Canonical forms and rotationally repetitive matrices for eigensolution of symmetric structures

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Abstract

In this paper, all canonical forms previously derived in literature for bilateral symmetry are derived from the formula for rotationally repetitive structures (systems) considering the rotation angle as 180 degrees. Different nodal numberings result in different patterns for matrices associated with bilaterally symmetric structures. This study shows that all these forms have the same nature and can be considered particular forms of circulant matrices associated with rotationally repetitive structures. In order to clarify this point, some numerical examples are investigated using both the classic approach and the canonical forms.

Keywords: canonical forms of matrices; graphs; regular structures; eigenvalues; Laplacian; bilateral symmetric systems; rotationally repetitive (circulant) matrices; dome structures.

1. Introduction

Eigenvalues and eigenvectors of matrices have a wide range of applications in different fields of engineering. In structural mechanics, these quantities are utilized for calculating natural frequencies of vibrating systems and buckling loads of structures [1, 2]. Besides, these quantities are used to performing free vibration and buckling analyses of circular cylindrical shells in composite structures, such as graphene-

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Another application of eigenvalues and eigenvectors is in graph theory, which has a long history. Some applications of eigenvalues and eigenvectors in graph theory are nodal ordering, graph partitioning, and decomposition of finite element meshes for use in parallel computing [6]. The reader can refer to the following books for computation of sparse matrices: Curtis and Reid [7], Pissanetzky [8], and Dongarra, et al. [9].

In algebraic graph theory, the eigenvalues and eigenvectors of Laplacian and adjacency matrices of graphs are used in understanding their characteristics. A significant contribution in algebraic graph theory is made by Fiedler [10], where the second eigenvector of the Laplacian matrix of a graph can be applied to the nodal numbering for profile reduction, and graph partitioning, and consequently, domain decomposition [11]. Further details can be found in books such as Biggs [12], Cvetkovic, et al. [13], Seidel [14], Chung and Graham [15] and Godsil and Royle [16]. Chen, et al. [17] introduced a systematic approach for deriving origami patterns by utilizing graph theory and mixed-integer linear programming (MILP). They showed that the proposed method is appropriate for developing origami patterns with degree-4 or degree-6 vertices.

In the past few decades, improvements in computers' capacity and the availability of memories have reduced the computational time. However, the fast computation of large matrices' eigenvalue problems with high sparsity is still considered a challenging problem. Therefore, researchers to use basic mathematical ideas for reducing the complexity of eigenproblems.

In order to accelerate the computation of eigenproblems, fast algorithms usually transform the matrix into simpler forms like diagonal, upper triangular, lower triangular matrices, or canonical forms. These transformations are known as "similarity transformations." Popular forms are Jordan form and Schur form. The term "similarity transformation" refers to a geometric similarity or matrix transformation in a likeness. In this paper, we use it with its second meaning.
The theory of groups has also been utilized for computational problems in structural mechanics. General group-theoretic methods for many problems are developed, where specific equilibrium matrices and stiffness matrices are given in symmetry-adapted coordinate systems and block-diagonal forms [18, 19]. Kangwai and Guest [20] utilized some new techniques for a symmetric structure to block-diagonalized their equilibrium matrices. Symmetry mentioned in structural mechanics often refers to either mirror symmetry or cyclic symmetry [21]. However, many space structures are neither cyclically symmetric nor mirror-symmetric [22]. Many seemingly asymmetric structures retain some kind of symmetry operations [22, 23]. Zingoni [24] presented a criterion for identifying the computationally efficient symmetry group, describing a system's symmetry properties, for a given problem in structures and solid mechanics. The efficiency of the criterion is shown by applying to a cubic configuration with octahedral symmetry. For improving the formal symmetry method for the dynamism of kinematically indefinite pin-jointed structures, the idea of graph products was combined with the symmetry-extended mobility rule [25]. Cable structures are one of the popular elements that are used for the design of symmetric structures [26]. The group theory was employed to recognize the vibration modes, the number of modes, and the pairs of modes of the same natural frequency for double-layer cable nets in order to reduce the computational effort for calculating their eigenvalue [27, 28].

Usually, symmetry occurs in engineering structures because of its ease in design and construction and aesthetic objects. In mathematics, "symmetry" means an invariant object under any of various transformations like reflecting, rotation or scaling. A plane structure is symmetric concerning an axis of symmetry in its plane if the reflection of the structure about the axis is identical in geometry, supports, and material properties to the structure itself. The post-buckling behavior of symmetric frames was studied by Plgnataro and Rizzi [29].

Kaveh and Sayarinejad [30] represented efficient and straightforward methods for calculating a matrix's eigensolution with particular forms. They proved that one could calculate the graph-based matrices' eigensolutions by uniting the eigenvalues and eigenvectors of the submatrices. These submatrices were
obtained by decomposing the graph-based matrix to small-dimension matrices. Also, they represented some methods for decomposing and healing the graph model of structures. According to these articles, the matrices associated with bilateral symmetric structures have the canonical Forms I, II, III, or IV. These forms' applications in mass-spring systems, symmetric frames, and buckling loads of symmetric plane frames [31] were studied in other articles.

Kaveh and Rahami [31, 32] presented other canonical forms for adjacency and Laplacian matrices associated with graph models and provided methods for decomposing regular structures. A structure is called 'regular' if its model can be considered a product graph [33]. Thomas [34] and Williams [35], [36] presented some results concerning the vibration of cyclically symmetric structures. Kaveh and Nemati [37] found the eigensolutions for buckling load and natural frequencies in vibrating systems and utilized the canonical form from linear algebra known as the circulant matrix [38]. In this method, the structure is decomposed into repeated substructures, and the eigenvalues and eigenvectors of the graph-based matrices are calculated by some simple operations on the eigenvalues and eigenvectors of the substructures. Kaveh et al.[39] utilized the algebraic graph theory for proposing a method of constructing preconditioners to employ in the entire design process of topology optimization. The efficiency of their approach was shown by applying it to the repetitive near-regular shell structures.

The different nodal numbering makes different patterns for matrices associated with bilateral symmetric structures. This study will show that all these forms can be considered individual cases of the form associated with rotationally repetitive structures. A few numerical examples are solved using the classic approach and the canonical forms to confirm the efficiency of the presented method.

This paper comprises six sections. A brief introduction containing the basic definitions from the theory of graphs and linear algebra are presented in Section 2. Section 3 discusses the matrices with canonical forms associated with bilateral symmetric structures and rotationally repetitive space structures. Different types of symmetry are discussed in Section 4. The proof of the relationship between
the canonical forms of bilateral symmetric structures and rotationally repetitive structures is provided in Section 5. Examples are investigated in Section 6, and the concluding remarks are presented in Section 7.

2. Basic definitions

2.1. Basic concepts from the theory of graphs

A graph \( G(N, M) \) is a set \( N \) of nodes (vertices) and a set \( M \) of elements members (edges) with a relation of incidence that relates a pair of nodes with each member. Two nodes are adjacent if they correspond to the same member. A graph \( G \) is called undirected if a member is an unordered pair of nodes.

2.2. Basic definitions from linear algebra

In linear algebra, two matrices, \( A \) and \( B \), are called similar if

\[
B = P^{-1}AP
\]

where \( P \) is a nonsingular matrix, the transformation \( A \rightarrow B \) is known as a similarity transformation of \( A \).

The characteristic polynomial of a square matrix \( A \) is a polynomial \( p(\lambda) = \det(A - \lambda I) \) that is invariant under matrix similarity and has the eigenvalues as roots. The scalar \( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( \nu \), known as an eigenvector, such that:

\[
A\nu = \lambda\nu \tag{2}
\]

\[
(\lambda I - A)\nu = 0 \tag{3}
\]

The characteristic polynomial of a diagonal matrix \( A \) can easily be defined. If the diagonal entries of \( A \) are \( a_1, a_2, a_3, \ldots, a_n \), then

\[
(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda)\ldots(a_n - \lambda) = 0 \tag{4}
\]

will be the characteristic polynomial of \( A \). According to Equation (4), it can be seen that the diagonal entries are also the eigenvalues of this matrix. The simplest matrix for finding the eigenvalues is a diagonal matrix. Similarly, eigenvalues of a triangular matrix are its diagonal entries.

A matrix can have complex eigenvalues since its characteristic polynomial can have real or complex roots. Every \( n \times n \) matrix has exactly \( n \) complex and real eigenvalues, counted with multiplicity.
2.3. Definitions from algebraic graph theory

The adjacency matrix \( A = [a_{ij}]_{n \times n} \) of a graph \( G \), with its nodes being labeled and contain \( n \) nodes, is defined as:

\[
a_{ij} = \begin{cases} 
1 & \text{if node } n_i \text{ is adjacent to } n_j \\
0 & \text{otherwise}
\end{cases} \tag{5}
\]

The degree matrix \( D = [d_{ii}]_{n \times n} \) is a diagonal matrix containing node degrees where \( d_{ii} \) is the degree of the \( i \)th node.

Laplacian matrix \( L = [l_{ij}]_{n \times n} \) is defined as Equation (6).

\[
L = D - A \tag{6}
\]

Therefore, the entries of \( L \) areas:

\[
l_{ij} = \begin{cases} 
-1 & \text{if node } n_i \text{ is adjacent to } n_j \\
\deg(n_i) & \text{if } i = j \\
0 & \text{otherwise}
\end{cases} \tag{7}
\]

For eigensolution of the adjacency matrix, consider the eigenproblem as:

\[
A \nu_i = \lambda_i \nu_i \tag{8}
\]

where \( \lambda_i \) is the \( i \)th eigenvalue and \( \nu_i \) is the corresponding eigenvector. For \( A \) being a real symmetric matrix, all the corresponding eigenvalues are real given as Equation (9).

\[
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n \tag{9}
\]

The largest eigenvalue \( \lambda_n \) has multiplicity equal to unity for the characteristic equation of \( A \). The corresponding eigenvector \( \nu_n \) is the only eigenvector with positive entries. This vector has many interesting properties employed in structural mechanics.

For eigensolution of the Laplacian matrix, consider the problem as:

\[
L \nu_i = \lambda_i \nu_i \tag{10}
\]
where \( \lambda_i \) is the \( i \)th eigenvalue and \( \nu_i \) is the corresponding eigenvector. Since \( A \) is a real symmetric matrix and all its eigenvalues are real, all the eigenvalues of \( L \) are also real. It can be shown that the matrix \( L \) is a positive semidefinite matrix with

\[
0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \lambda_n
\]

(11)

\[
\nu_1 = \{1,1,1,\ldots,1\}^T
\]

(12)

The second eigenvalue \( \lambda_2 \) of \( L \) is also known as "algebraic connectivity" of a graph, and its eigenvector \( \nu_2 \) is called the Fiedler vector. This vector has attractive properties.

The **Kronecker product** of two matrices is an operation on these matrices, which results in a block matrix. This operation is denoted by \( \otimes \),

The Kronecker product of two matrices \( A_{m \times n} \) and \( B_{p \times q} \) is the \( mp \times nq \) block matrix as:

\[
A_{m \times n} \otimes B_{p \times q} = \begin{bmatrix}
    a_{11}B & \cdots & a_{1n}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}_{mp \times nq}
\]

(13)

A **Hermitian matrix** is a square matrix that is equal to its conjugate transpose. A real matrix is Hermitian if and only if it is symmetric.

Being Hermitian is a necessary condition for a matrix to be diagonalizable.

\[
A \text{ is Hermitian} \iff a_{ij} = \overline{a_{ji}}
\]

(14)

For two matrices \( A_1 \) and \( A_2 \) to be diagonalizable simultaneously, these two matrices should be Hermitian and commutative \( A_1A_2 = A_2A_1 \).

3. **Canonical forms of a matrix**
In this section, all the canonical forms available for bilateral symmetry structures and rotationally repetitive structures are presented. Here $M$ is a matrix whose elements entirely consist of real numbers. The canonical Forms I, II, III, and IV was represented by Kaveh et al.[30].

3.1. Form I

In this form, $M$ is a block diagonal matrix having a pattern, as shown in Equation (15).

$$M_{2n} = \begin{bmatrix} A_n & 0 \\ 0 & A_n \end{bmatrix}_{2n \times 2n}$$  \hspace{1cm} (15)

Since $M$ is a diagonal matrix, its determinant can be calculated as:

$$\text{det}(M) = \text{det}(A) \times \text{det}(A)$$  \hspace{1cm} (16)

Moreover, the set of eigenvalues of submatrix $A$ is $\lambda_A$. Therefore, the eigenvalues of $M$ can be calculated as Equation (17).

$$\lambda_M = \{\lambda_A\} \cup \{\lambda_A\}$$  \hspace{1cm} (17)

This Form can be generalized by placing a large number of matrices with different dimensions on its main diagonal entries. Based on Equation (17), eigenvalues of this generalized matrix will be equal to the union of the eigenvalues of those matrices located on its main diagonal.

3.2. Form II

For this particular form, the matrix $M$ has a pattern, as shown in Equation (18).

$$M_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix}_{2n \times 2n}$$  \hspace{1cm} (18)

By adding the second column to the first one and then subtracting the first row from the second one, $M$ transforms to the upper triangular matrix as follows

$$M_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix} = \begin{bmatrix} C_n & B_n \\ 0 & D_n \end{bmatrix}$$  \hspace{1cm} (19)
in which

\[ C = [A] + [B] \]  \hspace{1cm} (20) \\
\[ D = [A] - [B] \]  \hspace{1cm} (21) 

Since \( M \) is a triangular matrix, its determinant is equal to the multiplication of two matrices that are located on the diagonal of \( M \). Therefore, the set of eigenvalues of \( M \) can be obtained as Equation (22).

\[ \lambda_M = \{ \lambda_C \} \cup \{ \lambda_D \} \]  \hspace{1cm} (22)

3.3. Form III

In this form, \( M \) has an \( N \times N \) submatrix, where \( N = 2n \). The pattern of Form II is augmented by \( k \) rows and \( k \) columns as formulated in Equation (23).

\[ M = \begin{bmatrix} A & B & P \\ B & A & P \\ Q & H & R \end{bmatrix} \]  \hspace{1cm} (23)

As seen, \( M \) is an \( (N + k) \times (N + k) \) matrix, and the entries of augmented columns are the same in each column. Similar to Form II, in this case, \( M \) can be factored using rows and columns permutation. By exchanging the second column and second row with the fourth column and fourth row respectively, then adding the fourth column to the first one. As the last step, by reducing the first row from the fourth one, \( M \) transforms into the upper triangular matrix as Equation (24).

\[ M = \begin{bmatrix} A + B & P & B \\ Q + H & R & H \\ 0 & 0 & A - B \end{bmatrix} = \begin{bmatrix} E & K \\ 0 & D \end{bmatrix} \]  \hspace{1cm} (24)

where

\[ E = \begin{bmatrix} A + B & P \\ Q + H & R \end{bmatrix} \]  \hspace{1cm} (25)
\[ D = [A] - [B] \]  \hspace{1cm} (26)

Since \( M \) is a triangular matrix, its determinant will be equal to the multiplication of two matrices that are located on diagonal entries. Therefore, the set of eigenvalues of \( M \) can be obtained as Equation (27).

\[ \lambda_M = \{ \lambda_e \} \cup \{ \lambda_D \} \]  \hspace{1cm} (27)

3.4. Form IV

\( M \) is a \( 6n \times 6n \) matrix having the following pattern

\[
M = \begin{bmatrix}
S - H & H - S \\
H - S & S - H \\
-H & S & H - S \\
H - S & S & -H \\
-H & S & H - S \\
H - S & S - H
\end{bmatrix}_{6n\times 6n}
\]  \hspace{1cm} (28)

\( S \) and \( H \) are an \( n \times n \) matrices. The characteristic polynomial of \( M \) can calculate as Equation (29).

\[
P_M(\lambda) = [\lambda \left(2H - 2S + \lambda\right)][\lambda^2 - 2S \lambda + SH - H^2][\lambda^2 - 2S \lambda + 3SH - 3H^2] \]  \hspace{1cm} (29)

The first term of Equation (29) can be considered as the characteristic polynomial of the matrix that is a matrix of Form \( II \) Equation (30).

\[
[E_1] = \begin{bmatrix}
S - H & H - S \\
H - S & S - H
\end{bmatrix} = \lambda_{\{2S - 2H\}} \cup \lambda_{\{S\},\, n} \]  \hspace{1cm} (30)

The second term of Equation (29) can be considered as the characteristic polynomial of the matrix in Equation (31).

\[
[E_2] = \begin{bmatrix}
S + H & -S \\
H - S & S - H
\end{bmatrix}
\]  \hspace{1cm} (31)

The third term of Equation (29) can be considered as the characteristic polynomial of the matrix in Equation (32).
\[
\begin{bmatrix}
E_3 & = & \begin{bmatrix}
2S & 3H \\
H - S & 0 \\
\end{bmatrix}
\end{bmatrix}
\] (32)

Hence, for finding the eigenvalues of a matrix in the form of Equation (28), calculating the eigenvalues of

\[
E_i \text{ for } i = 1, 2, and 3
\]
is sufficient.

\[
\lambda_M = \{\lambda_{E_1}\} \cup \{\lambda_{E_2}\} \cup \{\lambda_{E_3}\}
\] (33)

3.5. 
A canonical form associated with rotationally repetitive structures

An efficient eigensolution has been developed for determining the buckling load and free vibration of rotationally cyclic structures by Kaveh, A and Nemati [37]. This solution utilizes a canonical form, which is often involved in graph model matrices. This canonical form is presented in this section.

For rotationally repetitive structures, one can associate a canonical form as follows:

\[
M_{mn} = \begin{bmatrix}
J & L & L' \\
L' & J & L & \ldots \\
L & L' & J & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
L' & \ldots & J & L \\
\end{bmatrix}
\] (34)

where the matrix \(M\) is a symmetric block matrix, with \(n \times n\) blocks. The blocks of this matrix are \(J_m \times m\), \(L_m \times m\), and \(L^T_m \times m\). Thus this matrix generally has \(nm\times nm\) entries. Block \(J\) is located on the main diagonal, and blocks \(L\) and \(L^T\) are located on the upper and lower adjacent diagonals, and also in the lower-left corner and the upper right corner, respectively. Matrix \(M\) can be decomposed to the sum of three Kronecker products as

\[
M_{mn} = I_n \otimes J_m + H_n \otimes L_m + H_n^T \otimes L_m^T
\] (35)

where \(I\) is an \(n \times n\) identity matrix and \(H\) is an \(n \times n\) asymmetric matrix as
Since $H$ is an asymmetric matrix, block diagonalization of $M$ requires some considerations.

Here $H$ is a permutation matrix, and hence it is orthogonal. Thus, we have Equation (37).

$$HH^t = 1$$  \hspace{1cm} (37)

Matrix $H$ and its transpose have commutative property as Equation (38).

$$HH^t = H^tH$$  \hspace{1cm} (38)

According to Equations (37) and (38), the two matrices $H$ and $H^t$ can be diagonalized simultaneously.

Now by using a matrix such as $U = X \otimes I$, matrix $M$ can be diagonalized as Equation (39).

$$U^{-1}MU = U^{-1}(I_n \otimes J_m + H_n \otimes L_m + H_n^t \otimes L_m^t)U = (X \otimes I)^{-1}(I \otimes J + H \otimes L + H^t \otimes L')(X \otimes I)$$

$$= (X^{-1} \otimes I^{-1})(I \otimes J + H \otimes L + H^t \otimes L')(X \otimes I)$$

$$= (X^{-1} \otimes J + X^{-1}H \otimes L + X^{-1}H^t \otimes L')(X \otimes I)$$

$$= (I \otimes J + X^{-1}HX \otimes L + X^{-1}H^tX \otimes L')$$

Since a similarity transformation (see subsection 2.2) is employed; therefore, the eigenvalues do not change. In Equation (39) $I$ is a diagonal matrix, and it is sufficient to show that $X$ simultaneously diagonalizes $H$ and $H^t$.

The eigenvalues of the matrix $M$ can be found by using the union of the eigenvalues of n blocks as Equation (40).
\[ \lambda_M = \bigcup_{k=1}^{n} eig \left( J_m + \lambda_i \left( H_n \right) L_m + \lambda_i \left( H_n^T \right) L_m^T \right) \]  

(40)

Since the eigenvalues of the matrix \( H \) are needed for finding the eigenvalues of matrix \( M \), let find the eigenvalues of \( H \). \( H \)'s characteristic polynomial can be written as

\[ \lambda^n - 1 = 0 \]  

(41)

This equation has \( n \) real and complex roots which are identical to Equation (42).

\[ \cos(n\theta) + i \sin(n\theta) = 1 \]  

(42)

Obviously, if \( n \) is even or odd, then \(-1, 1\) or \(1 \) will be the only real roots of Equation (41), respectively. The complex and real roots are presented in Table 1.

Based on Table 1, for \( n \) being even or odd, two real roots and \((n - 2)\) complex roots or one real and \((n - 1)\) complex roots will be calculated, respectively.

4. Different kinds of symmetry

In this section, for different forms of the matrices associated with the symmetric structures, the corresponding graphs are introduced, Kaveh (2013).

4.1. Form I symmetry

Here, the symmetry axis does not pass through nodes and members. The model is a disjoint graph with at least two distinct subgraphs, as shown in Fig. 1.

After nodal numbering of one of the substructures, the second substructure's nodal numbers should be determined considering the symmetry of nodes, as numbered in Fig. 1.

4.2. Form II symmetry

In this case, the model has an even number of nodes, and the axis of symmetry passes through members; for instance, a sample structure of Form II is drawn in Fig. 2.
The structure is divided into two equal substructures. After nodal numbering of one of the substructures, the second substructure's nodal numbers should be determined considering the symmetry of nodes as Fig. 2.

4.3. **Form III symmetry**

In this case, the axis of symmetry passes through members and nodes, as Fig. 3. The structure is divided into two equal substructures. Firstly, the nodal numbering of one of the substructures except those located on the symmetry axis is performed. The nodal numbers of the second substructure should be determined considering the symmetry of nodes. Finally, the labels of the nodes located on the symmetry axis are assigned, as labeled in Fig. 3.

The number of the rows and columns that are added to the core of Form II are equal to the number of nodes that the axis of symmetry passes through them. This means in the \((N + K) \times (N + K)\) matrix \(M\) there are \(K\) nodes that the axis of symmetry passes through them.

4.4. **The form associated with rotationally repetitive structures**

A rotationally repetitive structure is a structure consisting of a cyclically symmetric configuration with an angle of cyclic symmetry equal to \(\theta\) as shown in Fig. 4.

A rotationally repetitive structure should be divided by some imaginary lines or surfaces into \(n = \frac{2\pi}{\theta}\) segments. The imaginary boundaries may not pass through any node so that the segment to which a given node belongs can be uniquely determined.

For nodal numbering, the difference between the number of an arbitrarily selected node in an arbitrary segment and the number of the corresponding node in the adjacent segment should be constant.

5. **The relations between the canonical forms**

5.1. **The relation between the Form I and Form II**

According to Equation (15) and Equation (18), Form I is a special form of Form II with the matrix \(B\) equaled to zero.
\[ M_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix}_{N \times N} \quad \rightarrow \quad M_{2n} = \begin{bmatrix} A_n & 0 \\ 0 & A_n \end{bmatrix}_{N \times N} \]  \hspace{1cm} (43)

### 5.2. The relation between the Form II and Form III

In this section, we are going to show that the canonical Form III can be obtained from the canonical Form II, according to Kaveh (2013). For reaching this purpose, first, some points should be considered as follows.

Consider a \( 2N \times 2N \) matrix \( L \), then add one arbitrary row and one arbitrary zero column to the matrix \( L \) as Equation (44):

\[
C = \begin{bmatrix} L_{2N \times 2N} & 0 \\ X & 0 \end{bmatrix}_{(2N+1) \times (2N+1)} \]  \hspace{1cm} (44)

According to that, all entries in \((2N + 1)\)th column of the matrix \( C \) are zero, the eigenvalues of the matrix \( C \) are equal to the union of the set of eigenvalues of matrix \( L \) and a set with one zero member.

This can be proven as follows:

The first \( 2N \) rows of \( C \) are respectively multiplied by \( k_1, k_2, \ldots, k_{2N} \), and the sum is equated to zero. Since any multiple of the last column will be zero, therefore the Equation (45) is obtained:

\[
KL + X = 0 \]  \hspace{1cm} (45)

If \( L \) is invertible (i.e., if \( det(L) \neq 0 \)), then \( K = -XL^{-1} \) and \( k_1, k_2, \ldots, k_{2N} \) can be found, and the last row of \( C \) becomes zero. However, if \( det(L) = 0 \) then, there are many sets of \( k_i \) which put the last row
of $C$ into zero. Therefore, one can conclude that there is always at least one transformation that makes the last row of $C$ as zero.

If the $k$th row of a matrix is multiplied in $m$ and the $k$th column is divided by $m$, the eigenvalues of this matrix stay unchanged. The reason is that the magnitude of the diagonal entry stays constant, and if it is expanded with respect to a row and column, the determinant of the submatrices stays unaltered.

In the following, an algorithm is provided for transforming the canonical Form $III$ to the Form $II$:

**Step 1.** A zero column and a zero row in the second column and second row are added as the following expression:

$$
\begin{bmatrix}
A & B & P \\
B & A & P \\
Q & H & R
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
A & 0 & B & P \\
0 & 0 & 0 & 0 \\
B & 0 & A & P \\
Q & 0 & H & R
\end{bmatrix}
$$

(46)

**Step 2.** $H-Q$ and $Q-H$ in the first and third entries of the second row are added. The sum of these entries is equal to zero; hence, adding these two entries has no effect on the matrix's eigenvalues.

$$
\begin{bmatrix}
A & 0 & B & P \\
0 & 0 & 0 & 0 \\
B & 0 & A & P \\
Q & 0 & H & R
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
A & 0 & B & P \\
(H-Q)/2 & 0 & (Q-H)/2 & 0 \\
B & 0 & A & P \\
Q & 0 & H & R
\end{bmatrix}
$$

(47)

**Step 3.** Half of the fourth column is added to column two, and half of the fourth row is added to the second row. Then interchange column two with column four.
Step 4. Column four is multiplied by 2, and row four is multiplied by $\frac{1}{2}$, resulting in Equation (49)

$$\begin{bmatrix} A & P & B & P/2 \\ H/2 & R/2 & Q/2 & R/4 \\ B & P & A & P/2 \\ Q & R & H & R/2 \end{bmatrix} \Rightarrow \begin{bmatrix} A & P & B & P \\ H/2 & R/2 & Q/2 & R/2 \\ B & P & A & P \\ Q/2 & R/2 & H/2 & R/2 \end{bmatrix} \Rightarrow \begin{bmatrix} M & N \\ N & M \end{bmatrix} = Form II \quad (49)$$

where

$$M + N = \begin{bmatrix} A+B & 2P \\ (Q+H)/2 & R \end{bmatrix}, \quad M - N = \begin{bmatrix} A-B & 0 \\ (H-Q)/2 & 0 \end{bmatrix} \quad (50)$$

In matrix $M + N$ by multiplying the second column by $\frac{1}{2}$, and multiplying the second row by 2, result in $E$ as Equation (51).

$$E = \begin{bmatrix} A+B & P \\ Q+H & R \end{bmatrix} \quad (51)$$

The right-hand side matrix $M - N$ in Equation (50) has the same eigenvalues as those of $A - B$ with the exception of having $2N$ extra zeros. $E$ and $D$ are the same matrices as defined in Form $III$ in the previous section.

5.3. The relation between the Form $IV$ and Form $III$
In this section, it is shown that the canonical Form IV is a special form of the canonical Form III. For this purpose, the characteristic polynomial of the canonical Form IV is considered as Equation (52).

\[
\det(M - \lambda I) = \begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & 0 \\
H - S & S - \lambda & -H & 0 & 0 & 0 \\
0 & -H & S - \lambda & H - S & 0 & 0 \\
0 & 0 & H - S & S - \lambda & -H & 0 \\
0 & 0 & 0 & -H & S - \lambda & H - S \\
0 & 0 & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix} = 0 \quad (52)
\]

Now by using the elementary matrix operation, we have an algorithm for transforming the canonical Form IV to the Form III. The algorithm is provided through the following steps.

**Step 1.** The third column is swapped with the sixth one and the fourth column with the fifth one.

\[
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & 0 \\
H - S & S - \lambda & -H & 0 & 0 & 0 \\
0 & -H & S - \lambda & H - S & 0 & 0 \\
0 & 0 & H - S & S - \lambda & -H & 0 \\
0 & 0 & 0 & -H & S - \lambda & H - S \\
0 & 0 & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix} \Rightarrow
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & 0 \\
H - S & S - \lambda & -H & 0 & 0 & 0 \\
0 & -H & S - \lambda & H - S & 0 & 0 \\
0 & 0 & H - S & S - \lambda & -H & 0 \\
0 & 0 & 0 & -H & S - \lambda & H - S \\
0 & 0 & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix} \quad (53)
\]

**Step 2.** The third row is swapped with the sixth one and the fourth row with the fifth one.
Step 3. The sixth column is replaced by itself plus the first and second columns, and the fifth column is replaced by itself plus the third and fourth columns.

\[
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & 0 \\
H - S & S - \lambda & 0 & 0 & 0 & -H \\
0 & -H & 0 & 0 & H - S & S - \lambda \\
0 & 0 & 0 & -H & S - \lambda & H - S \\
0 & 0 & H - S & S - \lambda & -H & 0 \\
0 & 0 & S - H - \lambda & -H & 0 & H - S \\
0 & -H & 0 & 0 & H - S & S - \lambda \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & 0 \\
H - S & S - \lambda & 0 & 0 & 0 & -H \\
0 & 0 & S - H - \lambda & H - S & 0 & 0 \\
0 & 0 & H - S & S - \lambda & -H & 0 \\
0 & 0 & 0 & -H & S - \lambda & H - S \\
0 & -H & 0 & 0 & H - S & S - \lambda \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & -\lambda \\
H - S & S - \lambda & 0 & 0 & 0 & -\lambda \\
0 & 0 & S - H - \lambda & H - S & -\lambda & 0 \\
0 & 0 & H - S & S - \lambda & -\lambda & 0 \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & -H & 0 & 0 & H - S & S - H - \lambda \\
\end{bmatrix}
\]
Step 4. The first row is replaced by itself minus the sixth row, the second row is replaced by itself minus the sixth row, the third row is replaced by itself minus the fifth row, and replace the fourth row by itself minus the fifth row.

\[
\begin{bmatrix}
S - H - \lambda & H - S & 0 & 0 & 0 & -\lambda \\
H - S & S - \lambda & 0 & 0 & 0 & -\lambda \\
0 & 0 & S - H - \lambda & H - S & -\lambda & 0 \\
0 & 0 & H - S & S - \lambda & -\lambda & 0 \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & -H & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
S - H - \lambda & 2H - S & 0 & 0 & S - H & H - S \\
H - S & S + H - \lambda & 0 & 0 & S - H & H - S \\
0 & 0 & S - H - \lambda & 2H - S & H - S & S - H \\
0 & 0 & H - S & S + H - \lambda & H - S & S - H \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & -H & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix}
\]

Step 5. The first row and the second row are multiplied by $-1$, and the first column and the second column are multiplied by $-1$.

\[
\begin{bmatrix}
S - H - \lambda & 2H - S & 0 & 0 & S - H & H - S \\
H - S & S + H - \lambda & 0 & 0 & S - H & H - S \\
0 & 0 & S - H - \lambda & 2H - S & H - S & S - H \\
0 & 0 & H - S & S + H - \lambda & H - S & S - H \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & -H & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
S - H - \lambda & 2H - S & 0 & 0 & H - S & S - H \\
H - S & S + H - \lambda & 0 & 0 & H - S & S - H \\
0 & 0 & S - H - \lambda & 2H - S & H - S & S - H \\
0 & 0 & H - S & S + H - \lambda & H - S & S - H \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & H & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix}
\]
The characteristic polynomial of the canonical Form IV could be transformed into a characteristic polynomial that is in the pattern of the canonical Form III as Equation (58).

\[
\begin{bmatrix}
S - H - \lambda & 2H - S & 0 & 0 & H - S & S - H \\
H - S & S + H - \lambda & 0 & 0 & H - S & S - H \\
0 & 0 & S - H - \lambda & 2H - S & H - S & S - H \\
0 & 0 & H - S & S + H - \lambda & H - S & S - H \\
0 & 0 & 0 & -H & S - H - \lambda & H - S \\
0 & H & 0 & 0 & H - S & S - H - \lambda
\end{bmatrix}
= \text{eig (Form III } - \lambda I \text{) } = 0
\]

\[
\begin{bmatrix}
S - H & 2H - S & 0 & 0 & H - S & S - H \\
H - S & S + H & 0 & 0 & H - S & S - H \\
0 & 0 & S - H & 2H - S & H - S & S - H \\
0 & 0 & H - S & S + H & H - S & S - H \\
0 & 0 & 0 & -H & S - H & H - S \\
0 & H & 0 & 0 & H - S & S - H
\end{bmatrix} = \begin{bmatrix}
A & B & P \\
B & A & P \\
Q & H & R
\end{bmatrix} = \text{Form III } (59)
\]

From Equation (59), we came to the conclusion that the canonical Form IV is a special form of the Form III.

5.4. The relation between the canonical Form II and the form associated with rotationally repetitive structures

If we have a bilaterally symmetric structure, we have a substructure that, by repeating it twice \((n = 2)\), we get the whole structure. It means that the rotation angle is 180°. We use \(n = 2\) in Table 1 and replace the results in Equations (34) and (40).

\[
n = 2 \text{ is even } \rightarrow \lambda_i \left( H'_2 \right) = \lambda_i \left( H_2 \right) = -1, +1, H_2 = H'_2 = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, \lambda_i \left( H'_n \right) = \lambda_i \left( H_n \right) \quad (60)
\]
According to Equation (61), we see that it is similar to what we have in the bilateral symmetric structures (Form II). Then we can conclude that a bilaterally symmetric structure is a special form of rotationally repetitive structure.

From Equation (59), we came to the conclusion that Form IV is a special form of Form III. According to Equations (43) and (49), we saw that the canonical Form III and Form I are a special form of the canonical Form II. From Equation (61), we saw that Form II is a special form of the canonical form that was presented for rotationally repetitive structures Equation (34). As a result, it turns out that all canonical forms that are presented for the bilateral symmetric structures are a special form of the rotationally repetitive structures.

6. Examples

This section shows that all forms associated with a particular structure with different nodal numbering cause the same eigenvalue for the whole structure. On the other hand, all these forms can compare with each other in time, memory, and the method of nodal numbering. $L(G)$ is the Laplacian matrix of an entire graph $G$ that is represented in each example. These examples are programed in MATLAB R2016b software and processed in a computer with Intel® Core™ i7_4510U CPU @ v2.00 GHz processor and 8.00GB RAM.

6.1. Example 1

Consider the symmetric frame, which has only one symmetry axes, as shown in Fig. 5. as can be seen, this is constrained against a motion in the sway direction and has only two rotation degrees of freedom. Both its mass and stiffness matrices are in Form II. For gaining its natural frequencies, the Equation (62) must
be solved. The natural frequencies are equal to the eigenvalues of the factors of Form \( II \), as expressed in Equation (63).

\[
\text{det}
\begin{bmatrix}
K - \lambda^2 M
\end{bmatrix} = 0
\] (62)

\[
\lambda_c = \sqrt{\frac{420EI}{mL^4}}, \quad \lambda_d = \sqrt{\frac{420EI}{11mL^4}}
\] (63)

6.2. Example 2

For the graph shown in Fig. 6, there are two axes of symmetry that one of them passes through members (Fig. 6) and the other passes through nodes (Fig. 7). By two different nodal numbering, we get to the canonical Form \( II \) and Form \( III \) for the Laplacian matrix.

The symmetry of Form \( II \)

The factors of this graph in Form \( II \), according to Equation (19) are as follows

\[
A = \begin{bmatrix}
3 & -1 & 0 & -1 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 3 & -1 \\
-1 & 0 & -1 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\] (64)

The eigenvalues of the whole graph are now calculated as the union of the eigenvalues for its factors as Equation (67).

\[
eig(A - B) = \{1.0968, 3.1939, 4.0000, 5.7093\}
\] (65)

\[
eig(A + B) = \{0.0000, 2.0000, 2.0000, 4.0000\}
\] (66)

\[
eig(L(G)) = \{1.0968, 3.1939, 4.0000, 5.7093, 0.0000, 2.0000, 2.0000, 4.0000\}
\] (67)

The symmetry of Form \( III \):
Consider a graph, as shown in Fig. 7.

The factors of this graph in Form III, according to Equation (24) are as follows:

\[ A - B = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 3 & -1 & 0 \\ 0 & -2 & 0 & -1 & 3 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix} \] \hspace{1cm} (68)

For the entire graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as Equation (71)

\[ \text{eig} \ (A - B) = \{2, 4\} \] \hspace{1cm} (69)

\[ \text{eig} \ (E) = \{0.0000, 2.0000, 3.1939, 4.0000, 1.0968, 5.7093\} \] \hspace{1cm} (70)

\[ \text{eig} \ (L(G)) = \{1.0968, 3.1939, 4.0000, 5.7093, 0.0000, 2.0000, 2, 4\} \] \hspace{1cm} (71)

As a result, the eigenvalues of this graph with two different nodal numberings that cause two different forms are the same.

6.3. Example 3

Consider the graph shown in Fig. 8. For this graph, there are two axes of symmetry that one of them passes through members (Fig. 8), and the other passes through nodes (Fig. 9). By tow different nodal numbering, we get to the canonical Form II and Form III for the Laplacian matrix.

The symmetry of Form II

The factors of this graph in Form II, according to Equation (19) are as follows
For the whole graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as Equation (75).

\[
eig(A - B) = \{0.2679, 1.2679, 2.0000, 3.0000, 3.2679, 3.7321, 4.7321, 5.0000, 6.7321\}
\]

(73)

\[
eig(A + B) = \{0.0000, 1.0000, 1.0000, 2.0000, 3.0000, 3.0000, 4.0000, 4.0000, 6.0000\}
\]

(74)

\[
eig(L(G)) = \{0.0000, 0.2679, 1.0000, 1.0000, 1.0000, 1.2679, 2.0000, 2.0000, 3.0000, 3.2679, 3.0000, 3.0000, 4.0000, 4.0000, 4.0000, 4.7321, 5.0000, 6.0000, 6.7321\}
\]

(75)

The symmetry of Form III

The factors of this graph in Form III, according to Equation (24) are as follows:
For the whole graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as

\[ \text{eig} (A - B) = \{1.0000, 1.2679, 2.0000, 3.0000, 4.0000, 4.7321\} \]  \hspace{1cm} (77)

\[ \text{eig} (E) = \left\{ \begin{array}{c} 6.7321, 6.0000, 0.0000, 0.2679, 5.0000, 1.0000, 4.0000, 2.0000, 3.7321, 3.2679, 3.0000, 3.0000 \end{array} \right\} \]  \hspace{1cm} (78)

The eigenvalues of the \( L(G) \) are calculated as the union of the above eigenvalues that are equal to the Equation (75).

**6.4. Example 4**

Here we consider a graph shown in Fig. 10. This example is taken from Kaveh and Sayarinejad (2005). For different nodal numbering of this graph, we have different forms for the Laplacian matrix.

**The symmetry of Form II**:  

This nodal numbering causes the canonical Form \( II \). According to Equation (19), for calculating the eigenvalues of the whole graph model, at first, the matrices \( A \) and \( B \) should be calculated. Then as Equation
(23), the eigenvalues of the whole graph can be calculated as the union of the eigenvalues for the matrices $(A - B)$ and $(A + B)$ as Equations (79) and (80)

$$eig (A - B) = \begin{cases} 0.58579, 0.85374, 1.58579, 2, 2.26795, 2.58579, 3, 3.41421, 3.58579, \\ 3.68216, 4, 4, 4.26795, 4.414421, 5, 5, 5.41421, 5.73205, 6, \\ 6.41421, 7, 7.14626, 7.73205 \end{cases}$$

$$eig (A + B) = \begin{cases} -1.05E -16, 0.26795, 0.58579, 0.85374, 1, 1.58579, 2, 2, 2.26795, \\ 2.58579, 3, 3, 3.41421, 3.58579, 3.68216, 3.73205, 4, 4.31784, 4.41421, \\ 5, 5.41421, 5.73205, 6.41421, 7.14626 \end{cases}$$

The eigenvalues of the whole graph are calculated as the union of the eigenvalues for its factors as

$$eig (L(G)) = \begin{cases} 0.58579, 0.85374, 1.58579, 2, 2.26795, 2.58579, 3, 3.41421, 3.58579, 3.68216, 4, 4, 4.26795, \\ 4.31784, 4.414421, 5, 5, 5.41421, 5.73205, 6, 6.41421, 7, 7.14626, 7.73205, -1.05E -16, \\ 0.26795, 0.58579, 0.85374, 1, 1.58579, 2, 2, 2.26795, 2.58579, 3, 3, 3.41421, 3.58579, \\ 3.68216, 3.73205, 4, 4.31784, 4.41421, 5, 5, 5.41421, 5.73205, 6.41421, 7.14626 \end{cases}$$

**The symmetry of Form III:**

Consider a graph model with the pattern in Form $III$, as shown in Fig. 11.

According to Equation (24), for calculating the eigenvalues of the entire graph model, at first, the matrices $A$, $B$, and $E$ should be calculated. Then as Equation (27), the eigenvalues of the considered graph is calculated as the union of the eigenvalues for the matrices $(A - B)$ and $E$

$$eig (A - B) = \begin{cases} 0.58579, 0.85374, 1.58579, 2, 2.26795, 2.58579, 3, 3.41421, 3.58579, \\ 3.68216, 4, 4, 4.31784, 4.414421, 5, 5, 5.41421, 5.73205, 6, \\ 6.41421, 7.14626 \end{cases}$$
The eigenvalues of the whole graph are now calculated as the union of the eigenvalues for its factors, and it is equal to Equation (81).

The symmetry of Form $IV$:

Consider a graph model with the pattern in Form $IV$, Fig. 12.

According to Equation (30), for calculating the eigenvalues of the entire graph model in Form $IV$, at first, the matrices $S$ and $H$ should be calculated. Then the eigenvalues of their linear combinations should be calculated as Equation (33).

$$
eig(E) = \left\{ -1.05 \times 10^{-16}, 0.26795, 0.58579, 0.85374, 1, 1.58579, 2, 2, 2.26795, \
2.58579, 3, 3, 3.41421, 3.58579, 3.68216, 3.73205, 4, 4, 4.26795, 4.31784, \
4.41421, 5, 5, 5.41421, 5.73205, 6, 6, 6.41421, 7, 7.14626, 7.732 \right\} \quad (83)$$

$$
The eigenvalues of the whole graph are now calculated as the union of the eigenvalues for its factors, and it is equal to Equation (81).

The symmetry of Form $IV$:

Consider a graph model with the pattern in Form $IV$, Fig. 12.

According to Equation (30), for calculating the eigenvalues of the entire graph model in Form $IV$, at first, the matrices $S$ and $H$ should be calculated. Then the eigenvalues of their linear combinations should be calculated as Equation (33).

$$
eig(S) = \{ 2, 2.58579, 2.58579, 4, 4, 5.41421, 5.41421, 6 \} \quad (84)$$

$$
eig(S - 2H) = \{ -1.25 \times 10^{-16}, 0.58578, 0.58578, 2, 2, 3.414214, 3.414214, 4 \} \quad (85)$$

$$
eig(S + H) = \{ 1, 1.58579, 1.58579, 3, 3, 4.41421, 4.41421, 5 \} \quad (86)$$

$$
eig(S - H) = \{ 3, 3.58579, 3.58579, 5, 5, 6.41421, 6.41421, 7 \} \quad (87)$$

$$
eig(S + \sqrt{3}H) = \left\{ 3.73205, 4.317837, 7.317837, 5.732051, \
5.732051, 7.14624, 7.14624, 7.73205 \right\} \quad (88)$$

$$
eig(S - \sqrt{3}H) = \left\{ 0.267949, 0.853736, 0.853736, 2.26795, \
2.26795, 3.68216, 3.68216, 4.26795 \right\} \quad (89)$$
The eigenvalues of the entire graph are now obtained as the union of the eigenvalues of its factors, which are equal to Equation (81).

**Form corresponding to rotationally repetitive symmetric structures:**

Consider a graph model with the pattern in the form associated with rotationally repetitive structures, Fig. 13.

\[
J_{6 \times 6} = \begin{bmatrix}
3 & -1 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & -1 & 3 \\
\end{bmatrix}
\]

(90)

\[
L_{6 \times 6} = L_{6 \times 6}^t = -I_{6 \times 6}
\]

(91)

\[
eig(H_{6 \times 6}) = eig(H_{6 \times 6}^t) = \{1, -1, (\frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i), \pm i, (\frac{-\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2} i)\}
\]

(92)

The eigenvalues of the entire graph are now obtained by replacing the \( J_{6 \times 6} \), \( L_{6 \times 6} \), \( L_{6 \times 6}^t \), \( eig(H_{6 \times 6}) \), and \( eig(H_{6 \times 6}^t) \) in the Equation (40), and in the end, it will be equal to Equation (83).

As you see, all these nodal numbering results in the same eigenvalues. The method of the nodal numbering of the canonical Forms II and III are harder than Form IV and the Form associated with rotationally repetitive structures.

**6.5. Example 5**

The finite element model of a dome is considered, as shown in Fig. 14. This model consists of 544 nodes and 2080 elements. Two different nodal numberings are assigned to this dome. The first nodal numbering
causes the canonical Form $II$, which is introduced for the bilateral symmetric structures. The second nodal numbering causes the form associated with rotationally repetitive structures. In a classic way, the Laplacians eigenvalues of the dome can be calculated by solving its polynomial equation. But with the new methods, the Laplacians eigenvalues of the entire dome can be obtained as the union of the eigenvalues of its factors in Form $II$ or in the form associated with rotationally repetitive structures. In the classic method, the eigenvalues of a $544 \times 544$ matrix are calculated.

In contrast, by the nodal numbering, which makes the matrix in Form $II$, the eigenvalues of the whole dome are obtained by the union of two $272 \times 274$ matrices. As a result of examining these two distinct methods, the computational time of the classic method is almost six times longer than that of Form $II$. Also, by the specific nodal numbering that makes the matrix in the form associated with rotationally repetitive structures, the eigenvalues are obtained by the union of 32 sets that are the $17\times17$ matrices' eigenvalues. The computational time of this form is almost four times shorter than that of Form $II$. All computational times are provided in Table 2. The computational time is calculated 100 times for each method, and finally, they are averaged.

6.6. Example 6

The finite element model of a cooling tower is considered, as shown in Fig. 15. This model consists of 900 nodes and 3510 elements. The specific nodal numbering is presented in Section 4.4, which causes the form associated with rotationally repetitive structures. In a classic way, the cooling tower's eigenvalues can be calculated by solving its polynomial equation and considering a $900 \times 900$ Laplacian matrix. On the other hand, with the unique nodal numbering that causes the form associated with rotationally repetitive structures, the entire tower's eigenvalues can be obtained as the union of 30 sets of the eigenvalues of $30 \times 30$ matrices. The computational time in the classic method is almost thirty times longer than in the new method, as shown in Table 3. The computational time is calculated 100 times for each method, and finally, they are averaged.
7. Concluding remarks

Symmetry results in the decomposition of the structures into smaller substructures. The matrices corresponding to the detached substructures have small dimensions in comparison with the dimensions of primary structures. In bilaterally symmetric structures, there are three kinds of symmetry. First, the axis of symmetry does not pass through members and nodes, Form $I$. Second, the axis of symmetry passes through members, Form $II$. Third, the axis of symmetry passes through nodes, Form $III$. In Form $II$, instead of finding the eigenvalues of an $n \times n$ matrix, one can calculate the eigenvalues of two $\frac{n}{2} \times \frac{n}{2}$ matrices. In Form $III$, instead of finding the eigenvalues of an $(n + k) \times (n + k)$ matrix, one can calculate the eigenvalues of an $\frac{n}{2} \times \frac{n}{2}$ and an $\left(\frac{n}{2} + k\right) \times \left(\frac{n}{2} + k\right)$ matrices. By decomposition of the rotationally repetitive structures into subsystems, large eigenproblems transform into much smaller problems. In fact, for a structure having $n$ rotationally repeating segments, instead of finding the eigenvalues of an $nm \times nm$ matrix, one can calculate the eigenvalues of $n$ number of $m \times m$ matrices. In structural mechanics, for calculating natural frequencies and buckling loads of vibrating regular systems, their matrices can be associated with one of the mentioned forms to reduce the computational load and time.

In the present paper, we proved that all these canonical forms for bilateral symmetry structures are interrelated. The only difference between them is the outward appearance of matrices associated with them, which is originated from the different methods of nodal numbering. Moreover, it is proved that all these forms are a special form of the form associated with rotationally repetitive structures.

Compliance with ethical standards:

Conflict of interest: No potential conflict of interest was reported by the authors.

References


Captions of the Figures

**Fig. 1.** A sample structure with the pattern in Form I

**Fig. 2.** A sample structure with the pattern in Form II

**Fig. 3.** A sample structure with the pattern in Form III

**Fig. 4.** A sample rotationally repetitive structure
Fig. 5. A sample of the symmetric frame with two DOFs for Example 1

Fig. 6. A sample graph model with the pattern in Form II for Example 2

Fig. 7. A sample graph model with the pattern in Form III for Example 2

Fig. 8. A sample graph model with the pattern in Form III for Example 3

Fig. 9. A sample graph model with the pattern in Form III for Example 3

Fig. 10. A sample graph model with the pattern in Form III for Example 4

Fig. 11. A sample graph model with the pattern in Form IV for Example 4

Fig. 12. A sample graph model with the pattern in Form IV for Example 4

Fig. 13. A sample graph model with the pattern in the form associated with rotationally repetitive structures for Example 4

Fig. 14. A dome for Example 5 (a) Three-dimensional view of the dome. (b) Top view of the dome

Fig. 15. A cooling tower for Example 6

Captions of the Tables

Table 1. Roots of the characteristic polynomial corresponding to the matrix $H$

Table 2. Comparison of the results for Example 5

Table 3. Comparison of the results for Example 6

**Table 1.** Roots of the characteristic polynomial corresponding to the matrix $H$

<table>
<thead>
<tr>
<th></th>
<th>Real roots</th>
<th>Complex roots</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$+1, -1$</td>
<td>$\cos \theta \pm i \sin \theta$ $\theta = \frac{2k \pi}{n}$</td>
</tr>
<tr>
<td>If $n$ is even</td>
<td></td>
<td>$k = 1, 2, 3, \ldots, \frac{n-2}{2}$</td>
</tr>
<tr>
<td>If $n$ is odd</td>
<td>$+1$</td>
<td>$\cos \theta \pm i \sin \theta$ $\theta = \frac{2k \pi}{n}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k = 1, 2, 3, \ldots, \frac{n-1}{2}$</td>
</tr>
</tbody>
</table>
Table 2.
Comparison of the results for Example 5

<table>
<thead>
<tr>
<th></th>
<th>Time (Sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Laplacian in classic form</td>
<td>10.4</td>
</tr>
<tr>
<td>The Laplacian in Form II</td>
<td>1.8</td>
</tr>
<tr>
<td>The Laplacian in rotationally repetitive form</td>
<td>0.44</td>
</tr>
</tbody>
</table>

Time ratio = \( \frac{\text{Form II}}{\text{rotationally repetitive form}} = 4.091 \)

Time ratio = \( \frac{\text{Classic}}{\text{Form II}} = 5.778 \)

Table 3.
Comparison of the results for Example 6

<table>
<thead>
<tr>
<th></th>
<th>Time (Sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Laplacian in classic form</td>
<td>17.2</td>
</tr>
<tr>
<td>The Laplacian in rotationally repetitive form</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Time ratio = \( \frac{\text{Classic}}{\text{rotationally repetitive form}} = 27.301 \)

Fig. 1. A sample structure with the pattern in Form I

Fig. 2. A sample structure with the pattern in Form II
Fig. 3. A sample structure with the pattern in Form III

Fig. 4. A sample rotationally repetitive structure

Fig. 5. A sample of the symmetric frame with two DOFs for Example 1

Fig. 6. A sample graph model with the pattern in Form II for Example 2
Fig. 7. A sample graph model with the pattern in Form III for Example 2

Fig. 8. A sample graph model with the pattern in Form II for Example 3

Fig. 9. A sample graph model with the pattern in Form III for Example 3

Fig. 10. A sample graph model with the pattern in of Form II for Example 4
Fig. 11. A sample graph model with the pattern in Form $III$ for Example 4

Fig. 12. A sample graph model with the pattern in Form $IV$ for Example 4

Fig. 13. A sample graph model with the pattern in the form associated with rotationally repetitive structures A cooling tower for Example 4
Fig. 14. A dome for Example 5 (a) Three-dimensional view of the dome. (b) Top view of the dome

Fig. 15. A cooling tower for Example 6

Ali Kaveh was born in 1948 in Tabriz, Iran. After graduation from University of Tabriz in 1969, he continued his studies on Structures at Imperial College of Science and Technology at London University, and received his M.Sc., DIC and Ph.D. in 1970 and 1974, respectively. He then joined the Iran University of Science and Technology. Professor Kaveh is the fellow of Iranian, World and European academies. He is the author of 690 international journal papers and 160 conference papers. He has authored 22 books in Persian and 15 books in English published mainly by Wiley and Springer. He is the editor-in-chief, editor and associate editor of 5 international journals.

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