Transient Solution of Multiple Vacation Queue with Discouragement and Feedback

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Abstract

The congestion problems with processor vacations have confronted with increasing intricacy, and their explicit transient solutions are exceptionally hard to compute. The transient solution is more significant for studying the dynamical behavior of computing systems over a finite period and predominantly utilizes within the state-of-the-art design architect for a real-time I/O system. Motivated from this, we adopt the mathematical concepts, namely continued fractions and generating function, to derive explicit expressions for transient-state probabilities. Transient-state probabilities of the processing delay problem with a single processor which adopts the multiple vacations policy to save power consumption and thermal trip error with discouragement and feedback are obtained in terms of modified Bessel functions using the properties of the confluent hypergeometric function. Due to the inaccessibility of processor, discouragement behaviors balking and reneging of the job requests are prone to exhibit. Routing back for the service feedback for the processed job request is also critical to maintaining the quality of service (QoS). For the glance of the I/O system performance, the expected value of the state of the computing system using stationary queue-size distribution is also derived.

Keyword: Multiple Vacations; Balking; Reneging; Feedback; Confluent hypergeometric function; Generating function; Modified Bessel function

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1 Introduction

Queues or waiting lines tend to exist at critical points in the job request/processor journey in I/O and computing systems. It can be experienced at service systems, computer and communication systems, manufacturing and production systems, check-in and check-out systems, etc. Despite where the queue is or its perseverance, it perpetually gets the opportunity to emphatically or adversely impact the experience of job requests and the proficiency of the processing system. Not only the benefits of the well-managed queue are many, but also the negative effects of poorly-managed are also significant for the improvement of the computing system. A well-managed queue can cut down on job request's impatience (timeout) behavior, improve the overall perception of the computing system, increase the conversion between job request and processor, optimal service

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allocation, and encourage the positive word of mouth. In contrast, the poorly-managed queue can turn off the satisfactory level; otherwise the correctly processed job request, diminish job request loyalty, decrease repeat request, decrease the efficiency and productivity.

The availability of the processing facility is one of the critical issues of queue management in the I/O system. The scheduling of vacation of the processing facility is an essential feature of interest in the well-managed computing system to balance between idle time, extra cost, unavailability, power consumption, thermal trip error, etc. In vacation of the processing facility, the processor is not processing the job requests and may be in sleep mode or may busy in performing some secondary job requests. In general, the schedule of vacation commences whenever there is no job request in the computing system for processing, and it takes repeated vacations until, at the end of the random duration of the vacation, there is at least one job request in the system for the seek of the processing facility.

When there is a distraught in the balance of the job request's waiting and processing facility, long waiting is experienced, job request delay, blocking, discouragements rise, and lead to job request loss. Whenever the balance between job requests and processing facility is ousted, the common reactions of job requests may be their impatience (timeout) behaviors namely, balking; a job request may decide that the queue is too long and leave the processing facility, and reneging; a job request may join the processing facility, but become impatient and leave the facility without taking intended process. A well-managed computing system deals with such issues strategically to prevent job request loss and delay. These reluctance behaviors of the job requests can be reduced with some proper strategies like publishing the estimated wait times, consider the virtual computing, monitors the queues in real-time, etc.

The feedback loop for the job requests about the satisfaction and dissatisfaction experienced in getting processing is critical. This information is an important resource for the processing facility provider to improve and address the needs and wants of the job requests. In collecting the feedback or inspection, a job request having received a unit of processing, routing back to the waiting line, under some decision rule, to receive another unit of processing, occurs. The feedback or inspection procedures provide a computing framework for requests, jobs, or packets that must be reworked. A broad class of such computing systems appears in computer modeling with round-robin models and foreground-background models. In this paper, such a computing system is referred to as a queue with feedback.

The design of a well-managed processing system gets perfect after much micro or macro analysis of experiences gained during trials of the developing phases. The transient analysis is a prominent tool to develop a stable computing system. It establishes the framework for design from the beginning state to a steady state. It is time-dependent tracking of the system's pros and cons to finalize the appropriate input parameters and conditions for better output of the system. The transient analysis is seldom available in the literature, even for the simple computing model despite its usefulness and necessity in designing a better service system. This gap motivates us to suggest a simple procedure to compute the transient solution for the complex computing system. For this purpose, we establish time-dependent queue-size distribution in terms of a continued fraction using properties of the confluent hypergeometric function.

The purpose of this article is threefold. Firstly, to study the computing system with the job request's discouragement and feedback and processor's multiple vacations. Second, to identify the conditions for the existence and to derive the explicit expression of transient queue-size distribution in terms of continued fraction and modified Bessel's function using properties of the confluent hypergeometric function. This objective is the kernel of this paper. And third objective is to compute the queue indices in terms of transient probabilities to give system performance at a glance for proper management and to validate the proposed procedure for some known results for existing models as a particular case.

The remaining of the paper is structured as follows. In the next section 2, the literature review and survey have been done to identify the research gap. In section 3, the governing forward differential-difference equations are given along with prominent assumptions and notations for the proposed computing model. The properties of the confluent hypergeometric function are highlighted in the next section 4. In section 5, the transient queue-size distribution of the proposed model has been generated. The expected value of the number of job requests in the system is derived using queue-size distribution in section 6. In section 7, some special cases have been illustrated to validate our findings. In the last section 8, we summarize the concluding remarks and highlight the future directions and limitation.

2 Literature Review

Today, we live in a world of internet of things (IoT) technology or fast-paced society. But there is still one thing that we all do, whether it is at the multiplex, mall, bank or even communication, computer computing, or in different ways of travel, is termed as wait. The well-managed computing systems can influence a lot in the consumer and communication environment. Mostly packets, jobs, or requests are familiar with the pros and cons associated with each computing system. Different state-of-the-art design of computing systems can greatly affect the experience of data, requests, jobs, or consumers and can also cause or decrease the problem of the queue or wait, such as jockeying, reneging and balking occurs by refusing the incoming job request to join the queue. Many pieces of researches have given vital development in the design of a better computing system with different types job request's oriented behavior and processor's oriented mechanism since the commencement of computing theory.

Specifically, many researchers have paid considerable attention to vacation queue and impatient behavior of job requests. An exhaustive survey of computing models with vacations is provided by Doshi [1], in which, an overview of some general decomposition results and associated methodology are given. Over the last decade, many relevant new results have been obtained for the single and multiple processor queues with vacations (*c.f.* Ke et al. [2]; Zhang and Hou [3]; Wu and Ke [4]; Jain et al. [5]; Wu and Ke [6]; Shin [7]; Yang and Wu [8] and Takhedmit and Abbas [9]). Thomasian [10] obtained the mean delay cycle for the $M/G/1$ queueing system with the vacationing server model (VSM) in which the delay cycle initiated with arrival during a vacation and ended with empty queue and a vacation restarted. Recently, Banik and Ghosh [11] analyzed the finite-buffer single processor computing system with vacation(s) with the assumptions that the arrivals followed a batch Markovian arrival process (BMAP) and got the processing according to a non-exhaustive type gated-limited service discipline. Shekhar et al. [12] used metaheuristic scheme to device optimal strategies for emergency vacationing queueing model.

The impatient (timeout) behavior of the customer(s) like, balking and reneging firstly investigated by Haight [13], [14]. After that, many researchers used these revenue cut-throat job request behavior in several real-time computing scenarios as mobile networking, computer communication, assembly system, and machine repair problem et cetera. (*c.f.* Ke and Wang [15]; Gans et al. [16]; Wang et al. [17]). Selvaraju and Goswami [18] derived the closed-form of expressions for the transient-state probability distribution for the queueing model with the impatient behavior of the customer(s) and two different vacation policies (single and multiple). Lately, researches determined equilibrium condition for various types of queueing situations with impatience behavior of customers (*c.f.* Guha et al. [19]; Bruneel and Maertens [20]; Guha et al. [21]; Yang and Wu [22] and Wang and Zhang [23]). Shekhar et al. [24] used matrix-geometric technique to compute the queue-size distribution for queueing model with Bernoulli scheduled vacation and retention of the reneged customer.

In the past, many academicians or researchers were interested in presenting different, ease to use, approaches to determine queue-size distribution for different computing models with realistic design parameters using many different mathematical concepts. The time-dependent probabilities can also be expressed in terms of Bessel functions (*c.f.* Parthasarathy and Sharafali [25]). Varshney et al. [26] used the diffusion approximation method to obtain transient probability distribution for the multi-server queueing model with balking. Al-Seedy et al. [27] obtained the closed-form queue-size distribution using generating function and some Bessel function properties for the multi-server queueing model. Ammar et al. [28] discussed the single-server queue with a finite waiting space and discouraged arrivals and reneging and obtained the time-dependent probabilities in terms of the eigenvalues of a symmetric tridiagonal matrix. Ammar [29] provided the elegant explicit solution for two heterogeneous servers queue with impatient behavior. Using generating function and Bessel function properties, Kumar and Sharma [30] explored the transient multi-server queue with balking and retention of reneging customers.

The feedback loop of the job requests is essential for the quality of service (QoS) in processing based computing systems. Atencia and Moreno [31] analyzed a discrete-time $Geo^[X]/G_H/1$ retrial queue where each call after service either immediately returned to the orbit for another service with some probability or left the system forever with complementary probability. Several papers (*c.f.* Mitrani and Robert [32]; Kumar et al. [33]; Liu and Gao [34]; Admas [35]; Upadhyaya [36]; Liu and Whitt [37]; Shekhar et al. [38]) have appeared in the literature in which the served customer or processed job provide the feedback on service completion instant due to unsatisfaction. Recently, Chang et al. [39] presented the analysis of an unreliable server retrial queue with customer's feedback and impatience to investigate the system viable economically. With the help of the supplementary variable technique, Rajadurai et al. [40] obtained steady-state probabilities via generating function for the system size for a single server feedback retrial queueing system with multiple working vacations and vacation interruption. Forghani and Fatemi Ghomi [41] explored the application of processor-based queueing system in different sectors.

3 Model description

In this paper, we consider the single processor computing system with first come first serve (FCFS) service discipline, and realistic behavior of job requests and processor. The following assumptions and notations are considered to structure the proposed computing system and to develop the governing equations.

- From the infinite population of the prospective job requests, jobs arrive at the computing system following the Poisson process with mean arrival rate λ.

- If the processor is available and idle, the job request immediately gets processing otherwise joins the waiting queue in infinite capacity space.

- The job request processing times are independent and identically distributed (i.i.d.) exponential random variables with parameter μ.

- For maintaining the state-of-the-art design and quality of the service, the processor seeks feedback from the served job requests. The processed job request leaves the computing system with the probability ξ or routs back the queue to provide feedback with complementary probability $1-\xi$.

- On the completion of the services for all waiting job requests, the processor opts a vacation to take rest or to diminish idle time or rendering service cost or thermal break error or power consumption. The processor's vacation time is an exponential random variate with the meantime $1/\theta$.

- At the epoch of vacation end, if the computing system is empty, the processor recommences another vacation for the random duration and repeatedly continues until the system has at least one job request waiting in the system.

- If the processor is not available for service due to vacation, some job requests may exhibit impatience (timeout) behavior. If the processor is on vacation, the job requests may balk with probability β from the system without joining the computing system. The job requests may also renege without being served after waiting for some random period when the processor is on vacation. The time-to-renege from the system follows an exponential distribution with meantime $1/\eta$.

For the modeling purpose, we define the state of the computing system as follows

$$S(t) \equiv \begin{cases} 0\,; & \text{the processor is in vacation state at any instant of time } t \\ 1\,; & \text{the processor is in busy state at any instant of time } t \end{cases}$$
$$N(t) \equiv \text{Number of the waiting job requests in the system at time } t.$$

Hence, $\{(S(t),\, N(t));\, t \geq 0\}$ is a continuous time Markov chain ($CTMC$) on the state space $\Omega = \{(0,0)\} \cup \{(s,n); s = 0,1 \;\&\; n = 1,2,...\}$. Therefore, the governing transient-state probabilities are defined as follows

$$P_{0,n}(t) = \text{Prob}\,\{S(t) = 0, N(t) = n\}\,; n \geq 0$$
$$P_{1,n}(t) = \text{Prob}\,\{S(t) = 1, N(t) = n\}\,; n \geq 1$$

Using above defined assumptions and notations, we develop the governing Chapman-Kolmogorov differential-difference equations of order and degree one for the studied model. On balancing the rates of inflow and outflow at each state, we have following set of differential equations

$$\frac{dP_{0,0}(t)}{dt} = -\,\lambda\, P_{0,0}(t) + \eta\, P_{0,1}(t) + \xi\mu\, P_{1,1}(t) \tag{1}$$

$$\frac{dP_{0,1}(t)}{dt} = -\,\left(\lambda\bar{\beta} + \eta + \theta\right) P_{0,1}(t) + \lambda\, P_{0,0}(t) + 2\eta\, P_{0,2}(t) \tag{2}$$

$$\frac{dP_{0,n}(t)}{dt} = -\,\left(\lambda\bar{\beta} + n\eta + \theta\right) P_{0,n}(t) + \lambda\bar{\beta}\, P_{0,n-1}(t) + (n+1)\,\eta\, P_{0,n+1}(t); \quad n \geq 2 \tag{3}$$

$$\frac{dP_{1,1}(t)}{dt} = -\,(\xi\mu + \lambda)\, P_{1,1}(t) + \theta\, P_{0,1}(t) + \xi\mu\, P_{1,2}(t) \tag{4}$$

$$\frac{dP_{1,n}(t)}{dt} = -\,(\xi\mu + \lambda)\, P_{1,n}(t) + \lambda\, P_{1,n-1}(t) + \xi\mu\, P_{1,n+1}(t) + \theta\, P_{0,n}(t); \quad n \geq 2 \tag{5}$$

The system of differential-difference equation (1)-(5) with initial condition $P_{0,0}(0) = 1$ and $P_{s,n}(0) = 0$; $s = 0, 1$ & $n \geq 1$, *i.e.* there is no job request in the computing system to be processed at time $t = 0$, can be solved using Laplace transformation, properties of confluent hypergeometric function, modified Bessel's function, generating function to compute the transient-state probabilities.

4 Confluent hypergeomtric function

In this section, we present the definition and properties of the special function, confluent hypergeometric function, which we use for deriving the explicit expression of transient queue-size distribution in terms of the continued fraction for the studied computing problem with processor's vacation and job request's feedback and discouragement. The confluent hypergeometric function is a solution of a confluent hypergeometric equation, which is a degenerate form of a hypergeometric differential equation where two of the three regular singularities merge into an irregular singularity. The confluent hypergeometric function ${}_1\mathrm{F}_1(a; c; z)$ is defined as infinite sum

$$\begin{aligned} {}_1\mathrm{F}_1(a; c; z) &= 1 + \frac{a}{c}\frac{z}{1!} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{z^2}{2!} + ... \\ &= \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!}; \quad c \neq 0, -1, -2, ... \end{aligned} \tag{6}$$

where $(\alpha)_k$ is the rising factorial function which can be represented as

$$(\alpha)_k = \frac{\Gamma\,(\alpha + 1)}{\Gamma\,(\alpha - k + 1)}; \quad k = 0, 1, 2, ...$$

From above definition, for $a = 0$, we have

$${}_1\mathrm{F}_1(0; c; z) = 1$$

To generate the values of the function for different values of a and c, the recurrence relation for the confluent hypergeometric function is

$$c(c-1)\ {}_1\mathrm{F}_1(a-1; c-1; z) - az\ {}_1\mathrm{F}_1(a+1; c+1; z) = c(c-1-z)\ {}_1\mathrm{F}_1(a; c; z) \tag{7}$$

We establish that the quotient of two hypergeometric function may be expressed as a continued fraction. It is useful for the algebra with confluent hypergeometric function. The following identity has been identified

$$\frac{{}_1\mathrm{F}_1(a+1;c+1;z)}{{}_1\mathrm{F}_1(a;c;z)} = \cfrac{c}{(c-z) + \cfrac{(a+1)z}{(c-z+1) + \cfrac{(a+2)z}{(c-z+2) + \ddots}}}$$

On simplification, it can be rewritten as

$$\frac{c\ {}_1\mathrm{F}_1(a;c;z)}{{}_1\mathrm{F}_1(a+1;c+1;z)} - (c-z) = \cfrac{(a+1)z}{(c-z+1) + \cfrac{(a+2)z}{(c-z+2) + \cfrac{(a+3)z}{(c-z+3) + \ddots}}} \tag{8}$$

For the confluent hypergeometric function, we have the following results also

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{y^k}{k!} {}_1\mathrm{F}_1(a+k;c+k;x) = {}_1\mathrm{F}_1(a;c;x+y). \tag{9}$$

Some values of a and c yield solutions that can be expressed in terms of other known functions. When a is a non-positive integer then Kummer's confluent hypergeometric function, if it is defined, is a generalized Laguerre polynomial. Just as the confluent differential equation is a limit of the hypergeometric differential equation as the singular point at 1 is moved towards the singular point at ∞, the confluent hypergeometric function can be given as a limit of the hypergeometric function.

5 Transient analysis

In this section, we derive the explicit expression for transient queue-size distribution for the computing system under consideration using the mathematical concepts of generating function, continued fractions, and some properties of confluent hypergeometric function. For this purpose, we use the following sequel.

5.1 Laplace Transform

Laplace transform is an integral transform that converts a function of a real variable t to a function of a complex variable s. It transforms differential equations into algebraic equations and convolution into multiplication. We define Laplace transformation of state probabilities and their derivatives as follows

$$\widetilde{P}_{i,j}(s) = L\left(P_{i,j}(t)\right) = \int_0^{\infty} e^{-s\,t} P_{i,j}(t)dt; \quad \forall\ i,j$$

$$L\left(\frac{dP_{i,j}(t)}{dt}\right) = s\,\widetilde{P}_{i,j}(s) - P_{i,j}(0); \quad \forall\ i,j$$

Using defined Laplace transform, the system of governing differential-difference equations is transformed as a system of linear equations as follows

$$s\,\widetilde{P}_{0,0}(s) - 1 = -\,\lambda\,\widetilde{P}_{0,0}(s) + \eta\,\widetilde{P}_{0,1}(s) + \xi\mu\,\widetilde{P}_{1,1}(s) \tag{10}$$

$$s\,\widetilde{P}_{0,1}(s) = -\,\left(\lambda\bar{\beta} + \eta + \theta\right)\widetilde{P}_{0,1}(s) + \lambda\,\widetilde{P}_{0,0}(s) + 2\eta\,\widetilde{P}_{0,2}(s) \tag{11}$$

$$s\,\widetilde{P}_{0,n}(s) = -\,\left(\lambda\bar{\beta} + n\eta + \theta\right)\widetilde{P}_{0,n}(s) + \lambda\bar{\beta}\,\widetilde{P}_{0,n-1}(s) + (n+1)\,\eta\,\widetilde{P}_{0,n+1}(s); \quad n \geq 2 \tag{12}$$

$$s\,\widetilde{P}_{1,1}(s) = -\,(\xi\mu + \lambda)\,\widetilde{P}_{1,1}(s) + \theta\,\widetilde{P}_{0,1}(s) + \xi\mu\,\widetilde{P}_{1,2}(s) \tag{13}$$

$$s\,\widetilde{P}_{1,n}(s) = -\,(\xi\mu + \lambda)\,\widetilde{P}_{1,n}(s) + \lambda\,\widetilde{P}_{1,n-1}(s) + \xi\mu\,\widetilde{P}_{1,n+1}(s) + \theta\,\widetilde{P}_{0,n}(s); \quad n \geq 2 \tag{14}$$

5.2 Evaluation of $\mathbf{P}_{1,n}(\mathbf{t})$, $\mathbf{n} \geq \mathbf{1}$ in terms of $\mathbf{P}_{0,n}(\mathbf{t})$

Define the generating function

$$P(z,t) = \sum_{n=1}^{\infty} P_{1,n}(t)\,z^n \tag{15}$$

Using some algebraic manipulation with $eq^n(5)$ and $eq^n(6)$, we establish the following partial differential equation

$$\frac{\partial P(z,t)}{\partial t} = \left\{-\,(\lambda + \xi\mu) + \lambda z + \frac{\xi\mu}{z}\right\} P(z,t) + \theta \sum_{n=1}^{\infty} P_{0,n}(t) z^n - \xi\mu\,P_{1,1}(t) \tag{16}$$

which on integrating gives

$$\begin{aligned} P(z,t) = \theta \int_0^t \sum_{m=1}^{\infty} P_{0,m}(y)\,z^m\,e^{-(\lambda+\xi\mu)(t-u)}\,e^{\left(\lambda z + \frac{\xi\mu}{z}\right)(t-u)}\,du \\ -\,\xi\mu \int_0^t P_{1,1}(u)\,e^{-(\lambda+\xi\mu)(t-u)}\,e^{\left(\lambda z + \frac{\xi\mu}{z}\right)(t-u)}\,du \end{aligned} \tag{17}$$

For $\alpha = 2\sqrt{\lambda\xi\mu}$ and $\gamma = \sqrt{\frac{\lambda}{\xi\mu}}$, it is well known from the theory of Bessel's function

$$e^{\left(\lambda z + \frac{\xi\mu}{z}\right)t} = \sum_{n=-\infty}^{\infty} (\gamma z)^n\, I_n(\alpha t)$$

where $I_n(.)$ is the modified Bessel function of the first kind. Comparing the coefficient of the like power of z, *i.e.* $z^n; n = 1, 2, \ldots$, on both sides of $eq^n(17)$, we have

$$\begin{aligned} P_{1,n}(t) = \theta \int_0^t \sum_{m=1}^{\infty} P_{0,m}(y)\,\gamma^{n-m}\,I_{n-m}\left(\alpha(t-u)\right) e^{-(\lambda+\xi\mu)(t-u)}\,du \\ +\,\xi\mu \int_0^t P_{1,1}(y)\,\gamma^n\,I_n\left(\alpha(t-u)\right)\,e^{-(\lambda+\xi\mu)(t-u)}\,du;\ n = 1, 2, \ldots \end{aligned} \tag{18}$$

The above equation also holds for $n = -1, -2, ...$ with left-hand side replaced by zero *i.e.* $P_{1,n}(t) = 0$. Using $I_{-n}(.) = I_n(.)$ for $n = 1, 2, ...$, we have

$$\begin{aligned} \theta \int_0^t \sum_{m=1}^{\infty} P_{0,m}(u)\, \gamma^{-n-m}\, I_{n+m}\left(\alpha(t-u)\right)\, e^{-(\lambda+\xi\mu)(t-u)}\, du \\ + \xi\mu \int_0^t P_{1,1}(u)\, \gamma^n\, I_n\left(\alpha(t-u)\right)\, e^{-(\lambda+\xi\mu)(t-u)}\, du = 0; \quad n = 1, 2, \ldots \end{aligned} \tag{19}$$

Hence, $eq^n(18)$ & $eq^n(19)$ give $P_{1,n}(t)$ in terms of $P_{0,n}(t)$

$$P_{1,n}(t) = \theta \int_0^t e^{-(\lambda+\xi\mu)(t-u)} \sum_{m=1}^{\infty} \gamma^{n-m}\, P_{0,m}(y) \times \{I_{n-m}\left(\alpha(t-u)\right) - I_{n+m}\left(\alpha(t-u)\right)\}\, du; \quad n = 1, 2, \ldots \tag{20}$$

5.3 Evaluation of $\mathbf{P}_{0,n}(\mathbf{t})$, $\mathbf{n} \geq \mathbf{2}$ in terms of $\mathbf{P}_{0,0}(\mathbf{t})$

In this sub-section, we obtain the expression for $P_{0,n}(t)$ as a continued fraction using the identities of confluent hypergeometric function. $Eq^n(12)$ is structured as

$$\frac{\widetilde{P}_{0,n}(s)}{\widetilde{P}_{0,n-1}(s)} = \frac{\lambda\bar{\beta}}{(s + \lambda + \theta + n\eta) - (n+1)\eta \left\{ \dfrac{\widetilde{P}_{0,n+1}(s)}{\widetilde{P}_{0,n}(s)} \right\}}$$

which can be further structured as a continued fraction as follows:

$$\frac{\widetilde{P}_{0,n}(s)}{\widetilde{P}_{0,n-1}(s)} = \cfrac{\lambda\bar{\beta}}{(s + \lambda\bar{\beta} + \theta + n\eta) - \cfrac{(n+1)\eta\lambda\bar{\beta}}{(s + \lambda\bar{\beta} + \theta + (n+1)\eta) - \cfrac{(n+2)\eta\lambda\bar{\beta}}{(s + \lambda\bar{\beta} + \theta + (n+2)\eta) - \ddots}}}$$

Using the identity $eq^n(8)$ of the confluent hypergeometric function discussed in section 4, the above expression can be expressed in terms of confluent hypergeometric function.

$$\frac{\widetilde{P}_{0,n}(s)}{\widetilde{P}_{0,n-1}(s)} = \frac{\lambda\bar{\beta}}{\eta} \frac{{}_1\mathrm{F}_1\left((n+1); \left(\dfrac{s+\theta}{\eta} + n + 1\right); \left(-\dfrac{\lambda\bar{\beta}}{\eta}\right)\right)}{\left(\frac{s+\theta}{\eta} + n\right)\, {}_1\mathrm{F}_1\left(n; \left(\dfrac{s+\theta}{\eta} + n\right); \left(-\dfrac{\lambda\bar{\beta}}{\eta}\right)\right)} \tag{21}$$

On iterative substitution and algebraic manipulation, we obtain the expression for $\widetilde{P}_{0,n}(s)$ in terms of $\widetilde{P}_{0,0}(s)$ as follows

$$\begin{aligned}\widetilde{P}_{0,n}(s) &= \left(\frac{\lambda\bar{\beta}}{\eta}\right)^{n} \frac{1}{\prod\limits_{i=1}^{n}\left(\frac{s+\theta}{\eta}+i\right)} \frac{{}_1\mathrm{F}_1\left((n+1);\left(\frac{s+\theta}{\eta}+n+1\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)}{{}_1\mathrm{F}_1\left(n;\left(\frac{s+\theta}{\eta}+n\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)}\, \widetilde{P}_{0,0}(s); \quad n=2,3,\ldots \\ &= \widetilde{\Psi}_n(s)\, \widetilde{P}_{0,0}(s); \quad n=2,3,\ldots\end{aligned} \tag{22}$$

On taking the inverse Laplace transform, we have

$$P_{0,n}(t) = \Psi_n(t) * P_{0,0}(t); \quad n = 2,3,\ldots\ldots \tag{23}$$

where $\Psi_n(t)$ is the inverse Laplace transform of $\widetilde{\Psi}_n(s)$, which will be evaluated in sub-section 5.6 and $*$ represents the convolution between two functions.

5.4 Evaluation of $\mathrm{P}_{0,0}(\mathrm{t})$

From $eq^n(10)$, we obtain

$$\widetilde{P}_{0,0}(s) = \frac{1}{(s+\lambda) - \eta\,\dfrac{\widetilde{P}_{0,1}(s)}{\widetilde{P}_{0,0}(s)} - \xi\mu\,\dfrac{\widetilde{P}_{1,1}(s)}{\widetilde{P}_{0,0}(s)}} \tag{24}$$

On putting $n=1$ in $eq^n(19)$, we have following expressions

$$P_{1,1}(t) = \theta \int_0^t e^{-(\lambda+\xi\mu)(t-u)} \sum_{m=1}^{\infty} \gamma^{1-m} P_{0,m}(y) \times \left\{ I_{1-m}\left(\alpha(t-u)\right) - I_{1+m}\left(\alpha(t-u)\right) \right\} du \tag{25}$$

and

$$P_{0,1}(t) = \Psi_2(t) * P_{0,0}(t) \tag{26}$$

respectively. Hence, $eq^n(25)$ & $eq^n(26)$ give

$$\widetilde{P}_{0,0}(s) = \sum_{k=0}^{\infty}\sum_{r=0}^{k} (-1)^k \theta^k \binom{k}{r} \left(\frac{\eta}{\theta}\right)^k \frac{\widetilde{\Psi}_2^r(s)}{(s+\lambda)^{k+1}} \left\{ \sum_{m=1}^{\infty} \left(\frac{\chi - \sqrt{\chi^2-\alpha^2}}{\alpha\gamma} \right)^m \widetilde{\Psi}_m(s) \right\}^{k-r} \tag{27}$$

where $\chi = (s + \lambda\beta + \xi\mu)$, which on inverse Laplace transform yields

$$\begin{aligned}P_{0,0}(t) = \xi\mu \sum_{k=0}^{\infty}\sum_{r=0}^{k} (-1)^k \theta^k \binom{k}{r} \left(\frac{\eta}{\theta}\right)^k e^{-\lambda t}\frac{t^k}{k!} * \Psi_2^r(t) * \\ \left\{ \sum_{m=1}^{\infty} \gamma^{1-m} \left(I_m(\alpha(t-y)) - I_{m+2}(\alpha(t-y))\right) \times e^{-(\lambda+\xi\mu)t} * \Psi_m(t) \right\}^{*(k-r)}\end{aligned} \tag{28}$$

where $*$ denotes the convolution, while $*(k-r)$ stands for the $(k-r)$ times fold convolution.

5.5 Evaluation of $\mathrm{P}_{0,1}(\mathrm{t})$

Similarly, we obtain the algebraic expression of $P_{0,1}(t)$. From $eq^n(11)$, we have

$$\begin{aligned}(s+\lambda\bar{\beta}+\eta+\theta)\widetilde{P}_{0,1}(s) &= \lambda\,\widetilde{P}_{0,0}(s)+2\eta\,\widetilde{P}_{0,2}(s)\\ \widetilde{P}_{0,1}(s) &= \frac{\lambda}{(s+\lambda\bar{\beta}+\eta+\theta)}\,\widetilde{P}_{0,0}(s)+\frac{2\eta}{(s+\lambda\bar{\beta}+\eta+\theta)}\,\widetilde{P}_{0,2}(s)\end{aligned} \tag{29}$$

which on inverse Laplace transform yields

$$P_{0,1}(t)=\lambda e^{-(\lambda\bar{\beta}+\theta+\eta)t}\,*P_{0,0}(t)+2\eta e^{-(\lambda\bar{\beta}+\theta+\eta)t}\,*P_{0,2}(t) \tag{30}$$

5.6 Evaluation of $\Psi_n(t); n \geq 2$

Finally, for obtaining all transient-state probabilities, we have to derive the expression for $\Psi_n(t)$. From $eq^n(22)$, we have

$$\widetilde{\Psi}_n(s)=\left(\frac{\lambda\bar{\beta}}{\eta}\right)^n\frac{1}{\prod\limits_{i=1}^{n}\left(\frac{s+\theta}{\eta}+i\right)}\frac{{}_1\mathrm{F}_1\left((n+1);\left(\frac{s+\theta}{\eta}+n+1\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)}{{}_1\mathrm{F}_1\left(n;\left(\frac{s+\theta}{\eta}+n\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)};\quad n\geq 2 \tag{31}$$

By using the definition of $eq^n(7)$ for confluent hypergeometric function, we have

$$\frac{{}_1\mathrm{F}_1\left((n+1);\left(\frac{s+\theta}{\eta}+n+1\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)}{\prod\limits_{i=1}^{n}\left(\frac{s+\theta}{\eta}+i\right)}=\eta^n\sum_{k=0}^{\infty}\frac{\binom{n+k}{k}(-\lambda\bar{\beta})^k}{\prod\limits_{i=1}^{n+k}(s+\theta+i\eta)};\quad n\geq 2 \tag{32}$$

which can be written into partial fractions as follows

$$\frac{{}_1\mathrm{F}_1\left((n+1);\left(\frac{s+\theta}{\eta}+n+1\right);\left(-\frac{\lambda\bar{\beta}}{\eta}\right)\right)}{\prod\limits_{i=1}^{n}\left(\frac{s+\theta}{\eta}+i\right)}$$

$$=\eta^n\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{-\lambda\bar{\beta}}{\eta}\right)^k\sum_{i=1}^{n+k}\frac{(-1)^{i-1}}{(i-1)!(n+k-i)!}\frac{1}{s+\theta+i\eta};\quad n\geq 2 \tag{33}$$

We also have,

$${}_1\mathrm{F}_1\left(1;\frac{s+\theta}{\eta}+1;\frac{-\lambda\bar{\beta}}{\eta}\right)=\sum_{k=0}^{\infty}\frac{(-\lambda\bar{\beta})^k}{\prod\limits_{i=1}^{k}(s+\theta+i\eta)}=\sum_{k=0}^{\infty}(-\lambda\bar{\beta})^k\,\widetilde{a}_k(s) \tag{34}$$

where

$$\widetilde{a}_0(s) = 1 \text{ and } \widetilde{a}_k(s) = \frac{1}{\prod_{i=1}^{k}(s+\theta+i\eta)} = \frac{1}{\eta^{k-1}}\sum_{r=1}^{k}\frac{(-1)^{r-1}}{(r-1)!(k-r)!}\frac{1}{s+\theta+r\eta}; \quad k=1,2,\ldots \tag{35}$$

Using the identity $eq^n(6)$ from section (4) for confluent hypergeometric function, we have

$$\left\{{}_1\mathrm{F}_1\left(n+1;\frac{s+\theta}{\eta}+n+1;\frac{-\lambda\bar{\beta}}{\eta}\right)\right\}^{-1} = \sum_{k=0}^{\infty}\widetilde{b}_k(s)(\lambda\bar{\beta})^k; \quad n\geq 2 \tag{36}$$

where

$$\widetilde{b}_k(s) = 1$$

and

$$\widetilde{b}_k(s) = \begin{vmatrix} \widetilde{a}_1(s) & 1 & & \ldots & & \\ \widetilde{a}_2(s) & \widetilde{a}_1(s) & 1 & \ldots & & \\ \widetilde{a}_3(s) & \widetilde{a}_2(s) & \widetilde{a}_1(s) & \ldots & & \\ \vdots & \vdots & \vdots & \ldots & & \\ \widetilde{a}_{k-1}(s) & \widetilde{a}_{k-2}(s) & \widetilde{a}_{k-3}(s) & \ldots & \widetilde{a}_1(s) & 1 \\ \widetilde{a}_k(s) & \widetilde{a}_{k-1}(s) & \widetilde{a}_{k-2}(s) & \ldots & \widetilde{a}_2(s) & \widetilde{a}_1(s) \end{vmatrix}; \quad k=1,2,\ldots$$
$$= \sum_{i=1}^{k}(-1)^{i-1}\widetilde{a}_i(s)\widetilde{a}_{k-1}(s); \quad k=1,2,\ldots \tag{37}$$

Hence, on substituting the results from $eq^n(33)$ and $eq^n(36)$, $eq^n(31)$ gives

$$\widetilde{\Psi}_n(s) = (\lambda\bar{\beta})^j\binom{n+j}{j}\widetilde{a}_{n+j}(s)\sum_{k=1}^{\infty}(\lambda\bar{\beta})^k\widetilde{b}_k(s); \quad n\geq 2 \tag{38}$$

which on inverse Laplace transform gives required expression as follows

$$\Psi_n(t) = (\lambda\bar{\beta})^j\binom{n+j}{j}a_{n+j}(t) * \sum_{k=1}^{\infty}(\lambda\bar{\beta})^k b_k(t); \quad n\geq 2 \tag{39}$$

where,

$$a_k(t) = \frac{1}{\eta^{k-1}}\sum_{r=1}^{k}\frac{(-1)^{r-1}}{(r-1)!(k-r)!}e^{-(\theta+r\eta)t}; \quad k=1,2,\ldots \tag{40}$$

$$b_k(t) = \sum_{i=1}^{k}(-1)^{i-1}a_i(t) * b_{k-1}(t); \quad k=2,3,\ldots \;\&\; b_1(t)=a_1(t) \tag{41}$$

6 Performance measure

Systematic observation of the state of the computing system is vital to enhance the performance and to improve decision making. For the computing system, it is primarily delivered through the expected value of the number of the job requests. Let

$$N(t) \equiv \text{ Number of job requests in the computing system at time } t.$$

Hence, the expected number of job requests in the computing system at time t is

$$E(N(t)) = \sum_{n=1}^{\infty} n\left(P_{0,n}(t) + P_{1,n}(t)\right) \tag{42}$$

Initially, at $t = 0$, we assume that there is no job request to be processed in the computing system. Hence,

$$E(N(0)) = \sum_{n=1}^{\infty} n\left(P_{0,n}(0) + P_{1,n}(0)\right) = 0$$

On differentiating $eq^n(42)$ with respect to t, we have

$$\frac{dE(N(t))}{dt} = \sum_{n=1}^{\infty} n\left(\frac{P_{0,n}(t)}{dt} + \frac{P_{1,n}(t)}{dt}\right) \tag{43}$$

On substituting the value from $eq^n(1)$ to $eq^n(5)$ and using some mathematical manipulation, we get following differential equation from $eq^n(43)$

$$\frac{dE(N(t))}{dt} = \lambda P_{0,0}(t) + \lambda\bar{\beta}\sum_{n=1}^{\infty} P_{0,n}(t) - \eta\sum_{n=1}^{\infty} nP_{0,n}(t) + (\lambda - \xi\mu)\sum_{n=1}^{\infty} P_{1,n}(t) \tag{44}$$

which on integration gives

$$E(N(t)) = \lambda\int_0^t P_{0,0}(y)dy + \lambda\bar{\beta}\sum_{n=1}^{\infty}\int_0^t P_{0,n}(y)dy - \eta\sum_{n=1}^{\infty}\int_0^t nP_{0,n}(y)dy + (\lambda - \xi\mu)\sum_{n=1}^{\infty}\int_0^t P_{1,n}(y)dy \tag{45}$$

where the explicit expressions for $P_{0,n}(t)$ and $P_{1,n}(t)$ are already derived in $eq^n(23)$ and $eq^n(20)$.

7 Special Cases

For the justification and docility of the studied computing system, we relax one or more assumptions and find that our model is similar to the models available in the existing literature. The analytical proof is not possible due to its complexity, even if we are relaxing some assumptions.

Case 1: For $\xi = 1$, i.e., the job request leaves the system after getting the processing without feedback loop. Our model and results coincide with the results of the model proposed by [42]. The model reduces to a single-server computing model with impatient behavior and multiple vacations.

Case 2: By limiting the vacation parameter as $\theta \to 0$ and setting the feedback parameter $\xi = 1$, our present model resembles computing model with balking and reneging which is proposed by [27]. They have also computed the transient solution for queue-size distribution in similar manner.

Case 3: If we set the balking probability $\beta = 0$ and reneging parameter $\eta = 0$, our model exhibits the result, which was obtained by [43]. They studied $M/M/1$ computing model with multiple vacations for server.

8 Conclusion

In this paper, we have carried out investigation for the time-dependent solution of a single processor Poisson queue in a computing system with job request feedback and discouragement (timeout) behaviors, namely balking and reneging and processor's multiple vacations. We have derived closed-form explicit expression for the queue-size distribution analytically by employing continued fractions, Laplace transforms, generating functions, and using the properties of confluent hypergeometric function. We have also established a mean for the state of the computing system for a glance to processing facility providers. The present study will provide the basic idea to queueing theorists to determine the transient solution of queueing problems and to service provider the design parameters for better services.

The present work can be extended in the future with more realistic phenomenons related to the waiting line problems like single and multiple vacations, Bernoulli vacation, working vacation, vacation interruption, imperfect service, multi-processor, processor breakdown, catastrophe, etc. Due to computational complexity involved, many variants of the queue may not be applicable to extend like jockeying, finite population, etc. For such variants, we must opt some numerical techniques for deriving the transient queue-size distribution.

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