Global Practical Stabilization of Discrete-time Switched Affine Systems via Switched Lyapunov Functions and State-dependent Switching Functions

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Abstract

This paper addresses the problem of global practical stabilization of discrete-time switched affine systems via switched Lyapunov functions with the objectives of achieving less conservative stability conditions and less conservative size of the ultimate invariant set of attraction. The main contribution is to propose a state-dependent switching controller synthesis that guarantees simultaneously the invariance and global attractive properties of a convergence set around a desired equilibrium point. This set is constructed by the intersection of a family of ellipsoids associated with each of switched quadratic Lyapunov functions. The global practical stability conditions are proposed as a set of Bilinear Matrix Inequalities (BMIs) for which an optimization problem is established to minimize the size of the ultimate invariant set of attraction. A DC-DC buck converter is considered to illustrate the effectiveness of the proposed stabilization and controller synthesis method.

Index Terms

Discrete-time switched affine systems, stabilization, Bilinear Matrix Inequalities (BMIs), switched Lyapunov functions, practical stability, switching power converters

I. INTRODUCTION

Switched systems are described by a set of continuous dynamics together with a controlled switching function that decides which continuous dynamic should be selected from this set at any time for the current continuous state evolution [1]. Switched systems not only appear during study of many real-world systems such as power systems and power electronics, automotive control, aircraft and air-traffic control, network control systems, but also they deserve investigation for theoretical reasons during study of dynamical systems subjected to sudden and abrupt parameter variations or investigation of multi-controller switching techniques [2], [3]. Although the literature on switched systems is rich and there are several books [2]–[6] and survey papers [7]–[10] in this context, however,

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most of the works in this domain assume a common equilibrium point for all isolated subsystems. On the other hand, one of the most important subclasses of switched systems is the switched affine systems that are very common in practice, especially in power electronics area [11]–[17].

In this class of switched systems, the equilibrium point varies discontinuously during switching among subsystems, and therefore, to achieve the asymptotic stability in a desired equilibrium point the switching frequency is approached to infinity, which is not realizable in practice [18]–[20]. One of the well-known solutions to the challenge of chattering phenomena, is to consider an upper bound to frequency of switching functions through time discretization. This is a motivation for the analysis and synthesis of switched affine systems in the discrete-time domain. Another positive side is that nowadays any controller implementation is made via computers and discrete-time samplers, and therefore, study of the discrete-time systems and their computer-based control are of particular importance.

However, due to the limitation on the switching frequency of the switching signal, the stability with respect to a set rather than a particular point can be achieved [21], [22]. As a result, many theoretical findings about the stability of switched systems with a common equilibrium point cannot be directly applied to the stability analysis of switched affine systems without any common equilibrium point. In this regard, the notion of practical stability has been proposed in the literature to analyze the stability of switched systems without any common equilibria [23]–[27].

There are two main types of Lyapunov functions for the stability analysis of switched systems, i.e., common and multiple Lyapunov functions. It is well known that there are many examples of stable switched systems that do not admit any common Lyapunov function [4], [7]. Therefore, to achieve less conservative stability conditions the rich and more complex classes of multiple Lyapunov functions have been utilized for the stabilization of switched systems of which we can mention to max-type Lyapunov functions [28], [29], min-type Lyapunov functions [19] and switched Lyapunov functions [30]–[34]. In this paper, we will use the switched quadratic Lyapunov functions to design the switching rules for the global practical stabilization of discrete-time switched affine systems.

A. Practical stabilization of switched affine systems

The problem of practical stabilization of switched systems without common equilibria or in particular case of switched affine systems has been investigated in the last years either in a general problem formulation or as in the case of particular problems most notably those related to the switching power converters [12], [35], [36]. There are very few works for the global practical stabilization of discrete-time switched affine systems via multiple Lyapunov functions and using state-dependent switching functions. Here, we review the most relevant works.

The local practical stability and stabilization of continuous-time nonlinear time-variant switched systems via a single Lyapunov-like function was proposed in [26], while the global practical asymptotic stabilizability of time-invariant switched nonlinear systems in continuous-time and discrete-time domains is investigated via a single quadratic Lyapunov function in [25] and [23], respectively. However, in these works no constructive and systematic way is proposed to compute the respective Lyapunov functions.

In [19] global practical stability conditions are proposed as a set of BMIs for discrete-time switched affine systems via min-type multiple Lyapunov functions. In this work, a single ellipsoidal set which contains the actual
convergence set is used as an invariant set of attraction, which may lead to more conservative results in size estimation of the ultimate convergence set. Moreover, the authors propose a more complex methodology with two different theorems each of which associated to the attractive and invariant properties to compute the final invariant set of attraction.

A set of BMI conditions was proposed in [34] for the global practical stabilization of continuous-time switched affine systems in the framework of sampled-data systems and using switched Lyapunov functions. However, only attractive property of the convergence set is guaranteed.

In [18] and [20], [34] the global practical stability conditions have been proposed for discrete-time and continuous-time switched affine systems respectively via a common quadratic Lyapunov function. One limitation of these works is that the invariant set of attraction must contain an equilibrium point that belongs to a predetermined set of attainable ones. This is an issue for the applicability of the proposed conditions because these equilibrium points are generated via a (Schur or Hurwitz) stable matrix calculated via the convex combination of each of affine subsystems. On the other hand, checking the existence of a stable matrix as a convex combination of a family of matrices needs special algorithms and is an NP-hard problem [28], [37]–[40]. This limitation can be relaxed by utilizing multiple Lyapunov functions and proposing less conservative stability conditions [20], [34].

The problem of robust and global practical stabilization of the switched affine systems in the discrete-time domain was addressed in [41]. The proposed switching functions in this work are established based on the existence of a common Lyapunov function, however, no constructive method is proposed for its computation. In [32] and [33], the local and global practical stabilization of continuous-time nonlinear switched systems have been investigated respectively, via time-dependent switchings rather than the state-dependent counterpart and using switched Lyapunov functions.

B. The contributions, objectives and organization of this paper

According to the preceding literature review and up to our best knowledge, it is the first time that the switched quadratic Lyapunov function is adopted in the context of the global practical stabilization of discrete-time switched affine systems via state-dependent switching rules guaranteeing invariant and attractive properties of the convergence set simultaneously. Utilizing multiple Lyapunov functions not only yields less conservative stability conditions but also using a family of corresponding ellipsoids around the desired equilibrium point instead of a single one and guidance of the state trajectories to their intersection, yields a less conservative size of the ultimate invariant set of attraction. In this regard, the theoretical foundations of the practical stability of the discrete-time switched systems without a common equilibrium point are presented through some basic definitions of various types of practical stability in Section II and Lemma 6 in Section III. Next, we propose a state-dependent switching rule (switching Algorithm 9) together with a set of BMI-based stability conditions via Theorems 12 and 14 by which the global practical stability of discrete-time switched affine systems are guaranteed. In Section IV, the optimization problems corresponding to the stability conditions of Theorems 12 and 14 are formulated to minimize the size of the invariant set of attraction. Finally, Section V discusses the applicability of the proposed stabilization method on a DC-DC buck converter as an illustrative example and concluding remarks are made in Section VI.

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Stabilization problem around the null equilibrium point by defining the error state vector $e \in \mathbb{R}^n$ isolated subsystems, namely, $x(\sigma)$

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Definition 1. The bounded set $V \subseteq D$ containing a ball $B_r = \{ e \in \mathbb{R}^n | ||e|| \leq r \}$, $r > 0$ is an invariant set of attraction on a given domain $D \subset \mathbb{R}^n$ for the system (2) by the switching function $\sigma(e(k))$ if the following conditions are simultaneously satisfied:
(a) $0_{n \times 1} \in \mathcal{V}$
(b) If $e(k) \in \mathcal{V}$ then $e(k + 1) = A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))} \in \mathcal{V}$.
(c) If $e(0) \in D - \mathcal{V}$ then there is $T = T(0) > 0$ such that $e(k) \in \mathcal{V}$ for $\forall k \geq T$.

Based on condition (b), the trajectories starting within $\mathcal{V}$, they never escape from it and therefore, according to (a) and boundedness of $\mathcal{V}$, they will remain around the null set point. If conditions (a) and (b) are satisfied it is called that the set $\mathcal{V}$ is invariant for system (2) under switching function $\sigma(e(k))$. Furthermore, according to condition (a) and (c), the trajectories starting outside $\mathcal{V}$ evolve in time towards the point $e(k) = 0_{n \times 1}$, but they never reach it. In this case, it is called that the set $\mathcal{V}$ is attractive for system (2) under switching function $\sigma(e(k))$.

Def. 1 falls into the context of practical stability by which it is meant that the trajectories either tend to the set $\mathcal{V}$ or remain inside it. Defs. 2, 3 and 4 clarify these notions.

**Definition 2.** System (2) is locally practically stable with respect to an invariant set of attraction $\mathcal{V}$ on the domain $D$ under switching function $\sigma(e(k))$ if there exist sets $\mathcal{V}$ and $D$ satisfying conditions of Def. 1 and $\mathcal{V} = D$.

**Definition 3.** System (2) is practically stable in the large with respect to an invariant set of attraction $\mathcal{V}$ on the domain $D$ under switching function $\sigma(e(k))$ if there exist sets $\mathcal{V}$ and $D$ satisfying conditions of Def. 1 and $\mathcal{V} \subseteq D$.

**Definition 4.** System (2) is practically stable in the whole or globally practically stable if it is practically stable in the large and that $D = \mathbb{R}^n$.

**Definition 5.** In Defs. 2–4, the set $D - \mathcal{V}$ is called as domain of attraction of the system (2) under switching rule $\sigma(e(k))$.

### III. STABILITY ANALYSIS AND CONTROLLER SYNTHESIS

In this section, the main results of this paper are presented. Lemma 6 states under what conditions system (2) is practically stable in the large in the sense of Def. 3 via Lyapunov functions.

**Lemma 6.** System (2) is practically stable in the large in a given domain $D \subset \mathbb{R}^n$ containing the origin in the sense of Def. 3 if there exist a bounded set $\mathcal{V} \subset D$ and a scalar function $v(e(k)) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that

(a) $0_{n \times 1} \in \mathcal{V}$
(b) If $e(k) \in \mathcal{V}$ then $e(k + 1) = A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))} \in \mathcal{V}$
(c) If $e(k) \in D - \mathcal{V}$ then $v(e(k + 1)) - v(e(k)) = \Delta v(e(k)) \leq -\phi(||e(k)||) < 0$ where $\phi(||e(k)||)$ is a nondecreasing scalar function such that $\phi(0) = 0$ and $\phi(||e(k)||) > 0$, $\forall e(k) \in D - \mathcal{V}$.
(d) $v(e(k)) \geq w(||e(k)||) > 0$ where $e(k) \in D - \mathcal{V}$ and $w(||e(k)||)$ is a continuous and nondecreasing scalar function such that $w(0) = 0$.

**Proof.** Conditions (a) and (b) are the same as conditions (a) and (b) of Def. 1 and therefore, fulfill the invariant property of the bounded set $\mathcal{V}$. To prove the attractive property of the set $\mathcal{V}$, according to condition (c) of Def. 1 it is necessary to show that starting any initial state $e(0) \in D - \mathcal{V}$, there exists a finite time $T = T(e(0)) > 0$ such
that for \( k \geq T \) the state \( e(k) \) eventually enters within the set \( \mathcal{V} \), i.e., \( \exists T > 0 \) such that \( e(k) \in \mathcal{V} \) for \( k \geq T \). We show this at two stages as follows. Since according to conditions (c) and (d), the sequence \( v(e(k)) \) is decreasing and lower bounded, then based on Weierstrass theorem \( \lim_{k \to \infty} v(e(k)) = h \geq 0 \). Therefore, the following two cases are introduced.

**Case 1.** \( h > 0 \): Our proof in this case is made by contradiction. Assume that the state trajectory \( e(k) \) never intersects \( \mathcal{V} \). Now since \( h > 0 \), this implies there exists \( \gamma > 0 \) such that \( \|e(k)\| > \gamma, \forall k \in \mathbb{Z}_{\geq 0} \). Then, since \( \phi(\|e(k)\|) \) is a nondecreasing function, we have \( v(e(k+1)) - v(e(k)) \leq -\phi(\gamma) < 0 \). In view of this, one can write

\[
v(e(k)) = v(e(0)) + \sum_{n=0}^{k-1} (v(e(n+1)) - v(e(n))) \leq v(e(0)) - k\phi(\gamma)
\]

The right side of Eq. (4) will be eventually negative when \( k \) takes large values. This leads to contradiction to the condition (d) where it is assumed \( v(e(k)) \) is positive definite on \( D - \mathcal{V} \).

**Case 2.** \( h = 0 \): On the other hand, if \( h = 0 \), then according to (d) one can write

\[
0 = \lim_{k \to \infty} v(e(k)) \geq \lim_{k \to \infty} w(\|e(k)\|) \geq 0
\]

Relation (5) implies that \( \lim_{k \to \infty} w(\|e(k)\|) = 0 \). Since \( w(\|e(k)\|) \) is a continuous, nondecreasing and positive definite function, it can be concluded that \( \lim_{k \to \infty} \|e(k)\| = 0 \). This means that \( \forall \epsilon > 0, \exists T(\epsilon) > 0 \) such that if \( k > T(\epsilon) \) then \( \|e(k)\| < \epsilon \). Therefore, there exists a finite time \( T = T(\epsilon) > 0 \) such that for \( k \geq T \) the state \( e(k) \) eventually enters within the set \( \mathcal{V} \). As a result, \( \mathcal{V} \) is an invariant set of attraction according to Def. 1, and therefore, according to Def. 3, system (2) is practically stable in the large on the domain \( D \) and under switching function \( \sigma(e(k)) \). Thus, the proof is completed.

Lemma 6 does not propose how one can choose the bounded set \( \mathcal{V} \). One approach for selecting \( \mathcal{V} \) among all possibilities, is to choose the bounded level sets of the function \( v(e(k)) \) as

\[
\mathcal{V} = \{e(k) \in \mathbb{R}^n | v(e(k)) \leq r\}
\]

with \( r > 0 \). In this regard, we need additional condition on the function \( v(e(k)) \) in such a way that guarantees the boundedness of its level sets \( \mathcal{V} \) defined in Eq. (6). This condition is stated as a growth condition presented in Eq. (7).

\[
\lim_{\|e(k)\| \to \infty} v(e(k)) \to \infty
\]

The function satisfying Relation (7) is said to be *radially unbounded*. Condition (7), implies that \( \forall r > 0, \exists R > 0 \) such that

\[
\|e(k)\| > R \Rightarrow v(e(k)) > r
\]

A contrapositive statement of Relation (8) is as

\[
v(e(k)) \leq r \Rightarrow \|e(k)\| \leq R
\]

which implies that \( \mathcal{V} \subset B_R \) where \( B_R \) denotes the ball defined as \( B_R = \{e(k) \in \mathbb{R}^n | \|e(k)\| \leq R\} \). As a result, the set \( \mathcal{V} \) defined in Eq. (6) is bounded.
A. State-dependent switching rule synthesis

(b) and (c) of Lemma 6 are fulfilled for system (2) as well.

subsection, we propose a switching rule and some conditions on matrices $P$ of the form as in (2). In this context, just replace condition (b) in Def. 1 and Lemma 6 by the following one:

(11)

Remark 7. Defs. 1–5, Lemma 6 and its proof can be applied to the nonlinear switched systems without a common equilibrium point where the functions $f_i(e(k))$, $i \in \mathbb{K}$ in $e(k+1) = f_i(e(k))$ do not necessarily need to be in the affine form as in (2). In this context, just replace condition (b) in Def. 1 and Lemma 6 by the following one:

(b) if $e(k) \in V$ then $e(k+1) = f_{\sigma(e(k))}(e(k)) \in V$

Remark 8. In Lemma 6, if $D = \mathbb{R}^n$, then according to Def. 4, system (2) is globally practically stable.

Note that the switched Lyapunov function $v(e(k)) = e(k)^TP_{\sigma(e(k))}e(k)$, with $P_i^T = P_i$, $i \in \mathbb{K}$ is a positive definite function and radially unbounded since $v(e(k)) \geq \min_{i \in \mathbb{K}} \lambda_{\min}(P_i)\|e(k)\|^2$. Moreover, $0_{n \times 1} \in V$, where the set $V$ is defined as in (3). It can be verified that the level set of the switched Lyapunov function $v(e(k)) = e(k)^TP_{\sigma(e(k))}e(k)$ is the union of the ellipsoids $E_i = \{e(k) \in \mathbb{R}^n|e(k)^TP_i e(k) \leq r\}$. By dividing both sides of $e(k)^TP_i e(k) \leq r$ to $r$ and substituting $P_i \leftarrow \frac{P_i}{r}$, the ellipsoids $E_i$ can be represented as $E_i = \{e(k) \in \mathbb{R}^n|e(k)^TP_i e(k) \leq 1\}$ with the same notation in (3). Since in the context of practical stability of switched systems it is targeted to obtain the invariant set of attraction with small size as far as possible [18], [19], [33], [42], therefore, in this paper, we select the intersection of the ellipsoids $E_i$ as in Eq. (3) instead of their union. Note that since

the boundedness of the right side of Relation (10) guarantees the boundedness of its left side. Furthermore, it is obvious that $0_{n \times 1} \in V$ where $V$ is given in (3). Therefore, the set $V$ defined in (3) is bounded and the switched Lyapunov function $v(e(k)) = e(k)^TP_{\sigma(e(k))}e(k)$ satisfies the conditions (a) and (d) of Lemma 6. In the next subsection, we propose a switching rule and some conditions on matrices $P_i = P_i^T > 0$ such that the conditions (b) and (c) of Lemma 6 are fulfilled for system (2) as well.

A. State-dependent switching rule synthesis

As discussed earlier, we are planning to design a switching rule $\sigma(e(k))$ for the system (2) that drives the state trajectories towards the set $V$ defined in Eq. (3) via a switched Lyapunov function $v(e(k)) = V_{\sigma(e(k))} = e(k)^TP_{e(k)}$, $P_i = P_i^T > 0$. In this paper, we use a min-type state feedback switching function described as follows:

Algorithm 9. Switching Law:

1) Set $k = 0$.

2) If the trajectory is outside the set $V$ given in (3), i.e., $e(k) \notin V$, then switch to the subsystem $\sigma(e(k))$ given by the following switching function

(11)

3) If the trajectory is inside the set $V$ defined in Eq. (3), i.e., $e(k) \in V$, then firstly construct the set $\mathcal{I} = \{j \in \mathbb{K}:(A_j e(k) + l_j)^TP_i (A_j e(k) + l_j) - e(k)^TP_j e(k)\}$.

Now:

3.1) If $|\mathcal{I}| = 1$, then switch to the subsystem $\sigma(e(k)) = j$, such that $j \in \mathcal{I}$.
3.2) If $|\mathcal{I}| > 1$, then compute $N$ switching indices $\sigma_i = \arg\min_{j \in \mathcal{I}} (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j)$, $\forall i \in \mathbb{K}$. If $\sigma_1 = \sigma_2 = \ldots = \sigma_N$, then switch to subsystem $\sigma(e(k)) = \sigma_j$, $j \in \mathbb{K}$. Otherwise, choose $\sigma(e(k)) = \arg\min_{j \in \mathcal{I}} \text{tr}(P_j^{-1})$.

4) Put $k = k + 1$ and go to item 2).

Set $\mathcal{I}$ in the above switching algorithm, contains all subsystem indices that by switching to them the state trajectory $e(k + 1)$ will remain in the invariant set of attraction $\mathcal{V}$. In item 3.2), if there is a degree of freedom for the controller in choosing suitable subsystems to keep the trajectory inside the set $\mathcal{V}$, then either the controller selects the subsystem that governs the state trajectory as close as possible towards to the null equilibrium point, or it selects the subsystem that moves the state trajectory towards to the ellipsoid $\mathcal{E}_i = \{ e(k) \in \mathbb{R}^n | e(k)^T P_i e(k) \leq 1 \}$ with minimum size in the sense of sum of the squares of the ellipsoid semi-axes [42]. Lemmas 10 and 11 are frequently used in the next derivations.

**Lemma 10.** For a set of functions $f_i : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, $i \in \mathbb{K}$ as $f_1(x), \ldots, f_i(x), \ldots, f_N(x)$, the following statements are equivalent.

- i) $\forall x \in D, \exists \lambda \in \mathbb{K}$, such that $f_i(x) < 0$.
- ii) $\forall x \in D, \exists (\lambda_1 \geq 0, \ldots, \lambda_i \geq 0, \ldots, \lambda_N \geq 0)$ such that $\sum_{i \in \mathbb{K}} \lambda_i f_i(x) < 0$, and $\sum_{i \in \mathbb{K}} \lambda_i > 0$.

**Proof.** i) $\Rightarrow$ ii) Let us assume, $\forall x \in D \subseteq \mathbb{R}^n$, at least one of the functions, say $f_l(x)$, $l \in \mathbb{K}$ satisfies the inequality $f_l(x) < 0$. Now, by choosing the set of parameters $\lambda_i > 0, \lambda_l = 0, i \neq l$, $i \in \mathbb{K}$, one can conclude $\sum_{i \in \mathbb{K}} \lambda_i f_i(x) = \lambda_l f_l(x) < 0$ and $\sum_{i \in \mathbb{K}} \lambda_i = \lambda_l > 0$.

ii) $\Rightarrow$ i) This can be shown by contradiction: Let us assume $\exists x \in D \subseteq \mathbb{R}^n$ such that all $f_l(x)$, $l \in \mathbb{K}$ fulfill $f_l(x) \geq 0$. By multiplication of these inequalities to $\lambda_l \geq 0$, such that $\sum_{i \in \mathbb{K}} \lambda_i > 0$, and summing all the terms one can reach $\sum_{i \in \mathbb{K}} \lambda_l f_l(x) \geq 0$. But this contradicts the fact that $\forall x \in D \subseteq \mathbb{R}^n$ there exists a set of nonnegative parameters $\lambda_i \geq 0$, $l \in \mathbb{K}$ such that $\sum_{i \in \mathbb{K}} \lambda_i f_l(x) < 0$. \hfill $\square$

Lemma 11 is a nonstrict form of Lemma 10.

**Lemma 11.** For a set of functions $f_i : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$, $i \in \mathbb{K}$ as $f_1(x), \ldots, f_i(x), \ldots, f_N(x)$, the following statements are equivalent.

- i) $\forall x \in D, \exists \lambda \in \mathbb{K}$ such that $f_i(x) \geq 0$.
- ii) $\forall x \in D, \exists (\lambda_1 \geq 0, \ldots, \lambda_i \geq 0, \ldots, \lambda_N \geq 0)$ such that $\sum_{i \in \mathbb{K}} \lambda_i f_i(x) \geq 0$, and $\sum_{i \in \mathbb{K}} \lambda_i > 0$.

**Proof.** The proof is similar to the proof of Lemma 10 and is omitted for the sake of brevity. \hfill $\square$

Theorem 12 provides sufficient conditions for which the switched affine system (2) under switching function (11) is globally practically stable in the sense of Def. 1.
Theorem 12. If there exist matrices $P_i^T = P_i > 0$ and nonnegative numbers $\beta_{hi} \geq 0$, $\lambda_{hj} \geq 0$, $\lambda_j \geq 0$, $\tau_{1h} \geq 0$, $\tau_{2h} \geq 0$, $i, j, h \in \mathbb{K}$, such that $\sum_{i \in \mathbb{K}} \beta_{hi} > 0$, $\sum_{j \in \mathbb{K}} \lambda_{hj} > 0$, $\sum_{j \in \mathbb{K}} \lambda_j > 0$ satisfying the system of inequalities

\[
\begin{bmatrix}
M_{1i} - \sum_{h \in \mathbb{K}} \tau_{1h} P_h & \ast \\
M_{2i} & M_{3i} + \sum_{h \in \mathbb{K}} \tau_{1h}
\end{bmatrix} \leq 0
\quad \forall i \in \mathbb{K}
\]

(12)

\[
\begin{bmatrix}
\tau_{2h} P_h + \tilde{M}_{1h} & \ast \\
\tilde{M}_{2h} & \tilde{M}_{3h} - \tau_{2h}
\end{bmatrix} \prec 0
\quad \forall h \in \mathbb{K}
\]

(13)

where

\[
M_{1i} = \sum_{j \in \mathbb{K}} \lambda_j A_j^T P_i A_j
\]

(14)

\[
M_{2i} = \sum_{j \in \mathbb{K}} \lambda_j j^T P_i A_j
\]

(15)

\[
M_{3i} = \sum_{j \in \mathbb{K}} \lambda_j (j^T P_i j - 1)
\]

(16)

\[
\tilde{M}_{1h} = \sum_{i \in \mathbb{K}} \sum_{j \in \mathbb{K}} \beta_{hi} \lambda_{hj} (A_j^T P_i A_j - P_j)
\]

(17)

\[
\tilde{M}_{2h} = \sum_{i \in \mathbb{K}} \sum_{j \in \mathbb{K}} \beta_{hi} \lambda_{hj} j^T P_i A_j
\]

(18)

\[
\tilde{M}_{3h} = \sum_{i \in \mathbb{K}} \sum_{j \in \mathbb{K}} \beta_{hi} \lambda_{hj} j^T P_i j
\]

(19)

then the switching strategy in Algorithm 9 assures that the system (2) to be globally practically stable in the sense of Def. 4 and the set $\mathcal{V}$ in (3) to be an invariant set of attraction on the domain $D = \mathbb{R}^n$ in the sense of Def. 1.

Proof: It is shown that matrix inequalities (12) imply the invariant property (conditions (a) and (b) of Def. 1), while matrix inequalities (13) yield the attractive property (conditions (a) and (c) of Def. 1). Pre-multiplying (12) by $[e(k)^T 1]^T$ and post-multiplying by $[e(k)^T 1]^T$ one can reach

\[
\begin{bmatrix}
e(k) \\ 1
\end{bmatrix}^T \begin{bmatrix}
M_{1i} - \sum_{h \in \mathbb{K}} \tau_{1h} P_h & \ast \\
M_{2i} & M_{3i} + \sum_{h \in \mathbb{K}} \tau_{1h}
\end{bmatrix} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix} \leq 0, \forall i \in \mathbb{K}
\]

(20)

Relation (20) can be rewritten as Relation (21).

\[
-\tau_{11} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix}^T \begin{bmatrix}
P_1 & \ast \\
0_{1 \times n} & -1
\end{bmatrix} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix} \ldots -\tau_{1h} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix}^T \begin{bmatrix}
P_h & \ast \\
0_{1 \times n} & -1
\end{bmatrix} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix} \ldots
\]

\[
-\tau_{1N} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix}^T \begin{bmatrix}
P_N & \ast \\
0_{1 \times n} & -1
\end{bmatrix} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix} + \begin{bmatrix}
e(k) \\ 1
\end{bmatrix}^T \begin{bmatrix}
M_{1i} & \ast \\
M_{2i} & M_{3i}
\end{bmatrix} \begin{bmatrix}
e(k) \\ 1
\end{bmatrix} \leq 0, \forall i \in \mathbb{K}
\]

(21)
Using S-procedure [43], [44], Relation (21) implies Relation (22).

\[
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
P_t & * \\
0_{1 \times n} & -1
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}
\leq 0, \ldots, \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
P_h & * \\
0_{1 \times n} & -1
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}
\leq 0, \forall i \in \mathbb{K}
\] (22)

By substituting \( M_{1i}, M_{2i}, \) and \( M_{3i} \) from Eqs. (14)-(16) into Relation (22) one can write

\[
e(k)^T P_t e(k) \leq 1, \ldots, e(k)^T P_h e(k) \leq 1, \ldots, \land e(k)^T P_N e(k) \leq 1
\]

\[
\sum_{j \in \mathbb{K}} \lambda_j (e(k)^T A_j^T P_t A_j e(k) + l_j^T P_t A_j e(k) + e(k)^T A_j^T P_j l_j + l_j^T P_j l_j - 1) \leq 0, \forall i \in \mathbb{K}
\] (23)

Since \( \lambda_j \geq 0, j \in \mathbb{K} \) and \( \sum_{j \in \mathbb{K}} \lambda_j > 0 \), according to Lemma 11, (23) is equivalent to

\[
e(k)^T P_t e(k) \leq 1, \ldots, e(k)^T P_h e(k) \leq 1, \ldots, \land e(k)^T P_N e(k) \leq 1
\]

\[
\exists j \in \mathbb{K} \quad \text{such that} \quad (e(k)^T A_j e(k) + l_j^T P_t A_j e(k) + e(k)^T A_j^T P_j l_j + l_j^T P_j l_j - 1) \leq 0, \forall i \in \mathbb{K}
\] (24)

After some algebra at right of Relation (24), one can reach

\[
e(k)^T P_t e(k) \leq 1, \ldots, e(k)^T P_h e(k) \leq 1, \ldots, \land e(k)^T P_N e(k) \leq 1
\]

\[
\exists j \in \mathbb{K} \quad \text{such that} \quad (A_j e(k) + l_j)^T P_t (A_j e(k) + l_j) \leq 1, \forall i \in \mathbb{K}
\] (25)

According to Relation (25) and based on the item 3) of the switching rule in Algorithm 9, one can conclude that \( |\mathcal{I}| \neq 0 \). As a result, Relation (25) can be rewritten as

\[
e(k)^T P_t e(k) \leq 1, \ldots, e(k)^T P_h e(k) \leq 1, \ldots, \land e(k)^T P_N e(k) \leq 1
\]

\[
\exists \sigma(e(k)) \in \mathbb{K} \quad \text{such that} \quad (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))})^T P_t (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))}) \leq 1, \forall i \in \mathbb{K}
\] (26)

Now using the definition of set \( \mathcal{V} \) in Eq. (3), Relation (26) can be rewritten as

\[
e(k) \in \mathcal{V} = \bigcup_{i=1}^{N} \mathcal{E}_i \Rightarrow e(k + 1) \in \mathcal{V} = \bigcap_{i=1}^{N} \mathcal{E}_i
\] (27)

Relation (27) implies that the set \( \mathcal{V} \) in Eq. (3) is an invariant set for the switched affine system (2) under switching law in Algorithm 9. In the sequel, we plan to prove the attractive of the set \( \mathcal{V} \) via matrix inequalities (13).

Pre-multiplying matrix inequalities (13) by \([e(k)^T 1]\) and post-multiplying by \([e(k)^T 1]^T\) one can obtain

\[
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
\tau_{22h} P_h + \tilde{M}_{1h} \\
\tilde{M}_{2h} & \tilde{M}_{3h} - \tau_{22h}
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}
< 0, \forall h \in \mathbb{K}
\] (28)

Relation (28) can be rewritten as Relation (29).

\[
-\tau_{22h}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
-P_h & * \\
0_{1 \times n} & 1
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}
+ \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
\tilde{M}_{1h} & * \\
\tilde{M}_{2h} & \tilde{M}_{3h}
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}
< 0, \forall h \in \mathbb{K}
\] (29)
Using S-procedure, from Relation (29) one can conclude (30), \( \forall e(k) \in \mathbb{R}^n \):

\[
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
-P_h & * \\
0_{1 \times n} & 1
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix} < 0 \Rightarrow \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T
\begin{bmatrix}
\tilde{M}_{1h} & * \\
\tilde{M}_{2h} & \tilde{M}_{3h}
\end{bmatrix}
\begin{bmatrix}
e(k) \\
1
\end{bmatrix} < 0, \forall h \in \mathbb{K}
\]

(30)

By substituting \( \tilde{M}_{1h}, \tilde{M}_{2h} \) and \( \tilde{M}_{3h} \) from Eqs. (17)-(19) into Relation (30) and after some algebra, one can reach

\[
e(k)^T P_h e(k) > 1 \Rightarrow \sum_{i \in \mathbb{K}} \beta_{hi} \left( \sum_{j \in \mathbb{K}} \lambda_{hj} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)] \right) < 0, \forall h \in \mathbb{K}
\]

(31)

Since \( \beta_{hi} \geq 0, i, h \in \mathbb{K} \), and \( \sum_{i \in \mathbb{K}} \beta_{hi} > 0 \) according to Lemma 10 and (31) one can conclude

\[
e(k)^T P_h e(k) > 1 \Rightarrow \exists i \in \mathbb{K} \text{ such that } \sum_{j \in \mathbb{K}} \lambda_{hj} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)] < 0, \forall h \in \mathbb{K}
\]

(32)

Again according to Lemma 10, since \( \lambda_{hj} \geq 0, j, h \in \mathbb{K} \) and \( \sum_{j \in \mathbb{K}} \lambda_{hj} > 0 \), Relation (32) implies

\[
e(k)^T P_h e(k) > 1 \Rightarrow \exists i, j \in \mathbb{K} \text{ such that } (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0, \forall h \in \mathbb{K}
\]

(33)

Now, according to the definition of the set \( V \) in (3), we have

\[
e(k) \notin V \Rightarrow \exists h \in \mathbb{K} \text{ such that } e(k)^T P_h e(k) > 1
\]

(34)

From Relations (33) and (34), one can infer

\[
e(k) \notin V \Rightarrow \exists i, j \in \mathbb{K} \text{ such that } (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0
\]

(35)

since \( e(k) \notin V \), according to Relation (35) and item 2) in the switching Algorithm 9, one can conclude there exist \( \sigma(e(k)) \), \( i, j \in \mathbb{K} \) satisfying the following expression

\[
(A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))})^T P_i (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))}) - e(k)^T P_{\sigma(e(k))} e(k) \\
\leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)
\]

(36)

From Relations (35) and (36) one can conclude that there exist \( \sigma(e(k)) \), \( i \in \mathbb{K} \) such that

\[
e(k) \notin V \Rightarrow \exists i, j, \sigma(e(k)) \in \mathbb{K} \text{ such that } (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))})^T P_i (A_{\sigma(e(k))} e(k) + l_{\sigma(e(k))}) - e(k)^T P_{\sigma(e(k))} e(k)
\leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0
\]

(37)

or equivalently

\[
e(k) \notin V \Rightarrow \exists i, j \in \mathbb{K} \text{ such that } \nu(e(k+1)) - v(e(k)) = \Delta v(e(k)) \leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0
\]

(38)

since \( v(e(k)) \geq \min_{i \in \mathbb{K}} \lambda_{\min}(P_i) \| e(k) \|^2 > 0 \) when \( e(k) \neq 0_{n \times 1} \), then according item (d) of Lemma 6 the continuous, nondecreasing and positive definite scalar function \( w(||e(k)||) \) can be taken as \( w(||e(k)||) = \min_{i \in \mathbb{K}} \lambda_{\min}(P_i) || e(k) ||^2. \)
Moreover, according to item (c) of Lemma 6 we still need to find a nondecreasing and positive definite function \( \phi(\|e(k)\|) \) such that \( \Delta v(e(k)) \leq \phi(\|e(k)\|) < 0 \) when \( e(k) \in D - V \). In this regard, we define

\[
\phi_{i,j}(e(k)) = e(k)^T P_j e(k) - (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j), (i,j) \in S
\]

\[
S = \{(i,j) \in \mathbb{K} \times \mathbb{K} | \Delta v(e(k)) \leq -\phi_{i,j}(e(k)) < 0 \}
\]

According to Relation (38), \(|S| \geq 1 \) and \( \phi_{i,j}(e(k)) > 0 \) when \( e(k) \notin V \). Now we define the function \( \phi(s) \) as

\[
\phi(s) = \inf_{s \leq \|e(k)\|} \min_{(i,j) \in S} \phi_{i,j}(e(k))
\]

(41)

The function \( \phi(\|e(k)\|) \) is nondecreasing and positive definite when \( e(k) \notin V \). Moreover, according to Eq. (41) and Relation (38), one can write

\[
\Delta v(e(k)) \leq -\phi_{i,j}(e(k)) \leq -\phi(\|e(k)\|) < 0, (i,j) \in S
\]

(42)

Relation (42) in conjunction with \( 0_{n \times 1} \in V \) imply the attractive property of the set \( V \) according to items (a), (c) and (d) of Lemma 6. Therefore, according to Relations (42) and (27), all conditions of Lemma 6 are fulfilled. Since during proof of Theorem 12 no restriction is imposed on the selection of \( e(k), \) i.e., \( e(k) \in \mathbb{R}^n \), as a result \( D = \mathbb{R}^n \) and according to Rem. 8 the switched affine system (2) is globally practically stable under switching Algorithm 9 and the proof is concluded.

Remark 13. The constraints in Relation (13) are highly non-convex due to the product of variables \( \{\beta_{hi}, \lambda_{hi}, P_i\} \) in (17)-(19). A more conservative, however simpler to solve conditions can be obtained by prefixing the variables \( \beta_{hi} \). In this regard, we define the matrix \( B \) whose elements are the parameters \( \beta_{hi} \geq 0, (h, i) \in \mathbb{K} \times \mathbb{K} \). Among all possibilities, one can choose matrix \( B \) as a simple form \( B = cI_n, \) with \( c > 0 \). Another alternative is \( B = c1_n, \) \( c > 0 \). With this approach all the conditions of Theorem 12 can be given as a set of BMIs without introducing the additional parameters \( \beta_{hi} \).

Theorem 14 presents another set of BMI conditions for the global practical stability of the system (2) under switching Algorithm 9 that compared to the proposed conditions in Theorem 12, are more conservative. However, the new conditions in Theorem 14 are intrinsically in BMIs form and do not require any prefixing stage of variables as in the case of Theorem 12 to come in the BMI form. Moreover, the number of unknown variables in the conditions of Theorem 12 is \( 2N^2 + 3N + Nn(n + 1)/2 \), while in Theorem 14 this number is \( 2N^2 + 2N + Nn(n + 1)/2 \). Therefore, the number of unknown parameters in the BMI conditions of Theorem 14 is less than that of proposed conditions in Theorem 12 by a quantity of \( N \).
\textbf{Theorem 14.} If there exist matrices $P_i^T = P_i > 0$ and nonnegative numbers $\lambda_{ij} \geq 0$, $\lambda_j \geq 0$, $\tau_{1h} \geq 0$, $\tau_{2hi} \geq 0$, $i, j, h \in \mathbb{K}$, such that $\sum_{j \in \mathbb{K}} \lambda_{ij} > 0$, $\sum_{j \in \mathbb{K}} \lambda_j > 0$, satisfying the system of inequalities

\[
\begin{bmatrix}
M_{1i} - \sum_{h \in \mathbb{K}} \tau_{1h} P_h & \ast \\
M_{2i} & M_{3i} + \sum_{h \in \mathbb{K}} \tau_{1h}
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
\tau_{2hi} P_h + \tilde{M}_{1hi} & \ast \\
\tilde{M}_{2hi} & \tilde{M}_{3hi} - \tau_{2hi}
\end{bmatrix} < 0
\]

where

\[
M_{1i} = \sum_{j \in \mathbb{K}} \lambda_j A_j^T P_i A_j
\]

\[
M_{2i} = \sum_{j \in \mathbb{K}} \lambda_j (l_j^T P_i l_j - 1)
\]

\[
M_{3i} = \sum_{j \in \mathbb{K}} \lambda_j (A_j^T P_i A_j - P_j)
\]

\[
\tilde{M}_{1hi} = \sum_{j \in \mathbb{K}} \lambda_{ij} (A_j^T P_i A_j - P_j)
\]

\[
\tilde{M}_{2hi} = \sum_{j \in \mathbb{K}} \lambda_{ij} l_j^T P_i l_j
\]

\[
\tilde{M}_{3hi} = \sum_{j \in \mathbb{K}} \lambda_{ij} l_j^T P_i l_j
\]

then the switching strategy in Algorithm 9 assures that the system (2) to be globally practically stable in the sense of Def. 4 and the set $\mathcal{V}$ in Eq. (3) to be an invariant set of attraction on the domain $D = \mathbb{R}^n$ in the sense of Def. 1.

\textbf{Proof:} Since the conditions in Relation (43) are the same as the conditions (12) in Theorem 12, the proof that the set $\mathcal{V}$ in Eq. (3) is invariant under switching Algorithm 9 for the system (2) is similar to the first part of the proof the Theorem 12 from (20)-(27). In the sequel, the attractive property of the set $\mathcal{V}$ is proved via conditions in (44). Pre-multiplying matrix inequality (44) by $[e(k)^T 1]$ and post-multiplying by $[e(k)^T 1]^T$ one can obtain

\[
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T \begin{bmatrix}
\tau_{2hi} P_h + \tilde{M}_{1hi} & \ast \\
\tilde{M}_{2hi} & \tilde{M}_{3hi} - \tau_{2hi}
\end{bmatrix} \begin{bmatrix}
e(k) \\
1
\end{bmatrix} < 0, \forall i, h \in \mathbb{K}
\]

Relation (51) can be rewritten as:

\[
-\tau_{2hi} \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T \begin{bmatrix}
-P_h & \ast \\
0_{1 \times n} & 1
\end{bmatrix} \begin{bmatrix}
e(k) \\
1
\end{bmatrix} + \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T \begin{bmatrix}
\tilde{M}_{1hi} & \ast \\
\tilde{M}_{2hi} & \tilde{M}_{3hi}
\end{bmatrix} \begin{bmatrix}
e(k) \\
1
\end{bmatrix} < 0, \forall i, h \in \mathbb{K}
\]

Using S-procedure, from Relation (52) one can reach to (53), $\forall e(k) \in \mathbb{R}^n$. 

\[
\begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T \begin{bmatrix}
-P_h & \ast \\
0_{1 \times n} & 1
\end{bmatrix} \begin{bmatrix}
e(k) \\
1
\end{bmatrix} < 0 \Rightarrow \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T \begin{bmatrix}
\tilde{M}_{1hi} & \ast \\
\tilde{M}_{2hi} & \tilde{M}_{3hi}
\end{bmatrix} \begin{bmatrix}
e(k) \\
1
\end{bmatrix}^T < 0, \forall i, h \in \mathbb{K}
\]
By substituting $\bar{M}_{1hi}, \bar{M}_{2hi}$ and $\bar{M}_{3hi}$ from Eqs. (48)-(50) into Relation (53) and after some algebra, one can reach
\begin{equation}
 e(k)^TP_h e(k) > 1 \Rightarrow \sum_{j \in \mathbb{K}} \lambda_{hj} [(A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k)] < 0, \forall i, h \in \mathbb{K} \tag{54}
\end{equation}
Since $\lambda_{hj} \geq 0, j, h \in \mathbb{K},$ and $\sum_{j \in \mathbb{K}} \lambda_{hj} > 0,$ according to Lemma 10, Relation (54) implies Relation (55).
\begin{equation}
 e(k)^TP_h e(k) > 1 \Rightarrow \exists j \in \mathbb{K} \text{ such that } (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0, \forall i, h \in \mathbb{K} \tag{55}
\end{equation}
According to the definition of the set $\mathcal{V}$ in (3), we have
\begin{equation}
 e(k) \notin \mathcal{V} \Rightarrow \exists h \in \mathbb{K} \text{ such that } e(k)^TP_h e(k) > 1 \tag{56}
\end{equation}
Now from Relations (55) and (56), one can reach
\begin{equation}
 e(k) \notin \mathcal{V} \Rightarrow \exists j \in \mathbb{K} \text{ such that } (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0, \forall i \in \mathbb{K} \tag{57}
\end{equation}
since $e(k) \notin \mathcal{V},$ according to Relation (57) and item 2) in the switching Algorithm 9, one can conclude there exists a $\sigma(e(k))$ satisfying the following expression:
\begin{equation}
 (A_{\sigma(e(k)))} e(k) + l_{\sigma(e(k)))})^T P_i (A_{\sigma(e(k)))} e(k) + l_{\sigma(e(k)))}) - e(k)^T P_{\sigma(e(k)))} e(k) \leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k), \forall i \in \mathbb{K} \tag{58}
\end{equation}
From Relations (57) and (58) one can reach
\begin{equation}
 e(k) \notin \mathcal{V} \Rightarrow \exists j \in \mathbb{K} \text{ such that } (A_{\sigma(e(k)))} e(k) + l_{\sigma(e(k)))})^T P_i (A_{\sigma(e(k)))} e(k) + l_{\sigma(e(k)))}) - e(k)^T P_{\sigma(e(k)))} e(k) \leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0, \forall i \in \mathbb{K} \tag{59}
\end{equation}
or equivalently
\begin{equation}
 e(k) \notin \mathcal{V} \Rightarrow \exists j \in \mathbb{K} \text{ such that } v(e(k + 1)) - v(e(k)) = \Delta v(e(k)) \leq (A_j e(k) + l_j)^T P_i (A_j e(k) + l_j) - e(k)^T P_j e(k) < 0, \forall i \in \mathbb{K} \tag{60}
\end{equation}
Similar to our argument in Theorem 12, it can be shown that there exist functions $u(\|e(k)\|)$ and $\phi(\|e(k)\|)$ that satisfy the respective properties in Lemma 6. This discussion is omitted here for the sake of the brevity. Therefore, the attractive property of the set $\mathcal{V}$ is inferred according to items (a), (c) and (d) of Lemma 6. Therefore, all conditions of Lemma 6 are fulfilled. Similar to Theorem 12, since no restriction is imposed on the selection of $e(k),$ namely, $e(k) \in \mathbb{R}^n,$ therefore $\mathcal{D} = \mathbb{R}^n$ and according to Rem. 8 the switched affine system (2) is globally practically stable under switching Algorithm 9 and the proof is completed.

**Remark 15.** Theorem 12 is less conservative than Theorem 14. As it can be seen from Relation (37) in Theorem 12, at any time the decreasing condition of Lyapunov function, i.e., $\Delta v(e(k)) < 0,$ is imposed at least on one of the matrices $P_i, i \in \mathbb{K},$ while according to Relation (59) in Theorem 14, it is applied to all matrices $P_i, i \in \mathbb{K}.$
IV. Minimization of the invariant set of attraction

As already discussed in Section III, we are interested to select the invariant set of attraction \( \mathcal{V} \) defined in (3) such that its size is minimized in some standard sense. As it can be seen from (3) this set is constructed from the intersection of \( N \) ellipsoids \( \mathcal{E}_i = \{ e \in \mathbb{R}^n \mid e(k)^T P_i e(k) \leq 1 \} \). Therefore, one approach to minimize the size of the set \( \mathcal{V} \) is to minimize the size of all ellipsoids \( \mathcal{E}_i \), \( i \in \mathbb{K} \). To minimize the size of the ellipsoid \( \mathcal{E}_i \) one can minimize the trace of the matrix \( P_i^{-1} \) that defines the sum of the squares of the ellipsoid \( \mathcal{E}_i \) semiaxes [42]. As a result, in this paper, we use the following optimization problem

\[
\inf_{P_i, \lambda_j, \lambda_{jh}, \tau_{1h}, \tau_{2h}} \sum_{i=1}^{i=N} tr(P_i^{-1})
\]

subject to \([ (12) - (13)] \) or \([ (43) - (44)] \)

\[
\lambda_j \geq 0, \lambda_{jh} \geq 0, \tau_{1h} \geq 0, \tau_{2h} \geq 0
\]

(61)

Since the tool YALMIP/MATLAB [45] in conjunction with the BMI solver PENBMI [46] are not able to handle the nonlinear terms \( tr(P_i^{-1}) \) in the objective function of (61), we consider an upper bound \( P_i^{-1} \leq t_i I_n \) with \( t_i > 0 \), \( P_i > 0 \) \( i \in \mathbb{K} \) which can be rewritten as \( I_n - t_i P_i \leq 0 \) with \( t_i > 0 \), \( P_i > 0 \) \( i \in \mathbb{K} \). Accordingly, (61) is replaced by the following optimization problem

\[
\inf_{P_i, \lambda_j, \lambda_{jh}, \tau_{1h}, \tau_{2h}, t_i} \sum_{i=1}^{i=N} t_i
\]

subject to \([ (12) - (13)] \) or \([ (43) - (44)] \)

\[
\lambda_j \geq 0, \lambda_{jh} \geq 0, \tau_{1h} \geq 0, \tau_{2h} \geq 0
\]

\[
I_n - t_i P_i \leq 0, t_i > 0, P_i = P_i^T > 0
\]

(62)

In this paper, the optimization problem in (62) is solved via the BMI solver PENBMI [46] interfaced by YALMIP [45].

During the numerical simulations in the next section, we realized that in some cases the optimization problem in (62) is still very nonlinear and non-convex to achieve suitable and acceptable results from the PENBMI package. In this regard, the optimization problem in (62) is accompanied with additional constraints to limit the search space and achieving desirable results from the solver, but at the price of a level of conservatism. These additional constraints are suggested as follows:

\[
\sum_{j \in \mathbb{K}} \lambda_{hj} = q_h, \forall h \in \mathbb{K}
\]

(63)

\[
\sum_{i \in \mathbb{K}} \lambda_j = q
\]

(64)

with \( q_h, q > 0 \). By taking limited and reasonable values of the parameters \( q_h \) and \( q \), one can control and limit the variation regions of non-negative variables \( \lambda_{hj} \) and \( \lambda_j \) in the conditions of Theorems 12 and 14. It should be noted that by addition of new constraints in Eqs. (63)-(64) to the conditions of Theorems 12 and 14, these theorems are still valid. The proof of this claim comes from Lemmas 10 and 11 where the constraint \( \sum_{i \in \mathbb{K}} \lambda_j > 0 \) is implied by more restrictive constraints in Eqs. (63) and (64).
V. APPLICATION

A DC-DC buck converter is shown in Fig. 1. By defining the state vector \( x(t) = [i_L(t), v_o(t)]^T \), the continuous dynamics in continuous conduction mode (CCM) associated with each mode are \( \dot{x}(t) = A_c x(t) + b_c, i \in \{1, 2\} \)

where

\[
A_c = \begin{bmatrix}
-\frac{E_b}{E} & -\frac{1}{L} \\
\frac{R}{C R + r C} \left(1 - C r C \frac{r e}{L}\right) & -\frac{1}{L} \\
\frac{1}{C R + r C} \left(1 + C r C \frac{R}{L}\right) & 0
\end{bmatrix}
\]


(65)

\[
b_c = \begin{bmatrix}
\frac{V_s}{L} \\
\frac{R}{C R + r C} \frac{r e}{L}
\end{bmatrix}, \quad A_{c2} = A_{c1}, b_{c2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

In (65), \( V_s = 50 \) V is the input DC voltage, \( R = 50 \) Ω is the load resistance, \( L = 2 \) mH is the inductor inductance, \( r_L = 0.5 \) Ω is the inductor resistance, \( C = 100 \) µF is the output capacitor capacitance and \( r_e = 0.1 \) Ω is the equivalent series resistance of the capacitor [47]. In the numerical simulations, to avoid ill conditioned matrix inequalities and make the problem more amenable for the numerical purposes, we use per unit parameters [34], [48]. In this regard, the base values for the chosen per unit system are \( v_{\text{base}} = 50 \) V, \( i_{\text{base}} = 2.5 \) A and \( T_{\text{base}} = 10 \) µs. The sampling time is set to \( T_s = 10 \) µs and as a result the maximum value of switching frequency is limited to \( \frac{1}{2T_s} = 50 \) kHz. The state-space matrices of the corresponding discrete-time system can be obtained as

\[
A_i = e^{A_i T_s}, b_i = \int_0^{T_s} e^{A_i t} dt b_{c3}
\]

where \( i \in \{1, 2\} \). Although the equilibrium point \( x_e \) can be chosen arbitrary at the expense of obtaining the set \( V \) with possibly greater size, in this work, the desired target is selected as the equilibrium point of the averaged system as \( x_e = (I_n - A_{\lambda r e f})^{-1} b_{\lambda r e f} = [0.2970, 0.7426]^T \) corresponding to \( \lambda_{r e f} = 0.75 \). Using the conditions of Theorem 12, i.e. (12)-(13), the solution of the optimization problem in (62) corresponding to \( B = 15 \) \( 1_2 \) and considering the new constraint in Eq. (64) as \( \lambda_1 + \lambda_2 = 6 \), yields \( t_1 = 0.0920 \) and \( t_2 = 0.2066 \), \( \lambda_{11} = 1.0184 E - 4 \), \( \lambda_{12} = 0.0381 E - 4 \), \( \lambda_{21} = 1.4846 E + 7 \), \( \lambda_{22} = 2.0762 E + 7 \), \( \lambda_1 = 4.4980 \), \( \lambda_2 = 1.4993 \), \( \tau_{11} = 5.9862 \), \( \tau_{12} = -0.0826 \), \( \tau_{21} = -0.9141 E - 4 \), \( \tau_{22} = 1.5199 E + 7 \) with the matrices \( P_1 \) and \( P_2 \) as

\[
P_1 = \begin{bmatrix}
10.8685 & -2.9654 \\
-2.9654 & 216.8589
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
4.9096 & -0.3452 \\
-0.3452 & 6.3089
\end{bmatrix}
\]

(67)

As it can be seen some of the variables have small deviations from the theoretical constraints, however, interestingly, these results are still acceptable in practice as it can be seen from the numerical experiments. The invariant set of attraction as well as the state trajectories \( x(k) \) corresponding to various initial conditions are shown in Fig. 2. Although it is rare in practice, however, to show the wide range of the attractive region and the validity of the obtained numerical solutions, the initial conditions are selected quite far from the equilibrium point \( x_e \). Fig. 3 illustrates an enlarged view of Fig. 2 around the attractive ellipsoids. Figure 4 shows the time trajectories of the inductor current, output voltage and switching function corresponding to the initial condition \( i_L(0) = v_o(0) = -25 \) p.u..

The optimization problem in (62) corresponding to the proposed stability conditions in Theorem 14, and the additional constraints in (63)-(64) as \( \lambda_1 + \lambda_2 = 1 \), \( \lambda_{11} + \lambda_{12} = 10 \), \( \lambda_{21} + \lambda_{22} = 10 \) yields \( t_1 = t_2 = 0.3755 \),
\[ \lambda_{11} = \lambda_{21} = 7.5000, \lambda_{12} = \lambda_{22} = 2.4999, \lambda_1 = 0.7500, \lambda_2 = 0.2499, \tau_{11} = 0.2410, \tau_{12} = 0.7539,\]
\[ \tau_{211} = \tau_{212} = \tau_{221} = \tau_{222} = 0.0498 \] with matrices \( P_1 \) and \( P_2 \) as
\[ P_1 = P_2 = \begin{bmatrix} 2.6627 & 0.0013 \\ 0.0013 & 53.4956 \end{bmatrix} \] (68)

Compared to the results obtained by the conditions of Theorem 12, the stability conditions of Theorem 14 yield more conservative results regarding the smallest size of the ultimate invariant set of attraction. This can be realized by the comparison of the values \( t_1 \) and \( t_2 \) obtained corresponding to the conditions provided by these two theorems. Figs. 5 and 6 illustrate the state trajectories of the converter under the stability conditions of Theorem 14.

Fig. 7 illustrates the state trajectories starting from the null initial conditions and invariant set of attractions corresponding to the conditions of the Theorems 12 and 14. As it can be seen, although the size of the ultimate invariant set of attraction estimated by the Theorem 12 is less conservative, however, from the performance point of view the results obtained from the stability conditions of Theorem 14 is more desirable.

VI. CONCLUSION

In this paper, a switched Lyapunov function approach is proposed for the global practical stability analysis and controller synthesis of discrete-time switched affine systems. The proposed conditions are BMI-based conditions that guarantee simultaneously the invariance and attractive properties of the ultimate convergence set. Two theorems are suggested with two different sets of sufficient conditions. In the first one, the proposed conditions are not in BMI form and as a result a prefixing of variables is suggested to convert them to the standard BMI form. The stability conditions at the second theorem are intrinsically in the BMI form, but are more conservative. The proposed design and stabilization method is carried out successfully on a DC-DC buck converter.

ACKNOWLEDGEMENT

The author would like to thank the anonymous reviewers for their valuable comments that helped to improve the paper.

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**Biography**

Mohammad Hejri received his B.Sc. degree from Tabriz University in 2000 and the M.Sc. degree from Sharif University of Technology, Tehran, Iran in 2002 both in electrical engineering. He received his Ph.D. degree in electrical engineering from Sharif University of Technology, Tehran and the University of Cagliari, Cagliari, Italy, in 2010 as a co-tutorship program. He has been with several industries and research centers such as Iran Tractor Foundry Company, Azerbaijan Regional Electric Company, Tabriz Oil and Refining Company and Iran’s Niroo (Energy) Research Institute (NRI). From 2010 to 2012, he was a Postdoctoral Research Associate with the Department of Electric Power and Energy Systems, School of Electrical Engineering, Royal Institute of Technology (KTH), Stockholm, Sweden. Since 2012, he has been an Assistant Professor with the Department of Electrical Engineering, Sahand University of Technology, Tabriz. His research interests include control theory with applications in power electronics, renewable energy and power systems.

**Figure captions**

Fig. 1. DC-DC buck converter.

Fig. 2. State trajectories corresponding to conditions of Theorem 12 and \( B = 15 \ 1_2 \).

Fig. 3. An enlarged view of Fig. 2 around the attractive ellipsoids.

Fig. 4. Time trajectories corresponding to the initial condition \( i_L(0) = v_o(0) = -25p.u.,(a) \) inductor current, (b) output voltage, (c) switching function.
Fig. 5. State trajectories corresponding to conditions of Theorem 14.

Fig. 6. An enlarged view of Fig. 5 around the attractive ellipsoids.

Fig. 7. State trajectories corresponding to conditions of Theorems 12 and 14.

**Figures with their captions and numbers**

Fig. 1. DC-DC buck converter.

![DC-DC buck converter diagram](image)

Fig. 2. State trajectories corresponding to conditions of Theorem 12 and $B = 15 \mathbf{1}_2$.
Fig. 3. An enlarged view of Fig. 2 around the attractive ellipsoids.

Fig. 4. Time trajectories corresponding to the initial condition $i_L(0) = v_o(0) = -25p.u.$ (a) inductor current, (b) output voltage, (c) switching function.
Fig. 5. State trajectories corresponding to conditions of Theorem 14.

Fig. 6. An enlarged view of Fig. 5 around the attractive ellipsoids.

Fig. 7. State trajectories corresponding to conditions of Theorems 12 and 14.