The business advantage of identifying and solving pseudo-continuous-integer periodical linear problems

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Abstract

Many applications of optimisation require the final value of the decision variables to be integer. In many cases the relaxed optimal solution

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does not satisfy the integrality constraint therefore, the problem must be solved by integer or mix-integer programming algorithms at a significant computational effort and most likely a worsen objective function value. The contribution of this paper is twofold: The identification of a type of problems in which the relaxed optimal objective function value can be kept in the implementation by a change in the planning horizon; and the identification of a multi-period based solution procedure. Three small instances are provided in order to illustrate the methodology as well as the economic impact involved. In addition, a fourth industrial size case is included for the benefit of practitioners. This work shows that business profit can be increased for pseudo-continuous-integer periodical linear problems by identifying optimal decision-making periods.

KEYWORDS

Business profit; Integer programming; linear programming; operations management; operations planning.

1 Introduction

In the competitive business environment, practitioners are interested in solutions that can be obtained and implemented in a simple way. They require that the generated solutions provide better profits (or lower costs) in order to promote business growth and competitiveness.

Linear, integer and mixed integer programming models are a flourishing field of optimisation. Nowadays, they are applied to an immense variety of real life problems in a number of disciplines [1; 2; 3; 4; 5; 6; 7; 8; 9; 10]; thanks to them efficient solutions were developed and improved through decades; and these are still progressing quickly. For linear models the well-known simplex method derived by [11], the interior point method developed by [12] and later improved by [13] are used to solve this class of problems. For integer programming models, there exists algorithms and methods such as branch and bound [14], cutting planes [15] and branch and cut[16]. It is important to remark that the integer and mixed integer programming models are complex problems and many of them are NP-hard. Therefore, the computational complexity is high with often long computational time requirements.

The process to obtain the solution of integer and mix integer models is far more complex than of simple linear programming ones. A particular network problem called transshipment has the unique characteristic that if all demands are integer, all vertex in the feasible region are integers, therefore the solution of the network simplex [11; 17] is integer without considering the problem as an integer model.

Often there is a loss in the objective function value when a problem is transformed from continuous to integer (or mix-integer). Hence, the search to reduce this loss is a continuous concern for practitioners and academics. There is scant treatment in the operational research academic literature in this regard, therefore this concern becomes the main motivation of this paper.

The contribution of this article is twofold: a) to identify a special class of problems which solution must be integer but the continuous solution is not, however an integer solution can be obtained by identifying special characteristics of the application; b) to provide a method to convert the continuous solution into an integer solution (that can be implemented) without loosing value in the objective function for these kind of problems.

This paper is organised as follows: Section 2 presents a class of problems (to be named pseudo-continuous-integer periodical linear problems) in which its special characteristics are the focus of this work. Section 1 includes the methodology to transform a continuous solution from this class of problems into an integer one. Section 4 presents three small numerical instances to illustrate the method, while the fourth one represents an industrial size case for the benefit of practitioners. Finally, Section 5 provides conclusions and further research.

2 Special characteristics of pseudo-continuous-integer periodical linear problems

Let us define a pseudo-continuous-integer periodical linear problem (PCIPLP) as one that satisfies the following five characteristics:

1. Pure integer programming problems with no binary variables.
2. Single period planning horizon that repeat identically over a non limited number of consecutive periods.
3. The time to make decisions can be transformed from every period to once every T periods, where T is an integer number to be determined.

4. Between two consecutive periods, fractional values (resources, demands, etc.) can be conveyed.

5. The objective function of the problem can change from a fixed periodical number to an average per period.

Many special applications in practice satisfy the above conditions, ranging from service management (public and private), transportation, production, order acceptance [18] and manufacturing, among others. Some illustrations are included in section 4.

3 Methodology

Let us consider an integer problem that meets the characteristics of the latter section. Let the integer one period problem be:

$$\begin{aligned} &\text{Opt } \ c^t x \\ \text{s.t.} \quad & \\ &Ax = b \\ &x \in Z^{n+} \end{aligned} \tag{1}$$

where $A \in R^{m \times n}$, $c \in R^n$, $b \in R^m$, $x \in Z^{n+}$ and Z^{n+} be the n dimensional set of non-negative integers.

The relaxed linear model associated to (1) is:

$$\text{Opt } c^t y$$

s.t.

$$Ay = b \tag{2}$$

$$y \geq 0$$

where $y \in R^n$. Let $I = \{1, , n\}$ be the index set, and y^* be the optimal solution of (2), with $y_i^* = d_i/e_i$, where $d_i, e_i \in Z^+ \ \forall i \in I$. It is important to remark that all optimal linear programming solutions involve rational numbers because of the computational nature of the algorithms [19]. Let T be the minimum common multiple of all e_i.

Multiplying by the scalar T both sides of the constraint $Ay = b$ by in (2), and making $w_i = Ty_i, \forall i \in I$. Then the one period model in (2) is transformed into a T periods model as:

$$\text{Opt } c^t w \tag{3}$$

s.t.

$$Aw = Tb \tag{4}$$

$$w \geq 0 \tag{5}$$

where $w \in R^n$, (3) represents the objective function value taking into account T periods and (4) represent the technical constraints for a T period problem.

Notice that the model in (2) is mathematically equivalent to (3) through (5) but the latter happens to have integer solutions for all its variables. Since both problems share the same A, c and $T > 0$ they share the same optimal basis (basic columns of matrix A) and the value for the dual variables. Thus, the sensitivity linear programming tools apply to (3) through (5). Hence, sensitivity analysis interpretations are valid on (3) through (5) as longs as the resultant solution remains with all integer values. The latter could be a challenging task.

Besides the problem in (3) through (5) has only a change in the time horizon of (1) as long as the latter satisfies all the characteristics listed in Section 2.

As a matter of fact all the constraints in (4) has just been moved parallel-wise their limits by the factor T from $Ay = b$ in (2).

The optimal solution of (3) through (5) is given by :

$$w_i^* = \left(\frac{T}{e_i}\right) d_i, \forall i \in I. \tag{6}$$

where $\frac{T}{e_i} \in Z^+$, thus $w_i^* \in Z^+$ is an optimal solution to (3) through (5).

Notice that the problem in (3) through (5) is a regular linear programming problem which happens to have integer solutions for all its variables. Hence, all sensitivity analysis techniques apply as long as the new solution remains integer.

In summary a problem that satisfies the conditions of a PCIPLP in Section 2, is re-formulated from a single period to a T period planning horizon with optimal solution as in (6) by solving the linear programming continuous problem in (2).

The pseodu-code in Table 1 transform the planning horizon for decision making from one period to T periods for PCIPLPs.

4 Numerical illustrations

Four PCIPLPs cases are presented in this section. Three of them are small instances to illustrate the methodology in simple terms, while the last one represents an industrial size order acceptance case to show that the methodology can be applied to larger problems.

4.1 A public service management instance

A small city of 15,000 inhabitants consumes an average of 1,200,000 litres of drinkable water per day. The city obtains water from the central purifying facility where water is treated by conventional filtration and chloro-hydration methods. In addition, two chemical compounds (softener and purifier) are included. The city is evaluating two potential suppliers of these chemical compounds. Supplier A offers packages with 4 kg of softener and 1.5 kg of purifier for $80 a package. Supplier B offers packages with 2 kg of softener and 4.5 kg of purifier for $100 a package. In order to keep water drinkable the city facility requires 75 kg of softener and 50 kg of purifier per day. The objective is to provide the daily levels of softener and purifier at minimum cost for the city [20].

Let x_A and x_B be the number of packages per day to buy from each supplier.

The relaxed linear programming model for the daily decision is:

$$
\begin{aligned}
& \min \quad 80y_A + 100y_B \\
& \text{s.t} \\
\text{Softener:} \quad & 4y_A + 2y_B \geq 75 \\
\text{Purifier:} \quad & 1.5y_A + 4.5y_B \geq 50 \\
& y_A, y_B \geq 0
\end{aligned} \tag{7}
$$

with optimal continuous solution $y_A = 95/6$ and $y_B = 35/6$ and optimal daily cost of \$ 1,850.00.

The city can not buy fractional packages from the suppliers. If the integrality constraint is added for both variables, the solution changes to $x_A = 16$ and $x_B = 6$ with optimal solution of \$ 1,880.00, these represents and increment of \$ 30.00 per day which is approximately 1.62% of increase in the daily cost.

Since the daily requirements of softener and purifier are fixed and the minimum common multiple in the denominator of the decision variables is $T = 6$, the city can buy every six days 95 packages (w_A^*) from supplier A and 35 packages (w_B^*) from supplier B.

Because the packages have separated containers for every element (softener and purifier), the city has to measure every day 75 kg of softener and 50 kg of purifier and apply it to the city water supply. The comprehensive purchase will cost \$ 11,100.00 every six days meanwhile the average daily cost remains at its minimum at \$ 1,850.00.

The problem in (7) is the numerical version of this problem of (2), while the following problem (8), is equivalent of (3) through (5):

$$
\begin{aligned}
& \min \quad 80w_A + 100w_B \\
& \text{s.t} \\
\text{Softener T:} \quad & 4w_A + 2w_B \quad \geq 75 \times 6 = 450 \\
\text{Purifier T:} \quad & 1.5w_A + 4.5w_B \quad \geq 50 \times 6 = 300 \\
& w_A, w_B \geq 0
\end{aligned} \tag{8}
$$

Figure 1 shows the three solutions x^*, y^* and w^*. One can notice that the feasible region of (8) is limited by the constraints Softener T and Purifier T, has just re-scaled (moved parallel-wise) the boundaries of both constraints by a factor of $T = 6$ from the initial problem in (7), limited by the original constraints Softener and Purifier.

Regarding sensitivity analysis, the dual variables of (8) are: Softener T $= -14$ and Purifier T $= -16$ (both integers). Thus, if one right hand side is moved, it must be in such a way that the new solution remains integer. Or one can multiply both right hand sides by multiples of $T = 6$ and keep the same dual variables and an integer solution.

In addition the dual variables of (7) are: Softener $= -14$ and Purifier $= -16$ which have the same values than the ones in (8).

4.2 A production mix problem

A small manufacturing facility produces two products. Three machines are used in the manufacturing process, each with 44 available hours per week. The single period is considered as a week.

Table 2 shows operational information, were columns 2-4 represents the number of manufacturing hours to produce a unit of each product. The last column in the table represents the marginal contribution obtained per product.

The problem consists of finding the optimal production mix to maximise the sum of marginal contributions obtained by the production plan. In practice is not possible to manufacture and deliver a fraction of a unit.

Let x_i represents the number of units of product i to be produce per week, where $i = 1, 2$. The relaxed linear programming model follows:

$$\begin{aligned} & \max \quad 50y_1 + 105y_2 \\ & \text{s.t} \\ \text{Compression:} \quad & 0.5y_1 + 0.9y_2 \leq 44 \\ \text{Cut:} \quad & 0.49y_1 + 0.7y_2 \leq 44 \\ \text{Polish:} \quad & 0.21y_1 + 0.39y_2 \leq 44 \\ & y_1, y_2 \geq 0 \end{aligned} \tag{9}$$

The optimal continuous solution is $y_1 = a_1/b_1 = 0$, $y_2 = a_2/b_2 = 440/9 = 48 + 8/9 \approx 48.8888...$, with an objective function value of $\$5,133 + 1/3$. Thus,

$T = 9$.

If the integrality constrain is added to (9), the solution transforms to $x_1 = 1$, $x_2 = 48$, with objective function value $\$5,090.00$. This integer solution translates to \$ 43.33... less per period.

Considering that the original period of the problem is a week, and that the manufacturing facility works for an undetermined number of weeks. A practical solution is to manufacture 48 units of product two, at the end of the first 44 hour week the 49-th unit will be $8/9 \approx 0.8888$ (88.88%) finished. Assuming that the process can be stop at no loss of any kind, and resume in the next consecutive period, at the beginning of the second period the 49-th unit (which needs only 1/9 of the work) is considered to continue the process until finishing it, and the manufacturing process continues. In this way in eight out of nine weeks the manufacturing delivers 49 units of product 2 and in one out of nine weeks only 48 units of product 2 will be delivered. Hence, in 9 weeks 440 units of product two are delivered, which is an integer number.

In this way the average production per week is $440/9 \approx 48.88$ units of product two and the average objective function value is \$ 5,133.33 per week with all constraints met. This is possible because an inventory of partially-finished units could be conveyed from one period to the next. Notice that the five characteristics of PCIPLPs are met.

This process can be extended to any number of products and resources under the same modelling.

Notice, that the proposed solution is neither the solution for (9) nor for the integer version of (9), but satisfies the practitioner requirements in periods of nine weeks.

4.3 An airline opening route decision

An airline company is analysing the opening of four new routes to be assigned to their newly adquired fleet. Tables 3, 4 and 5 show the related data [20]. The relaxed model is defines as:

$$
\begin{array}{rcl}
\min & 250,000y_{1,1} + 280,000y_{1,2} + 120,000y_{1,3} + 80,000y_{1,4} + & \\
& 245,000y_{2,1} + 400,000y_{2,2} + 125,000y_{2,3} + 200,000y_{2,4} + & \\
& 252,000y_{3,1} + 315,000y_{3,2} + 162,000y_{3,3} + 156,000y_{3,4} + & \\
& 2,500nfp_1 + 3,000nfp_2 + 2,800nfp_3 + 2,950nfp_4 & \\
\text{s.t} & & \\
\text{Fleet 1} & y_{1,1} + y_{1,2} + y_{1,3} + y_{1,4} & \leq 5 \\
\text{Fleet 2} & y_{2,1} + y_{2,2} + y_{2,3} + y_{2,4} & \leq 3 \\
\text{Fleet 3} & y_{3,1} + y_{3,2} + y_{3,3} + y_{3,4} & \leq 2 \\
\text{Seats route 1} & 1,000y_{1,1} + 1,050_{y2,1} + 1,080y_{3,1} + nfp_1 - es_1 & = 2,500 \\
\text{Seats route 2} & 800y_{1,2} + 750y_{2,2} + 840y_{3,2} + nfp_2 - es_2 & = 2,000 \\
\text{Seats route 3} & 600y_{1,3} + 750y_{2,3} + 720y_{3,3} + nfp_3 - es_3 & = 2,200 \\
\text{Seats route 4} & 400y_{1,4} + 600y_{2,4} + 720y_{3,4} + nfp_4 - es_4 & = 1,800 \\
& y_{i,j}, nfp_j, es_i & \geq 0 \\
& \forall i = 1,2,3 \text{ and } j = 1,2,3,4 &
\end{array}
$$

where $y_{i,j}$ represents the number of airplanes from fleet i to be assigned to fly route j per a day; nfp_j defines the number of passengers (per day) under demand for route j (passenger not flown); and es_i is equal to the number of empty seats flown in a day for the j-th route.

The optimal continuous solution is: $y_{1,1} = 5/2$, $y_{1,2} = 5/2$, $y_{2,3} = 44/15$, $y_{2,4} = 1/15$, $y_{3,4} = 2$, and $nfp_4 = 320$ as seen in Table 6, which shows the non-zero elements of the solution. The minimum daily cost is $\$2,961,000.00$. If the integrality constraint is included the objective function rises to $\$4,259,000.00$, i.e. approximately an increase in the daily cost of 43.84%.

From Table 6 the minimum common multiple of $\{2, 15\}$ is $T = 30$. Taking into account that the flight plan contemplates 30 days now, Table 7 shows a solution for that period, maintaining a 30 day cost of $ 88,830,000.00 i.e. an average daily cost at $\$2,961,000.00$.

4.4 Industrial size order acceptance case

An automotive Original Equipment Manufacturer (OEM) is asked to quote orders (products) for a new automotive platform. The potential contract includes a non determined long number of periods. The OEM has manufacturing technical capabilities to quote ninety three orders (m=93). Each order to be quoted is expected to be manufactured on a highly specialised manufacturing cell. For the purpose of this case, the manufacturing cell can be considered as a single machine. This manufacturing cell works three shifts of eight hours each. The manufacturing cell utilisation factor is 85 percent, making 36,720 working minutes available per month (available time, AT= 36,720).

Table 8, shows the monthly demand for each order, set up time (in minutes), the marginal contribution of each unit in usd, the set up cost in usd, and the manufacturing standard time per unit, respectively. Five main raw materials (m=5) are needed for every unit in each order. The current monthly availability of these raw materials are: $RM = \{9,000; 8,000; 7,000; 6,000; 3,000\}$. The unitary requirement of each raw material is also included in Table 8.

The relaxed mathematical formulation of the problem in a general form follows:

$$\begin{aligned}
& \max \ \textstyle\sum_{i \in I} MC_i y_i - \sum_{i \in I} SUC_i a_i \\
\text{s.t} & \\
\text{Demand} \quad & y_i \leq a_i d_i, \forall i \in I \\
\text{Time availability} \quad & \textstyle\sum_{i \in I} (SUT_i a_i + ST_i y_i) \leq AT \\
\text{Row material availability} \quad & \textstyle\sum_{i \in I} c_{i,j} y_i \leq RM_j, \forall j \in J \\
& a_i \in B, x_i \in R, \forall i \in I
\end{aligned} \tag{10}$$

where $I = \{1, , n\}$ is the set of n orders, $J = \{1, , m\}$ is the set of m raw materials, $d_i, MC_i, SUC_i, SUT_i, ST_i$, represent demand, marginal contribution, set up cost, set up time for the $i-th$ order respectively; $c_{i,j}$ defines the number of units from raw material j that each product in order i requires, AT states the available manufacturing time per period and RM_j contains the availability of the $j-th$ raw material per period. The decision variables y_i defines the number of units to accept from the $i-th$ order, while a_i are auxiliary variables to model the set ups.

If all orders at full demand are accepted an operational profit (marginal contribution minus set up costs) of 328,436.00 usd is achieved, but that solution requires 120,449 minutes per month (while only 36,700 are available) from the manufacturing cell, while the raw materials requirements for this solution are: 29,330 units of raw material one (while 9,000 are available) ; 28,392 units of raw material two (only 8,000 available); 23,791 units of raw material three (only 7,000 available); 20,021 units of raw material four (only 6,000 available) and finally, 15,529 units of raw material five (only 3,000 available).

Since accepting all orders is not feasible, the order acceptance problem consists on maximising the average monthly operational profit by deciding which orders to accept, and for those accepted orders at which manufacturing level to run (units per month).

Table 9 shows the optimal relaxed solution (up to two digits after the decimal point), which makes 142,850.34 usd of operational profit while the integer solution makes 137,314 usd. Notice that the relaxed solution increases the monthly operational profit by 5,536.34 usd (approximately 4.03 % increase).

In the relaxed solution, orders 20 (14.32 units), 38 (113.26 units) and 43 (44.11 units) have not integer production units. Therefore order 20 will deliver 15 units in 32 percent of the months while 14 units in the rest; order 38 will deliver 114 units in 26 percent of the months and 113 in the rest; finally, order 43 will deliver 45 units in 11 percent of the months and 44 in the rest.

For a literature review of order acceptance please refer to [21].

5 Conclusions and further research

Many real world problems requires to formulate and solve an integer programming model; and the integer solution produces in general a significant worsen objective function value compared with the objective function value of the corresponding relaxed linear programming model. Thus, it is important to explore the possibility of generating a feasible integer solution which maintains the objective value of the relaxed problem at least in average per decision period.

This research shows a class of integer programming problems named PCIPLP, in which the relaxed solution is used as the basis to construct a feasible integer solution maintaining the value of the relaxed solution by proposing a change in the problem planning horizon. One significant fact of our approach is that the integer solution provided is not an optimal solution for neither the original integer model proposed nor its relaxed version, since a change in the problem planning horizon is made. The integer solution provided can be implemented in practice and maintains on average (per period) the objective function value of the relaxed problem.

The set of PCIPLPs consist on integer single period problems that repeats indefinitely, but elements from one period can be conveyed to the next, thus a change in the problem planning horizon from one period to T periods is feasible, where T is computed as the minimum common multiple of the optimum continuous decision variable denominator values.

Further research pends ahead: (a) The identification of some other kind of integer problems that can be treated in a similar way. (b) The identification of a set of mix-integer problems with similar characteristics. (c) The search for classes of non-linear integer (or mix-integer) programming problems that can be solved in the same manner. (d) the exploration of the effectiveness of this methodology for solving large scale problems. (e) Since sensitivity analysis can be performed on (3) through (5) and its results will be practical as long as the resultant solution remains fully integer, hence new techniques must be developed and new characteristics must be found to achieve sensitivity analysis in a more practical matter.

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Biographies

Federico Trigos holds a PhD in Industrial Engineering with dissertation in large scale optimization (Georgia Institute of Technology, USA), a Master of Finance (EGADE Business School, Tecnológico de Monterrey, México), a Master of Science in Industrial Engineering with major in Production Distribution and Material Handling (Georgia Institute of Technology, USA), a Master in Engineering with major in Operations Research (Tecnológico de Monterrey Campus Monterrey, México) and a Bachelor of Industrial and Systems Engineering (Tecnológico de Monterrey Campus Toluca, México). Dr. Federico Trigos has lectured in several professional and research congresses in the United States, Europe, Asia and Latin America. He has been a visiting professor at the Southern Illinois University at Edwardsville and the Iowa State University, both in the USA. He has been a research visiting scholar at the University of Bristol and the University of Southampton, both in the U.K. He has also been recognized by the Massachusetts Institute of Technology (USA) as an International Faculty Fellow. He held the presidency of the Mexican Institute of Industrial and Systems Engineering (based in Monterrey N.L., México) in 2015. Dr. Trigos is an active researcher publishing and leading at EGADE Business School areas related with the sustainable development of quantitative tools to optimize strategic organizational performance. His research interests include: Industrial and business statistics, simulation, mathematical programming, engineering economics, corporate and family business finance as well as financial investments. He is currently a member of the Mexican National Researchers System (SNI), and the Mexican Academy of Science (AMC). He is an active member of research groups both in Mexico, and the United Kingdom.

Leopoldo Eduardo Cárdenas-Barrón is currently a Professor at School of Engineering and Sciences at Tecnológico de Monterrey, Campus Monterrey, México. He is also a faculty member in the Department of Industrial and Systems Engineering at Tecnológico de Monterrey. He was the associate director of the Industrial and Systems Engineering programme from 1999 to 2005. Moreover, he was also the associate director of the Department of Industrial and Systems Engineering from 2005 to 2009. His research areas include primarily related to inventory planning and control, logistics, and supply chain. He has published papers and technical notes in several international journal. He has co-authored one book in the field of Simulation in Spanish.

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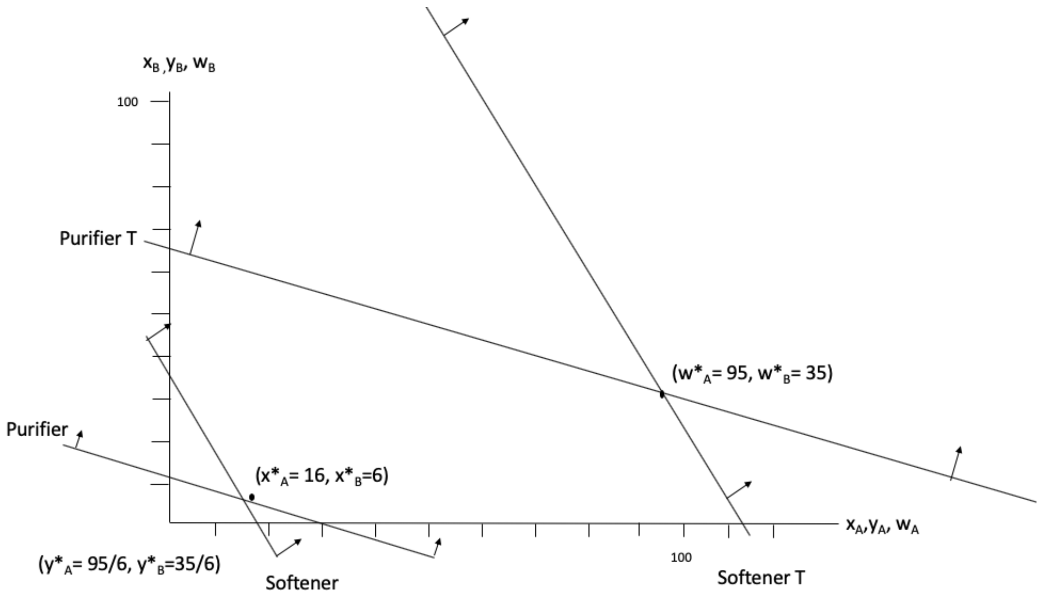

Figure 1: Geometrical representation of the three solution spaces for the illustration in section 4.1

Table 1: Method for solving the single period problem in (1) through the multiple-period problem in (3) through (5).

Step 1:	Solve to optimality the continuous linear programming model in (2) as: Opt $c^t y$ s.t. $Ay = b$, $y \geq 0$.
Step 2:	Let the optimal solution of (2) be $y_i^* = d_i/e_i, \forall i \in I$, where d_i, $e_i \in Z^+$.
Step 3:	Determine T as the minimum common multiple for all $e_i, \forall i \in I$, where T determines the planning horizon.
Step 4:	Define the optimal solution of the problem in (3) through (5) as: Opt $c^t w$ s.t. $Aw = Tb$, $w \geq 0$ as $w_i^* = \left(\frac{T}{d_i}\right) e_i$, where $w_i^* \in Z^+, \forall i \in I$.
Step 5:	Sensitivity analysis, since the problem in (3) through (5) is a continuous linear programming problem with optimal solution $w_i^* \in Z^+$, all regular sensitivity analysis techniques are available to the problem in (3) through (5) as long as the new solution remains integer.

Table 2: Manufacturing operational data for production mix problem.

Product	Compression (hrs.)	Cut (Hrs.)	Polish (hrs.)	Marginal contribution ($)
1	0.5	0.49	0.21	50
2	0.9	0.7	0.39	105

Table 3: Airline fleet capacity and cost per route for the airline case.

Fleet	Seat capacity	Availability	Route 1	Route2	Route 3	Route 3
1	200	5	5	4	3	2
2	150	3	7	5	5	4
3	120	2	9	7	6	6

Table 4: Demand and opportunity cost per route for the airline case.

Route	Passenger demand per day	Opportunity cost per empty seat
1	2,500	2,500
2	2,000	3,000
3	2,200	2,800
4	1,800	2,950

Table 5: Operational cost per route for the airline case.

Fleet	Route 1	Route 2	Route 3	Route 4
1	50,000	70,000	40,000	40,000
2	35,000	80,000	25,000	50,000
3	28,000	45,000	27,000	26,000

Table 6: Optimal continuous daily solution for the airline case, with cost of $\$2,961,000.00$.

Fleet	Route 1	Route 2	Route 3	Route 4
1	5/2	5/2		
2			44/15	1/15
3				2
nfp				320
es				

Table 7: Optimal continuous solution for the airline case, with T = 30, and average daily cost of $\$2,961,000.00$.

Fleet	Route 1	Route 2	Route 3	Route 4
1	75	75		
2			88	2
3				60
nfp				9,600
es				

Table 8: Industrial size order acceptance problem data.

Order	Demand	Set up	Marg. Cont.	Set up	std. Time		Raw	materials	per unit		Order	Demand	Set up	Marg. Cont.	Set up	std. Time		Raw	materials	per unit	
(i)	(d_i)	(SUT_i)	(MC_i)	(SUC_i)	(ST_i)	$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$	(i)	(d_i)	(SUT_i)	(MC_i)	(SUC_i)	(ST_i)	$c_{i,1}$	$c_{i,2}$	$c_{i,3}$	$c_{i,4}$	$c_{i,5}$
1	62	86	68	66	14	1	6	3	3	1	47	123	24	27	29	6	6	5	2	4	3
2	129	63	26	100	14	7	1	3	4	3	48	39	44	59	10	6	3	6	3	2	3
3	114	82	23	84	18	3	4	2	3	2	49	45	11	62	32	16	7	3	5	2	1
4	94	89	15	77	17	4	3	5	1	1	50	67	31	41	91	8	4	2	2	4	2
5	20	52	73	70	11	2	2	1	4	1	51	50	29	10	68	18	7	3	5	1	2
6	27	57	25	50	23	2	4	1	4	2	52	23	74	30	38	11	3	3	5	1	1
7	125	67	70	98	18	1	4	1	4	1	53	128	11	37	58	18	3	5	3	1	2
8	95	82	7	92	20	1	4	3	2	1	54	113	68	70	82	16	4	4	5	3	3
9	74	80	68	95	11	1	4	5	2	3	55	57	49	43	26	13	7	2	4	3	2
10	143	26	29	34	20	2	6	2	4	1	56	125	67	64	57	6	3	1	5	4	1
11	113	61	70	10	7	3	6	2	4	3	57	103	86	60	42	8	2	1	5	1	3
12	20	8	79	96	21	1	1	4	1	3	58	76	50	80	16	13	4	5	2	3	2
13	21	20	59	95	9	4	1	5	3	3	59	121	78	74	35	11	4	6	4	4	2
14	130	45	28	64	23	5	2	1	1	2	60	137	33	52	26	16	6	2	4	1	1
15	114	64	34	91	23	3	6	4	2	1	61	118	71	45	77	7	3	6	3	4	1
16	100	96	25	38	12	6	5	1	2	1	62	74	19	16	83	23	2	4	3	2	1
17	124	87	62	23	9	5	5	2	3	1	63	143	56	68	59	14	2	6	4	2	2
18	136	89	77	10	7	1	2	4	4	3	64	22	24	63	94	24	4	5	3	3	2
19	34	61	50	46	8	4	4	1	2	2	65	54	18	37	67	10	3	3	3	4	3
20	70	99	56	12	8	6	4	3	2	2	66	47	99	28	12	15	1	2	1	3	3
21	148	24	47	82	16	7	4	5	1	2	67	148	98	15	92	24	7	6	1	1	3
22	88	32	82	92	13	6	5	5	3	3	68	20	33	55	19	14	1	4	3	1	1
23	103	56	76	31	7	6	6	2	2	2	69	54	27	8	76	7	1	3	5	2	2
24	132	7	65	69	10	3	6	1	2	3	70	114	77	81	26	24	4	1	5	1	1
25	95	73	6	61	18	3	3	4	4	3	71	62	42	11	60	9	4	1	3	2	2
26	42	45	19	7	19	2	6	2	2	1	72	43	64	20	47	21	1	6	5	1	2
27	130	35	31	51	8	2	2	4	4	3	73	67	14	19	41	16	3	4	2	1	1
28	143	73	29	22	19	2	3	2	4	2	74	53	82	34	52	19	4	5	3	3	3
29	127	25	8	40	8	6	4	5	1	2	75	141	33	71	44	22	1	3	2	3	3
30	33	41	16	7	13	1	5	2	4	3	76	81	60	14	83	23	4	6	2	3	2
31	55	34	21	90	7	3	5	1	3	1	77	19	70	23	69	8	2	2	4	3	3
32	52	94	25	62	9	1	4	1	1	3	78	141	31	73	61	14	6	1	4	4	2
33	19	35	29	19	22	7	6	1	4	3	79	18	69	59	50	22	6	1	4	1	3
34	138	53	27	91	22	4	3	4	4	3	80	119	77	15	76	18	3	2	1	1	1
35	73	49	68	38	19	3	4	1	1	1	81	19	23	77	94	16	5	3	1	1	1
36	76	32	59	89	12	3	1	1	3	2	82	23	59	53	51	21	4	1	5	3	2
37	93	58	42	66	6	3	5	3	2	1	83	42	80	49	26	24	5	4	4	2	3
38	116	57	48	85	22	2	6	2	4	1	84	146	45	83	81	23	5	2	5	4	1
39	127	35	21	44	6	4	1	3	1	2	85	47	82	38	65	19	4	1	3	3	3
40	63	20	67	66	14	5	1	3	4	1	86	112	60	8	94	7	1	2	2	4	1
41	142	50	22	59	12	4	6	2	1	3	87	46	7	76	32	23	7	3	3	2	2
42	96	58	13	78	18	5	6	4	1	1	88	149	84	14	28	6	7	2	2	3	3
43	128	53	54	62	21	7	4	5	3	1	89	29	18	80	50	10	1	1	5	4	1
44	78	69	66	65	15	4	1	5	1	3	90	67	41	8	20	14	1	2	1	1	3
45	50	73	13	11	25	7	1	2	1	1	91	36	24	49	43	19	3	3	3	4	2
46	69	91	78	71	18	3	1	4	1	3	92	114	60	9	80	24	2	4	4	4	2
											93	83	58	28	43	10	2	6	2	1	2

Table 9: Relaxed versus integer solutions.

Op. profit		$142,850.34	$137,314.00
Order	Demand (u)	Relaxed	Integer
1	62	62	62
5	20	20	20
7	125	125	100
15	114	0	63
17	124	124	100
18	136	0	10
20	70	**14.32**	70
23	103	103	100
35	73	73	73
36	76	76	75
37	93	93	93
38	116	**113.26**	100
40	63	63	63
43	128	**44.11**	100
46	69	69	69
49	45	45	45
56	125	125	100
58	76	76	76
59	121	121	100
60	137	137	100
61	118	0	98
63	143	143	100
64	22	0	22
68	20	20	20
70	114	114	100
78	141	141	100
81	19	19	19
84	146	146	100
87	46	46	46
89	29	29	29