Monitoring Process Mean Using a Second Order Filter: Signal and System Approach

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Abstract

In statistical process control one objective is to control the stability of a process. A process is stable when its mean is in control and variance bounded. Different control charts were introduced for monitoring the mean and variance of a process by plotting suitable test statistics on the chart. In this research design of a system which converts the sample mean to a test statistics was proposed. The second order filter, a special class of the Linear Time Invariant (LTI) systems, was used to design the converting system. It was shown that design of a low pass filter was better for detecting a level (mean) change in the process. Markov chain approach was also followed to construct appropriate control chart and to estimate its control limits. Simulated data under normality assumption for different scenarios were used to compare the proposed control chart with Shewhart and Exponentially Weighted Moving Average charts by means of ARL and PFS criteria. Existing data from the Central Bank of Iran was also applied to evaluate the suggested method. The signal to noise ratio was used to assess the performance of this method at different stages. Results indicate that the proposed method detects shifts more rapidly.

Key Words: Control Charts, Linear Filters, LTI Systems, Markov Chain, Statistical Process Control.

1- Introduction

Statistical process control techniques are tools used to reduce the process dispersion and to improve its stability. In statistical process control it is usually assumed that the output of the process, is normally distributed with in control mean $\mu_0$ and in-control standard deviation $\sigma_0$. Control charts are one of the quality control tools which are frequently used to investigate the process stability. In simplest case where the quality characteristic is normally distributed, it is of interest to investigate if the process mean and standard deviation remain at in-control levels.
According to Montgomery [1] one may use Shewhart control charts to monitor the process parameters. Construction and implementation of Shewhart charts requires introduction of some test statistics for monitoring the stability of the process. Then the probability distribution of the test statistics under the assumption of process being in-control must be determined. Using this probability distribution, the values of test statistics which may be generated when the process is in-control are determined. It is expected that when the process is in-control, test statistics falls inside a specific interval. The upper and lower bounds of this interval are called Upper Control Limit (UCL) and Lower Control Limit (LCL), respectively. Determination of the control limits requires a desired value for Average Run Length (ARL). Average run length is the mean number of samples needs to be taken from the process until the test statistics exceeds its control limits. Control limits are defined to obtain a predetermined in control ARL, such as 370. In this research it is assumed that the process standard deviation remains in-control when monitoring the process mean. One of the most popular control charts for monitoring the process mean, is the $\bar{X}$ chart. For implementing this chart samples of size $n$ are taken from the process and the sample mean, $\bar{X}$ is computed for each one of them. As long as $\bar{X} \in \left( \mu_0 \pm 3\sigma_0 / \sqrt{n} \right)$ the process mean is judged to be in-control. According to Montgomery [1] Shewhart $\bar{X}$ chart is suitable to detect large shifts and is less sensitive to small shifts. Some authors including Jiang et.al. [2], Wu and Spedding [3], Zhang [4], Chang and Aw [5] and Harris and Ross [6] proposed other charts for monitoring the process mean which are sensitive to small and moderate changes. Two of such charts are Exponentially Weighted Moving Average (EWMA) and Cumulative Sum (CUSUM). In these two charts other test statistics rather sample mean is plotted on the chart. It was shown that EWMA control chart is highly robust against non-normality of the process distribution. Thus, this chart has been adopted to construct the proposed control chart. Saif [7] proposed a framework to integrate the control charts with automation process control.

Monitoring the process mean may be categorized into two approaches. In first approach monitoring procedure is directly based on plotting the sample mean on the appropriate chart. The second approach consists of monitoring some appropriate functions of the sample mean. These functions must be chosen so that the corresponding charts would find small to moderate changes rapidly.

These two approaches could be considered as a unified approach. In this unified approach one may design a function which is called system. This system takes $\bar{X}_t$, $t = 1, 2, ..., s$ as input and
transforms them to output shown by $Y_t$, which is the charting test statistics. The system along with its input and output is shown in Figure 1.

Figure 1

Using this unified approach one may study the behavior of monitoring procedure more precisely. In other words, selecting appropriate test statistics to monitor the process will be reduced to selecting an appropriate system. Process monitoring scheme is usually designed only based on statistical considerations. In this strategy, the behavior of processes in time domain is usually assessed. However, some important features of the processes may not be discovered if processes are considered only in time domain. Accordingly, any change in the process mean may not be detected effectively.

On the other hand, considering statistical process monitoring from system view point, enables practitioners to better understand the processes. The reason is that, using system approach one could study the process in frequency domain. This domain contains information about the process which may not be evident in the time domain. Thus, it is possible to monitor complicated process more precisely using system approach. Choosing an appropriate system shown in Figure 1, is very important to construct a control scheme. Designing such a system requires that the process be studied in frequency domain, as well. In this research, the system approach is followed to construct a unified scheme to monitor processes.

Various authors proposed different methods for monitoring process mean including Rabyk and Schmid [8] Shokrizadeh et. al. [9], Yang and Arnold [10], Chen et. al. [11], Apley and Shi [12] and Lu and Reynolds [13]. However, signal and system approach was not explicitly used to define a monitoring scheme. Previous studies have not tried the hybrid method discussed in this study which adopts the second order filter and the Markov chain approach as a means of detecting changes in process mean. In this research signal and system approach is applied to define control scheme with suitable properties. The proposed monitoring procedure must be designed so that the resulting chart has desirable properties.

The rest of this paper is organized as follow. In Section 2 some preliminaries about system theory are provided. In Section 3, LTI systems are introduced. The proposed monitoring procedure is presented in Section 4. In the proposed method the test statistics is obtained by using a suitable LTI system. Average run length of the proposed procedure is obtained using Markov chain approach in Section 5. In Section 6 the performance of the proposed method is
evaluated via simulation. Results of monitoring a real stochastic process by proposed control scheme is provided in Section 7. Finally, discussions and conclusion are made. Symbols and notations used in this research are provided in Table 1.

Table 1

2- Theory and System Design

The test statistics for EWMA chart is given in Equation 1.

\[ Z_t = \lambda \bar{X}_t + (1-\lambda)Z_{t-1}; \quad Z_0 = \mu_0; \quad t = 1, 2, \ldots \]  

(1)

The upper and the lower control limits for this chart are as follows:

\[ UCL = \mu_0 + L\sigma_0 \sqrt{\frac{\lambda}{(2-\lambda)n}} \]

\[ LCL = \mu_0 - L\sigma_0 \sqrt{\frac{\lambda}{(2-\lambda)n}} \]  

(2)

where the charting parameters \( 0 \leq \lambda \leq 1 \) and \( L > 0 \) are designed by the analyst for desired properties, such as having a predetermined in control ARL. As long as \( Z_t \in (LCL, UCL) \) the process is considered to be in control.

According to Zhang et al. [14], CUSUM chart is based on plotting the following statistics on the chart:

\[ S_0^- = 0; \quad S_t^- = \max \left\{ 0, S_{t-1}^- + \frac{\sqrt{n(\bar{X}_t - \mu_0)}}{\sigma_0} \right\}; \quad t = 1, 2, \ldots \]  

(3)

\[ S_0^+ = 0; \quad S_t^+ = \min \left\{ 0, S_{t-1}^+ + \frac{\sqrt{n(\bar{X}_t - \mu_0)}}{\sigma_0} \right\}; \quad t = 1, 2, \ldots \]  

(4)

For a given \( h > 0 \) the process mean is said to be in control at time \( t \) if \( S_t^+ < h \) and \( S_t^- > -h \). The parameter \( h > 0 \) is the charting parameter chosen to have a predetermined in control ARL.

Considering EWMA and CUSUM control charts one may conclude that it is possible to use a system which transforms \( \bar{X}_t \) values to generate some other test statistics. These test statistics then may be used to monitor the process mean rather than \( \bar{X}_t \) values.

A class of systems is called the Linear Time Invariant (LTI) which is important and suitable for defining test statistics, \( Y_t \). In this research input and output of a LTI system at time \( t \) is
shown by $x[t]$ and $y[t]$, respectively. In system theory $x[t]$ and $y[t]$ are called input and output signals, respectively. For the sake of simplicity the relation between $x[t]$ and $y[t]$ is shown by $x[t] \rightarrow y[t]$. The LTI systems as stated by Oppenheim et. al. [15] are systems which transforms input to output using two important properties. These properties are linearity and time invariance. For defining linearity suppose that $x_1[t]$ and $x_2[t]$ are two arbitrary input signals of a system with corresponding output signals $y_1[t]$ and $y_2[t]$, respectively, i.e. $x_1[t] \rightarrow y_1[t]$ and $x_2[t] \rightarrow y_2[t]$. The system is said to be linear if for every $a_1$ and $a_2$, $a_1 x_1[t] + a_2 x_2[t] \rightarrow a_1 y_1[t] + a_2 y_2[t]$. A system intuitively is said to be time invariant if its behavior does not change with time. More formally if $x[t] \rightarrow y[t]$ represents a system, this system is time invariant if for every $t_0 \in \mathbb{R}$, $x[t-t_0] \rightarrow y[t-t_0]$. In this study the LTI systems are used to define a suitable test statistics for monitoring the process mean.

### 3- Statistical Monitoring Procedure: the LTI System

As mentioned in Section 2, LTI systems have two important properties, i.e. linearity and time invariance. These properties make it easy to obtain a closed form for output signal of a system in terms of input signal. Outputs of a system may be called system responses. To show how to obtain LTI system response, it is required to define two important input signals. These two input signals are unit impulse, $\delta[t]$, and unit step, $u[t]$, which are defined as follow:

\[
\delta[t] = \begin{cases} 
1 & t = 0 \\
0 & t \in \mathbb{R} - \{0\}
\end{cases} \quad (5)
\]

\[
u[t] = \begin{cases} 
1 & t \in \mathbb{Z}^+ \cup \{0\} \\
0 & t \in \mathbb{Z}^-
\end{cases} \quad (6)
\]

Let $h[t]; t \in \mathbb{Z}$ be the system response to the unit impulse, i.e. $\delta[t] \rightarrow h[t]$. In literature of signals and systems $h[t]$ is called impulse response. According to Oppenheim et. al. [15] the output of a LTI system to any input signal $x[t]$ can be expressed as follow:

\[
y[t] = \sum_{k=-\infty}^{\infty} h[k] x[t-k] = h[t] \ast x[t]; \; t, k \in \mathbb{Z}
\]

where $y[t]$ is the system response to $x[t]$ and $(\ast)$ is the convolution operator.
Based on Equation 7, having \( h[t] \) one can find system response to any input signal. Equation 7 represents output of a system in time domain. In Figure 2 relation between input and output of a LTI system in time domain is shown:

**Figure 2**

It is possible to represent a system in frequency domain as well. To this end one may take Fourier transformation of both sides of Equation 7. It is shown in Appendix A that taking the Fourier Transformation of Equation 7 one may conclude that:

\[
Y(e^{jw}) = H(e^{jw})X(e^{jw})
\]  \hspace{1cm} (8)

In which \( j = \sqrt{-1} \), \( w \) is the frequency variable, \( Y(e^{jw}) \), \( H(e^{jw}) \) and \( X(e^{jw}) \) are the Fourier transformation of \( y[t] \), \( h[t] \) and \( x[t] \), respectively which are shown below:

\[
Y(e^{jw}) = \sum_{t=-\infty}^{\infty} y[t]e^{-jwt}
\]  \hspace{1cm} (9)

\[
H(e^{jw}) = \sum_{t=-\infty}^{\infty} h[t]e^{-jwt}
\]  \hspace{1cm} (10)

\[
X(e^{jw}) = \sum_{t=-\infty}^{\infty} x[t]e^{-jwt}
\]  \hspace{1cm} (11)

The function \( H(e^{jw}) \) is usually called the frequency response of system. According to Equation 8, one may multiply \( H(e^{jw}) \) and \( X(e^{jw}) \) to obtain \( Y(e^{jw}) \). Then taking inverse Fourier transformation of \( Y(e^{jw}) \) yields \( y[t] \). Representation of a LTI system in frequency domain is illustrated in Figure 3:

**Figure 3**

LTI systems are usually called filters. In designing a filter it is very important to define \( h[t] \) or its Fourier transformation \( H(e^{jw}) \) properly. Generally, three types of filters, including low pass, high pass and medium pass filters could be defined. Low pass filters only permit low frequencies pass through the filter. In high pass filters only high frequencies pass the filters and low frequencies are vanished. Median pass filters only send moderate frequencies out.
One type of frequently used LTI systems or filters are those represented by linear difference equations with constant coefficients. More formally let $x[t]$ and $y[t]$ be input and output signals of a system which are related to each other according to Equation (12):

$$\sum_{k=0}^{M} b_k y[t-k] = \sum_{k=0}^{N} a_k x[t-k]$$

Equation 12 is a linear difference equation with constant coefficients. Solving Equation 12 needs some auxiliary conditions. It is rational to assume that if $x[t] = 0$ for $t < \tau$ then $y[t] = 0$ for $t < \tau$. This auxiliary condition is called initial rest. Based on Oppenheim et. al. [15] any system which is represented by Equation 12 with initial rest condition, specifies a LTI system which is a linear filter in turn. Designing linear filters has been studied by various authors, especially in the field of communication systems. Wang et. al. [16] proposed linear feedback control loops to detect sensor faults. Chen et. al. [17] studied the problem of detecting faults in linear stochastic dynamic systems. Zuo et. al. [18] designed a linear filter to estimate missing measurements. Liu and Shi [19] studied optimal linear filtering for correlated data. Liu et. al. [20] studied optimal design of linear filters. Eijnden et. al. [21] proposed a hybrid low-pass filter to control nonlinear motion.

Two frequently used filters are the first order and the second order filters. In this paper the second order filter is used to monitor the process mean. The first and the second order filters are reviewed briefly in subsection 3.1.

### 3-1- The First and the Second Order Filters:

The simplest linear filter which is used to design a system is a first order system with initial rest condition:

$$y[t] = \phi_1 y[t-1] + x[t]; \quad t \in \mathbb{I}$$

In which $\phi_1$ is a constant. The block diagram of the first order system is shown in Figure 4:

![Figure 4](image)

In Figure 4, D represents the difference operator. From Equation 13 it is evident that for a given input signal $x[t]$, the output signal $y[t]$ depends on the value of $y[t-1]$. Accordingly, the corresponding system is called a first order filter.

In designing filters, usually first order systems are connected serially or parallel. Deciding which types of connection to be used depends on the application of the resulting filter.
Another frequently used filter, is the second order filter which is represented by Equation 14:

$$y[t] = \phi_1 y[t-1] + \phi_2 y[t-2] + x[t]; \quad t \in \mathbb{Z}$$

(14)

in which $\phi_1$ and $\phi_2$ are some constants. Block diagram of the second order filter is shown in Figure 5:

![Figure 5](image)

It could be seen from Equation 14 that for a given input signal $x[t]$, the output signal $y[t]$ depends on the values of $y[t-1]$ and $y[t-2]$. Because of this dependency, the system which is represented by Equation 14 is called the second order filter.

As stated before the behavior of each LTI system, including the first order and the second order systems, in time domain is determined by impulse response, $h[t]$. In frequency domain the behavior of a filter is modeled by its frequency response, $H(e^{jw})$. In this study the authors attempted to use the second order filter to design a unified control system as shown in Figure 1 to transform $\bar{X}_t$ to $Y_t$. Designing this second order filter requires determination of the values of $\phi_1$ and $\phi_2$ in Equation 14. These coefficients should be determined so that the resulting test statistics, $Y_t$, becomes sensitive to a certain type of change in process mean. In this study it is assumed that the process mean is $\mu_0$ by time $t_0$. At time $t_0$ the process mean changes to an out of control level shown by $\mu_0 + \gamma \sigma_0$, where $\gamma$ is the magnitude of shift in process mean in terms of process standard deviation. The unified controlling system should be designed in order to detect this change rapidly. In other words, the values of $\phi_1$ and $\phi_2$ in Equation 14 must be determined so that the change in process mean is magnified in the sequence of $Y_t$ s. Note that mean and variance of filter output, $Y_t$, will not be in general equal to those of $\bar{X}_t$. One may define Signal to Noise Ratio (SNR) of $\bar{X}_t$ and $Y_t$ by

$$SNR_{\bar{X}_t} = \frac{E^2[\bar{X}_t]}{\text{var}(\bar{X}_t)}$$

and

$$SNR_{Y_t} = \left(\frac{E^2[Y_t]}{\text{var}(Y_t)}\right),$$

respectively. If $SNR_{Y_t} > SNR_{\bar{X}_t}$, changes in process mean will be unfolded more prominently using $Y_t$. Then, out of control conditions will be detected more rapidly. Thus, it is better to use a filter which increases $SNR_{Y_t}$ comparing to $SNR_{\bar{X}_t}$. In this study, a second order low pass filter is proposed to transform $\bar{X}_t$ to $Y_t$. Use of low pass filters is
logical, since these filters remove high frequency components and smooth out input signals. Using the ratio $\frac{SNR_Y}{SNR_X}$, one may compare levels of dispersion and mean before and after filtering. Thus the ability of filter in removing noises may be studied properly. It is important to study the behavior of a second order system in time and frequency domains for designing the filter properly. In next subsection, the behavior of the second order system in terms of $h(t)$ and $H(e^{j\omega})$ is investigated. This investigation makes it clear to design an appropriate unified control system.

3-2- Behavior of the Second Order Filters

To study the behavior of a second order system, its impulse response, $h(t)$, and frequency response, $H(e^{j\omega})$ must be determined. It is better to specify $H(e^{j\omega})$ at first. According to Oppenheim et. al. [15] in order to have frequency response, system must be stable. Intuitively speaking a system is stable if its responses to bounded inputs remain finite, i.e. $|x(t)| < \infty \Rightarrow |y(t)| < \infty$.

A second order system shown by Equation 14 is stable if the following conditions are met:

$$\phi_1 + \phi_2 < 1; \quad \phi_2 - \phi_1 < 1; \quad |\phi_2| < 1$$

(15)

Thus, if a second order filter is used to transform $X_t$ to $Y_t$, as long as conditions of Equation 15 are satisfied and $|X_t| < \infty$, the output signal $Y_t$ will be bounded. Using Markov inequality one may conclude that the sufficient condition for $|X_t| < \infty$ is that $E[|X_t|] < \infty$. On the other hand $E[|X_t|] \leq E[|X|]$. Thus, in the stability region of a second order filter, if $E[|X|] < \infty$ then $|Y_t| < \infty$.

Suppose that a second order system defined by Equation 14 is stable. For determining $H(e^{j\omega})$ one may take Fourier transformation of both sides of Equation 14 which results the following:

$$Y(e^{j\omega}) = \phi_1 e^{-j\omega}Y(e^{j\omega}) + \phi_2 e^{-2j\omega}Y(e^{j\omega}) + X(e^{j\omega})$$

(16)

Rearranging Equation 16 yields Equation 17:
\[
\frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \phi_1 e^{-j\omega} - \phi_2 e^{-2j\omega}} \tag{17}
\]

According to Equation 8, Equation 17 may be written as:

\[
H(e^{j\omega}) = \frac{1}{1 - \phi_1 e^{-j\omega} - \phi_2 e^{-2j\omega}} \tag{18}
\]

Equation 18 is called the frequency response of system. Taking the inverse Fourier transformation of both sides of Equation 18 one may obtain impulse response of system. The impulse response of the second order filter is obtained in Appendix B.

In Appendix B it was shown that the behavior of \( h[t] \) depends on the sign of \( \phi_1^2 + 4\phi_2 \). Thus, two cases are possible for computing \( h[t] \) which are explained in the following.

**Case I, \( \phi_1^2 + 4\phi_2 > 0 \):**

If \( \phi_1^2 + 4\phi_2 > 0 \) the impulse response of system is equal to:

\[
h[t] = \frac{x_2}{x_2 - x_1} \left( \frac{1}{x_1} \right) u[t] + \frac{x_1}{x_1 - x_2} \left( \frac{1}{x_2} \right) u[t]; \ t \in \mathbb{I} \tag{19}
\]

In which

\[
x_1 = \frac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \tag{20}
\]

\[
x_2 = \frac{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}
\]

where \( u[t] \) is a unit step function defined in Equation 6. The stability conditions of the system require \( |x_1| > 1 \) and \( |x_2| > 1 \). Conversely, given \( x_1 \) and \( x_2 \), one could compute \( \phi_1 \) and \( \phi_2 \) as follow:

\[
\phi_1 = \frac{1}{x_1} + \frac{1}{x_2}
\]

\[
\phi_2 = -\frac{1}{x_1 x_2} \tag{21}
\]
Note that $x_1^{-1}$ and $x_2^{-1}$ are called poles of system. If $\phi_1^2 + 4\phi_2 > 0$, system has two real poles. The behavior of second order filter is determined by location of these poles. According to Oppenheim et. al. [15], when poles move toward 1, the corresponding system behaves as a low pass filter. If poles move toward -1 the system will be a high pass filter.

It could be seen that for case $\phi_1^2 + 4\phi_2 > 0$ the impulse response of system damps out without oscillation as $t \to +\infty$. Thus, to design a second order system for this case, it is important to define the values of $\phi_1$ and $\phi_2$. As an example, consider a second order filter with poles $x_1^{-1} = 0.9$ and $x_2^{-1} = 0.8$. Using Equation 21, the values of $\phi_1$ and $\phi_2$ were computed as 1.7 and -0.72, respectively. The impulse response of this system is shown in Figure 6. Note that these values for $\phi_1$ and $\phi_2$ are in non-oscillation region.

Figure 6

Case II, $\phi_1^2 + 4\phi_2 < 0$:

In case $\phi_1^2 + 4\phi_2 < 0$ the impulse response of a system becomes:

$$h[t] = \frac{\sin((t+1)\theta)}{\sin(\theta)} \rho' u[t] \quad t \in \mathbb{Z}$$  \hspace{1cm} (22)

where

$$2\rho \cos \theta = \phi_1 \quad \rho^2 = \phi_2$$  \hspace{1cm} (23)

For case $\phi_1^2 + 4\phi_2 < 0$ the impulse response of system damps with oscillation as $t \to +\infty$.

Let $x_1$ and $x_2$ be as follow:

$$x_1 = \frac{-\phi_1 + j\sqrt{-\phi_1^2 - 4\phi_2}}{2\phi_2}$$

$$x_2 = \frac{-\phi_1 - j\sqrt{-\phi_1^2 - 4\phi_2}}{2\phi_2}$$  \hspace{1cm} (24)

According to Oppenheim et. al. [15], in this case $x_1^{-1}$ and $x_2^{-1}$ are called poles of system. In order the system to be stable, it is required that $|x_1| > 1$ and $|x_2| > 1$. Again, given $x_1$ and $x_2$, the values of $\phi_1$ and $\phi_2$ could be obtained using Equation 21. According to Oppenheim et. al.
[15], if poles move toward 1, the system will be a low pass filter. If they move toward -1 the system will be a high pass filter. As \( x_1 \to j \) and \( x_2 \to -j \) the resulting system is a medium pass filter. As an example, consider a system with poles \( x_1^{-1} = 0.9 + 0.1j \) and \( x_2^{-1} = 0.9 - 0.1j \). Using Equations 21 and 23 following values were obtained: \( \phi_1 = 1.8 \) and \( \phi_2 = -0.82 \), \( \rho = \sqrt{-\phi_2} = 0.91 \) and \( \theta = \cos^{-1}\left(\frac{\phi_1}{2\rho}\right) = 0.11 \). The impulse response of this system is shown in Figure 7.

Figure 7

In summary, for designing a second order system, the values of its poles must be chosen. This values are set in a way that the resulting filter poses some suitable features. Based on poles values and using Equation 21, \( \phi_1 \) and \( \phi_2 \) values are determined. When monitoring the process mean, a desired feature that the filter is better to have is being sensitive to step change. Based on this desired feature, suitable second order filter can be designed. Designed filter could be used to transform sample means \( \bar{X}_t \) to test statistics, \( Y_t \) s, as shown in Figure 1. Proper design of filter makes \( Y_t \) sensitive to step changes. Based on this guideline a second order filter is used to define a unified control system here. The proposed monitoring method using second order filter is introduced in Section 4.

4- Proposed Monitoring Method

4-1- Underlying Logic

In this section a unified control system is designed to monitor the process mean. More formally, suppose that when the process is in-control, the quality characteristic of interest is normally distributed with mean \( \mu_0 \) and variance \( \sigma_0^2 \). It is assumed that when the process is out of control, the process mean jumps to a new level \( \mu_0 + \gamma \sigma_0 \). It is important to detect this change as soon as possible. Let \( t_0 \) be the change point of the process mean. Let \( \bar{X}_t \) be the sample mean of a random sample of size \( n \) taken at time \( t \). \( \bar{X}_t \) can be represented as:
\[
\bar{X}_t = \begin{cases} 
\mu_0 + \frac{\sigma_0}{\sqrt{n}} \epsilon_t, & t = 0, 1, \ldots, t_0 - 1 \\
\mu_0 + \gamma \sigma_0 + \frac{\sigma_0}{\sqrt{n}} \epsilon_t, & t = t_0, t_0 + 1, \ldots
\end{cases}
\]  
(25)

In which \( \epsilon_t \) s are Normally Independently Distributed [NID] with mean 0 and variance 1, i.e. \( \epsilon_t \sim NID(0,1) \). Expected value of \( \bar{X}_t \) in Equation 25 could be represented as:

\[
E[\bar{X}_t] = \nu[t] = \mu_0 + (\gamma \sigma_0) u[t-t_0]
\]  
(26)

Suppose that the sequence of \( \bar{X}_t \) s enter into a stable second order system. Thus

\[
Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \bar{X}_t, \quad t = 1, 2, \ldots; \quad Y_0 = Y_{-1} = \frac{\mu_0}{1 - \phi_1 - \phi_2}
\]  
(27)

Note that it is required to have the initial values for \( Y_0, Y_{-1} \) to compute \( Y_t \) for \( t > 0 \). Reasonable initial values for \( Y_0 \) and \( Y_{-1} \) are the expected value of the stochastic process \( Y_t \) which is \( \frac{\mu_0}{1 - \phi_1 - \phi_2} \). In Equation 27 this initial value was used. According to Box et. al. [22], when \( \bar{X}_t \) is normally distributed, then \( Y_t \) follows normal distribution.

Equation 27 defines an autoregressive of order 2 process, which is called AR(2) in time series analysis. It is suitable to determine the effect of step change on the mean value of \( Y_t \). Let \( E[Y_t] = \lambda[t] \). Taking expectation from both sides of Equation 27 results:

\[
\lambda[t] = \phi_1 \lambda[t-1] + \phi_2 \lambda[t-2] + \nu[t]
\]  
(28)

Equation 28 defines a LTI system with input signal \( \nu[t] \) and response signal \( \lambda[t] \). Let \( h[t] \) be the impulse response of the system. When \( \phi_1^2 + 4\phi_2 > 0 \) then \( h[t] \) is given by Equation 19. For \( \phi_1^2 + 4\phi_2 < 0 \), \( h[t] \) is obtained according to Equation 22. Based on Equation 7, the response of the system to the input signal \( \nu[t] \) is given by:

\[
\lambda[t] = h[t] \ast \nu[t] = \sum_{k=-\infty}^{\infty} h[k] \nu[t-k]
\]  
(29)

Determination of Equation 29 was provided in Appendix C. Solution of Equation 29 is summarized in Table 2.

Table 2
A numerical example is here provided to assess the response of the system to the level shift.

4-2- Test Evaluation of Process Parameters

Suppose that the quality characteristic of interest is normally distributed with in-control mean $\mu_0 = 0$ and variance $\sigma^2 = 1$. At time $t_0 = 15$ a positive shift of one standard deviation ($\gamma = 1$) occurs so $\mu_0 + \gamma \sigma_0 = 1$. A second order filter with two poles $x_1^{-1} = 0.65$ and $x_2^{-1} = -0.15$ is considered. In this pattern of poles, $x_1^{-1}$ is closer to 1 than $x_2^{-1}$ to -1. Thus, the system behaves as a low pass filter. Using Equation 21 values of $\phi_1$ and $\phi_2$ are computed as 0.5 and 0.1, respectively. This filter is used to convert $X_i$ to $Y_i$. In this case $\phi_1^2 + 4\phi_2 = 0.65 > 1$. Thus, the response of filter does not oscillate. It is expected that when $t \to \infty$, $\lambda[t] \to \frac{\mu_0 + \gamma \sigma_0}{1 - \phi_1 - \phi_2} = 2.5$. Based on Table 2, $\lambda[t]$ is determined and shown in Figure 8-a, along with $v[t]$. From Figure 8-a it is obvious that before the change point, $E[X_i]$ and $E[Y_i]$ remains at their constant levels. At change point, $E[X_i]$ and $E[Y_i]$ jump to a new level 1. After change point, $E[X_i]$ remains at 1. While, $E[Y_i]$ tends to 2.5. This behavior makes it easy to detect a level change in $E[X_i]$. In other words, using a proper second order filter magnifies the level change and it would be detected sooner.

Now consider the case in which $x_1^{-1}, x_2^{-1} = 0.25 \pm 0.58 j$. These values of poles are chosen arbitrarily in stability region to have oscillatory impulse response. Using Equation 21, $\phi_1 = 0.5$, $\phi_2 = -0.4$ and $\phi_1^2 + 4\phi_2 = -1.35 < 0$. In this case when $t \to \infty$, $\lambda[t] \to \frac{\mu_0 + \gamma \sigma_0}{1 - \phi_1 - \phi_2} = 1.1$. According to Table 2, $\lambda[t]$ is computed and shown in Figure 8-b along with $v[t]$.

Figure 8 reveals that using a proper second order filter magnifies the change in process level.

4-3- Proposed Control Chart

Considering the example presented in section 3.2, it is logical to use $Y_i$ instead of only $\bar{X}_i$ for monitoring the process mean. So, in proposed method a second order filter with proper parameters, $\phi_1$ and $\phi_2$ must be defined. Then, using this system, $\bar{X}_i$ s are filtered to obtain
test statistics $Y_i$ s. The centered test statistics $Y_i - \frac{\mu_0}{1 - \phi_1 - \phi_2}$ are then compared with control limits $\pm L\sigma_Y$ in which $L$ is the coefficient of the proposed control chart and $\sigma_Y$ is the in-control standard deviation of $Y_i$. Note that according to Equation 27, the sequence of $Y_i$ s generates a stationary AR(2) process. So, $\text{var}(Y_i)$ is the variance of an AR(2) process. According to Box et. al. [22] the variance for an AR(2) process, when the process is in-control is as follow:

$$\sigma_Y^2 = \frac{(1 - \phi_2)\sigma_0^2}{n(1 + \phi_2)(1 - \phi_1 - \phi_2)(1 - \phi_2 + \phi_1)}$$

(30)

The value of $L$ in $\pm L\sigma_Y$ is determined so that the in control ARL equals to a pre specified value. The approach for determining $L$ is discussed in next section. As an illustration of the proposed method application, the mentioned example in Section 3-2, Case I, is considered. The proposed method with $\phi_1 = 1.7$ and $\phi_2 = -0.72$ as well as the Shewhart control chart is used to detect any change in process mean. The control limits for the two monitoring methods are set in a way that the in-control ARL becomes 370. Note that, in the proposed method the test statistics are not independent. Thus, the distribution of the run length is not geometric and the traditional method of computing ARL does not work anymore. In Section 4 a Markov chain approach is applied to compute the in-control ARL. In Table 3 characteristics of the two monitoring methods are presented:

Table 3

For comparing the performance of the two methods, 50 observations were generated from the following model.

$$X_i = \begin{cases} 
\epsilon_i & t < 15 \\
1 + \epsilon_i & t \geq 15 
\end{cases}$$

(31)

where $\epsilon_i$ s were independent standard normal random variables. Note that here $n = 1$ so $\bar{X}_i = X_i$. The sequence of $X_i - \mu_0$ along with the corresponding Shewhart control limits are shown in Figure 9:

Figure 9
As could be seen from Figure 9, the Shewhart control chart does not issue any out of control signal.

In Figure 10 the sequence of $Y_t - \mu_0 / (1 - \phi_1 - \phi_2)$ along with the proposed control limits are shown.

**Figure 10**

Considering Figure 10 reveals that the proposed method signals out of control at 18th sample. Because of using a second order filter, a level shift is magnified and the chart signals rapidly.

It is useful to compare SNR for two control schemes in this example. When $t \to \infty$, $E[Y_t] \to \frac{E[X_t]}{1 - \phi_1 - \phi_2} = 50$. According to Equation 30, variance of $Y_t$ equals to 89.8. Thus,

$$SNR_{Y_t} = \left(\frac{\text{var}(Y_t)}{\text{var}(Y_t)}\right) = 27.84.$$  

For Shewhart control chart as $t \to \infty$, according to Equation 29, $E[X_t] = 1$. It is obvious that variance of $X_t$ equals to 1. Thus $SNR_{X_t} = \left(\frac{E^2[X_t]}{\text{var}(X_t)}\right) = 1$, for $t \to \infty$. In other words, before using the proposed filter, SNR equals to 1. While, after using this filter SNR raised to the value of 27.84. By removing noise and smoothing the input signal, the proposed second order filter increases SNR in comparison to Shewhart control chart. This property helps to detect change in the process mean more rapidly.

Using the proposed method for monitoring the process mean, requires determination of the values of $\phi_1$ and $\phi_2$. The precise determination of the values of $\phi_1$ and $\phi_2$ is left as further research. However, some guidelines are provided in the next subsection.

**4-4: Determination of $\phi_1$ and $\phi_2$**

$\phi_1$ and $\phi_2$ are two real numbers which characterize the behavior of the filter. These two parameters must be chosen so that the conditions in 15 are satisfied. These conditions are stability conditions for a second order system or stationary conditions for an AR(2) process. These conditions could be plotted in $\phi_1$ and $\phi_2$ space as shown in Figure 11.
If one chooses a point in non-oscillatory region, according to Equation 19 the impulse response of the system becomes non-oscillatory and decays exponentially. While, if a point in oscillatory region is chosen, the corresponding impulse response, provided in Equation 22, oscillates and damps. According to Equation 29, for choosing the values of \( \phi_1 \) and \( \phi_2 \) the response of the system to \( v[t] = \mu_0 + (\gamma \sigma_0)u[t-t_0] \) must be determined. This response could be found in time domain using Equation 29. However, it is better to study behavior of second order system in frequency domain. Without loss of generality assume \( \mu_0 = 0 \), it is convenient to study the behavior of level shift \( v[t] \) in frequency domain. The Fourier transformation of \( v[t] \) equals to:

\[
V(e^{jw}) = (\gamma \sigma_0) \frac{e^{-jw_0}}{1 - e^{-jw}}
\]  

(32)

The function shown in Equation 32 is complex valued. The magnitude of this function is:

\[
|V(e^{jw})| = \frac{|\gamma| \sigma_0}{\sqrt{2(1 - \cos(w))}}
\]  

(33)

Note that \( |V(e^{jw})| \) is a periodic function of \( w \). One period of this function is shown in Figure 12 for \( w \in [-\pi, \pi) \) when \( |\gamma| \sigma_0 = 1 \):

Figure 12

Figure 12 shows that the low frequencies near \( w = 0 \) have large contribution in \( v[t] \). Thus, for magnifying level shift a low pass filter must be designed. For studying the effect of using a second order system for magnifying level shift, it is more suitable to investigate system in frequency domain. Taking Fourier transformation of both sides of Equation 29 results:

\[
\Lambda(e^{jw}) = H(e^{jw})V(e^{jw})
\]  

(34)

in which \( \Lambda(e^{jw}) \) is the Fourier transformation of \( \lambda[t] \) and \( H(j\omega) \) is the frequency response of system shown in Equation 18. Substituting Equation 32 in Equation 34 and taking absolute value yields:
\[
\left| \Lambda(e^{jw}) \right| = \frac{|\gamma| \sigma_0}{\sqrt{1 - \phi_1 e^{-jw} - \phi_2 e^{-j2w}}} \sqrt{2(1 - \cos(w))}
\] (35)

A low pass filter requires large absolute values for \( w \uparrow 0 \) in order to magnify low frequencies. Thus, according to Equation 35 for \( w \uparrow 0 \) it is appropriate to minimize \( |1 - \phi_1 - \phi_2| \). This value is minimized when \( \phi_1 + \phi_2 = 1 \). However, according to the stability conditions for a system \( \phi_1 + \phi_2 \) must be less than 1. Thus, for determining the values of \( \phi_1 \) and \( \phi_2 \) it is better to choose these values in stability region near the border of \( \phi_1 + \phi_2 = 1 \). This choice makes the system sensitive to level changes. In next section, Markov chain approach is used to determine the in-control ARL of the proposed method. Determination of the in-control ARL is necessary for specifying the control limit.

Note that according to Box et. al. [22], the main purpose of process control is to cancel out the entropy and disorganization of processes. Process control techniques are categorized into two classes. The first class is Statistical Process Control (SPC). The main objective of SPC is to detect when the process is out of control by process monitoring and remove assignable causes of variations. The second class of statistical process control methods is Automatic Process Control (APC). In APC, it is attempted to compensate disturbances by process adjustment. The main goal of process adjustment is to maintain the output of a process at a desired level by manipulating a control variable. This manipulation is usually performed through feedback and feedforward control loops. Note that, the proposed method which is a combination of second order filters and control charts, falls into the SPC category. More formally, the purpose of the proposed method is to monitor the process but not adjust the process using feedback or feedforward control loops. However, according to English and Case [23] SPC methods could be combined with APC tools to control the processes more effectively. For example, the proposed method may be placed within a feedback control loop to better adjust the process. The block diagram of this structure is shown in Figure 13.

Figure 13

In Figure 13, \( R_t \) is the desired level of process. Disturbances are added to the output of process to make \( X_t \). Using the proposed second order filter, \( X_t \) is transformed to \( Y_t \). Statistical stability of \( Y_t \) is then monitored by means of proposed control chart. If the control chart shows
that the process is in-control, the control signal $C_i$ will be $R_i$. Otherwise, for out of control situations, $C_i = Y_i$. The control error, $E_i = R_i - C_i$ is then computed. Afterwards, a Proportional-Integral-Derivative (PID) controller adjusts the process based on $E_i$ value. For more details about PID controllers and feedback control loops one may refer to Smith and Corripio [24]. Thus, it is possible to use the proposed method within feedback control loops.

5- Markov Chain Approach for Determining the in Control ARL

In statistical process control performances of control charts are usually evaluated using ARL criterion. Control charts are designed to have large in-control ARL. However, the control charts are desired to have small ARL when the process is out of control. In traditional Shewhart control charts, test statistics used to monitor the process are independent. Thus, the distribution of the run length (RL) is geometric and computation of ARL is straightforward. However, in some cases including EWMA chart, CUSUM chart and the proposed control chart in this study, the test statistics are correlated. Computation of the ARL for these control charts are somehow complicated. Three different approaches may be used to compute ARL of a control chart. These include analytical, simulation and Markov chain. The ARL of different control charts such as EWMA and CUSUM charts in Chang and Wu [25], Fu and Spring [26] and Bohm and Hackl [27] and ARMA chart in Jiang [28] were computed. In this research Markov chain approach was used to compute the in-control ARL. Suppose two parameters of the second order system, namely $\phi_1$, $\phi_2$ and the parameter of control limits, $L$, are determined. Let $z_t = (Y_t, Y_{t-1})'$ which may be expressed as:

$$z_t = \begin{pmatrix} Y_t \\ Y_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{t-2} \\ Y_{t-1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{X}_t$$

$$= \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} z_{t-1} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overline{X}_t$$

(36)

According to Equation 36, the conditional distribution of $z_t$ given $z_{t-1}$ only depends on the distribution of $\overline{X}_t$. Thus, $z_t$ has Markov property and generates a Markov process with continues state space. This Markov process may be approximated by a Markov chain with finite state space.

For a given $L>0$ each side of control chart is segmented into $m+1$ state where one state is used to show the values exceeding the control limits. The value of $m$ could be determined by
analyst. Note that for a fixed value of $L$, the larger the value of $m$ the more precise the approximation of $ARL$. The total number of states is then equal to $2m+3$. For each state in the control chart, a strip with width $\Delta = \frac{2L\sigma_y}{2m+1}$ is then defined. Total number of strips inside the control chart is then $2m+1$. The reason that the number of strips is 2 units less than the number of states is that 2 out of $2m+3$ states are used to show the situations in which the test statistics exceeds the control limits.

The segmentation of the proposed control chart along with the strips for $m=3$ is shown in Figure 14.

Figure 14

One may discretize the value of $Y_t$ based on the segment in which the $Y_t$ falls. According to the notation used by Chang and Wu [25] let $D_0(Y_t)$ be discretized value of $Y_t$ defined as follow:

$$D_0(Y_t) = \begin{cases} 
  i & (i-0.5)\Delta \leq Y_t \leq (i+0.5)\Delta \text{ for } i = -m,-m+1,\ldots,m \\
  m+1 & Y_t > (m+0.5)\Delta \\
  -m-1 & Y_t < -(m+0.5)\Delta
\end{cases} \tag{37}$$

By discretizing $Y_t$, the vector $z_t$ is discretize and is shown by:

$$D_0(z_t) = \left(D_0(Y_t), D_0(Y_{t-1})\right)^\prime \tag{38}$$

where $D_0(Y_t)$ is defined in Equation 37. Since the $z_t$ s have Markov property, the vector $D_0(z_t)$ constructs a Markov chain with finite state space. The state space of this Markov chain is as:

$$S = \{(S_1,S_2)^\prime, \alpha\} \tag{39}$$

where $S_1, S_2 \in \{-m,-m+1,\ldots,m\}$ and $\alpha$ is used to show the state in which the process is out of control. This state is an absorbing state which is:

$$\alpha = \begin{cases} 
  \left(\begin{array}{c} 
m+1 \\
  S_1 \\
m+1 \\
  S_1 \\
-m-1 \\
  S_1 \\
m+1 \\
  S_1 \\
m+1 \\
  S_1
\end{array}\right) & S_1 \in \{-m,-m+1,\ldots,m\}
\end{cases} \tag{40}$$
If the $z_t$ process moves to each elements of set $\alpha$, it will remains in this set. Thus, it is an absorbing state.

Let $P = [p_{ab}]$ be the transition matrix of $D_0(z_t)$, where $p_{ab}$ is the probability of $D_0(z_t)$ moving from state $a$ to state $b$. The method of computing $p_{ab}$s was provided in Appendix D. After constructing $P$, one could compute the in-control ARL of the control chart using formula provided in Appendix E. For different values of $\phi_1$, $\phi_2$ and the in-control ARL, the values of $L$ for the proposed control chart are provided in Table 4. These values are used in next section to assess the performance of the proposed control chart.

Table 4

As stated in Section 3, values of system poles, $x_1^{-1}$ and $x_2^{-1}$, determine the system to be low pass, high pass or medium pass filter. Thus, depending on the values of system poles, the proposed second order filter could be low pass, high pass or medium pass. The system relation shown in Equation 36 is valid in all of these cases. As a result, the conditional distribution of $z_t$ given $z_{t-1}$ only depends on the distribution of $\bar{X}_t$, as long as $|x_1| > 1$ and $|x_2| > 1$. Thus, the stochastic process $z_t$ is a Markov process, whether the proposed second order filter is low pass, high pass or medium pass. So, the hybrid method discussed in this study could be used even in cases where the proposed filter is high pass or medium pass. Note that the main purpose of this study is to detect shifts in process mean. To reduce noises, it is recommended to choose system poles near to 1 and design a low pass filter.

6- Simulation Studies

In this section performance of the proposed control chart to detect level shifts in the process mean is evaluated by means of simulation in MATLAB R2018b software. The proposed control chart is compared with two frequently used control charts, including Shewhart $\bar{X}$ chart and $EWMA$ chart. Shewhart $\bar{X}$ and $EWMA$ control charts are described in Section 1. The Markov chain approach was used to design the control charts proposed by Chang and Wu [25] and Bohm and Hackl [27]. However, these charts are used to monitor autocorrelated processes. The proposed method was designed to monitor independent sequence of observations. Thus, control charts to monitor autocorrelated observations were excluded from comparison.
The proposed control chart to be discussed here is shown by $AR(2)$. Two criteria were used in this section to compare the control charts. The first one is ARL, and the second criterion is the percentage of times when a chart signals before change occurs, i.e. Percentage of False Signals (PFS). The smaller the value of this later criterion the better a control chart. The control limits of the charts were specified so that the in-control ARL be equal to 370. In Equation 2, using the Markov chain approach, $L = 2.085$ and $\lambda = 0.15$ results an in control ARL to be equal to 370. An EWMA chart with $\lambda = 0.15$ is special case of $AR(2)$ model with $\phi_1 = 0.85$ and $\phi_2 = 0$. Introduction of a second parameter $\phi_2$ into a first order system, results a second order model, $AR(2)$. Let $\phi_1 = 0.85$ and $\phi_2 = 0.14$ for simulation study. For these values of parameters, $L = 1.86$ was selected from Table 4. According to Equation 20, by these values for $\phi_1$ and $\phi_2$, $x_1 = 1.0088$ and $x_2 = -7.08$ were obtained. Thus, poles of system are equal to $x_1^{-1} = 0.99$ and $x_2^{-1} = -0.14$. As could be seen, one pole is very near to 1. Thus, the corresponding system is a low pass filter. In each simulation run the data were generated from the following model:

$$X_t = \begin{cases} \epsilon_t & t < t_0 \\ \gamma + \epsilon_t & t \geq t_0 \end{cases}$$

(41)

where $\epsilon_t$ s are independent standard normal random variables, $t_0$ is the time instance at which level shift in process mean occurs and $\gamma$ is the value of the level shift. Note that in this section individual observations were used to control the process, i.e. $n = 1$. As it was stated in Section 3-1, it is possible to use $\frac{SNR_y}{SNR_x}$ to study the effect of proposed filter on input signal. Note that $E[X_t] = \gamma, E[Y_t] = \frac{\gamma}{1-\phi_1-\phi_2}$ for $t \geq t_0$. Using Equation 30, one may compute $\text{var}(Y_t)$ for the proposed filter which is equal to 27.96. It is possible to compute SNR values for different values of $\gamma$. For example, if $\gamma = 1$ then $SNR_{X_t} = 1$ and $SNR_{Y_t} = 3.58$ for $t \geq t_0$. Thus, after shifts occurred, the proposed low pass filter smooths out input signals. This will help the control scheme to detect shifts rapidly.

After generating each observation, test statistics for each control chart was computed. For each control chart, generating a new observation was continued until an observation exceeds its control limits and then the run length was recorded. For each values of $t_0 = 20, 50, 75$ and
\( \gamma = -2(0.1)^2 \) this procedure repeated 10000 times. The mean and standard deviation of the run lengths for each control chart as well as the proportion of the false signals were then computed. Results for ARL criterion are shown in Figure 15:

Figure 15: 15-a, 15-b and 15-c
In Figure 15, the 95% confidence intervals of ARL values are also shown using vertical error bars. These intervals are defined as \( \text{ARL} \pm 2 \frac{SDRL}{\sqrt{10000}} \) in which \( SDRL \) is the estimated standard deviation of run length. Figure 15 shows that the ARL of the three control charts reduces when \( |\gamma| \) increases. But, the reduction rate of ARL for \( AR(2) \) and \( EWMA \) charts is steeper compared to the Shewhart chart. This shows that \( EWMA \) and \( AR(2) \) detects the level change more rapidly than the Shewhart chart. Comparing the \( AR(2) \) and the \( EWMA \) charts, one could find that for small changes the \( AR(2) \) chart in the average detects the level shifts before the \( EWMA \) charts. For large changes the \( EWMA \) chart performs slightly better than the \( AR(2) \). The \( AR(2) \) chart and the \( EWMA \) charts outperform Shewhart chart in terms of ARL. From Figure 15 it is obvious that as \( |\gamma| \) increases, the \( SDRL \) decreases for all control charts. This indicates that as shifts become larger, the standard deviation of run length decreases and shifts are detected more precisely. Note that the confidence level is the same for estimating ARL of the three control charts. However, changes in \( SDRL \) values result in changes in width of confidence intervals. Note that the performance of the proposed method in detecting shifts in process could be enhanced by assigning suitable values to \( \phi_1 \) and \( \phi_2 \). The appropriate values of these parameters may be determined using some optimization methods. Using this approach, the behavior of the proposed method could be modified to accommodate larger mean changes. Optimizing the performance of the proposed method is left as an area for further research.

Results for PFS criterion are shown in Figure 16:

Figure 16: 16-a, 16-b and 16-c
In Figure 16, the 95% confidence intervals of PFS values are also shown. These intervals are defined as \( \text{PFS} \pm 2 \frac{SDPFS}{\sqrt{10000}} \) in which \( SDPFS \) is the estimated standard deviation of PFS. Considering PFS curves shown in Figure 16 reveals that PFS of the three charts is relatively
constant. But, the probability of false alarms for $AR(2)$ is much less than the other two charts. Considering the two criteria shows that the $AR(2)$ chart detects changes more rapidly and precisely. The $AR(2)$ chart outperforms EWMA in terms of PFS. In terms of PFS the proposed $AR(2)$ chart is much better than the other two charts. In summary results of simulation studies showed that the level shifts in process mean are detected more rapidly and precisely when the proposed $AR(2)$ chart is used.

7. Case Study – Existing Data

In this section, application of proposed control scheme to monitor a real process is shown. The stochastic process which was monitored is Euro exchange rate to Iranian Riyal. Time series data of this stochastic process may be found in the website of Islamic Republic of Iran Central Bank, https://cbi.ir/exratesadv/exratesadv_en.aspx. Euro exchange rates from 01/08/2019 to 30/07/2020, a total of 365 observations, were used as a process to be monitored. The main purpose of monitoring this process was to detect if the mean of Euro exchange rate shifted. Three control charts including Shewhart, EWMA and AR(2) were used in this section to monitor the process. Parameters of these charts are same as those in Section 6. To construct control charts it is required to compute in-control process mean and standard deviation. To remove the effects of outliers, robust estimators of process mean and standard deviation were used to estimate these parameters. Process mean was estimated using sample median of 365 observations which was equal to 46525 Riyals. To estimate the process standard deviation, Median Absolute Deviation from sample median (MAD) was used. For this data set $MAD = 549.6$ Riyals. Using these estimated parameters and parameters introduced in Section 6, control limits of each control chart were computed. These charts, along with corresponding sequences of test statistics are shown in Figure 17.

Figure 17: 17-a, 17-b, 17-c

Considering Figure 17-a, it is obvious that Shewhart control chart detects change in process mean at point 357. Whereas, some abrupt changes are evident before this point. In fact, presence of heavy noise in observations made Shewhart control chart insensitive to detect changes in process mean. In EWMA control chart, Figure 17-b, fluctuations were removed to some extent. In this chart, a few initial observations fall outside the control limits. While, it seems that the process mean is initially in-control and these observations fall outside control limits due to noises.
Considering AR(2) control chart, Figure 17-c, it is evident that noises were removed. In fact, by using a low pass filter, input signal was smoothed out. As a result, changes in process mean were magnified and were detected using proposed AR(2) control chart.

8. Results and Discussion

In this study a unified approach, based on signal and system theory was proposed to monitor the process mean. The most of approaches to monitor processes are designed only based on statistical properties of processes in time domain. Some important features of processes may be overlooked if the processes are only studied in time domain. Thus, it was proposed in this research to study the process in frequency domain, as well. In this research the test statistics for monitoring the process was considered as an output of a system. Using signal theory, the behavior of this transforming system was studied in frequency domain. In this domain, some features of the process are unfolded. Thus, one could design the control scheme more efficiently.

The problem of monitoring process mean was addressed in this paper. It was attempted to design a control chart sensitive to level shifts in process mean. The design of such control chart is based on the signal and system approach. More accurately, a unified control scheme based on the linear time invariant systems was developed. This control scheme is called a filter. It was suggested to use a second order system to filter out the sample means and then to control the process via the proposed control chart. This filter has two parameters which must be predetermined in order to detect the level shifts rapidly. To determine the values of these parameters the behavior of the second order system was investigated in frequency domain. It was shown that the level shift only has low frequency components. Thus, it is required to design a filter which magnifies the low frequencies. Based on this fact the values of the system parameters are then chosen. After designing the filter, control limits of the proposed control chart must be determined. To adjust the control limits it was recommended to use the Markov chain approach. The proposed control chart was compared with EWMA and the Shewhart control charts by means of simulation. It must be noted that the EWMA chart is a special case of the proposed AR(2) chart with $\phi_1 = 1 - \lambda$ and $\phi_2 = 0$. The Shewhart chart also could be considered as a special case of AR(2) chart when $\phi_1 = \phi_2 = 0$. The simulation results showed that the proposed control chart detects the level shifts more rapidly and precisely. In addition to the second order filters, the other filters may be used to monitor the process. The proposed method is able to detect shifts in process mean rapidly with low frequency components. If out
of control conditions with high frequency components need to be detected, it is suggested to use other filters which magnify high frequencies

9. Conclusion

In this paper monitoring process mean was considered. A second order low pass filter was proposed to remove noises from sequence of sample means. This filter is a LTI system which is used to transform sample mean to a test statistics. It was shown that by using this filter, any changes in process mean were magnified while at the same time, noises were filtered out. As a consequence, the SNR value increases which results in detecting shifts rapidly. By choosing poles of proposed system appropriately, low pass, high pass or medium pass filter could be obtained. All of these second order filters could be used to remove nuisance components of signals. As a result, changes in process could be detected rapidly. The proposed second order filter has two parameters which must be determined. Performance of the proposed control chart depends on the values of these parameters. To identify appropriate values of these parameters some optimization methods, such as gradient based search or metaheuristic algorithms, should be used. This is an open area which could be studied to better design the proposed control chart.

The filtering approach proposed in this research may be also used to control correlated observations. These areas are left as further research. The proposed monitoring scheme may be used to monitor various stochastic processes, including chemical, financial and electrical processes.

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References


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Appendices

Appendix A: Fourier Transformation of LTI System Response

Substituting Equation 7 in Equation 9 results the following:

\[ Y(e^{jw}) = \sum_{t=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]x[t-k]e^{-jkw} \]  \hspace{1cm} (A-1)

Let \( t-k = t' \in \mathbb{R} \), thus:

\[ Y(e^{jw}) = \sum_{t'=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]x[t']e^{-j(w+k)t'} \]  \hspace{1cm} (A-2)

\[ = \sum_{t'=-\infty}^{\infty} x[t']e^{-jw} \sum_{k=-\infty}^{\infty} h[k]e^{-jkw} \]

Using Equations 10 and 11, one may have:

\[ Y(e^{jw}) = X(e^{jw})H(e^{jw}) \]  \hspace{1cm} (A-3)

Appendix B: Impulse Response of Second Order Filter

To obtain the inverse Fourier transformation of \( H(e^{jw}) \) it is better to first factorize \( H(e^{jw}) \). Let \( x = e^{-jw} \), thus the denominator of \( H(e^{jw}) \) becomes \( D(x) = 1-\phi_1 x - \phi_2 x^2 \). Suppose \( x_1 \) and \( x_2 \) denote the roots of \( D(x) \). It is easy to verify that these roots are:
\[ x_1 = \frac{-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \]  
\[ x_2 = \frac{-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \]  

(B-1)

Note that \( D(x) \) may be written as following:

\[
D(x) = 1 - \phi_1 x - \phi_2 x^2 \\
= -\phi_2 \left( x^2 + \frac{\phi_1}{\phi_2} x - \frac{1}{\phi_2} \right) \\
= -\phi_2 (x - x_1)(x - x_2) \\
= -\phi_2 x_1 x_2 \left(1 - \frac{x}{x_1}\right) \left(1 - \frac{x}{x_2}\right)
\]  

(B-2)

According to Equation B-1, \( \phi_2 x_1 x_2 = -1 \) thus, \( D(x) \) could be written as:

\[
D(x) = \left(1 - \frac{x}{x_1}\right) \left(1 - \frac{x}{x_2}\right)
\]  

(B-3)

If \( \phi_1^2 + 4\phi_2 \neq 0 \) two roots are distinct. For the sake of simplicity suppose that this is the case.

Now \( H(e^{jw}) \) could be written as:

\[
H(e^{jw}) = \frac{1}{(1 - x_1^{-1} e^{-jw})(1 - x_2^{-1} e^{-jw})}
\]  

(B-4)

Then \( H(e^{jw}) \) can be factorized in the following manner:

\[
H(e^{jw}) = \frac{1}{1 - x_1^{-1} x_2^{-1} e^{-jw}} + \frac{1}{1 - x_1^{-1} x_2^{-1} e^{-jw}}
\]  

(B-5)

Taking inverse Fourier transformation of both sides of B-5, one may obtain the following:

\[
h[t] = \frac{x_2}{x_2 - x_1} \left( \frac{1}{x_1} \right) u[t] + \frac{x_1}{x_1 - x_2} \left( \frac{1}{x_2} \right) u[t]
\]  

(B-6)

In which \( u[t] \) is a unit step function defined in Equation 6. According to stability of system, \( |x_1| > 1 \) and \( |x_2| > 1 \).

Based on the sign of \( \phi_1^2 + 4\phi_2 \) in Equation B-1 two cases could be considered. If \( \phi_1^2 + 4\phi_2 > 0 \) two roots, \( x_1 \) and \( x_2 \) are real numbers and \( h[t] \) will tend to zero without oscillation as \( t \to +\infty \).
If \( \phi_1^2 + 4\phi_2 < 0 \) two roots are complex conjugate numbers. In this case \( x_1 \) and \( x_2 \) could be written as:

\[
x_1 = \frac{-\phi_1 + j\sqrt{-\phi_1^2 - 4\phi_2}}{2\phi_2} = re^{j\theta} = r \cos \theta + jr \sin \theta
\]

\[
x_2 = \frac{-\phi_1 - j\sqrt{-\phi_1^2 - 4\phi_2}}{2\phi_2} = re^{-j\theta} = r \cos \theta - jr \sin \theta
\]

where \( r = |x_1| = |x_2| \) and \( \theta = \Box x_1 \).

It could be shown that

\[
x_1 + x_2 = -\frac{\phi_1}{\phi_2} = 2r \cos \theta \quad (B-8)
\]

\[
x_1 - x_2 = j\sqrt{-\phi_1^2 - 4\phi_2} = 2jr \sin \theta \quad (B-9)
\]

Let \( r^{-1} = \rho \). Solving Equations B-8 and B-9 for \( \phi_1 \) and \( \phi_2 \) results the following:

\[
\phi_1 = 2\rho \cos \theta
\]

\[
\phi_2 = -\rho^2
\]

Substituting Equations B-7 and B-9 in B-6 results in:

\[
h(t) = \left( -\frac{r^{t-1}e^{-j\theta(t+1)}}{2jr \sin \theta} + \frac{r^{1+t}e^{j\theta(t+1)}}{2jr \sin \theta} \right) u(t)
\]

Note that:

\[
e^{j\theta(t+1)} - e^{-j\theta(t+1)} = 2j \sin(\theta(t+1)) \quad (B-12)
\]

Substituting B-12 in B-11 results the following:

\[
h(t) = \frac{\sin ((t+1)\theta)}{\sin(\theta)} \rho^t u(t)
\]

Thus, if \( \phi_1^2 + 4\phi_2 < 0 \) the impulse response of the second order system oscillates and damps as \( t \to +\infty \).

**Appendix C: Response of Unified Controlling System to Level Shift**

Define \( A_1 \) and \( A_2 \) as following:
\[ A_1 = \frac{x_2}{x_2 - x_1} \]
\[ A_2 = \frac{x_1}{x_1 - x_2} \]

**Case I:** \( \phi_1^2 + 4\phi_2 > 0 \)

Let \( \phi_1^2 + 4\phi_2 > 0 \). Substituting Equations 19 and 26 in Equation 29, results in:

\[
\lambda(t) = \sum_{k=0}^{\infty} \left[ A_1 \left( \frac{1}{x_1} \right)^k + A_2 \left( \frac{1}{x_2} \right)^k \right] \left( \mu_0 + (\gamma\sigma_0)u[t - t_0 - k] \right) \quad (C-2)
\]

For \( t < t_0 \)

\[
\lambda(t) = \mu_0 \sum_{k=0}^{\infty} \left[ A_1 \left( \frac{1}{x_1} \right)^k + A_2 \left( \frac{1}{x_2} \right)^k \right] \quad (C-3)
\]

After some algebraic computation of Equation C-3 one may obtain:

\[
\lambda(t) = \frac{\mu_0}{1 - \phi_1 - \phi_2} \quad t < t_0 \quad (C-4)
\]

For \( t \geq t_0 \) Equation C-2 becomes:

\[
\lambda(t) = \sum_{k=0}^{\infty} \left[ A_1 \left( \frac{1}{x_1} \right)^k + A_2 \left( \frac{1}{x_2} \right)^k \right] \left( \mu_0 + \gamma\sigma_0 \right) \quad (C-5)
\]

Simplifying Equation C-5 yields:

\[
\lambda(t) = (\mu_0 + \gamma\sigma_0) \left\{ A_1 \frac{x_1 - \left( \frac{1}{x_1} \right)^{t_0}}{x_1 - 1} + A_2 \frac{x_2 - \left( \frac{1}{x_2} \right)^{t_0}}{x_2 - 1} \right\} \quad t \geq t_0 \quad (C-6)
\]

Note that if \( t \rightarrow +\infty \), the control system reveals steady state behavior. This behavior is modeled by \( \lim_{t \rightarrow +\infty} \lambda(t) \) according to following:

\[
\lim_{t \rightarrow +\infty} \lambda(t) = \frac{\mu_0 + \gamma\sigma_0}{1 - \phi_1 - \phi_2} \quad (C-7)
\]

**Case II:** \( \phi_1^2 + 4\phi_2 < 0 \):
Now suppose $\phi_1^2 + 4\phi_2 < 0$. Substituting Equations 22 and 26 in Equation 29 results in:

$$\lambda[t] = \sum_{k=0}^{\infty} \sin((k+1)\theta) \frac{\rho^k}{\sin(\theta)} \left( \mu_0 + (\gamma\sigma_0)u[t-t_0-k] \right)$$  \hspace{1cm} (C-8)

For $t < t_0$ Equation C-8 becomes:

$$\lambda[t] = \frac{\mu_0}{\sin(\theta)} \sum_{k=0}^{\infty} \sin((k+1)\theta) \rho^k; \hspace{0.5cm} t < t_0$$  \hspace{1cm} (C-9)

Using Euler formula Equation C-9 is simplified as:

$$\lambda[t] = \frac{\mu_0}{1-2\rho\cos\theta+\rho^2} = \frac{\mu_0}{1-\phi_1 - \phi_2}; \hspace{0.5cm} t < t_0$$  \hspace{1cm} (C-10)

For $t \geq t_0$ Equation C-8 becomes:

$$\lambda[t] = \frac{\mu_0 + \gamma\sigma_0}{\sin(\theta)} \sum_{k=0}^{t_0-\infty} \sin((k+1)\theta) \rho^k; \hspace{0.5cm} t \geq t_0$$  \hspace{1cm} (C-11)

**Appendix D: Computing in Control $p_{ab}$**

For computing $p_{ab}$, let $a = (S_{ia},S_{2a})'$ and $b = (S_{ib},S_{2b})'$ be two states in state space. Thus, $p_{ab}$ is equal to:

$$p_{ab} = \Pr\left(D_0(z) = b | D_0(z_{i-1}) = a\right) = \Pr\left(D_0(Y_i) = S_{ib}, D_0(Y_{i-1}) = S_{2b} | D_0(Y_{i-1}) = S_{1a}, D_0(Y_{i-2}) = S_{2a}\right)$$  \hspace{1cm} (D-1)

If $S_{1a} \neq S_{2b}$, $p_{ab} = 0$. Due to this fact, only some transitions are possible in the intended Markov chain. These transitions are shown below:

$$a = \left(\begin{array}{c} S_{ia} \\ S_{2a} \end{array}\right) \rightarrow b = \left(\begin{array}{c} S_{ib} \\ S_{1a} \end{array}\right); \hspace{0.5cm} S_{ia}, S_{2a}, S_{ib} \in \{-m,-m+1,...,m\}$$  \hspace{1cm} (D-2)

$$a = \left(\begin{array}{c} m+1 \\ S_{ia} \end{array}\right), \left(\begin{array}{c} -m-1 \\ S_{ia} \end{array}\right), \left(\begin{array}{c} m+1 \\ S_{ia} \end{array}\right), \left(\begin{array}{c} -m-1 \\ S_{ia} \end{array}\right); \hspace{0.5cm} S_{1a}, S_{2a} \in \{-m,-m+1,...,m\}$$  \hspace{1cm} (D-3)

$$a = \alpha \rightarrow b = \alpha$$  \hspace{1cm} (D-4)

Probability of transition shown in D-2 is equal to:
\[ \Pr\left( D_0(Y_r) = S_{1b}, D_0(Y_{r-1}) = S_{1a}, D_0(Y_{r-2}) = S_{2a} \right) \]  
(D-5)

For small values of \( \Delta \), Equation D-5 becomes:

\[
p_{ab} = \Pr\left( (S_{1b} - 0.5)\Delta \leq Y_r \leq (S_{1b} + 0.5)\Delta \mid Y_{r-1} = S_{1a}, Y_{r-2} = S_{2a} \Delta \right)
\]

\[
= \Pr\left( (S_{1b} - 0.5)\Delta \leq \phi Y_{r-1} + \phi_2 Y_{r-2} + \bar{X}_r \leq (S_{1b} + 0.5)\Delta \mid Y_{r-1} = S_{1a} \Delta, Y_{r-2} = S_{2a} \Delta \right)
\]

\[
= \Pr\left( (S_{1b} - 0.5)\Delta \leq \phi S_{1a} \Delta + \phi_2 S_{2a} \Delta + \bar{X}_r \leq (S_{1b} + 0.5)\Delta \right)
\]

\[
= \Pr\left( \begin{bmatrix} S_{1b} - 0.5 - (\phi S_{1a} - \phi_2 S_{2a}) \Delta \sqrt{n} \leq Z \leq (S_{1b} + 0.5 - (\phi S_{1a} - \phi_2 S_{2a}) \Delta \sqrt{n} \end{bmatrix} \right)
\]

where \( Z \) is standard normally distributed.

To compute the probability of transitions shown in D-3, note that:

\[
\Pr\left\{ a = \begin{bmatrix} S_{1a} \\ S_{2a} \end{bmatrix} \rightarrow \begin{bmatrix} m+1 \\ m+1 \end{bmatrix} \right\} = \Pr\left\{ a = \begin{bmatrix} S_{1a} \\ S_{2a} \end{bmatrix} \rightarrow \begin{bmatrix} -m-1 \\ -m-1 \end{bmatrix} \right\} = 0
\]
(D-7)

For \( S_{1a}, S_{2a} \in \{-m, -m+1, \ldots, m\} \).

Other transition probabilities of D-3 for small \( \Delta \) are computed as follow:

\[
\Pr\left\{ a = \begin{bmatrix} S_{1a} \\ S_{2a} \end{bmatrix} \rightarrow \begin{bmatrix} m+1 \\ S_{1a} \end{bmatrix} \right\} = \Pr\left\{ D_0(Y_r) = m+1, D_0(Y_{r-1}) = S_{1a}, Y_{r-1} = S_{1a} \Delta, Y_{r-2} = S_{2a} \Delta \right\}
\]

\[
= \Pr\left\{ Y_r > (m+0.5)\Delta \mid Y_{r-1} = S_{1a} \Delta, Y_{r-2} = S_{2a} \Delta \right\}
\]

\[
= \Pr\left\{ \phi Y_{r-1} + \phi_2 Y_{r-2} + \bar{X}_r > (m+0.5)\Delta \mid Y_{r-1} = S_{1a} \Delta, Y_{r-2} = S_{2a} \Delta \right\}
\]

\[
= \Pr\left\{ \phi S_{1a} \Delta + \phi_2 S_{2a} \Delta + \bar{X}_r > (m+0.5)\Delta \right\}
\]

(D-8)

\[
= \Pr\left\{ Z > \frac{(m+0.5) \phi S_{1a} - \phi_2 S_{2a} \Delta \sqrt{n}}{\sigma_0} \right\}
\]

and:

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The probability of transition of $D-4$ is equal to 1.

Other transition probabilities except those shown in D-2 to D-5 are equal to zero.

**Appendix E: Computing in Control ARL**

Let $P$ be the transition matrix whose computation procedure is explained in Appendix D. According to Chang and Wu [25], $P$ could be written as:

$$ P_{(2m+2)(2m+2)} = \begin{pmatrix} A_{(2m+1)\times(2m+1)} & B_{(2m+1)\times 1} \\ \mathbf{0}'_{(2m+1)\times 1} & 1 \end{pmatrix} $$  \hspace{1cm} (E-1)

where $A_{(2m+1)\times(2m+1)}$ is the transition matrix of transient states, $B_{(2m+1)\times 1}$ is a vector whose entries are the probability of process moving to absorbing state, $\alpha$. Let $RL$ be the in-control run length of the proposed chart defined as follow:

$$ RL = \inf \left\{ t : D_\alpha (z_\alpha) = \alpha \right\} $$  \hspace{1cm} (E-2)

Thus, the in control ARL of the model can be defined as follow:

$$ ARL = E[RL] = \sum_{t=0}^{\infty} \Pr(RL > t) $$  \hspace{1cm} (E-3)

According to Fu and Spring [26], Equation E-3 could be written as follow:

$$ ARL = \pi_0' (I - A)^{-1} 1 $$  \hspace{1cm} (E-4)

where $\pi_0$ is a vector whose elements are the probability of $z_\alpha$ being in each state, $I$ is the identity matrix, $1$ is the vector whose elements are all equal to one.

**Figures Caption:**

Figure 1: The unified controlling system

Figure 2: Time representation of a LTI system
Figure 3: Frequency representation of a LTI system
Figure 4: Block diagram of first order system
Figure 5: Block diagram of first second order system
Figure 6: Impulse response with $\phi_1 = 1.7$ and $\phi_2 = -0.72$
Figure 7: Impulse response with $\rho = 0.91$ and $\theta = 0.11$
Figure 8: Expected value of $X_t$ and $Y_t$: a) $\phi_1 = 0.5$ and $\phi_2 = 0.1$, b) $\phi_1 = 0.5$ and $\phi_2 = -0.4$
Figure 9: Shewhart control chart
Figure 10: Proposed control chart
Figure 11: Stability region of second order filter
Figure 12: magnitude of Fourier transformation of level shift
Figure 13: Combination of the proposed method with feedback control loop
Figure 14: Segmentation of proposed control chart
Figure 15: ARL of control charts: a) $t_0 = 20$, b) $t_0 = 50$, c) $t_0 = 75$
Figure 16: PFS of control charts: a) $t_0 = 20$, b) $t_0 = 50$, c) $t_0 = 75$
Figure 17: Control Charts for Monitoring Exchange Rate: a) Shewhart, b) EWMA, c) AR(2)

Tables Title:

Table 1: Nomenclatures
Table 2: Response of system to level shift
Table 3: Characteristics of proposed method and Shewhart control chart
Table 4: Values of $L$ for the proposed control chart ($m=10$)
Table 1: Nomenclatures

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<thead>
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<th>Symbol</th>
<th>Definition</th>
<th>Symbol</th>
<th>Definition</th>
<th>Symbol</th>
<th>Definition</th>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mu_0)</td>
<td>In-control process mean</td>
<td>(\delta[t])</td>
<td>Unit impulse function</td>
<td>(\phi_1)</td>
<td>Coefficient of (y[t-1])</td>
<td>(m)</td>
<td>Parameter used to determine states of proposed control chart.</td>
</tr>
<tr>
<td>(\sigma_0)</td>
<td>In-control process standard deviation</td>
<td>(u[t])</td>
<td>Unit step function</td>
<td>(\phi_2)</td>
<td>Coefficient of (y[t-2])</td>
<td>(\Delta)</td>
<td>Width of control chart strips</td>
</tr>
<tr>
<td>(n)</td>
<td>Sample size to compute sample mean</td>
<td>(\mathbb{Z}^+)</td>
<td>Set of positive integer numbers</td>
<td>(t_0)</td>
<td>Change point in process mean</td>
<td>(D_0(.))</td>
<td>Discretizing function</td>
</tr>
<tr>
<td>(\mathbb{Z})</td>
<td>Set of integer numbers</td>
<td>(\mathbb{Z}^-)</td>
<td>Set of negative integer numbers</td>
<td>(\gamma)</td>
<td>Magnitude of shift in process mean</td>
<td>(S)</td>
<td>Markov chain state space</td>
</tr>
<tr>
<td>(t \in \mathbb{Z})</td>
<td>Time index</td>
<td>(Y_t)</td>
<td>Proposed test statistic to be monitored.</td>
<td>(E[.])</td>
<td>Expectation operator</td>
<td>(P)</td>
<td>Transition probability matrix</td>
</tr>
<tr>
<td>(\overline{X}_t)</td>
<td>Sample mean at time (t)</td>
<td>(h[t])</td>
<td>Impulse response of LTI system</td>
<td>(x_1, x_2)</td>
<td>Inverse of second order filter poles</td>
<td>(r)</td>
<td>Magnitude of (x_1, x_2)</td>
</tr>
<tr>
<td>(\lambda \in [0,1])</td>
<td>Parameter of EWMA control chart</td>
<td>(t', k \in \mathbb{Z})</td>
<td>Shifts in time index</td>
<td>(\nu[t])</td>
<td>Expected value of (\overline{X}_t)</td>
<td>(\rho)</td>
<td>Inverse of (r)</td>
</tr>
<tr>
<td>(L)</td>
<td>Coefficient of control chart</td>
<td>(w)</td>
<td>Frequency variable</td>
<td>(p_{ab})</td>
<td>One Step Transition Probability From State (a) to State (b).</td>
<td>(\theta = \mathbb{I}_1 \times x_1)</td>
<td>Phase of (x_1)</td>
</tr>
<tr>
<td>(j = \sqrt{-1})</td>
<td>Imaginary Unit</td>
<td>(H(e^{jw}))</td>
<td>Fourier transform of (h[t])</td>
<td>(\lambda[t])</td>
<td>Expected value of (Y_t)</td>
<td>(P(\cdot</td>
<td>\cdot))</td>
</tr>
<tr>
<td>(x[t])</td>
<td>Input signal of LTI system</td>
<td>(X(e^{jw}))</td>
<td>Fourier transform of (x[t])</td>
<td>(V(e^{jw}))</td>
<td>Fourier transform of (\nu[t])</td>
<td>(I)</td>
<td>Identity Matrix</td>
</tr>
<tr>
<td>(y[t])</td>
<td>Output signal of LTI system</td>
<td>(Y(e^{jw}))</td>
<td>Fourier transform of (y[t])</td>
<td>(\Lambda(e^{jw}))</td>
<td>Fourier transform of (\lambda[t])</td>
<td>(\alpha)</td>
<td>Absorbing State</td>
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</tbody>
</table>
Table 2: Response of system to level shift

<table>
<thead>
<tr>
<th>Condition</th>
<th>Test Statistics</th>
<th>Control Limits</th>
<th>In control Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t &lt; t_0$</td>
<td>$\lambda[t] = \frac{\mu_0}{1-\phi_1-\phi_2}$</td>
<td>$\pm 2.9\sigma_y$</td>
<td>$</td>
</tr>
<tr>
<td>$t \geq t_0$</td>
<td>$\lambda[t] = (\mu_0 + \gamma\sigma_0) \left{ \frac{1}{x_1} \right}^{t-t_0} + \frac{1}{x_2-1} \right}^{t-t_0}$</td>
<td>$\frac{3\sigma_0}{\sqrt{n}}$</td>
<td>$</td>
</tr>
</tbody>
</table>

Table 3: Characteristics of proposed method and Shewhart control chart

<table>
<thead>
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<th>Method</th>
<th>Test Statistics</th>
<th>Control Limits</th>
<th>In control Region</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed Method</td>
<td>$Y_i - \frac{\mu_0}{1-\phi_1-\phi_2}$</td>
<td>$\pm 2.9\sigma_y$</td>
<td>$</td>
</tr>
<tr>
<td>Shewhart</td>
<td>$\bar{X}_i - \mu_0$</td>
<td>$\pm \frac{3\sigma_0}{\sqrt{n}}$</td>
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Table 4: Values of $L$ for the proposed control chart ($m = 10$)

<table>
<thead>
<tr>
<th>ARL</th>
<th>$\phi_1^2 + 4\phi_2 &gt; 0$</th>
<th>$\phi_1^2 + 4\phi_2 &lt; 0$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_1 = 0.2$ $\phi_2 = 0.79$</td>
<td>$\phi_1 = 1.8$ $\phi_2 = -0.85$</td>
</tr>
<tr>
<td>200</td>
<td>1.39</td>
<td>2.63</td>
</tr>
<tr>
<td>250</td>
<td>1.50</td>
<td>2.79</td>
</tr>
<tr>
<td>300</td>
<td>1.61</td>
<td>2.89</td>
</tr>
<tr>
<td>370</td>
<td>1.71</td>
<td>3.05</td>
</tr>
<tr>
<td>500</td>
<td>1.88</td>
<td>3.25</td>
</tr>
<tr>
<td></td>
<td>$\phi_1 = 0.85$ $\phi_2 = 0.14$</td>
<td>$\phi_1 = 1.5$ $\phi_2 = -0.6$</td>
</tr>
<tr>
<td>200</td>
<td>1.51</td>
<td>2.54</td>
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<tr>
<td>250</td>
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<tr>
<td>300</td>
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<td>370</td>
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<td>2.92</td>
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