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Canonical forms and rotationally repetitive matrices for eigensolution of symmetric structures

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Dome structures.

Abstract. In this paper, all canonical forms that exist in the literature for bilateral symmetry are derived from the formula for rotationally repetitive structures (systems) considering the rotation angle as 180 degrees. Different nodal numberings lead to different patterns for matrices associated with bilaterally symmetric structures. This study shows that all these forms are of the same nature and can be considered as particular forms of circulant matrices associated with rotationally repetitive structures. Simply put, some numerical examples are investigated using both the classic approach and the canonical forms.

1. Introduction

Eigenvalues and eigenvectors of matrices have a wide range of applications in different fields of engineering. In structural mechanics, these quantities are utilized for calculating natural frequencies of vibrating systems and buckling loads of structures [1,2]. Besides, these quantities are used in performing free vibration and buckling analysis of circular cylindrical shells in composite structures such as graphene-plates and graphene foam [3,4]. Using repeated eigenvalues, Choi et al. [5] developed a generalized steepest descent method for structural optimization.

Another application of eigenvalues and eigenvectors is in graph theory, which has a long history. Some applications of eigenvalues and eigenvectors in graph theory are nodal ordering, graph partitioning, and decomposition of finite element meshes for use in parallel computing [6]. The reader can refer to [7–9].

In algebraic graph theory, the eigenvalues and eigenvectors of Laplacian and adjacency matrices of graphs are used in understanding their characteristics. A significant contribution to algebraic graph theory was made by Fiedler [10], in which the second eigenvector of the Laplacian matrix of a graph could be applied to the nodal numbering for profile reduction, graph partitioning, and consequently, domain decomposition [11]. Further details can be found in such books as Biggs [12], Cvetkovic et al. [13], Seidel [14], Chung and Graham [15], and Godsil and Royle [16]. Chen et al. [17] introduced a systematic approach to deriving origami patterns employing graph theory and Mixed-Integer Linear Programming (MILP). They showed that the proposed method could be appropriate for developing origami patterns with vertices of degree 4 or degree 6.

In the past few decades, improvements in com-

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puter capacity and the availability of memories have shortened the computational time. However, fast computation of large matrix eigenvalue problems with high sparsity is still considered a challenging problem. Therefore, researchers use basic mathematical ideas for reducing the complexity of eigenproblems.

To accelerate the computation of eigenproblems, fast algorithms usually transform the matrix into simpler forms like diagonal, upper triangular, lower triangular matrices, or canonical forms. These transformations are known as "similarity transformations." Popular forms are *Jordan* form and *Schur* form. The term "similarity transformation" refers to a geometric similarity or matrix transformation in likeness. This study uses it with its second meaning.

The theory of groups has also been applied to computational problems in structural mechanics. General group-theoretic methods for many problems are developed, where specific equilibrium matrices and stiffness matrices are given in symmetry-adapted coordinate systems and block-diagonal forms [18,19]. Kangwai and Guest [20] employed some new techniques for a symmetric structure to block-diagonalize their equilibrium matrices. Symmetry mentioned in structural mechanics often refers to either mirror symmetry or cyclic symmetry [21]. However, many space structures are neither cyclically symmetric nor mirror-symmetric [22]. Many seemingly asymmetric structures retain some kind of symmetry operations [22,23]. Zingoni [24] presented a criterion for identifying the computationally efficient symmetry group and describing the symmetry properties of the system for a given problem in structures and solid mechanics. The efficiency of the criterion is shown by applying it to a cubic configuration with octahedral symmetry. To improve the formal symmetry method for the dynamism of kinematically indefinite pin-jointed structures, the idea of graph products was combined with the symmetry-extended mobility rule [25]. Cable structures are one of the popular elements that are used to design symmetric structures [26]. The group theory was employed to recognize the vibration modes, the number of modes, and the pairs of modes of the same natural frequency for double-layer cable nets to reduce the computational effort for calculating their eigenvalue [27,28].

Usually, symmetry occurs in engineering structures because of its ease of design and construction and aesthetic objects. In mathematics, "symmetry" means an invariant object under any of various transformations like reflecting, rotation, or scaling. A plane structure is symmetric with respect to an axis of symmetry on its plane if the reflection of the structure about the axis is identical in geometry, supports, and material properties to the structure itself. The post-buckling behavior of symmetric frames was studied by Plgnataro and Rizzi [29].

Kaveh and Sayarinejad [30] represented efficient and straightforward methods for calculating the eigensolution of a matrix with particular forms. They proved that one could calculate the eigensolutions of graph-based matrices by uniting the eigenvalues and eigenvectors of the submatrices. These submatrices were obtained by decomposing the graph-based matrix into small-dimension matrices. Also, they represented some methods for decomposing and healing the graph model of structures. According to these articles, the matrices associated with bilateral symmetric structures had canonical forms I, II, III, or IV. The applications of these forms in mass-spring systems, symmetric frames, and buckling loads of symmetric plane frames [31] were studied in other papers.

Kaveh and Rahami [31,32] presented other canonical forms for adjacency and Laplacian matrices associated with graph models and provided methods for decomposing regular structures. A structure is called 'regular' if its model can be considered a product graph [33]. Thomas [34] and Williams [35,36] presented some results concerning the vibration of cyclically symmetric structures. Kaveh and Nemati [37] found the eigensolutions for buckling load and natural frequencies in vibrating systems and utilized the canonical form from linear algebra known as the circulant matrix [38]. In this method, the structure is decomposed into repeated substructures and the eigenvalues and eigenvectors of the graph-based matrices are calculated by some simple operations on the eigenvalues and eigenvectors of the substructures. Kaveh et al. [39] utilized the algebraic graph theory for proposing a method of constructing preconditioners to apply to the entire design process of topology optimization. The efficiency of their approach was shown by applying it to the repetitive near-regular shell structures.

The different nodal numbering makes different patterns for matrices associated with bilateral symmetric structures. This study will show that all these forms can be considered individual cases of the form associated with rotationally repetitive structures. A few numerical examples are solved using the classic approach and the canonical forms to confirm the efficiency of the presented method.

This paper comprises six sections. A brief introduction containing the basic definitions from the theory of graphs and linear algebra is presented in Section 2. Section 3 discusses the matrices with canonical forms associated with bilateral symmetric structures and rotationally repetitive space structures. Different types of symmetry are discussed in Section 4. The proof of the relationship between the canonical forms of bilateral symmetric structures and rotationally repetitive structures is provided in Section 5. Examples are investigated in Section 6 and the concluding remarks are presented in Section 7.

2. Basic definitions

2.1. Basic concepts from the theory of graphs

A graph $G(N, M)$ comprises a set of nodes (vertices) and a set of elements members (edges) with a relation of incidence that relates a pair of nodes to each member. Two nodes are adjacent if they correspond to the same member. A graph G is called undirected if a member is an unordered pair of nodes.

2.2. Basic definitions from linear algebra

In linear algebra, two matrices, A and B, are called similar if:

$$B = P^{-1}AP, \tag{1}$$

where P is a nonsingular matrix and the transformation $A \rightarrow B$ is known as a similarity transformation of A.

The characteristic polynomial of a square matrix A is a polynomial $p(\lambda) = \det(A - \lambda I)$, which is invariant under matrix similarity and has the eigenvalues as roots. The scaler λ is an eigenvalue of A if there is a nonzero vector ν, known as an eigenvector, such that:

$$A\nu = \lambda\nu, \tag{2}$$

$$(\lambda I - A)\nu = 0. \tag{3}$$

The characteristic polynomial of a diagonal matrix A can easily be defined. If the diagonal entries of A are $a_1, a_2, a_3, \cdots, a_n$, then:

$$(a_1 - \lambda)(a_2 - \lambda)(a_3 - \lambda), \cdots, (a_n - \lambda) = 0, \tag{4}$$

will be the characteristic polynomial of A. According to Eq. (4), it can be seen that the diagonal entries are also the eigenvalues of this matrix. The simplest matrix for finding the eigenvalues is a diagonal matrix. Similarly, eigenvalues of a triangular matrix are its diagonal entries.

A matrix can have complex eigenvalues since its characteristic polynomial can have real or complex roots. Every $n \times n$ matrix has exactly n complex and real eigenvalues, counted with multiplicity.

2.3. Definitions from algebraic graph theory

The adjacency matrix $A = [a_{ij}]_{n \times n}$ of a graph G, with its nodes being labeled and containing n nodes, is defined as:

$$a_{ij} = \begin{cases} 1 & \text{if node } n_i \text{ is adjacent to } n_j \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The degree matrix $D = [d_{ij}]_{n \times n}$ is a diagonal matrix containing node degrees where d_{ii} is the degree of the ith node.

Laplacian matrix $L = [l_{ij}]_{n \times n}$ is defined as:

$$L = D - A. \tag{6}$$

Therefore, the entries of L areas:

$$l_{ij} = \begin{cases} -1 & \text{if node } n_i \text{ is adjacent to } n_j \\ \deg(n_i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

For the eigensolution of the adjacency matrix, consider the eigenproblem as follows:

$$A\nu_i = \lambda_i \nu_i, \tag{8}$$

where λ_i is the ith eigenvalue and ν_i is the corresponding eigenvector. For A being a real symmetric matrix, all the corresponding eigenvalues are real given as Eq. (9):

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n. \tag{9}$$

The largest eigenvalue λ_n has multiplicity equal to unity for the characteristic equation of A. The corresponding eigenvector ν_n is the only eigenvector with positive entries. This vector has many interesting properties employed in structural mechanics.

For the eigensolution of the Laplacian matrix, consider the problem as follows:

$$L\nu_i = \lambda_i \nu_i, \tag{10}$$

where λ_i is the ith eigenvalue and ν_i is the corresponding eigenvector. Since A is a real symmetric matrix and all its eigenvalues are real, all the eigenvalues of L are also real. It can be shown that the matrix L is a positive semidefinite matrix with:

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n, \tag{11}$$

$$\nu_1 = \{1, 1, 1, \cdots, 1\}^t. \tag{12}$$

The second eigenvalue λ_2 of L is also known as "algebraic connectivity" of a graph, and its eigenvector ν_2 is called the Fiedler vector. This vector has attractive properties.

The **Kronecker product** of two matrices is an operation on these matrices, which results in a block matrix. This operation is denoted by $\otimes$.

The Kronecker product of two matrices $A_{m \times n}$ and $B_{p \times q}$ is the $mp \times nq$ block matrix as:

$$A_{m \times n} \otimes B_{p \times q} = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}_{mp \times nq}. \tag{13}$$

A **Hermitian matrix** is a square matrix that is equal to its conjugate transpose. A real matrix is Hermitian if and only if it is symmetric.

Being Hermitian is a necessary condition for a matrix to be diagonalizable:

$$A \text{ is Hermitian} \Leftrightarrow a_{ij} = \overline{a_{ji}}. \tag{14}$$

For two matrices A_1 and A_2 to be diagonalizable simultaneously, these two matrices should be Hermitian and commutative $A_1A_2 = A_2A_1$.

3. Canonical forms of a matrix

In this section, all the canonical forms available for bilateral symmetry structures and rotationally repetitive structures are presented. Herein, M is a matrix whose elements entirely consist of real numbers. Canonical Forms I, II, III, and IV were represented by Kaveh and Sayarinejad [30].

3.1. Form I

In this form, M is a block diagonal matrix having a pattern shown in Eq. (15):

$$M_{2n} = \begin{bmatrix} A_n & 0 \\ 0 & A_n \end{bmatrix}_{2n\times 2n}. \tag{15}$$

Since M is a diagonal matrix, its determinant can be calculated as:

$$\det(M) = \det(A) \times \det(A). \tag{16}$$

Moreover, the set of eigenvalues of submatrix A is λ_A. Therefore, the eigenvalues of M can be calculated as Eq. (17).

$$\lambda_M = \{\lambda_A\} \cup \{\lambda_A\}. \tag{17}$$

This form can be generalized by placing a large number of matrices with different dimensions on its main diagonal entries. Based on Eq. (17), eigenvalues of this generalized matrix will be equal to the union of the eigenvalues of those matrices located on its main diagonal.

3.2. Form II

For this particular form, the matrix M has a pattern, as shown in Eq. (18):

$$M_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix}_{2n\times 2n}. \tag{18}$$

By adding the second column to the first one and then, subtracting the first row from the second one, M transforms to the upper triangular matrix as follows:

$$M_{2n} = \begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix} = \begin{bmatrix} C_n & B_n \\ 0 & D_n \end{bmatrix}, \tag{19}$$

in which:

$$C = [A] + [B], \tag{20}$$

$$D = [A] - [B]. \tag{21}$$

Since M is a triangular matrix, its determinant is equal to the multiplication of two matrices that are located on the diagonal of M. Therefore, the set of eigenvalues of M can be obtained as Eq. (22):

$$\lambda_M = \{\lambda_C\} \cup \{\lambda_D\}. \tag{22}$$

3.3. Form III

In this form, M has an $N \times N$ submatrix, where $N = 2n$. The pattern of Form II is augmented by k rows and k columns as formulated in Eq. (23):

$$M = \begin{bmatrix} A & B & P \\ B & A & P \\ Q & H & R \end{bmatrix}. \tag{23}$$

As seen, M is an $(N + k) \times (N + k)$ matrix, and the entries of augmented columns are the same in each column. Similar to Form II, in this case, M can be factored using row and column permutation.

By exchanging the second column and second row with the fourth column and fourth row, respectively, the fourth column is added to the first one. At the last step, by reducing the first row from the fourth one, M transforms into the upper triangular matrix as Eq. (24):

$$M = \begin{bmatrix} A+B & P & B \\ Q+H & R & H \\ 0 & 0 & A-B \end{bmatrix} = \begin{bmatrix} E & K \\ 0 & D \end{bmatrix}, \tag{24}$$

where:

$$E = \begin{bmatrix} A+B & P \\ Q+H & R \end{bmatrix}, \tag{25}$$

$$D = [A] - [B]. \tag{26}$$

Since M is a triangular matrix, its determinant will be equal to the multiplication of two matrices that are located on diagonal entries. Therefore, the set of eigenvalues of M can be obtained as Eq. (27):

$$\lambda_M = \{\lambda_E\} \cup \{\lambda_D\}. \tag{27}$$

3.4. Form IV

M is a $6n \times 6n$ matrix with the following pattern:

$$M=\begin{bmatrix} S-H & H-S & & & & \\ H-S & S & -H & & & \\ & -H & S & H-S & & \\ & & H-S & S & -H & \\ & & & -H & S & H-S \\ & & & & H-S & S-H \end{bmatrix}_{6n\times 6n}. \tag{28}$$

S and H are $n \times n$ matrices. The characteristic polynomial of M can be calculated through Eq. (29):

$$P_M(\lambda) =[\lambda(2H - 2S + \lambda)] \left[\lambda^2 - 2S\lambda + SH - H^2\right] \left[\lambda^2 - 2S\lambda + 3SH - 3H^2\right]. \tag{29}$$

The first term of Eq. (29) can be considered as the characteristic polynomial of the matrix that is a matrix of Form II Eq. (30):

$$[E_1] = \begin{bmatrix} S-H & H-S \\ H-S & S-H \end{bmatrix} = \lambda_{(2S-2H)} \cup \lambda_{[0]_{n\times n}}. \tag{30}$$

The second term of Eq. (29) can be considered as the characteristic polynomial of the matrix in Eq. (31):

$$[E_2] = \begin{bmatrix} S+H & -S \\ H-S & S-H \end{bmatrix}. \tag{31}$$

The third term of Eq. (29) can be considered as the characteristic polynomial of the matrix in Eq. (32):

$$[E_3] = \begin{bmatrix} 2S & 3H \\ H-S & 0 \end{bmatrix}. \tag{32}$$

Hence, for finding the eigenvalues of a matrix in the form of Eq. (28), calculating the eigenvalues of E_i for $i = 1, 2$ and 3 is sufficient:

$$\lambda_M = \{\lambda_{E_1}\} \cup \{\lambda_{E_2}\} \cup \{\lambda_{E_3}\}. \tag{33}$$

3.5. A canonical form associated with rotationally repetitive structures

An efficient eigensolution was developed for determining the buckling load and free vibration of rotationally cyclic structures by Kaveh and Nemati [37]. This solution applies a canonical form, which is often involved in graph model matrices. This canonical form is presented in this section.

For rotationally repetitive structures, one can associate a canonical form as follows:

$$M_{mn} = \begin{bmatrix} J & L & & & L^t \\ L^t & J & L & \cdots & \\ & L^t & J & & \\ & \vdots & & \ddots & \vdots \\ L & & & \cdots & \begin{matrix} J & L \\ L^t & J \end{matrix} \end{bmatrix}, \tag{34}$$

where the matrix M is a symmetric block matrix, with $n \times n$ blocks. The blocks of this matrix are $J_m \times m$, $L_m \times m$, and $L^T_m \times m$. Thus, this matrix generally has $nm \times nm$ entries. Block J is located on the main diagonal and blocks L and L^T are located on the upper and lower adjacent diagonals and also in the lower-left corner and the upper right corner, respectively. Matrix M can be decomposed to the sum of three Kronecker products as follows:

$$M_{mn} = I_n \otimes J_m + H_n \otimes L_m + H^t_n \otimes L^t_m, \tag{35}$$

where I is an $n \times n$ identity matrix and H is an $n \times n$ asymmetric matrix as:

$$H_n = \begin{bmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & \cdots & \\ & 0 & 0 & & \\ & \vdots & & \ddots & \vdots \\ 1 & & & \cdots & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{bmatrix}. \tag{36}$$

Since H is an asymmetric matrix, block diagonalization of M requires some considerations.

Here, H is a permutation matrix and hence, it is orthogonal. Thus, we have Eq. (37):

$$HH^t = 1. \tag{37}$$

Matrix H and its transpose are characterized by commutative property as in Eq. (38):

$$HH^t = H^t H. \tag{38}$$

According to Eqs. (37) and (38), the two matrices H and H^t can be diagonalized simultaneously. Now, by using a matrix such as $U = X \otimes I$, matrix M can be diagonalized as in Eq. (39):

$$\begin{aligned} U^{-1}MU &= U^{-1}\left(I_n \otimes J_m + H_n \otimes L_m + H^t_n \otimes L^t_m\right) U \\ &= (X \otimes I)^{-1}\left(I \otimes J + H \otimes L + H^t \otimes L^t\right)(X \otimes I) \\ &= \left(X^{-1} \otimes I^{-1}\right)\left(I \otimes J + H \otimes L + H^t \otimes L^t\right)(X \otimes I) \\ &= \left(X^{-1} \otimes J + X^{-1}H \otimes L + X^{-1}H^t \otimes L^t\right)(X \otimes I) \\ &= \left(I \otimes J + X^{-1}HX \otimes L + X^{-1}H^t X \otimes L^t\right). \end{aligned} \tag{39}$$

Since a similarity transformation (see Subsection 2.2) is employed, the eigenvalues do not change. In Eq. (39), I is a diagonal matrix and it is sufficient to show that X simultaneously diagonalizes H and H^t.

The eigenvalues of the matrix M can be found by using the union of the eigenvalues of n blocks as Eq. (40):

$$\lambda_M = \cup^n_{k=1} \mathrm{eig}\left(J_m + \lambda_i(H_n)L_m + \lambda_i(H^t_n)L^t_m\right). \tag{40}$$

Since the eigenvalues of the matrix H are needed for finding the eigenvalues of matrix M, let us find the eigenvalues of H. H's characteristic polynomial can be written as follows:

$$\lambda^n - 1 = 0. \tag{41}$$

This equation has n real and complex roots that are identical to those in Eq. (42):

$$\cos(n\theta) + i\sin(n\theta) = 1. \tag{42}$$

Obviously, if n is even or odd, then -1, 1, or 1 will be the only real roots of Eq. (41), respectively. The complex and real roots are presented in Table 1.

Based on Table 1, for n being even or odd, two real roots and $(n-2)$ complex roots or one real and $(n-1)$ complex roots will be calculated, respectively.

4. Different kinds of symmetry

In this section, for different forms of the matrices associated with the symmetric structures, the corresponding graphs are introduced [6].

Table 1. Roots of the characteristic polynomial corresponding to the matrix H.

	Real roots	Complex roots
If n is even	$+1, -1$	$\cos\theta \pm i\sin\theta \quad \theta = \frac{2k\pi}{n}$ $k = 1, 2, 3, \cdots, \frac{n-2}{2}$
If n is odd	$+1$	$\cos\theta \pm i\sin\theta \quad \theta = \frac{2k\pi}{n}$ $k = 1, 2, 3, \cdots, \frac{n-1}{2}$

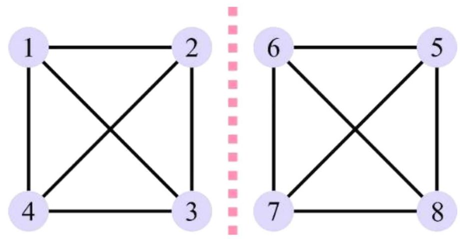


Figure 1. A sample structure with the pattern in Form I.

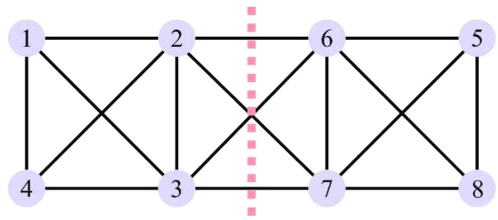


Figure 2. A sample structure with the pattern in Form II.

4.1. Form I symmetry

Here, the symmetry axis does not pass through nodes and members. The model is a disjoint graph with at least two distinct subgraphs, as shown in Figure 1.

After nodal numbering of one of the substructures, the nodal numbers of the second substructure should be determined considering the symmetry of nodes, as numbered in Figure 1.

4.2. Form II symmetry

In this case, the model has an even number of nodes and the axis of symmetry passes through members; for instance, a sample structure of Form II is drawn in Figure 2.

The structure is divided into two equal substructures. After nodal numbering of one of the substructures, the nodal numbers of the second substructure should be determined considering the symmetry of nodes, as shown in Figure 2.

4.3. Form III symmetry

In this case, the axis of symmetry passes through members and nodes (see Figure 3). The structure is divided into two equal substructures. Firstly, the nodal numbering of one of the substructures except those located on the symmetry axis is performed. The nodal numbers of the second substructure should

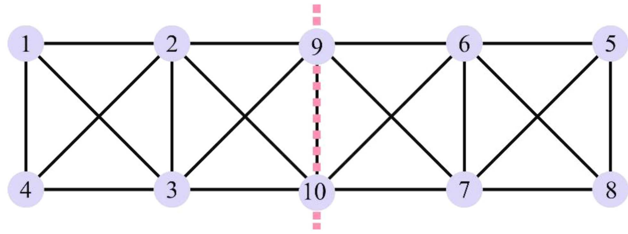


Figure 3. A sample structure with the pattern in Form III.

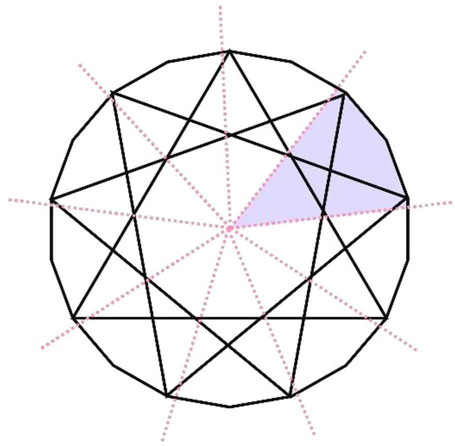


Figure 4. A sample rotationally repetitive structure.

be determined considering the symmetry of nodes. Finally, the labels of the nodes located on the symmetry axis are assigned, as labeled in Figure 3.

The number of the rows and columns added to the core of Form II is equal to the number of nodes through which the axis of symmetry passes. This means that in the $(N+K) \times (N+K)$ matrix M, there are K nodes through which the axis of symmetry passes.

4.4. The form associated with rotationally repetitive structures

A rotationally repetitive structure is a structure consisting of a cyclically symmetric configuration with an angle of cyclic symmetry equal to θ, as shown in Figure 4.

A rotationally repetitive structure should be divided by some imaginary lines or surfaces into $n = \frac{2\pi}{\theta}$ segments. The imaginary boundaries may not pass through any node so that the segment to which a given node belongs can be uniquely determined.

For nodal numbering, the difference between the number of an arbitrarily selected node in an arbitrary segment and the number of the corresponding node in the adjacent segment should be constant.

5. The relations between the canonical forms

5.1. The relation between Forms I and II

According to Eqs. (15) and (18), Form I is a particular form of Form II with the matrix B equal to zero.

$$M_{2n}=\begin{bmatrix}A_n & B_n\\ B_n & A_n\end{bmatrix}_{N\times N}\xrightarrow{B=0} M_{2n}=\begin{bmatrix}A_n & 0\\ 0 & A_n\end{bmatrix}_{N\times N}. \tag{43}$$

5.2. The relation between Form II and Form III

In this section, an attempt is made to show that the canonical Form III can be obtained from the canonical Form II, according to Kaveh [6]. To this end, first, the following points should be considered.

Consider a $2N \times 2N$ matrix L; then, add one arbitrary row and one arbitrary zero column to the matrix L as in Eq. (44):

$$C=\begin{bmatrix}L_{2N\times 2N} & 0\\ X & 0\end{bmatrix}_{(2N+1)\times(2N+1)}. \tag{44}$$

Accordingly, all entries in the $(2N+1)$th column of the matrix C are zero, the eigenvalues of the matrix C are equal to the union of the set of eigenvalues of matrix L and a set with one zero member. This can be proven as follows.

The first $2N$ rows of C are multiplied by $k_1, k_2, \cdots, k_{2N}$, respectively, and the sum is equated to zero. Since any multiple of the last column will be zero; therefore, Eq. (45) is obtained:

$$KL + X = 0. \tag{45}$$

If L is invertible (i.e., if $\det(L) \neq 0$), then $K = -XL^{(-1)}$ and $k_1, k_2, \cdots, k_{2N}$ can be found, and the last row of C becomes zero. However, if $\det(L) = 0$, then there are many sets of k_i that put the last row of C into zero. Therefore, one can conclude that there is always at least one transformation that makes the last row of C zero.

If the kth row of a matrix is multiplied by m and the kth column is divided by m, the eigenvalues of this matrix remain unchanged. The reason is that the magnitude of the diagonal entry stays constant, and if it is expanded with respect to the row and column, the determinant of the submatrices remains unaltered.

In the following, an algorithm is provided for transforming the canonical Form III to Form II:

Step 1. A zero column and a zero row in the second column and second row are added as in the following expression:

$$\text{Form III}=\begin{bmatrix}A & B & P\\ B & A & P\\ Q & H & R\end{bmatrix}\Rightarrow\begin{bmatrix}A & 0 & B & P\\ 0 & 0 & 0 & 0\\ B & 0 & A & P\\ Q & 0 & H & R\end{bmatrix}. \tag{46}$$

Step 2. $\frac{H-Q}{2}$ and $\frac{Q-H}{2}$ in the first and third entries of the second row are added. The sum of these entries is equal to zero; hence, adding these two entries does not affect the eigenvalues of the matrix.

$$\begin{bmatrix}A & 0 & B & P\\ 0 & 0 & 0 & 0\\ B & 0 & A & P\\ Q & 0 & H & R\end{bmatrix}\Rightarrow\begin{bmatrix}A & 0 & B & P\\ (H-Q)/2 & 0 & (Q-H)/2 & 0\\ B & 0 & A & P\\ Q & 0 & H & R\end{bmatrix}. \tag{47}$$

Step 3. Half of the fourth column is added to the first column and half of the fourth row is added to the second row. Then, the second interchange column is added to column 4.

$$\begin{bmatrix}A & 0 & B & P\\ (H-Q)/2 & 0 & (Q-H)/2 & 0\\ B & 0 & A & P\\ Q & 0 & H & R\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}A & P/2 & B & P\\ (H-Q)/2 & 0 & (Q-H)/2 & 0\\ B & P/2 & A & P\\ Q & R/2 & H & R\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}A & P/2 & B & P\\ H/2 & R/4 & Q/2 & R/2\\ B & P/2 & A & P\\ Q & R/2 & H & R\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}A & P & B & P/2\\ H/2 & R/2 & Q/2 & R/4\\ B & P & A & P/2\\ Q & R & H & R/2\end{bmatrix}. \tag{48}$$

Step 4. Column 4 is multiplied by 2 and row 4 is multiplied by 1/2, resulting in Eq. (49):

$$\begin{bmatrix}A & P & B & P/2\\ H/2 & R/2 & Q/2 & R/4\\ B & P & A & P/2\\ Q & R & H & R/2\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}A & P & B & P\\ H/2 & R/2 & Q/2 & R/2\\ B & P & A & P\\ Q/2 & R/2 & H/2 & R/2\end{bmatrix}\Rightarrow\begin{bmatrix}M & N\\ N & M\end{bmatrix}$$

$$= \text{Form II}, \tag{49}$$

where:

$$M+N=\begin{bmatrix}A+B & 2P\\ (Q+H)/2 & R\end{bmatrix},$$

$$M-N=\begin{bmatrix}A-B & 0\\ (H-Q)/2 & 0\end{bmatrix}. \tag{50}$$

In matrix $M+N$, multiplying the second column by 1/2 and multiplying the second row by 2 result in E as in Eq. (51):

$$E=\begin{bmatrix}A+B & P\\ Q+H & R\end{bmatrix}. \tag{51}$$

$$\det(M - I\lambda) = \begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & -H & 0 & 0 & 0 \\ 0 & -H & S-\lambda & H-S & 0 & 0 \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & 0 & 0 & 0 & H-S & S-H-\lambda \end{bmatrix} = 0. \tag{52}$$

Box I

$$\begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & -H & 0 & 0 & 0 \\ 0 & -H & S-\lambda & H-S & 0 & 0 \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & 0 & 0 & 0 & H-S & S-H-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & 0 & 0 & 0 & -H \\ 0 & -H & 0 & 0 & H-S & S-\lambda \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & S-H-\lambda & H-S & 0 & 0 \end{bmatrix}. \tag{53}$$

Box II

The right-hand side of matrix $M-N$ in Eq. (50) has the same eigenvalues as those of $A-B$ except for $2N$ extra zeros. E and D are the same matrices as defined in Form III in the previous section.

5.3. The relation between Form IV and Form III

In this section, it is shown that the canonical Form IV is a special form of the canonical Form III. For this purpose, the characteristic polynomial of the canonical Form IV is considered as in Eq. (52) shown in Box I.

Now, by using the elementary matrix operation, we have an algorithm that transforms the canonical Form IV to Form III.

The algorithm is provided through the following steps:

Step 1. The third column is swapped with the sixth one and the fourth column with the fifth one Eq. (53) is shown in Box II.

Step 2. The third row is swapped with the sixth one and the fourth row with the fifth one Eq. (54) is shown in Box III.

Step 3. The sixth column is replaced by itself plus the first and second columns and the fifth column is replaced by itself plus the third and fourth columns Eq. (55) is shown in Box IV.

Step 4. The first row is replaced by itself minus the sixth row, the second row is replaced by itself minus the sixth row, the third row is replaced by itself minus the fifth row, and the fourth row is replaced by itself minus the fifth row Eq. (56) is shown in Box V.

Step 5. The first and second rows are multiplied by -1 and the first and second columns are multiplied by -1 Eq. (57) is shown in Box VI.

The characteristic polynomial of the canonical Form IV could be transformed into a characteristic polynomial that is in the pattern of the canonical Form III as in Eq. (58), shown in Box VII.

From Eq. (59), shown in Box VIII, it can be concluded that the canonical Form IV is a special form of Form III.

$$\begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & 0 & 0 & 0 & -H \\ 0 & -H & 0 & 0 & H-S & S-\lambda \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & S-H-\lambda & H-S & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & 0 & 0 & 0 & -H \\ 0 & 0 & S-H-\lambda & H-S & 0 & 0 \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-\lambda \end{bmatrix}. \tag{54}$$

Box III

$$\begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & 0 \\ H-S & S-\lambda & 0 & 0 & 0 & -H \\ 0 & 0 & S-H-\lambda & H-S & 0 & 0 \\ 0 & 0 & H-S & S-\lambda & -H & 0 \\ 0 & 0 & 0 & -H & S-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-\lambda \end{bmatrix} \Rightarrow \begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & -\lambda \\ H-S & S-\lambda & 0 & 0 & 0 & -\lambda \\ 0 & 0 & S-H-\lambda & H-S & -\lambda & 0 \\ 0 & 0 & H-S & S-\lambda & -\lambda & 0 \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix}. \tag{55}$$

Box IV

$$\begin{bmatrix} S-H-\lambda & H-S & 0 & 0 & 0 & -\lambda \\ H-S & S-\lambda & 0 & 0 & 0 & -\lambda \\ 0 & 0 & S-H-\lambda & H-S & -\lambda & 0 \\ 0 & 0 & H-S & S-\lambda & -\lambda & 0 \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} S-H-\lambda & 2H-S & 0 & 0 & S-H & H-S \\ H-S & S+H-\lambda & 0 & 0 & S-H & H-S \\ 0 & 0 & S-H-\lambda & 2H-S & H-S & S-H \\ 0 & 0 & H-S & S+H-\lambda & H-S & S-H \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix}. \tag{56}$$

Box V

$$\begin{bmatrix} S-H-\lambda & 2H-S & 0 & 0 & S-H & H-S \\ H-S & S+H-\lambda & 0 & 0 & S-H & H-S \\ 0 & 0 & S-H-\lambda & 2H-S & H-S & S-H \\ 0 & 0 & H-S & S+H-\lambda & H-S & S-H \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & -H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} S-H-\lambda & 2H-S & 0 & 0 & H-S & S-H \\ H-S & S+H-\lambda & 0 & 0 & H-S & S-H \\ 0 & 0 & S-H-\lambda & 2H-S & H-S & S-H \\ 0 & 0 & H-S & S+H-\lambda & H-S & S-H \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix}. \tag{57}$$

Box VI

$$\begin{bmatrix} S-H-\lambda & 2H-S & 0 & 0 & H-S & S-H \\ H-S & S+H-\lambda & 0 & 0 & H-S & S-H \\ 0 & 0 & S-H-\lambda & 2H-S & H-S & S-H \\ 0 & 0 & H-S & S+H-\lambda & H-S & S-H \\ 0 & 0 & 0 & -H & S-H-\lambda & H-S \\ 0 & H & 0 & 0 & H-S & S-H-\lambda \end{bmatrix} = \mathrm{eig}(\text{Form } III-\lambda I)=0, \tag{58}$$

Box VII

$$\begin{bmatrix} S-H & 2H-S & 0 & 0 & H-S & S-H \\ H-S & S+H & 0 & 0 & H-S & S-H \\ 0 & 0 & S-H & 2H-S & H-S & S-H \\ 0 & 0 & H-S & S+H & H-S & S-H \\ 0 & 0 & 0 & -H & S-H & H-S \\ 0 & H & 0 & 0 & H-S & S-H \end{bmatrix} = \begin{bmatrix} A & B & P \\ B & A & P \\ Q & H & R \end{bmatrix} = \text{Form } III. \tag{59}$$

Box VIII

5.4. *The relation between the canonical Form II and the form associated with rotationally repetitive structures*

If there is a bilaterally symmetric structure, we have a substructure that by repeating it twice ($n = 2$), the whole structure can be obtained. It is implied that the rotation angle is 180°. We use $n = 2$ in Table 1 and replace the results in Eqs. (34) and (40).

$$n = 2 \text{ is even } \rightarrow \lambda_i(H_2^t) = \lambda_i(H_2) = -1, +1,$$

$$H_2 = H_2^t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\lambda_i(H_n^t) = \lambda_i(H_n), \tag{60}$$

$$\begin{aligned} M_{m2} =& I_2 \otimes J_m + H_2 \otimes L_m + H_2^t \otimes L_m^t \\ =& \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix} + \begin{bmatrix} 0 & L \\ L & 0 \end{bmatrix} + \begin{bmatrix} 0 & L^t \\ L^t & 0 \end{bmatrix} \\ =& \begin{bmatrix} J & L+L^t \\ L+L^t & J \end{bmatrix} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}. \end{aligned} \tag{61}$$

According to Eq. (61), the mentioned form is similar to what we have in the bilateral symmetric structures (Form II). Then, it can be concluded that a bilaterally symmetric structure is a special form of a rotationally repetitive structure.

From Eq. (59), it becomes clear that Form IV is a special type of Form III. According to Eqs. (43) and (49), the canonical Form III and Form I are a special form of the canonical Form II. From Eq. (61), Form II is a special form of the canonical form and it was presented for rotationally repetitive structures as in Eq. (34). As a result, it appears that all canonical forms that are presented for the bilateral symmetric structures are a special form of the rotationally repetitive structures.

6. Examples

This section shows that all forms associated with a particular structure with different nodal numbering cause the same eigenvalue for the whole structure. On the other hand, all these forms can be compared with each other in terms of time, memory, and the method of nodal numbering. $L(G)$ is the Laplacian matrix of an entire graph G which is represented in each example. These examples are programed in MATLAB R2016b software and processed in a computer with Intel® Core$^{\text{TM}}$i7_4510U CPU @ v2.00 GHz processor and 8.00 GB RAM.

6.1. *Example 1*

Consider the symmetric frame that has only one symmetry axis, as shown in Figure 5. As can be seen, it is constrained against a motion in the sway direction and has only two rotation degrees of freedom. Both mass and stiffness matrices of the mentioned frame are in Form II. To determine its natural frequencies, Eq. (62) must be solved. The natural frequencies are equal to the eigenvalues of the factors of Form II, as expressed in Eq. (63):

$$\det\left[K - \lambda^2 M\right] = 0, \tag{62}$$

$$\lambda_C = \sqrt{\frac{420EI}{mL^4}}, \qquad \lambda_D = \sqrt{\frac{420EI}{11mL^4}}. \tag{63}$$

6.2. *Example 2*

For the graph shown in Figure 6, there are two axes

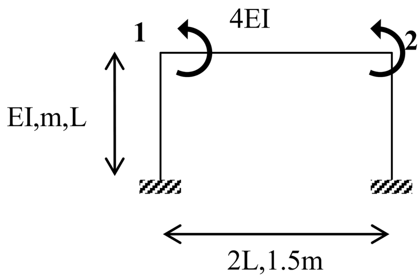


Figure 5. A sample of the symmetric frame with two DOFs for Example 1.

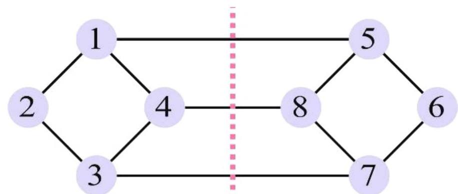


Figure 6. A sample graph model with the pattern in Form II for Example 2.

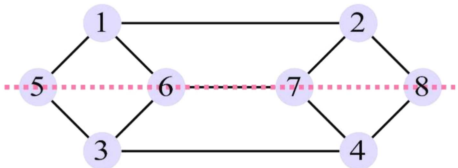


Figure 7. A sample graph model with the pattern in Form III for Example 2.

of symmetry one of which passes through members (Figure 6) and the other passes through nodes (Figure 7). By two different nodal numberings, we get to the canonical Forms II and III for the Laplacian matrix.

The symmetry of Form II:
The factors of this graph in Form II, according to Eq. (19), are given below:

$$A=\begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix},$$

$$B=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{64}$$

The eigenvalues of the whole graph are now calculated as the union of the eigenvalues for its factors as Eq. (67):

$$eig(A-B)=\{1.0968, 3.1939, 4.0000, 5.7093\}, \tag{65}$$

$$eig(A+B)=\{0.0000, 2.0000, 2.0000, 4.0000\}, \tag{66}$$

$$eig(L(G))=\{1.0968, 3.1939, 4.0000, 5.7093, 0.0000, 2.0000, 2.0000, 4.0000\}. \tag{67}$$

The symmetry of Form III
Consider a graph shown in Figure 7. The factors of this graph in Form III, according to Eq. (24), are as follows:

$$A-B=\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix},$$

$$E=\begin{bmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & -1 & -1 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 3 & -1 & 0 \\ 0 & -2 & 0 & -1 & 3 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{68}$$

For the entire graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as Eq. (71):

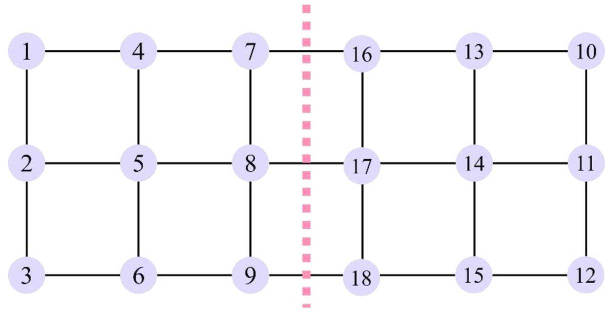


Figure 8. A sample graph model with the pattern in Form II for Example 3.

$$eig(A-B)=\{2,4\}, \tag{69}$$

$$eig(E)=\{0.0000, 2.0000, 3.1939, 4.0000, 1.0968, 5.7093\}, \tag{70}$$

$$eig(L(G))=\{1.0968, 3.1939, 4.0000, 5.7093, 0.0000, 2.0000, 2, 4\}. \tag{71}$$

As a result, the eigenvalues of this graph with two different nodal numberings that cause two different forms are the same.

6.3. Example 3

Consider the graph shown in Figure 8. For this graph, there are two axes of symmetry one of which passes through members (Figure 8) and the other passes through nodes (Figure 9). With two different nodal numberings, we get to the canonical Forms II and III for the Laplacian matrix.

The symmetry of Form II
The factors of this graph in Form II, according to Eq. (19), are as follows:

$$A=\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{bmatrix},$$

$$B=\begin{bmatrix} 0 & 0 & & 0 & \\ 0 & 0 & & 0 & \\ & & -1 & & \\ 0 & 0 & & -1 & \\ & & & & -1 \end{bmatrix}. \tag{72}$$

For the whole graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as in Eq. (75):

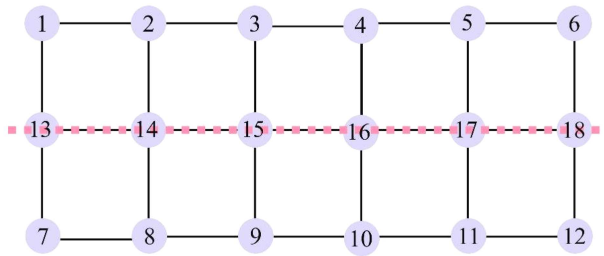


Figure 9. A sample graph model with the pattern in Form III for Example 3.

$$eig(A-B) = \{0.2679, 1.2679, 2.0000, 3.0000, 3.2679, 3.7321, 4.7321, 5.0000, 6.7321\}, \tag{73}$$

$$eig(A+B) = \{0.0000, 1.0000, 1.0000, 2.0000, 3.0000, 3.0000, 4.0000, 4.0000, 6.0000\}, \tag{74}$$

$$eig(L(G)) = \{0.0000, 0.2679, 1.0000, 1.0000, 1.2679, 2.0000, 2.0000, 3.0000, 3.2679, 3.0000, 3.0000, 4.0000, 4.0000, 4.7321, 5.0000, 6.0000, 6.7321\}. \tag{75}$$

The symmetry of Form III:
The factors of this graph in Form III, according to Eq. (24), are as follows:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$B = [0]_{6\times 6}, \qquad P = H = Q = -I_{6\times 6},$$

$$R = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 \end{bmatrix}. \tag{76}$$

For the whole graph, the eigenvalues are calculated as the union of the eigenvalues for its factors as follows:

$$eig(A-B) = \{1.0000, 1.2679, 2.0000, 3.0000, 4.000, 4.7321\}, \tag{77}$$

$$eig(E) = \{6.7321, 6.0000, 0.0000, 0.2679, 5.0000, 1.0000, 4.0000, 2.0000, 3.7321, 3.2679, 3.0000, 3.0000\}. \tag{78}$$

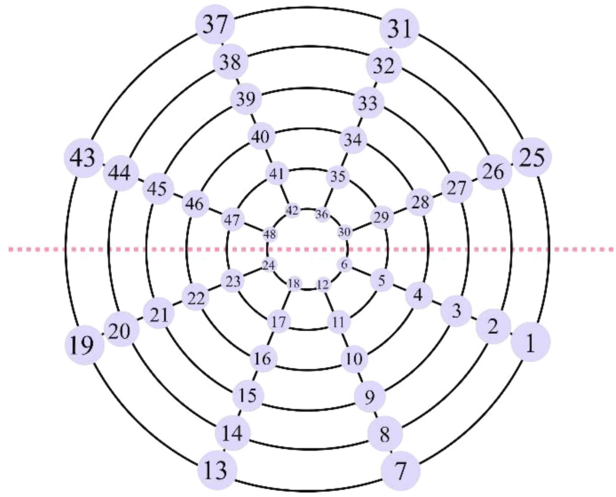


Figure 10. A sample graph model with the pattern in of Form II for Example 4.

The eigenvalues of the $L(G)$ are calculated as the union of the above eigenvalues that are equal to Eq. (75).

6.4. Example 4

Here, a graph is considered, as shown in Figure 10. This example is taken from the study of Kaveh and Sayarinejad [40]. For different nodal numberings of this graph, we have different forms for the Laplacian matrix.

The symmetry of Form II:
This nodal numbering causes the canonical Form II. According to Eq. (19), for calculating the eigenvalues of the whole graph model, at first, the matrices A and B should be calculated. Then, the eigenvalues of the whole graph can be calculated through Eq. (23) as the union of the eigenvalues for the matrices $(A-B)$ and $(A+B)$ as Eqs. (79) and (80):

$$eig(A-B) = \{0.58579, 0.85374, 1.58579, 2, 2.26795, 2.58579, 3, 3.41421, 3.58579, 3.68216, 4, 4, 4.26795, 4.31784, 4.414421, 5, 5, 5.41421, 5.73205, 6, 6.41421, 7, 7.14626, 7.73205\}, \tag{79}$$

$$eig(A+B) = \{-1.05E-16, 0.26795, 0.58579, 0.85374, 1, 1.58579, 2, 2, 2.26795, 2.58579, 3, 3, 3.41421, 3.58579, 3.68216, 3.73205, 4, 4.31784, 4.41421, 5, 5.41421, 5.73205, 6.41421, 7.14626\}. \tag{80}$$

The eigenvalues of the whole graph are calculated as the union of the eigenvalues for its factors as follows:

$$eig(L(G)) = \{0.58579, 0.85374, 1.58579, 2,$$

$$2.26795, 2.58579, 3, 3.41421, 3.58579, 3.68216,$$
$$4, 4, 4.26795, 4.31784, 4.414421, 5, 5, 5.41421,$$
$$5.73205, 6, 6.41421, 7, 7.14626, 7.73205,$$
$$-1.05E-16, 0.26795, 0.58579, 0.85374, 1,$$
$$1.58579, 2, 2, 2.26795, 2.58579, 3, 3, 3.41421,$$
$$3.58579, 3.68216, 3.73205, 4, 4.31784, 4.41421,$$
$$5, 5.41421, 5.73205, 6.41421, 7.14626\}. \quad (81)$$

The symmetry of Form III:
Consider a graph model with the pattern in Form III, as shown in Figure 11.

According to Eq. (24), for calculating the eigenvalues of the entire graph model, at first, the matrices A, B, and E should be calculated. Then, through Eq. (27), the eigenvalues of the considered graph are calculated as the union of the eigenvalues for the matrices $(A-B)$ and E.

$$eig(A-B) = \{0.58579, 0.85374, 1.58579, 2,$$
$$2.26795, 2.58579, 3, 3.41421, 3.58579, 3.68216,$$
$$4, 4, 4.31784, 4.414421, 5, 5.41421, 5.73205,$$
$$6.41421, 7.14626\}, \quad (82)$$

$$eig(E) = \{-1.05E-16, 0.26795, 0.58579, 0.85374,$$
$$1, 1.58579, 2, 2, 2.26795, 2.58579, 3, 3, 3.41421,$$
$$3.58579, 3.68216, 3.73205, 4, 4, 4.26795, 4.31784,$$
$$4.41421, 5, 5, 5.41421, 5.73205, 6, 6.41421, 7,$$
$$7.14626, 7.732\}. \quad (83)$$

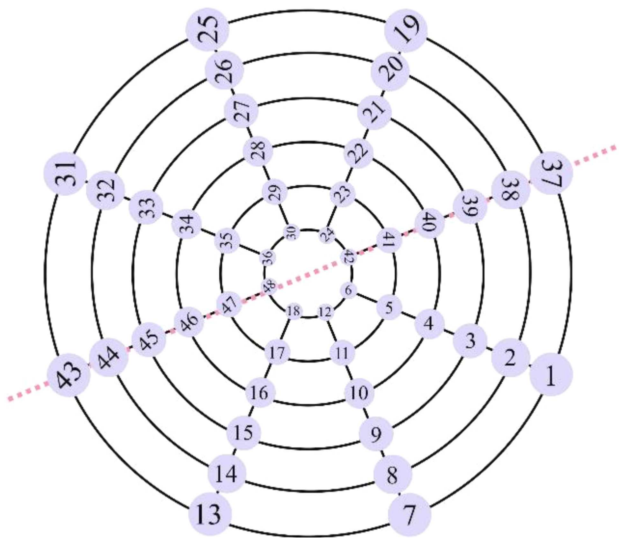


Figure 11. A sample graph model with the pattern in Form III for Example 4.

The eigenvalues of the whole graph are now calculated as the union of the eigenvalues for its factors and it is equal to Eq. (81).

The symmetry of Form IV:
Consider a graph model with the pattern in Form IV, Figure 12.

According to Eq. (30), for calculating the eigenvalues of the entire graph model in Form IV, at first, the matrices S and H should be calculated. Then, the eigenvalues of their linear combinations should be calculated using Eq. (33).

$$eig(S) = \{2, 2.58579, 2.58579, 4, 4, 5.4142, 5.4142, 6\}, \quad (84)$$

$$eig(S-2H) = \{-1.25E-16, 0.58578, 0.58578,$$
$$2, 2, 3.414214, 3.414214, 4\}, \quad (85)$$

$$eig(S+H) = \{1, 1.58579, 1.58579, 3, 3, 4.41421,$$
$$4.41421, 5\}, \quad (86)$$

$$eig(S-H) = \{3, 3.58579, 3.58579, 5, 5, 6.41421,$$
$$6.41421, 7\}, \quad (87)$$

$$eig\left(S+\sqrt{3}H\right) = \{3.73205, 4.317837, 7.317837,$$
$$5.732051, 5.732051, 7.14624, 7.14624, 7.73205\}, (88)$$

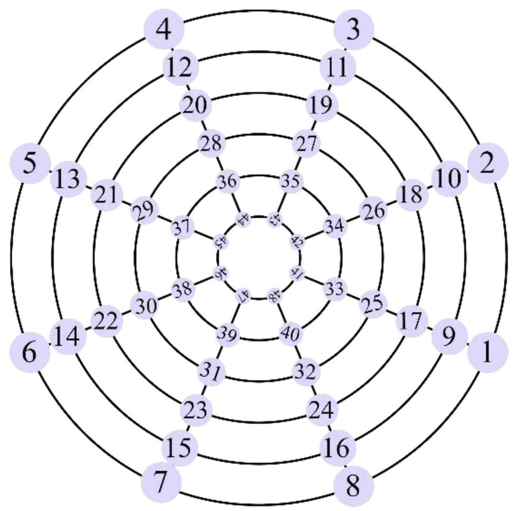


Figure 12. A sample graph model with the pattern in Form IV for Example 4.

$$eig\left(S-\sqrt{3}H\right)=\{0.267949, 0.853736, 0.853736,$$

$$2.26795, 2.26795, 3.68216, 3.68216, 4.26795\}. \quad (89)$$

The eigenvalues of the entire graph are now obtained as the union of the eigenvalues of its factors, which are equal to the outcome of Eq. (81).

Form corresponding to rotationally repetitive symmetric structures:

Consider a graph model with the pattern in the form associated with rotationally repetitive structures (Figure 13).

$$J_{6\times6}=\begin{bmatrix}3&-1&0&0&0&0\\-1&4&-1&0&0&0\\0&-1&4&-1&0&0\\0&0&-1&4&-1&0\\0&0&0&-1&4&-1\\0&0&0&0&-1&3\end{bmatrix}, \quad (90)$$

$$L_{6\times6}=L^t_{6\times6}=-I_{6\times6}, \quad (91)$$

$$eig(H_{6\times6})=\mathrm{eig}(H^t_{6\times6})$$

$$=\left\{1,-1,\left(\frac{\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\right),\pm i,\left(\frac{-\sqrt{2}}{2}\pm\frac{\sqrt{2}}{2}i\right)\right\}. \quad (92)$$

The eigenvalues of the entire graph are now obtained by replacing $J_{6\times6}$, $L_{6\times6}$ $L^t_{6\times6}$, $eig(H_{6\times6})$, and $eig(H^t_{6\times6})$ in Eq. (40) and in the end, they will be equal to the outcome of Eq. (83).

As is clear, all these nodal numberings result in the same eigenvalues. The method of the nodal numbering of the canonical Forms II and III is harder than Form IV and the form associated with rotationally repetitive structures.

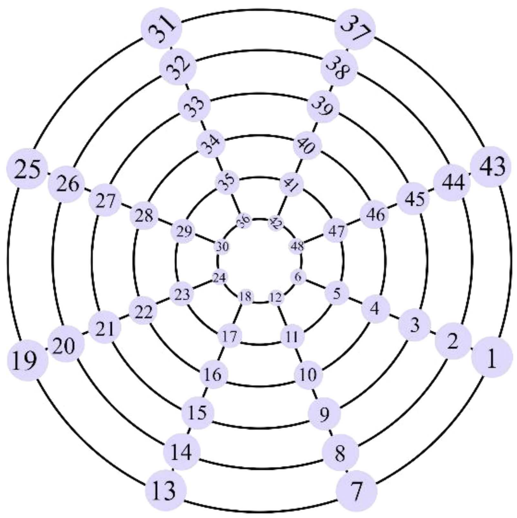


Figure 13. A sample graph model with the pattern in the form associated with rotationally repetitive structures for Example 4.

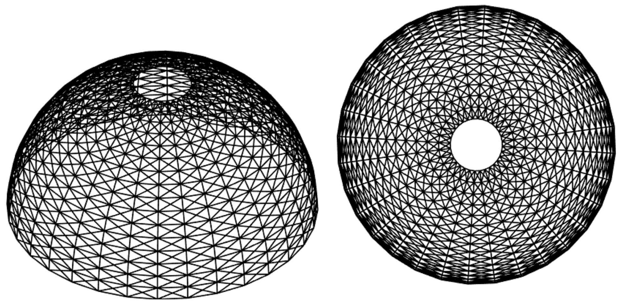
6.5. Example 5

The finite element model of a dome is considered, as shown in Figure 14. This model consists of 544 nodes and 2080 elements. Two different nodal numberings are assigned to this dome. The first nodal numbering causes the canonical Form II, which is introduced for the bilateral symmetric structures. The second nodal numbering causes the form associated with rotationally repetitive structures. In a conventional fashion, the Laplacian eigenvalues of the dome can be calculated by solving the corresponding polynomial equation. However, by using the new methods, the Laplacian eigenvalues of the entire dome can be obtained as the union of the eigenvalues of its factors in Form II or in the form associated with rotationally repetitive structures. In the classic method, the eigenvalues of a 544×544 matrix are calculated.

In contrast, through nodal numbering that makes the matrix in Form II, the eigenvalues of the whole dome are obtained by a union of two 272×274 matrices. As a result of examining these two distinct methods, the computational time of the classic method is almost six times longer than that of Form II. Also, by the specific nodal numbering that makes the matrix in the form associated with rotationally repetitive structures, the eigenvalues are obtained by the union of 32 sets that are equal to the eigenvalues of 17×17 matrices. The computational time of this form is almost four times shorter than that of Form II. All computational times are provided in Table 2. The computational time(s) is calculated 100 times for each method and finally, they are averaged.

6.6. Example 6

As shown in Figure 15, the finite element model of a cooling tower is considered. This model consists of 900 nodes and 3510 elements. The specific nodal numbering is presented in Section 4.4, which creates the form associated with rotationally repetitive structures. In a conventional fashion, the eigenvalues of the cooling tower can be calculated by solving its polynomial equation and considering a 900×900 Laplacian matrix. On the other hand, with unique nodal numbering that



(a) (b)

Figure 14. A dome for Example 5: (a) Three-dimensional view of the dome and (b) top view of the dome.

Table 2. Comparison of the results for Example 5.

	Time (sec)
The Laplacian in the classic form	10.4
The Laplacian in Form II	1.8
The Laplacian in the rotationally repetitive form	0.44
Time ratio $= \frac{\text{Form II}}{\text{rotationally repetitive form}} = 4.091$	Time ratio $= \frac{\text{Classic}}{\text{Form II}} = 5.778$

Table 3. Comparison of the results for Example 6.

	Time (sec)
The Laplacian in the classic form	17.2
The Laplacian in the rotationally repetitive form	0.63
Time ratio $= \frac{\text{classic}}{\text{rotationally repetitive form}} = 27.301$	

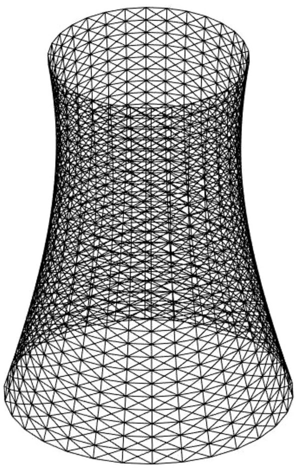


Figure 15. A cooling tower for Example 6.

causes the form associated with rotationally repetitive structures, the eigenvalues of the entire tower can be obtained as the union of 30 sets of the eigenvalues of 30×30 matrices. The computational time in the classic method is almost thirty times longer than that in the new method, as shown in Table 3. The computational time(s) is calculated 100 times for each method and finally, they are averaged.

7. Concluding remarks

Symmetry results in the decomposition of structures into smaller substructures. The matrices corresponding to the detached substructures have smaller dimensions than the dimensions of primary structures. In bilaterally symmetric structures, there are three kinds of symmetry. First, the axis of symmetry does not pass through members and nodes, Form I. Second, the axis of symmetry passes through members, Form II. Third, the axis of symmetry passes through nodes, Form III. In Form II, instead of finding the eigenvalues of an $n \times n$ matrix, one can calculate the eigenvalues of two $\frac{n}{2} \times \frac{n}{2}$ matrices. In Form III, instead of finding the eigenvalues of an $(n+k) \times (n+k)$ matrix, one can calculate the eigenvalues of an $\frac{n}{2} \times \frac{n}{2}$ and an $(\frac{n}{2} + k) \times (\frac{n}{2} + k)$ matrices. With the decomposition of the rotationally repetitive structures into subsystems, large eigenproblems transform into much smaller problems. In fact, for a structure with n rotationally repetitive segments, instead of finding the eigenvalues of an $nm \times nm$ matrix, one can calculate the eigenvalues of n number of $m \times m$ matrices. In structural mechanics, to calculate natural frequencies and buckling loads of vibrating regular systems, their matrices can be associated with one of the mentioned forms to reduce the computational load and time.

In the present paper, it is proved that all these canonical forms for bilateral symmetry structures are interrelated. The only difference between them is the outward appearance of matrices associated with them, which is originated from the different methods of nodal numbering. Moreover, it is proved that all these forms represent special forms associated with rotationally repetitive structures.

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