted by $\varepsilon$. The disk is located in $z=0$ plane. Most appropriate choice here is to choose system of cylindrical coordinates such that the components of velocity $(u,v,w)$ are considered along $(r,\theta,z)$ direction. Flow is assumed to be axi-symmetric therefore derivatives are neglected along tangential component. Electric field is ignored because of low Reynolds number. Uniform potency
magnetic field is employed in the axial direction. Heat and mass transfer features are inspected by utilizing modified form of Fick and Fourier’s laws namely Cattaneo-Christov mass and heat flux theories.

Temperature and concentration at wall is denoted by $T_w$ and $C_w$ whereas ambient temperature and concentration is determined by $T_\infty$ and $C_\infty$. Fig. 1 explains the geometrical configuration.

Considering the above assumptions, the mathematical model is [2, 35, 36,37]:

\[
\frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{1}
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{u}{r} \right) + \frac{\partial^2 u}{\partial z^2} \right) - \frac{\sigma_e \beta_0^2 u}{\rho} - \frac{\mu \phi_i}{\rho k_i} u, \tag{2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{u v}{r} + \frac{w}{r} \frac{\partial v}{\partial z} = \nu \left( \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \right) - \frac{\sigma_e \beta_0^2 v}{\rho} - \frac{\mu \phi_i}{\rho k_i} v, \tag{3}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right) - \frac{\mu \phi_i}{\rho k_i} w, \tag{4}
\]

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) -
\left. \left( \frac{\partial^2 T}{\partial t^2} + u \frac{\partial u}{\partial r} \frac{\partial T}{\partial r} + w \frac{\partial w}{\partial z} \frac{\partial T}{\partial z} + u \frac{\partial u}{\partial r} \frac{\partial T}{\partial r} + w \frac{\partial w}{\partial z} \frac{\partial T}{\partial z} + 2uw \frac{\partial^2 T}{\partial r \partial z} + u^2 \frac{\partial^2 T}{\partial r^2} \right) \right|_{t=0} \tag{5}
\]

\[
\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial r} + w \frac{\partial C}{\partial z} = D_B \left( \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} \right) -
\left. \left( \frac{\partial^2 C}{\partial t^2} + u \frac{\partial u}{\partial r} \frac{\partial C}{\partial r} + w \frac{\partial w}{\partial z} \frac{\partial C}{\partial z} + u \frac{\partial u}{\partial r} \frac{\partial C}{\partial r} + w \frac{\partial w}{\partial z} \frac{\partial C}{\partial z} + 2uw \frac{\partial^2 C}{\partial r \partial z} + u^2 \frac{\partial^2 C}{\partial r^2} \right) \right|_{t=0} \tag{6}
\]
The associated boundary conditions for \((t > 0)\) are [37]:

\[
\begin{align*}
  \{ u(r,z;t) &= ar(1 + \varepsilon \sin \omega t), v(r,z;t) = \gamma r, w(r,z;t) = 0, \\
  T(r,z;t) &= T_w, C(r,z;t) = C_w \quad \text{at} \quad z = 0, \\
  u(r,z;t) \rightarrow 0, v(r,z;t) \rightarrow 0, w(r,z;t) \rightarrow 0, \\
  T(r,z;t) \rightarrow T_w, C(r,z;t) \rightarrow C_w, \quad \text{when} \quad z \rightarrow \infty,
\end{align*}
\]

(7)

and initial conditions [37]:

\[
u = 0, v = 0, w = 0 \quad \text{at} \quad t = 0 \quad \text{for all} \quad r \quad \text{and} \quad z.
\]

(8)

Here \(\nu, \rho \) and \(\mu\) represent the kinematic viscosity, density and dynamic viscosity, \(\beta_0\) the magnetic flux density, \(\sigma\) the electrical conductivity, \(\phi\) and \(k\) the porosity and permeability of porous medium, \(\alpha = \frac{k}{\rho c_p}\) co-efficient of heat diffusion, \(D_b\) Brownian diffusion co-efficient, \(\tau_0\) heat flux relaxation time, \(\omega\) the angular frequency and \(\tau_1\) is the concentration relaxation time.

The self-similar variables are expressed as [37]:

\[
\begin{align*}
  \{ u &= a r F(\eta^*, t^*), v = a r G(\eta^*, t^*), w = -2\sqrt{a \nu F(\eta^*, t^*)}, p = \rho a \nu P(\eta^*, t^*) \\
  \theta(\eta^*, t^*) &= \frac{T - T_w}{T_{\infty} - T_w}, \phi(\eta^*, t^*) = \frac{C - C_w}{C_{\infty} - C_w}, \eta^* = \sqrt{\frac{a}{\nu}} z, t^* = \omega t.
\end{align*}
\]

(9)

Self-similar transformations reduce equations (1)-(7) into the following:

\[
\begin{align*}
  F''(t') + 2FF' - StF' + G^2 - (M^2 + P)F' &= 0, \\
  G'' - 2FG + 2FG' - (M^2 + P)G - StG &= 0, \\
  \theta'' + Pr(2F\theta' - St\theta' - \lambda)(4FF'\theta' + St^2\theta'' + 4F^2\theta') &= 0,
\end{align*}
\]

(10) (11) (12)
\[ \phi'' + Sc \left( 2F \phi' - St \phi_t - \lambda_2 \left( 4FF' \phi' + St^2 \phi_{tt} + 4F^2 \phi'' \right) \right) = 0, \tag{13} \]

and boundary conditions transform to the following:

\[
\begin{aligned}
F(0, t) &= 0, F'(0, t) = 1 + \varepsilon \sin t, G(0, t) = \Omega, \theta(0, t) = 1, \phi(0, t) = 1, P(0, t) = 0, \\
F'(\infty, t) &\to 0, G(\infty, t) \to 0, \theta(\infty, t) \to 0, \phi(\infty, t) \to 0,
\end{aligned} \tag{14}
\]

where prime \(^{(\prime)}\) denotes the differentiation w.r.t. the dimensionless variable \(z\), the subscript \(t\) represents the differentiation w.r.t. dimensionless time \(t\), \(St = \frac{\omega}{a}\) the unsteady parameter, \(M = \sqrt{\frac{\sigma \beta_0^2}{a \rho}}\) the magnetic parameter, \(P = \frac{\mu \phi_1}{a \rho k_1}\) the porosity parameter, \(Pr = \frac{\nu}{\alpha}\) the Prandtl number, \(\Omega = \frac{\gamma}{a}\) the swirl parameter, \(\lambda_1 = a \tau_0\) the thermal relaxation time parameter, \(\lambda_2 = a \tau_1\) the concentration relaxation time parameter and \(Sc = \frac{\nu}{D_B}\) stands for Schmidt number.

Shear stresses at the disk surface are defined as:

\[
\begin{aligned}
\tau_{x\varepsilon} &= \mu \frac{\partial u}{\partial \varepsilon} \bigg|_{z=0} = \mu a \text{Re}_x \frac{1}{2} F''(0, t), \\
\tau_{z\theta} &= \mu \frac{\partial v}{\partial \varepsilon} \bigg|_{z=0} = \mu a \text{Re}_x \frac{1}{2} G'(0, t),
\end{aligned} \tag{15}
\]

here \(\text{Re}_x = \frac{ar^2}{\nu}\) stands for Local Reynolds number.

3. Numerical Solution
The resulting system of equations is of second order in $G$, $\theta$ and $\phi$ and third order in third order in $F$. Equation (10) is converted into second order using technique of order reduction as follows:

$$F' = X,$$  \hfill (16)

$$X'' + 2FX' - X^2 + G^2 - StX_t - \left( M^2 + P \right)X = 0,$$  \hfill (17)

$$G'' - 2XG + 2FG' - StG_t - \left( M^2 + P \right)G = 0,$$  \hfill (18)

$$\theta'' + Pr\left( 2F\theta' - St\theta_t - \lambda_1 \left( St^2 \theta_{xx} + 4FX\theta' + 4F^2 \theta'' \right) \right) = 0,$$  \hfill (19)

$$\phi'' + Sc\left( 2F\phi' - St\phi_t - \lambda_2 \left( St^2 \phi_{xx} + 4FX\phi' + 4F^2 \phi'' \right) \right) = 0.$$  \hfill (20)

Boundary conditions (14) in terms of (16) become:

$$\begin{cases} F(0,t) = 0, X(0,t) = 1 + \varepsilon \sin \tau, G(0,t) = \Omega, \theta(0,t) = 1, \phi(0,t) = 1, P(0,t) = 0, \\ X(\infty, t) \rightarrow 0, G(\infty, t) \rightarrow 0, \theta(\infty, t) \rightarrow 0, \phi(\infty, t) \rightarrow 0, \end{cases}$$  \hfill (21)

Domain truncation technique is now implemented executed in order to obtain finite domain $[0, N]$ from semi-infinite domain $[0, \infty)$. Boundary conditions which are utilized at infinity are mentioned at end point $N$ of domain. The uniformly distributed $R + 1$ points that are discretize with variable $\eta$ s.t. the step-size is chosen to be $\delta \eta = \frac{N}{R}$. The resulting system is then solved through numerical way by implementing finite differences discretization scheme. The backward difference formulas of first-second orders are approximated with respect to time as”
\[
\begin{align*}
\frac{\partial X}{\partial t} & \approx X_{i,j} - X_{i,j-1}, \quad \frac{\partial G}{\partial t} \approx G_{i,j} - G_{i,j-1}, \quad \frac{\partial \theta}{\partial t} \approx \theta_{i,j} - \theta_{i,j-1}, \quad \frac{\partial^2 \theta}{\partial t^2} \approx \frac{\theta_{i,j} - 2\theta_{i,j-1} + \theta_{i,j-2}}{\partial^2 t}, \\
\frac{\partial \phi}{\partial t} & \approx \phi_{i,j} - \phi_{i,j-1}, \quad \frac{\partial^2 \phi}{\partial t^2} \approx \frac{\phi_{i,j} - 2\phi_{i,j-1} + \phi_{i,j-2}}{\partial^2 t}.
\end{align*}
\]  

(22)

Central difference approximations of first and second order are approximated with respect to \( \eta \) as:

\[
\begin{align*}
\frac{\partial Y}{\partial \eta} & \approx \frac{Y_{i+1,j} - Y_{i-1,j}}{2\delta \eta}, \quad \frac{\partial G}{\partial \eta} \approx \frac{G_{i+1,j} - G_{i-1,j}}{2\delta \eta}, \quad \frac{\partial \theta}{\partial \eta} \approx \frac{\theta_{i+1,j} - \theta_{i-1,j}}{2\delta \eta}, \quad \frac{\partial \phi}{\partial \eta} \approx \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\delta \eta}, \\
\frac{\partial^2 Y}{\partial \eta^2} & \approx \frac{Y_{i+1,j} - 2Y_{i,j} + Y_{i-1,j}}{\delta \eta^2}, \quad \frac{\partial^2 G}{\partial \eta^2} \approx \frac{G_{i+1,j} - 2G_{i,j} + G_{i-1,j}}{\delta \eta^2}, \quad \frac{\partial^2 \theta}{\partial \eta^2} \approx \frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{\delta \eta^2}, \quad \frac{\partial^2 \phi}{\partial \eta^2} \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\delta \eta^2},
\end{align*}
\]  

(23)

here \( i, j \) taken along spatial and time level, respectively.

The Eqs. (16)-(21) in view of Eqs. (22) and (23) are written as:

\[
\begin{align*}
\left(- X_{i,j} - \frac{2}{\delta \eta^2} - \frac{St}{\partial t} - M^2 - P\right) Y_{i,j} + \left(\frac{1}{\delta \eta^2} + \frac{F_{i,j}}{\partial \eta}\right) X_{i+1,j} + \left(\frac{1}{\delta \eta^2} - \frac{F_{i,j}}{\delta \eta}\right) X_{i-1,j} \\
+ G_{i,j}^2 + \frac{St}{\partial \eta} X_{i,j-1} = 0,
\end{align*}
\]  

(24)

\[
\begin{align*}
\left(- \frac{2}{\delta \eta^2} - X_{i,j} - \frac{St}{\partial t} - M^2 - P\right) G_{i,j} + \left(\frac{1}{\delta \eta^2} + \frac{F_{i,j}}{\partial \eta}\right) G_{i+1,j} + \left(\frac{1}{\delta \eta^2} - \frac{F_{i,j}}{\delta \eta}\right) G_{i-1,j} + \frac{St}{\partial \eta} G_{i,j-1} = 0,
\end{align*}
\]  

(25)

10
\[
\left( \frac{1}{\eta^2} \frac{\partial \theta}{\partial \eta} - \frac{2 \eta \frac{\partial \theta}{\partial \eta} + 4 \eta^2 \left( \frac{\partial^2 \theta}{\partial \eta^2} \right) \right) \theta_{i+1,j} + \left( \frac{\partial \theta}{\partial t} + \frac{2 \eta \frac{\partial \theta}{\partial \eta} \frac{\partial^2 \theta}{\partial \eta^2} \right) \theta_{i,j+1}
- \left( \frac{\eta^2}{d^2} \right) \theta_{i-1,j} + \left( \frac{\partial \theta}{\partial \eta} - \frac{2 \eta \frac{\partial \theta}{\partial \eta} + 4 \eta^2 \left( \frac{\partial^2 \theta}{\partial \eta^2} \right) \right) \theta_{i,j}
- \left( \frac{2 \eta^2}{d^2} \right) \theta_{j+2} + \left( \frac{\partial \theta}{\partial \eta} - \frac{2 \eta \frac{\partial \theta}{\partial \eta} + 4 \eta^2 \left( \frac{\partial^2 \theta}{\partial \eta^2} \right) \right) \theta_{i-1,j}
- \left( \frac{2 \eta^2}{d^2} \right) \theta_{j-2} + \left( \frac{\partial \theta}{\partial \eta} - \frac{2 \eta \frac{\partial \theta}{\partial \eta} + 4 \eta^2 \left( \frac{\partial^2 \theta}{\partial \eta^2} \right) \right) \theta_{i,j-1}
= 0,
\]

here boundary conditions [21] take the following form:

\[
\begin{align*}
F_{0,j} &= 0, \quad Y_{0,j} = 1 + \varepsilon \sin \gamma \xi, \quad G_{0,j} = \Omega, \quad \theta_{0,j} = 1, \quad \phi_{0,j} = 1, \\
Y_{N,j} &= G_{N,j} = \theta_{N,j} = \phi_{N,j} = 0.
\end{align*}
\]

SOR (successive over relaxation) is enforced for an iterative process to obtain numerical solution. Further equation (16) is integrated using Milne and Adam Moulton methods [38]. In our numerical method the optima value of successive over relaxation parameter is taken \(\omega = 1.95\). A spectral radius convergence criterion is expressed by [39]:

\[
\rho(G) = \begin{cases} 
\frac{1}{4} \left( \omega^2 + \sqrt{(\omega^2 - 4(\omega - 1))^2} \right) & \text{for } 0 < \omega \leq \omega_{opt}, \\
\omega - 1 & \text{for } \omega_{opt} < \omega < 2
\end{cases}
\]

here the optimal relaxation parameter is calculated by:
\[ \omega_{\text{opt}} = 1 + \left( \frac{\rho(G_{\text{Jac}})}{1 + \sqrt{1 - \left( \rho(G_{\text{Jac}}) \right)^2}} \right)^2. \]  

(30)

Where \( \rho(G_{\text{Jac}}) \) denotes spectral radius of Jacobi method. In current process of execution, we used time scales \( t_k = k \delta t, k = 0,1,2,\ldots \).

4. Discussion of results

Numerical analysis is implemented to study the unsteady, hydromagnetic flow problem of viscous fluid by implementing modified forms of heat and mass flux theories subjected to oscillatory disk. The disk takes its place in porous space. Analysis is carried out to discuss the results of dimensionless quantities through graphical representations and in tabular view forms. The physical parameters under consideration are magnetic parameter, porosity parameter, swirl parameter, constant amplitude, Prandtl number, Schmidt number and thermal and concentration relaxation time parameters. The graphs of various magnetic parameter values \( M = 0,1,1.5,2 \) on \( F'(0.25,t) \) and \( F(0.25,t) \) are demonstrated in Figs. 2 and 3 respectively. The fluid flow becomes hydrodynamic under the situation when \( M = 0 \). By introducing the magnetic field, a force termed as Lorentz force evolves in flow field. Such force is responsible for producing resistance and hence reducing velocity graphs. Furthermore in Fig. 2, the axial velocity \( F(0.25,t) \) curves are more closed to each other and domain of oscillation is small in comparison to radial velocity \( F'(0.25,t) \) profiles in Fig. 3. The demonstration of porosity parameter on both axial and radial velocities is pictured in Figs. 4 and 5. Similar observation is perceived in graphical results for varying porosity parameter as in the scenario of magnetic parameter. The porosity parameter has also the tendency to lower down the velocity profiles. Here we picked \( \eta = 0.25 \). The dominance
of swirl parameter on $F'(0.25,t)$ and $F(0.25,t)$ is prescribed in Figs. 6 and 7 respectively. Here in case of swirl parameter the domain of oscillatory profiles is slightly larger for both velocity curves in comparison to the oscillatory domain to porosity and magnetic parameters. The axial curves of velocity are more closer to each other than for radial velocity profiles. As the swirl parameter and stretching rate keeps inverse relation therefore amplifying values of $\Omega$ heighten the graphical results. At initial level of time we observed small amplitudes related to oscillatory curves however with increasing time the amplitude increased due to fluid particles rapid thrust.

Various constant amplitude values on flow velocities are depicting in Figs. 8 and 9. Both figures revealed the case of linearly rotating disk for $\varepsilon = 0$. In such case amplitude diminished. The non-zero declining miscellaneous values shrinks the amplitude of oscillations. Moreover, amplitude interval for $F'(0.25,t)$ is larger than that of $F(0.25,t)$. Prandtl number illustration for distinct values of temperature filed is pictured in Fig. 10. A rise in Prandtl number causes a reduction in temperature curves. For smaller values of Prandtl number the amplitudes of oscillatory temperature curves are very small however for larger $\Pr$ the amplitude of oscillations increases.

A pictorial representation of thermal relaxation time parameter on $\theta(0.25,t)$ is shown in Fig. 11. The increasing nature of $\lambda_1$ demands additional time for transformation of energy of molecules from one molecule to another therefore profiles convey the decreasing behavior for increased $\lambda_1$. The oscillatory temperature curves are close to each other. Fig. 11 also portrays the zooming figure in time interval $15 \leq t \leq 25$. Here phase difference is a notable feature in oscillatory temperature curve. The phase difference increases with an increase in $\lambda_1$. Similar observation is eminent for $\phi(0.25,t)$ from Fig. 12 in scenario of enhancing $\lambda_2$. The consequences of various Schmidt number on $\phi(0.25,t)$ are illustrated in Fig. 13. Schmidt number is responsible in
decaying the oscillatory concentration curves. For smaller Schmidt number the amplitude of oscillations is small while amplitude tends to increase for larger $Sc$.

Figs. 14 and 15 are sketched for fixed values of $M = 0.5, P = 1, \varepsilon = 0.3, \Omega = 3, \text{Pr} = 0.4, \text{Sc} = \lambda_1 = \lambda_2 = 0.3, \ St = 0$ and $\eta = \pi/2$. The velocity curves are presented in Fig. 14 while temperature and concentration curves are plotted in Fig. 15. Both the figures (14 and 15) represents steady fluid flow phenomenon subject to linearly rotating disk. Three-dimensional demonstration of flow behavior is exhibited from Figs. 16-20. Fig. 16-18 depicting the oscillatory flow phenomenon of axial, radial and tangential velocities while Figs. 19 and 20 revealed flow behavior on temperature and concentration fields respectively. Fig. 21 described the time series radial velocity curves considering numerous distances $\eta = 0.15, 0.35, 0.55, 0.75$ from disk surface. Because of oscillatory disk the curves amplitude is bigger nearby disk surface while away from disk the amplitudes tend to diminish.

Table 1 is constructed to examine the numerical values of magnetic and porosity parameters on frictional torque and radial shear stress taking into account various levels of time $t = \pi/6, \pi/4, \pi/2, 3\pi/4$. Magnetic parameter ranges from 0 to 2 and we considered the porosity parameter values from 0 to 3. The cases $M = 0$ and $P = 0$ represent hydrodynamic fluid flow which does not saturate the porous space. Increasing variations in both parameters enlarges frictional torque as well as radial shear stress in account to different time levels. Prandtl number and thermal relaxation time parameter influence on heat transfer rate is narrated in Table 2. The improved $\text{Pr}$ and $\lambda_1$ grown up the rate of heat transfer for four different time scales.

Table 3 illustrate the dominance of Schmidt number and concentration relaxation time parameter on mass transfer rate under $t = \pi/6, \pi/4, \pi/2, 3\pi/4$. Expanding values in both parameters results in strengthen the mass transfer rate. In Table 4, the comparison of numerical values of
shear and couple stresses by varying $\varepsilon$ and $\Omega$ at distinct time levels is given. Excellent comparison endorses our numerically computed results.

5. Conclusions

The analysis is made through numerical technique for the problem of MHD viscous fluid flow through rotating disk. The fluid is caused due to disk oscillation. Cattaneo-Christov heat and mass flux models are implemented in energy and mass equation respectively. The acquired system after using self-similar transformations is solved through SOR method. Key findings are pointed as:

1. The velocity graphs $F(0.25, t)$ and $F'(0.25,t)$ depicting declining nature for increased magnetic and porosity parameters and increasing trend for enhanced swirl parameter such that the distance between profiles $F'(0.25, t)$ are larger as compared to $F(0.25, t)$.

2. The oscillatory amplitudes magnify for larger constant amplitude and when constant amplitude is zero, the amplitude diminished.

3. Thermal and concentration relaxation time parameters have reducing trend in temperature and concentration fields respectively. Furthermore phase-log in oscillatory curves is also noticed in such profiles.

4. Time concentration profiles reduced for larger Schmidt number and small oscillations are observed in profiles.

5. Intensifying values of thermal relaxation time parameter and Prandtl number strengthen heat transfer rate against $t = \pi/6, \pi/4, \pi/2, 3\pi/4$. 
6. Mass transfer rate enlarged for enhanced concentration relaxation time parameter and Schmidt number considering various time levels.

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**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$\gamma$</td>
<td>Angular speed of disk</td>
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<tr>
<td>$\varepsilon$</td>
<td>Amplitude</td>
</tr>
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<td>$(u,v,w)$</td>
<td>Velocity components</td>
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<tr>
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<td>Wall concentration</td>
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<td>Ambient temperature</td>
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<td>$C_\infty$</td>
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<td>$\nu$</td>
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<td>Density</td>
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<tr>
<td>$\mu$</td>
<td>Viscosity</td>
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<td>$\sigma_\varepsilon$</td>
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<td>$\phi$</td>
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<tr>
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<td>Permeability of porous medium</td>
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<td>$\alpha = \frac{k}{\rho c_p}$</td>
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<td>$\tau_0$</td>
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<td>Time</td>
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<tr>
<td>$a$</td>
<td>Stretching strength</td>
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<tr>
<td>$St = \frac{\omega}{a}$</td>
<td>Unsteady parameter</td>
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<tr>
<td>$M = \sqrt{\frac{\sigma_\varepsilon \beta_0^2}{a \rho}}$</td>
<td>Magnetic parameter</td>
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</table>
\[ P = \frac{\mu \phi_i}{\alpha \rho k_i} : \text{Porosity parameter} \]
\[ \Pr = \frac{\nu}{\alpha} : \text{Prandtl number} \]
\[ \Omega = \frac{\gamma}{a} : \text{Swirl parameter} \]
\[ \lambda_1 = a \tau_0 : \text{Thermal relaxation time parameter} \]
\[ \lambda_2 = a \tau_1 : \text{Concentration relaxation time parameter} \]
\[ S_c = \frac{\nu}{D_B} : \text{Schmidt number} \]
\[ \text{Re}_x = \frac{a r^2}{\nu} : \text{Local Reynolds number} \]
\[ \rho(G_{Jac}) : \text{Spectral radius of Jacobi method} \]

**References**


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Figure Captions
Fig. 1: Geometry of the problem.

Fig. 2: Out-turns of $M$ on $F$ at $\eta = 0.25$.

Fig. 3: Out-turns of $M$ on $F'$ at $\eta = 0.25$.

Fig. 4: Out-turns of $P$ on $F$ at $\eta = 0.25$.

Fig. 5: Out-turns of $P$ on $F'$ at $\eta = 0.25$.

Fig. 6: Out-turns of $\Omega$ on $F$ at $\eta = 0.25$.

Fig. 7: Out-turns of $\Omega$ on $F'$ at $\eta = 0.25$.

Fig. 8: Out-turns of $\varepsilon$ on $F$ at $\eta = 0.25$.

Fig. 9: Out-turns of $\varepsilon$ on $F'$ at $\eta = 0.25$.

Fig. 10: Out-turns of $Pr$ on $\theta$ at $\eta = 0.25$.

Fig. 11: Out-turns of $\lambda_1$ on $\theta$ at $\eta = 0.25$.

Fig. 12: Out-turns of $\lambda_2$ on $\phi$ at $\eta = 0.25$.

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Fig. 14: $F,F'$ and $G$ at $\eta = \pi/2$.

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Fig. 16: Flow behavior at $\eta,t$ and $F$.

Fig. 17: Flow behavior at $\eta,t$ and $F'$.

Fig. 18: Flow behavior at $\eta,t$ and $G$.

Fig. 19: Flow behavior at $\eta,t$ and $\theta$.

Fig. 20: Flow behavior at $\eta,t$ and $\phi$.

Fig. 21: Various $\eta$ on $F'$ versus $t$. 
Table Captions

Table 1: Numerical values of $\tau_r$ and $\tau_{z\theta}$ for $M$ and $P$ considering four times levels.

Table 2: Numerical values of $-\theta'(0,t)$ for Pr and $\lambda_1$ considering four times levels.

Table 3: Numerical values of $-\phi'(0,t)$ for Sc and $\lambda_2$ considering four times levels.

Table 4: Comparison of $\tau_r$ and $\tau_{z\theta}$ for $\varepsilon$ and $\Omega$ considering distinct time levels.
\[ w = -2\sqrt{\alpha F(\eta^*, t)} \]

Fig. 1: Geometry of the problem

\[ u = aF'(\eta^*, t) \]

\[ v = aG(\eta^*, t) \]

Fig. 2: Out-turns of \( M \) on \( F \) at \( \eta = 0.25 \)

Fig. 3: Out-turns of \( M \) on \( F' \) at \( \eta = 0.25 \)
Fig. 4: Out-turns of $P$ on $F$ at $\eta = 0.25$

Fig. 5: Out-turns of $P$ on $F'$ at $\eta = 0.25$

Fig. 6: Out-turns of $\Omega$ on $F$ at $\eta = 0.25$

Fig. 7: Out-turns of $\Omega$ on $F'$ at $\eta = 0.25$
Fig. 16: Flow behavior at $\eta, t$ and $F$

Fig. 17: Flow behavior at $\eta, t$ and $F'$

Fig. 18: Flow behavior at $\eta, t$ and $G$

Fig. 19: Flow behavior at $\eta, t$ and $\theta$
Table 1: Numerical values of $\tau_{rz}$ and $\tau_{z\theta}$ for $M$ and $P$ considering four times levels.

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Table 4: Comparison of $\tau_{rz}$ and $\tau_{z\theta}$ for $\varepsilon$ and $\Omega$ considering distinct time levels

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<th>$\varepsilon$</th>
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Fig. 20: Flow behavior at $\eta, t$ and $\phi$

Fig. 21: Various $\eta$ on $F'$ versus $t$
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