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Linguistic Z-number Muirhead mean operators and their applications in ethical-financial portfolio selection

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Abstract

Traditionally the performance of firms is evaluated by financial criteria, but this study presents a new qualitative comprehensive framework that incorporates the ethical criteria into the portfolio models and it is widely matched with the preferences of socially responsible investors. The increase of corporate deceptions has caused investors or fund managers consider the ethical assessments in their investment management. Therefore, it is essential to develop models that capture the ethical criteria along with the financial criteria in the investment processes. In this study, a multi-stage methodology is proposed and linguistic Z-numbers are applied to represent the evaluation information, and the Muirhead mean (MM) aggregation operators are employed to fuse the input data under linguistic Z-number environment. Hence, we firstly develop four linguistic Z-number Muirhead mean operators. Then, using the max-score rule and the score-accuracy trade-off rule, three qualitative portfolio models are proposed. These models have been aimed to maximize the financial performance of portfolio as main objective and have been distinguished by the ethical goal that the investor follows. The obtained results of numerical example validates the capability of the models for constructing more diversified portfolio based on a trade-off between financial and ethical criteria according to investors’ preferences.

Keywords: Portfolio selection, Ethical and financial investment, Z-number, Reliability, Aggregation operators.

1. Introduction

Modern portfolio theory, which is introduced by Markowitz [1], has only emphasized on financial criteria to manage the assets and selects the optimal portfolios. In other words, the traditional approaches of selecting assets are one-dimensional and they just concentrate on financial criteria as the basis of evaluation. However, there is increasing evidence that traditional portfolio theory cannot capture all related data only in form of financial criteria [2]. On the other hand, there are some growing samples of environmental hazards, corporate deceptions and the social pressure to consider the social responsibilities in investment processes which have reinforced the assessment of ethical criteria in addition to financial criteria in investment management. Nowadays considering the ethical and social principles such as environment, civil rights protection, transparency, and credibility of the corporations is an unavoidable necessity in the investment processes.

Thus, both financial criteria and ethical criteria should be captured simultaneously to construct more diversified portfolio according to investors’ preferences. An ethical-financial demeanor in the investment management called ethical investment or socially responsible investment (SRI) has become widespread over the world since 1960, and compliance of environment, social and governance (ESG) requirements has considerably increased in the last decade. Nowadays, many asset owners, asset managers, and firms publish ESG and sustainability information in their annual reports. According to the latest Global
Sustainable Investment Alliance (GSIA) report 2016, global sustainable investment assets reached $22.89 trillion [3]. Therefore, the increasing number of SRI bonds shows growing interest of issuers and investors for sustainable development plans. However, a serious challenge in the ethical investment is the conflicts between the ethical and financial objectives that might have negative impacts on financial objectives such as profitability of investment. Hence, in order to better utilize the high potential of SRI markets and more accurate investigation of the influences of ESG criteria on the portfolio management, some authors combined the ethical approaches with the investment processes and analyzed their influences over investors’ decisions [4-8]. Therefore, to create a trade-off between ethical and financial criteria in portfolio selection problems, this study combines ESG and financial criteria with the asset allocation problem.

Financial markets always encounter uncertainty and information is imperfect, vague and uncertain. Hence, acquiring exact quantitative estimation for input parameters is difficult and sometimes impossible. In addition, since there are some firms which have recently been constituted and are newly added to financial markets, the quantitative information about them is not adequately accessible. These limitations in the evaluation of the firm performance can be omitted by the utilization of fuzzy set theory [9] and its extended forms such as the interval-valued fuzzy sets [10], the type-2 fuzzy sets [11], the intuitionistic fuzzy sets [12], the hesitant fuzzy sets [13] and the interval-valued hesitant fuzzy sets [14]. The fuzzy set theory and its developed forms have widely been used to model portfolio selection problems [15-17]. Although fuzzy set theory and its extended forms have created considerable improvement in the representation of human’s knowledge, they have shortcomings in the expression of the reliability of information. Thus, to fill this limitation, Zadeh [18] introduced the concept of Z-numbers. Totally, the studies focused on Z-numbers can be categorized into two fields. The first field is the basic studies comprising conversion techniques [19-20], arithmetic operations [21-23], ranking methods [24-28] and development researches [29-30]. The second field can be assigned the utilization of Z-numbers in the optimization and decision making problems [31-37]. The main advantage of the Z-numbers is to model the uncertainty and reliability of relevant information, simultaneously. With increasing complexity of the financial markets, this property of Z-numbers can help investors and fund managers make more fruitful decisions. Therefore, this study applies the concept of Z-number to represent the information evaluation related to the ethical-financial criteria.

Due to the fuzziness, uncertainty and ambiguity of financial markets, linguistic terms help DMs or experts to assess real information. For instance, when assessing the financial performance or the ethical performance of a firm, expert can apply linguist terms such as “low”, “medium” and “high”. The prominent feature of linguistic terms is that they usually include both vagueness and uncertainty. The fuzziness and the randomness are the most significant aspects of uncertainty in linguistic terms [38]. The fuzziness and the randomness available in linguistic terms completely match with the possibilistic and probabilistic constraints in Z-numbers [39]. Hence, Peng et al. [40] introduced uncertain hesitant linguistic Z-numbers. Wang et al. [39] improved the concept of Z-numbers and introduced a subclass of Z-numbers called linguistic Z-numbers. A linguistic Z-number can represent the fuzzy constraint by linguistic terms such as “low”, “medium” and “high” and it can characterize the measure of reliability by linguistic terms such as “seldom”, “frequently” and “usually”. For instance, linguistic Z-number (medium, usually) can be applied to assess the financial performance of a firm or estimate the future return of an asset. The main advantages of linguistic Z-numbers are that they not only reduce the loss of information due to creating a more flexible, holistic and accurate structure but also they capture the possibilistic and probabilistic constraints, simultaneously. These properties cause the linguistic Z-numbers to be beneficial for evaluating the ethical-financial information in financial markets. Therefore, this study applies linguistic Z-numbers to evaluate the ethical and financial performance of each firm.

One of the most useful and powerful tools in decision-making problems under various uncertain environments is aggregation operator. The main advantage of aggregation operators in comparison with traditional decision-making techniques is that they can obtain the comprehensive values for each alternative and then they rank alternatives. It may be noted that the decision-making techniques modeled based on aggregation operators are preferable in comparison with the traditional techniques [41]. During the last decade, the aggregation operators have become an interesting idea of investigation. Some
aggregation operators such as prioritized average (PA) operator [42] have been developed in positions where the input arguments are exact. Hence, many scholars extended aggregation operators under various types of fuzzy environment [43-51]. So far, there are only two aggregation operators under linguistic Z-number environment such as the hesitant uncertain linguistic Z-numbers power weighted average (HULZPW) operator, and the hesitant uncertain linguistic Z-numbers power weighted geometric (HULZPG) operator [40]. All mentioned aggregation operators are inefficient at considering the interrelationship between the input arguments and all arguments are supposed independent. Hence, some operators such as Bonferroni mean (BM) operator [52], Heronian mean (HM) operator [41, 53] and geometric Bonferroni mean (GBM) operator [54] were developed to consider the interrelationship among the input arguments. However, BM operator, HM operator, and GBM operator have a shortcoming. They only take into account the interrelationship among two arguments and they are unable to capture the interrelationship between all input arguments. So, it is imperative to develop a more holistic operator which is able to consider the interrelationship between all input arguments. Muirhead mean (MM) operator [55] can capture the interrelationship between every input argument. Recently, Liu and Teng [56] proposed some MM operators under probabilistic linguistic environment. However, there is no MM operator under Z-number environment. Therefore, this study applies Muirhead mean (MM) operator to fuse linguistic Z-numbers information.

On the basis of above discussions, since linguistic Z-numbers can simultaneously capture the possibilistic and probabilistic constraints in the evaluation information modeling and can reduce the loss of information due to creating a more flexible, holistic and accurate structure, they can represent the assessment information more flexibly and efficiently. Moreover, MM operator can aggregate the linguistic Z-number information and linguistic scale functions (LSFs) can be applied to describe various semantic measures. Therefore, the integration of the asset allocation problems with linguistic Z-numbers and MM aggregation operators is very useful. The main motivations of this study are briefly highlighted as follows:

1. Applying linguistic Z-numbers to represent the information evaluation relevant to the ethical and financial performances of firms. This feature causes the proposed approach and the proposed models become more general and more flexible in comparison with the traditional portfolio models due to capturing the reliability of evaluation information in the modeling process.
2. Developing MM operator under linguistic Z-number environment and introducing linguistic Z-number Muirhead mean (LZMM) operator, linguistic Z-number weighted Muirhead mean (LZWMM) operator, linguistic Z-number dual Muirhead mean (LZDMM) operator and linguistic Z-number dual weighted Muirhead mean (LZDWMM) operator.
3. Proposing three qualitative ethical-financial portfolio models based on LZMM, LZWMM, LZDMM and LZDWMM operators under linguistic Z-number environment. The proposed models consider investors’ preferences to construct more diversified portfolios and they are suitable for both general socially responsible investors and risky socially responsible investors.

The rest of this paper is structured as follows. Section 2 includes the necessary prerequisite definitions. In Section 3, we introduce some aggregation operators under linguistic Z-number environment and present their properties. A multi-stage methodology is proposed to assign the suitable assets in the portfolio and three new qualitative portfolio models are developed under linguistic Z-number environment in Section 4. Section 5 provides the required actual data as a case study and the results and sensitivity analysis are shown. Finally, concluding remarks and future work suggestions are presented in Section 6.

2. Preliminaries

2.1. Linguistic terms set and linguistic scale functions (LSFs)
Suppose \( t_i \in T \) be a conceivable value of linguistic variable where \( T = \{ t_i \mid l = 0,1,\ldots,2m \} \). \( T \) should include the following properties [57, 58]:

a) \( T \) is ordered: \( t_i < t_k \) if and only if \( l < k \).

b) \( T \) conforms negation operator: \( \text{neg}(t_i) = t_{2m-l} \).

Clearly, \( T \) is a discrete linguistic term set. Since the computational results do not usually match the members of \( T \), Xu [59, 60] has introduced a continuous linguistic term set \( \bar{T} \) where \( \bar{T} = \{ t_i \mid i \in [0,1] \} \) in order to prevent the loss of obtained information.

Since computation with linguistic terms (LTs) is placed within the category of computing with words (CWW), carrying out the arithmetic operation with them is not easy, some functions called linguistic scale functions (LSFs) have been defined to simplify the computation under linguistic environment [46, 60]. In order to have more flexible and efficient information, various semantics are devoted to linguistic terms by using LSFs under various circumstances. There is a strictly monotonically ascending connection between each \( t_i \in T \) and its label [46, 60].

Definition 1. Let \( T = \{ t_i \mid l = 0,1,\ldots,2m \} \) be a set of discrete linguistic terms with odd cardinality. The linguistic scale function \( f \) is defined as follows [46]:

\[
f: t_i \rightarrow \theta_i \quad (l = 0,1,\ldots,2m)
\]

where \( \theta_i \) is a positive real number and \( 0 \leq \theta_0 \leq \theta_1 \leq \ldots \leq \theta_{2m} \). \( \theta_1 \) characterizes the priorities of DMs when \( t_i \in T \) is chosen to describe their opinions.

Some LSFs are introduced as follows [39, 46]:

**LSF1:** \[
f_1(t_i) = \theta_i = \frac{l}{2m} \quad (l = 0,1,\ldots,2m) \quad \text{and} \quad \theta \in [0,1]
\]

**LSF2:** \[
f_2(t_i) = \theta_i = \left( \frac{l}{2m} \right)^m \quad (l = 0,1,\ldots,2m)
\]

**LSF3:** \[
f_3(t_i) = \theta_i = \left( \frac{l}{2m} \right)^\frac{1}{m} \quad (l = 0,1,\ldots,2m)
\]

**LSF4:**

\[
f_4(t_i) = \theta_i = \begin{cases} 
\frac{m^\alpha - (m-l)^\alpha}{2m^\alpha} & (l = 0,1,\ldots,m) \\
\frac{m^\beta + (l-m)^\beta}{2m^\beta} & (l = m+1,\ldots,2m)
\end{cases}
\]

where \( \alpha \) and \( \beta \in (0,1] \).

2.2. Z-numbers and linguistic Z-numbers

This subsection covers the definition of Z-numbers, uncertain linguistic Z-numbers and their operation.
2.2.1. Z-numbers

Uncertainty is an inseparable feature of the real problems. Always, DMs use their uncertain data, knowledge and experiments to select the best solutions. In order to make more beneficial decisions, this information must be reliable. Hence, Zadeh [18] introduced the concept of Z-numbers to better represent uncertain data with incorporation of partial reliability and fuzziness.

Definition 2. [18] (Z-number). An ordered pair of fuzzy numbers as \((\tilde{A}, \tilde{B})\) shows a Z-number such that the values, which can be assigned to an uncertain variable \(X\), are represented by using the first component as a fuzzy constraint and a soft restriction on a partial reliability of the first component is determined by the second component, \(\tilde{B}\). Often \(\tilde{A}\) and \(\tilde{B}\) are expressed by using linguistic terms.

2.2.2. Linguistic Z-numbers

Following to Zadeh [18], two forms of Z-numbers have been developed to better describe uncertain data with consideration of partial reliability. Peng and Wang [40] introduced hesitant uncertain linguistic Z-numbers by using the concept of Z-numbers and linguistic terms. Wang et al. [39] extended a new form of Z-numbers called linguistic Z-numbers in order to measure the reliability of the real phenomena and describe the qualitative data, simultaneously.

Definition 3. [39] (Linguistic Z-numbers). Consider a universe of discourse \(U\). Let two finite discrete linguistic term sets representing different preference data are defined as \(T = \{t_0, t_1, \ldots, t_m\}\) and \(T' = \{t'_0, t'_1, \ldots, t'_n\}\) where \(m\) and \(n\) are nonnegative integers. Therefore, a linguistic Z-number set in \(U\) is defined as follows:

\[
Z = \left\{ \left( u, A_{\varphi(u)}, B_{\varphi(u)} \right) \mid u \in U \right\}
\]

where \(A_{\varphi(u)}\) is a fuzzy constraint on the values which can be assigned to the uncertain variable and \(B_{\varphi(u)}\) characterizes a reliability measure of the first component. \(A_{\varphi(u)}\) and \(B_{\varphi(u)}\) are described by using uncertain linguistic terms.

2.2.3. The arithmetic operations over uncertain linguistic Z-numbers

Wang, Cao and Zhang [39] developed some arithmetic operations for LZNs. The proposed operations maintain both the flexibility of linguistic term sets and the reliability value of Z-numbers.

Definition 4. [39]. Suppose two linguistic Z-numbers are defined as \(z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \) and \(z_j = \left( A_{\varphi(j)}, B_{\varphi(j)} \right) \). \(f^*\) and \(g^*\) functions can be selected from between \(f_1(t_i), f_2(t_i), f_3(t_i)\) and \(f_4(t_i)\). Hence, some operations under linguistic Z-number environment are defined as follows:

\[
\text{neg} \left( z_i \right) = f^{-1} \left( f^* \left( A_{2m} \right) - f^* \left( A_{\varphi(i)} \right) \right) \cdot g^{-1} \left( g^* \left( B_{2n} \right) - g^* \left( B_{\varphi(i)} \right) \right)
\]

\[
z_i + z_j = f^{-1} \left( f^* \left( A_{\varphi(i)} \right) + f^* \left( A_{\varphi(j)} \right) \right) \cdot g^{-1} \left( f^* \left( A_{\varphi(i)} \right) \times g^* \left( B_{\varphi(i)} \right) + f^* \left( A_{\varphi(j)} \right) \times g^* \left( B_{\varphi(j)} \right) \right) \]

(1)
\[ \rho z_i = \left( f^{-1} \left( \rho f^* \left( A_{2(i)} \right) \right), B_{\phi(i)} \right) \quad \rho \geq 0 \]  
(8)

\[ z_i \times z_j = \left( f^{-1} \left( f^* \left( A_{2(i)} \right) \right) f^{-1} \left( f^* \left( A_{2(j)} \right) \right), g^{-1} \left( g^* \left( B_{\phi(i)} \right) g^* \left( B_{\phi(j)} \right) \right) \right) \]  
(9)

\[ z_i^p = \left( f^{-1} \left( f^* \left( A_{2(i)} \right) \right)^p, g^{-1} \left( g^* \left( B_{\phi(i)} \right)^p \right) \right) \quad \rho \geq 0 \]  
(10)

**Definition 5.** [39]. Suppose \( z_i = \left( A_{2(i)}, B_{\phi(i)} \right) \) be a linguistic Z-numbers. Then, the score function of linguistic Z-number is equal to:

\[ S(z_i) = f^* \left( A_{2(i)} \right) \times g^* \left( B_{\phi(i)} \right) \]  
(11)

The accuracy function of linguistic Z-number is as follows:

\[ A(z_i) = f^* \left( A_{2(i)} \right) \times \left( 1 - g^* \left( B_{\phi(i)} \right) \right) \]  
(12)

By using the score and accuracy functions, a comparison technique is defined for two LZNs as follows [39]:

I. If \( S(z_i) > S(z_j) \), then \( z_i > z_j \)

II. If \( S(z_i) = S(z_j) \), then

- If \( A(z_i) > A(z_j) \), then \( z_i > z_j \)
- If \( A(z_i) < A(z_j) \), then \( z_i < z_j \)

### 2.3. Muirhead mean (MM) operator

**Definition 6.** [55]. Let \( a_i (i = 1, \ldots, k) \) be a set of nonnegative real numbers and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then, the MM operator is defined as follows:

\[
\text{MM}^P \left( a_1, a_2, \ldots, a_k \right) = \left( \frac{1}{k!} \sum_{\theta \in S_k} \prod_{i=1}^{k} \theta_{a_i}^{P_{\theta(i)}} \right)^{\frac{1}{k!}}
\]
(13)

where \( \left( \theta(1), \theta(2), \ldots, \theta(k) \right) \in S_k \), and \( S_k \) is a set of all permutation of \( (1,2,\ldots,k) \).

**Definition 7.** [61]. Let \( a_i (i = 1, \ldots, k) \) be a set of nonnegative real numbers and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then, the Dual Muirhead mean (DMM) operator is defined as follows:

\[
\text{DMM}^P \left( a_1, a_2, \ldots, a_k \right) = \frac{1}{\sum_{\theta \in S_k} \prod_{i=1}^{k} P_{\theta(i)}^{a_i}} \left( \prod_{\theta \in S_k} \sum_{i=1}^{k} P_{\theta(i)} \right)^{\frac{1}{k!}}
\]
(14)

where \( \left( \theta(1), \theta(2), \ldots, \theta(k) \right) \in S_k \), and \( S_k \) is a set of all permutation of \( (1,2,\ldots,k) \).

### 3. Linguistic Z-number Muirhead mean aggregation operators
In this section, four types of Muirhead mean (MM) aggregation operators are developed under linguistic Z-number environment. The description of MM operators and its related properties and theorems are shown in the following.

3.1. Linguistic Z-number Muirhead mean (LZMM) operator

**Definition 8.** Suppose \( Z = \{ z_i = (A_{z(i)}, B_{z(i)}) | i = 1, \ldots, k \} \) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then the linguistic Z-number Muirhead mean (LZMM) operator can be defined as follows:

\[
LZMM^P(Z_1, Z_2, \ldots, Z_k) = \left( \frac{1}{k!} \sum_{\theta \in S_k} \prod_{i=1}^{k} Z_{\theta(i)}^p \right) \sum_{i=1}^{k} P_i
\]

(15)

where \( \theta (1), \theta (2), \ldots, \theta (k) \in S_k \) and \( S_k \) is a set of all permutation of \( (1,2,\ldots,k) \).

**Theorem 1.** Let \( z_i = (A_{z(i)}, B_{z(i)}) (i = 1, \ldots, k) \) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then, the aggregated result acquired based on LZMM operator is a LZN and it is indicated as follows:

\[
LZMM^P(Z_1, Z_2, \ldots, Z_k) = \left( f^{-1} \left( \frac{1}{k!} \sum_{\theta \in S_k} \prod_{i=1}^{k} f' (A_{z(\theta(i))})^P_i \right) \right) \cdot \left( \frac{\sum_{\theta \in S_k} \prod_{i=1}^{k} f' (A_{z(\theta(i))})^P_i \cdot \prod_{i=1}^{k} \theta (i)}{\sum_{\theta \in S_k} \prod_{i=1}^{k} f' (A_{z(\theta(i))})^P_i} \right)
\]

(16)

The proof of Theorems 1 is given in Appendix A.

**Theorem 2.** (Commutativity) Let \( z_i = (A_{z(i)}, B_{z(i)}) \) and \( z'_i = (A_{z(i)}, B_{z(i)}) \) be two sets of LZNs, where \( z'_i = (A_{z(i)}, B_{z(i)}) (i = 1, \ldots, k) \) is any permutation of \( z_i = (A_{z(i)}, B_{z(i)}) (i = 1, \ldots, k) \), and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then

\[
LZMM^P(Z_1, Z_2, \ldots, Z_k) = LZMM^P(Z'_1, Z'_2, \ldots, Z'_k)
\]

(17)

**Proof.**

\[
LZMM^P(Z_1, Z_2, \ldots, Z_k) = \left( \frac{1}{k!} \sum_{\theta \in S_k} \prod_{i=1}^{k} (Z_{\theta(i)})^p \right)^{\frac{1}{P_i}} \sum_{i=1}^{k} P_i = \left( \frac{1}{k!} \sum_{\theta \in S_k} \prod_{i=1}^{k} (Z_{\theta(i)})^p \right)^{\frac{1}{P_i}} = LZMM^P(Z_1, Z_2, \ldots, Z_k)
\]

3.2. Linguistic Z-number weighted Muirhead mean (LZWMM) operator
Definition 9. Suppose \( Z = \left\{ z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \mid i = 1, \ldots, k \right\} \) be a set of LZNs, and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Let \( w_i \in [0,1] \) be the weight of the \( i^{th} \) input parameter such that \( \sum_{i=1}^{k} w_i = 1 \). Then the linguistic Z-number weighted Muirhead mean (LZWMM) operator can be defined as follows:

\[
\text{LZWMM}^p (Z_1, Z_2, \ldots, Z_k) = \left( \frac{1}{k!} \sum_{\sigma \in S_k} \left( \prod_{i=1}^{k} \left( kw_i \left( \sigma^{-1} \right) Z_{\sigma(i)} \right)^{w_i} \right) \right)^{1/k} \sum_{\sigma \in S_k}
\]

where \( (\theta(1), \theta(2), \ldots, \theta(k)) \in S_k \), and \( S_k \) is a set of all permutation of \((1,2,\ldots,k)\).

Theorem 3. Let \( z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \) (i = 1, \ldots, k) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Let \( w_i \in [0,1] \) be the weight of the \( i^{th} \) input parameter such that \( \sum_{i=1}^{k} w_i = 1 \). Then, the aggregated result acquired based on LZWMM operator is a LZN and it is indicated as follows:

\[
\text{LZWMM}^p (Z_1, Z_2, \ldots, Z_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \left( \prod_{i=1}^{k} \left( kw_i \left( \sigma^{-1} \right) Z_{\sigma(i)} \right)^{w_i} \right)^{1/k} \sum_{\sigma \in S_k}
\]

Similarly, Theorem 3 can easily be proven by using Definition 4 and the mathematical induction technique.

Theorem 4. (Commutativity) Let \( z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \) and \( z_i' = \left( A'_{\varphi(i)}, B'_{\varphi(i)} \right) \) be two sets of LZNs, where \( z_i' = \left( A'_{\varphi(i)}, B'_{\varphi(i)} \right) \) (i = 1, \ldots, k) is any permutation of \( z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \) (i = 1, \ldots, k), and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then

\[
\text{LZWMM}^p (Z_1, Z_2, \ldots, Z_k) = \text{LZWMM}^p (Z_1, Z'_2, \ldots, Z'_k)
\]

3.3. Linguistic Z-number dual Muirhead mean (LZDMM) operator

Definition 10. Suppose \( Z = \left\{ z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) \mid i = 1, \ldots, k \right\} \) be a set of LZNs, and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then the linguistic Z-number dual Muirhead mean (LZDMM) operator can be defined as follows:

\[
\text{LZDMM}^p (Z_1, Z_2, \ldots, Z_k) = \frac{1}{\sum_{i=1}^{k} P_i \sum_{\sigma \in S_k} \left( P_i Z_{\sigma(i)} \right)^{w_i} \sum_{\sigma \in S_k}}
\]

where \( (\theta(1), \theta(2), \ldots, \theta(k)) \in S_k \), and \( S_k \) is a set of all permutation of \((1,2,\ldots,k)\).
Theorem 5. Let \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then, the aggregated result acquired based on LZDMM operator is a LZN and it is shown as follows:

\[
LZDMM^p(Z_1, Z_2, \ldots, Z_k) = \left( f'^{-1} \left( \frac{1}{\sum_{i=1}^{k} p_i \left( \prod_{\varphi(i) \in S_1} \left( \sum_{i=1}^{k} p_i f^r \left( A_{\varphi(i)} \right) \right)^{m_{\varphi(i)}} \right) \right)^{\frac{1}{m}} \right), g'^{-1} \left( \frac{\sum_{i=1}^{k} p_i f^r \left( A_{\varphi(i)} \right) x g^r \left( B_{\varphi(i)} \right) \right)^{\frac{1}{m}} \right)^\frac{1}{m}
\]

(22)

The proof of Theorems 5 is given in Appendix B.

Theorem 6. (Commutativity) Let \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) and \( z'_i = (A'_{\varphi(i)}, B'_{\varphi(i)}) \) \((i = 1, \ldots, k)\) be two sets of LZNs, where \( z'_i = (A'_{\varphi(i)}, B'_{\varphi(i)}) \) \((i = 1, \ldots, k)\) is any permutation of \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\), and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then

\[
LZDMM^p(Z_1, Z_2, \ldots, Z_k) = LZDMM^p(Z'_1, Z'_2, \ldots, Z'_k)
\]

(23)

3.4. Linguistic Z-number dual weighted Muirhead mean (LZDWM) operator

Definition 11. Suppose \( Z = \left\{ z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \right\} | i = 1, \ldots, k \) be a set of LZNs, and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Let \( w_i \in [0,1] \) be the weight of the \( i^{th} \) input parameter such that \( \sum_{i=1}^{k} w_i = 1 \). Then the linguistic Z-number dual weighted Muirhead mean (LZDWM) operator can be defined as follows:

\[
LZDWM^p(Z_1, Z_2, \ldots, Z_k) = \frac{1}{\sum_{i=1}^{k} P_i} \left( \prod_{i=1}^{k} \sum_{S_1} \left( P_i f (Z_{\varphi(i)}) \right)^{m_{\varphi(i)}} \right)^{\frac{1}{m}}
\]

(24)

Where \( \left( \theta(1), \theta(2), \ldots, \theta(k) \right) \in S_k \), and \( S_k \) is a set of all permutation of \( \{1,2,\ldots,k\} \).

Theorem 7. Let \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Let \( w_i \in [0,1] \) be the weight of the \( i^{th} \) input parameter such that \( \sum_{i=1}^{k} w_i = 1 \). Then, the aggregated result acquired based on LZDWM operator is a LZN and it is shown as follows:

\[
LZDWM^p(Z_1, Z_2, \ldots, Z_k) = \left( f'^{-1} \left( \frac{1}{\sum_{i=1}^{k} p_i \left( \prod_{\varphi(i) \in S_1} \left( \sum_{i=1}^{k} p_i f^r \left( A_{\varphi(i)} \right) \right)^{m_{\varphi(i)}} \right) \right)^{\frac{1}{m}} \right), g'^{-1} \left( \frac{\sum_{i=1}^{k} p_i f^r \left( A_{\varphi(i)} \right) x g^r \left( B_{\varphi(i)} \right) \right)^{\frac{1}{m}} \right)^\frac{1}{m}
\]

(25)

Similarly, Theorem 7 can easily be proven by using Definition 4 and the mathematical induction technique.
**Theorem 8.** (Commutativity) Let \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) and \( \tilde{z}_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\) be two sets of LZNs, where \( \tilde{z}_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\) is any permutation of \( z_i = (A_{\varphi(i)}, B_{\varphi(i)}) \) \((i = 1, \ldots, k)\), and \( P = (P_1, P_2, \ldots, P_k) \in R^k \) be a parametric vector. Then

\[
LZDWMMP (Z_1, Z_2, \ldots, Z_k) = LZDWMMP (\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_k)
\]  
(26)

4. Ethical–financial portfolio selection under linguistic Z-number environment

In this section, a qualitative framework is proposed to construct the ethical-financial portfolios based on the mentioned aggregation operations under linguistic Z-number environment. Expert’s experience and opinions are the most important source of data to assess the ethical-financial performance of assets that can be applied to describe uncertainty and reliability of information, simultaneously. The proposed aggregation operators are powerful tools to incorporate expert’s opinions under linguistic Z-number environment. Figure 1 shows a total schematic of the proposed method.

![Typical flowchart for the ethical-financial portfolio selection based on the linguistic Z-number aggregation operators.](image)

This approach is composed of three stages. In the first stage, the firms are evaluated based on their ethical and financial criteria, and then, experts express their opinions about the ethical-financial performance of each firm as linguistic Z-number. In the second stage, the ethical and financial criteria are separately aggregated by using the proposed aggregation operators, and their score and accuracy values are computed. Finally, three qualitative portfolio selection models are developed according to expert’s opinions and investor’s preferences in the third stage.

4.1. The first stage: Ethical and financial assessment of assets

Assume that there are \( n \) risky assets as \( \{x_1, x_2, \ldots, x_n\} \) that investors can devote their capital to them. These assets are separately evaluated by expert based on \( m \) qualitative financial criteria \( \{f_1, f_2, \ldots, f_m\} \) and \( r \) qualitative ethical criteria \( \{e_1, e_2, \ldots, e_r\} \). The evaluation information can be expressed by the linguistic Z-numbers as \( z_{ij}^f \) and \( z_{ik}^e \) \((i = 1, \ldots, n, j = 1, \ldots, m \text{ and } k = 1, \ldots, r)\), respectively, and it can be represented as two linguistic Z-number matrices \( Z_f = [z_{ij}^f]_{n \times m} \) and \( Z_e = [z_{ik}^e]_{n \times r} \).

4.2. The second stage: computing the comprehensive value of each asset and its corresponding score and accuracy values

In this stage, the financial and ethical comprehensive values of each asset are separately obtained based on the proposed aggregation operators \( \tilde{z}_i^f = (\tilde{A}_{\varphi(i)}^f, \tilde{B}_{\varphi(i)}^f) \) and \( \tilde{z}_i^e = (\tilde{A}_{\varphi(i)}^e, \tilde{B}_{\varphi(i)}^e) \) \((i = 1, \ldots, n)\), respectively. The aggregated values are calculated under the proposed aggregation operators as follows:
\[ \tilde{z}_i^f = LZMM^p \left(z_{i1}^f, \ldots, z_{im}^f \right) \]
\[ \tilde{z}_i^e = LZMM^p \left(z_{i1}^e, \ldots, z_{ik}^e \right) \]
Or
\[ \tilde{z}_i^f = LZWMM^p \left(z_{i1}^f, \ldots, z_{im}^f \right) \]
\[ \tilde{z}_i^e = LZWMM^p \left(z_{i1}^e, \ldots, z_{ik}^e \right) \]
Or
\[ \tilde{z}_i^f = LZDMM^p \left(z_{i1}^f, \ldots, z_{im}^f \right) \]
\[ \tilde{z}_i^e = LZDMM^p \left(z_{i1}^e, \ldots, z_{ik}^e \right) \]
Or
\[ \tilde{z}_i^f = LZDWMM^p \left(z_{i1}^f, \ldots, z_{im}^f \right) \]
\[ \tilde{z}_i^e = LZDWMM^p \left(z_{i1}^e, \ldots, z_{ik}^e \right) \]

Consequently, two linguistic Z-number matrices \( Z^f = \left[ z_{ij}^f \right]_{n \times m} \) and \( Z^e = \left[ z_{ik}^e \right]_{n \times r} \) can be transformed into two qualitative column vectors \( \tilde{Z}^f = \left[ \tilde{z}_i^f \right]_{n \times 1} \) and \( \tilde{Z}^e = \left[ \tilde{z}_i^e \right]_{n \times 1} \) by fusing all the financial linguistic Z-numbers and all the ethical linguistic Z-numbers on one line, respectively. Then, the financial score values, the financial accuracy values, the ethical score values and the ethical accuracy values are separately obtained according to Definition 5 as follows:

\[
S \left( \tilde{z}_i^f \right) = f^* \left( \overline{A}_{\tilde{z}_i^f} \right) \times g^* \left( \overline{B}_{\tilde{z}_i^f} \right) \\
A \left( \tilde{z}_i^f \right) = f^* \left( \overline{A}_{\tilde{z}_i^f} \right) \times \left( 1 - g^* \left( \overline{B}_{\tilde{z}_i^f} \right) \right) \\
S \left( \tilde{z}_i^e \right) = f^* \left( \overline{A}_{\tilde{z}_i^e} \right) \times g^* \left( \overline{B}_{\tilde{z}_i^e} \right) \\
A \left( \tilde{z}_i^e \right) = f^* \left( \overline{A}_{\tilde{z}_i^e} \right) \times \left( 1 - g^* \left( \overline{B}_{\tilde{z}_i^e} \right) \right)
\]

The obtained values can be represented as four column vectors \( S^f = \left[ S \left( \tilde{z}_i^f \right) \right]_{n \times 1} \), \( A^f = \left[ A \left( \tilde{z}_i^f \right) \right]_{n \times 1} \), \( S^e = \left[ S \left( \tilde{z}_i^e \right) \right]_{n \times 1} \) and \( A^e = \left[ A \left( \tilde{z}_i^e \right) \right]_{n \times 1} \).

4.3. The third stage: portfolio selection based on aggregation operators under linguistic Z-number environment

In this stage, three qualitative portfolio models are proposed to allocate the optimal assets. These models can be considered as the alternative procedures of constructing more diversified portfolios based on a trade-off between ethical and financial criteria. In the proposed optimization models, the diversification of portfolio can be handled by considering lower and upper bounds of portion of the capital budget in each asset along with a predefined number of assets that can be devoted to the selected portfolio. Moreover, the portfolios can be selected by considering financial goal as an objective function and ethical goal as a constraint. Besides, the portfolios can be chosen based on a bi-objective optimization model. Furthermore, the trade-off among the financial and ethical criteria can be tested by using a bi-objective optimization portfolio model along with an admissible accuracy level of portfolio as a risk constraint. In this study, these three circumstances are considered to formulate the qualitative portfolio models. Hong and Choi [62] indicated that the relation between the score function and the accuracy function is equivalent to the relation between the mean and variance of quantitative information in
statistics. Therefore, the score and accuracy values can be applied to measure the expected return and risk values of the portfolios under linguistic Z-number environment. In the following, we formulate three qualitative portfolio selection models based on the max-score rule and the score-accuracy trade-off rule. Firstly, the objective function and constraints, which are common in all the three proposed qualitative portfolio models, are introduced as follows:

- **Objective function (Financial goal)**
  The financial objective function based on the max-score rule is defined as follows:

  \[
  \text{Max } \sum_{i=1}^{n} \left( \left( S\left( \tilde{Z}_i \right) \right)x_i \right) = \sum_{i=1}^{n} \left( \left( f^* \left( \tilde{X}_i \right) \right)x_i \right)
  \]

- **Constraints**

  \[
  \sum_{i=1}^{s} x_i = 1 \quad (28)
  \]

  \[
  \sum_{i=1}^{s} y_i = h \quad (29)
  \]

  \[
  l_i y_i \leq x_i \leq u_i y_i, \quad i = 1, \ldots, s \quad (30)
  \]

  \[
  y_i \in \{0, 1\}, \quad i = 1, \ldots, s \quad (31)
  \]

  \[
  x_i \geq 0, \quad i = 1, \ldots, s \quad (32)
  \]

Constraint (28) is the budget constraint. Constraint (29), which is called cardinality constraint, guarantees the portfolio is confined to preserve a predefined number of assets such as \( h \). \( l_i (\geq 0) \) is the minimum fraction of total capital which can be invested in the \( i^{th} \) asset and \( u_i (0 \leq l_i \leq u_i) \) is the maximum fraction of total investment which can be assigned to the \( i^{th} \) asset. Let \( x_i \) is the weight of the \( i^{th} \) asset in the portfolio and \( y_i \) is a binary variable which is equal to one when the corresponding asset is allocated to portfolio, otherwise it is zero. Eventually, constraint (32) shows the prohibition of short selling.

Now, we propose the three hybrid optimization models. The first two models (Model 1 and Model 2) are formulated based on the max-score rule. These proposed models can be used in situations where the investors wish to obtain the maximum financial goal with a desirable ethical level. Therefore, the first two models are appropriate for the general socially responsible investors. Similarly, the third model (Model 3) is formulated based on the score-accuracy trade-off rule. This model can be applied in circumstances where the investors wish to obtain both the maximum financial goal and the maximum ethical goal with an admissible accuracy level as the risk of portfolio. Therefore, the third model is suitable for the risky socially responsible investors.

4.3.1. Model (1): The portfolio selection optimization model

To combine the ethical criteria and portfolio selection problem, an ethical constraint is added to this model as follows:

- **Ethical constraint**
If investors or fund managers consider a desirable level for the ethical performance of the portfolio, an ethical constraint can be added to the portfolio selection model. The ethical constraint is formulated based on max-score rule as follows:

\[
\sum_{i=1}^{n} \left( S\left( \pi_i^e \right) \right) x_i = \sum_{i=1}^{n} \left( f^* \left( \tilde{A}^e_{\phi(i)} \right) \times g^* \left( \tilde{B}^e_{\phi(i)} \right) \right) x_i \geq \gamma
\]  

(33)

where \( \gamma \in \left[ 0, \max_{1:isn} \left( S\left( \pi_i^e \right) \right) \right] \) is a minimum desirable level of the ethical performance of the portfolio that investors determine. Three cases may occur for determining desirable ethical level \( \gamma \):

1. If \( \gamma > \max_{1:isn} \left( S\left( \pi_i^e \right) \right) \), then, no portfolio is constructed because no feasible solution can be detected.
2. If \( \gamma = \max_{1:isn} \left( S\left( \pi_i^e \right) \right) \), then, only one portfolio can be generated.
3. If \( 0 \leq \gamma \leq \max_{1:isn} \left( S\left( \pi_i^e \right) \right) \), then, the greater the \( \gamma \), the greater the effect of desirable ethical level in the portfolio performance. Contrariwise, the lower the lower the effect of desirable ethical level in the portfolio performance.

Now, the portfolio selection optimization model (Model 1) is formulated as follows:

Model (1):  \[ \text{Max} \sum_{i=1}^{n} \left( S\left( \pi_i^f \right) \right) x_i = \sum_{i=1}^{n} \left( f^* \left( \tilde{A}^f_{\phi(i)} \right) \times g^* \left( \tilde{B}^f_{\phi(i)} \right) \right) x_i \]  

(34)

S.t. Constraints (28)-(33)

Model (1) can be used by the general socially responsible investors as they wish to obtain the maximum financial goal with the desirable ethical level \( \gamma \).

4.3.2. Model (2): The portfolio selection optimization model

When investors or fund managers want to choose their portfolios based on the maximization of both the financial performance and the ethical performance simultaneously, a bi-objective portfolio optimization model can be applied. Thus, Model (2) can be formulated by considering both the financial criteria and the ethical criteria as follows:

Model (2):  \[ \text{Max} \sum_{i=1}^{n} \left( S\left( \pi_i^f \right) \right) x_i = \sum_{i=1}^{n} \left( f^* \left( \tilde{A}^f_{\phi(i)} \right) \times g^* \left( \tilde{B}^f_{\phi(i)} \right) \right) x_i \]  

(35)

\[ \text{Max} \sum_{i=1}^{n} \left( S\left( \pi_i^e \right) \right) x_i = \sum_{i=1}^{n} \left( f^* \left( \tilde{A}^e_{\phi(i)} \right) \times g^* \left( \tilde{B}^e_{\phi(i)} \right) \right) x_i \]  

(36)

S.t. Constraints (28)-(32)

Model (2) can be used by the general socially responsible investors as they wish to obtain both the maximum financial goal and the maximum ethical goal, simultaneously.

4.3.3. Model (3): The portfolio selection optimization model

In Subsection 4.3.2, a bi-objective qualitative portfolio optimization models was presented based on the max-score rule under linguistic Z-number environment. As mentioned above, this model is suitable for the general socially responsible investors as they want to maximize both the financial goal and the ethical goal in the selected portfolio, simultaneously. Hence, we can further extend these models by introducing the score-accuracy trade-off rule. As obvious, this extended qualitative portfolio model can be
applied by the risky socially responsible investors as they wish to acquire both the maximum financial goal and the maximum ethical goal along with an admissible accuracy level as the portfolio risk in the investment processes. Therefore, in this subsection, a bi-objective qualitative portfolio optimization model (Model (3)) is formulated for the risky socially responsible investors based on the score-accuracy trade-off rule under linguistic Z-number environment. To obtain the optimal portfolio for the risky socially responsible investors, Model (3) can be formulated as follows:

Model (3): \[ \begin{align*}
\text{Max} & \sum_{i=1}^{n} \left( \left( S\left( \frac{Z_e^i}{f}\right) \right) x_i \right) = \sum_{i=1}^{n} \left( \left( f^* \left( \frac{A_e^i}{f_{\phi(\phi)}(i)} \right) \times g^* \left( B_e^i \right) \right) x_i \right). \\
\text{Max} & \sum_{i=1}^{n} \left( \left( S\left( \frac{Z_e^i}{f}\right) \right) x_i \right) = \sum_{i=1}^{n} \left( \left( f^* \left( \frac{A_e^i}{f_{\phi(\phi)}(i)} \right) \times g^* \left( B_e^i \right) \right) x_i \right)
\end{align*} \]

Subject to

\[ \begin{align*}
\sum_{i=1}^{n} \left( \left( A\left( \frac{Z_e^i}{f}\right) \right) x_i \right) = \sum_{i=1}^{n} \left( \left( f^* \left( \frac{A_e^i}{f_{\phi(\phi)}(i)} \right) \times \left( 1 - g^* \left( B_e^i \right) \right) \right) x_i \right) \geq \delta
\end{align*} \]

\[ \begin{align*}
\sum_{i=1}^{n} \left( \left( A\left( \frac{Z_e^i}{f}\right) \right) x_i \right) = \sum_{i=1}^{n} \left( \left( f^* \left( \frac{A_e^i}{f_{\phi(\phi)}(i)} \right) \times \left( 1 - g^* \left( B_e^i \right) \right) \right) x_i \right) \geq \varphi
\end{align*} \]

Constraints (28)-(32)

where \( \delta \in \left[ 0, \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \right] \) is a minimum admissible level of the financial accuracy value and \( \varphi \in \left[ 0, \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \right] \) is a minimum admissible level of the ethical accuracy value. The \( \delta \)-value and \( \varphi \)-value are determined by investors or fund managers. Three cases may occur for determining the \( \delta \)-value and \( \varphi \)-value:

1. If \( \delta > \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \) and \( \varphi > \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \), then, no portfolio is constructed because no feasible solution can be detected.
2. If \( \delta = \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \) and \( \varphi = \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \), then, only one portfolio can be generated.
3. If \( 0 \leq \delta \leq \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \) and \( 0 \leq \varphi \leq \max_{1 \leq i \leq n} \left( A\left( \frac{Z_e^i}{f}\right) \right) \), then, the greater the \( \delta \)-value, the greater the effect of admissible level of the financial accuracy value in the portfolio performance. Similarly, the greater the \( \varphi \)-value, the greater the effect of admissible level of the ethical accuracy value in the portfolio performance.

5. Case study and computational results

In this section, the Thomson Reuters ASSET database has been used as information source. Thomson Reuters ASSET provides the standardized and simplified ESG information. This ESG database covers over 6000 companies and involves 400 ESG metrics. Authors such as Miras-Rodríguez et al. [63], Ferrero- Ferrero et al. [64] and Escrig-Olmedo et al. [65] applied Thomson Reuters ASSET database in their studies. Thomson Reuters is suggesting an ESG score to measure the ESG performance of each company based on its published data and categorizing the criteria into 10 topics within three groups which includes social performance (workforce, human rights, community, product responsibility), environmental performance (resource use, emissions and innovation), and corporate governance performance (management, shareholders, CSR strategy). In order to examine the proposed qualitative
framework, this study applies the information existing in the ESG database as of September 2018 for the water and other utilities industry companies. This example includes 20 listed firms.

### 5.1. The first stage: ESG and financial assessment of assets

The database presented by Thomson Reuters involves criteria for each foresaid domain. However, there are various restrictions including the balancing of scores between domains [64]. Therefore, the Thomson Reuters information can be used as input of a fuzzy system in order to evaluate ESG and financial criteria [64]. In this stage, in order to better represent the intricate concepts, linguistic variables, which are shown in Tables (1)-(4) [64], are applied for evaluating performance of each asset with respect to the ESG and financial criteria. On the other hand, Expert’s experience and knowledge can efficiently be indicated by using fuzzy if-then rules [66]. Thus, these rules are applied in this study in order to better assess the performance of firms with respect to the ethical and financial criteria based on linguistic variables.

In this study, it is assumed that investors want to compare the performance of these 20 firms using four financial criteria: short term returns (STR), long term returns (LTR), external reputation (ER) and liquidity (L) and evaluate the performance of these 20 firms with respect to three ethical criteria: environmental (E), social (S), corporate governance (G). Then, one expert evaluates the performance of each firm with respect to these mentioned criteria, and his/her opinions are captured using the linguistic terms provided in Tables (1)-(5). The linguistic term sets $T_s = \{t^0, t^1, t^2, t^3, t^4\}$, $T_e = \{t^0, t^1, t^2, t^3, t^4\}$, $T_g = \{t^0, t^1, t^2, t^3, t^4\}$, which are shown in Table (1)-(3), can be used to assess the performance of firms with respect to social, environmental and corporate governance criteria, respectively. Also, the linguistic terms set $T_f = \{t^0, t^1, t^2, t^3, t^4\}$, which is shown in Table (4), can be employed to evaluate the performance of firms with respect to financial criteria. Moreover, the linguistic terms set $T' = \{t_0, t_1, t_2, t_3, t_4\}$, which is indicated in Table (5), can be used to determine the reliability measure of the related information.

**Table 1. Linguistic variables for evaluating social performance of firms ($T_s$)**

**Table 2. Linguistic variables for evaluating environmental performance of firms ($T_e$)**

**Table 3. Linguistic variables for evaluating corporate governance performance of firms ($T_g$)**

**Table 4. Linguistic variables for evaluating financial performance of firms ($T_f$)**

**Table 5. Linguistic variables for evaluating expert’s reliability ($T'$)**

So, one expert evaluates the performance of the assets $\{x_1, x_2, \ldots, x_{20}\}$ with respect to both the financial criteria $\{f_1, f_2, f_3, f_4\} = \{\text{STR, LTR, ER, L}\}$ and the ethical criteria $\{e_1, e_2, e_3\} = \{\text{E, S, G}\}$. The assessment information is represented by the linguistic Z-numbers as $z_{ij}^f$ and $z_{ik}^e$ ($i = 1, \ldots, 20$, $j = 1, 2, 3, 4$ and $k = 1, 2, 3$), and then two linguistic Z-number matrices $Z' = \left[ z_{ij}^f \right]_{20 \times 4}$ and $Z'' = \left[ z_{ik}^e \right]_{20 \times 3}$ are generated based on the $z_{ij}^f$ and $z_{ik}^e$. These matrices characterize the performance of assets with respect to the financial criteria and the ESG criteria and the results are shown
in Tables (6) and (7), respectively. The information expressed to assess the performance of each firm with respect to the mentioned criteria simulates expert’s opinions and decision-maker’s preferences. Therefore, the performance of the proposed models as a qualitative technique for combining the priorities of each decision-maker or investor in the asset allocation process is examined.

Table (6). The evaluation data of assets relative to financial criteria which is expressed by LZNs

5.2. The second stage: calculating the comprehensive value of each asset and its corresponding score and accuracy values

In this stage, two linguistic Z-number matrices $Z^f = \begin{bmatrix} z^f_{ij} \end{bmatrix}_{20 \times 4}$ and $Z^e = \begin{bmatrix} z^e_{ij} \end{bmatrix}_{20 \times 3}$ are transformed into two qualitative column vectors $\tilde{Z}^f = \begin{bmatrix} \tilde{z}^f_{i} \end{bmatrix}_{20 \times 1}$ and $\tilde{Z}^e = \begin{bmatrix} \tilde{z}^e_{i} \end{bmatrix}_{20 \times 1}$ by fusing all the financial linguistic Z-numbers and all the ESG linguistic Z-numbers on one line based on LZMM, LZWMM, LZDMM and LZDWMM operators, respectively. Therefore, two qualitative vectors $\tilde{Z}^f = \begin{bmatrix} \tilde{z}^f_{i} \end{bmatrix}_{20 \times 1}$ and $\tilde{Z}^e = \begin{bmatrix} \tilde{z}^e_{i} \end{bmatrix}_{20 \times 1}$, which are shown in Tables (8)-(11), are calculated based on LZMM operator, LZWMM operator, LZDMM operator, and LZDWMM operator, respectively. Then, the financial score values, the financial accuracy values, the ESG score values and the ESG accuracy values are separately obtained according to Definition 5, and the resulted values are listed in Tables (8)-(11).

Table (8). The aggregated values and their corresponding score and accuracy values which resulted by LZMM operator for $f^+(x_i) = LSF\,1$ and $g^+(x_i) = LSF\,1$.

Table (9). The aggregated values and their corresponding score and accuracy values which resulted by LZWMM operator for $f^+(x_i) = LSF\,1$, $g^+(x_i) = LSF\,1$, $w^f = (0.3,0.3,0.25,0.15)$ and $w^{ESG} = (0.4,0.3,0.3)$.

Table (10). The aggregated values and their corresponding score and accuracy values which resulted by LZDMM operator for $f^+(x_i) = LSF\,1$ and $g^+(x_i) = LSF\,1$.

Table (11). The aggregated values and their corresponding score and accuracy values which resulted by LZDWMM operator for $f^+(x_i) = LSF\,1$ and $g^+(x_i) = LSF\,1$.

As clear in Tables (8)-(11), if we consider only the financial criteria to calculate the score values and accuracy values of assets then the ranking of assets according to their financial performance in the descending order is as follows:

- LZMM operator
As seen, some assets (such as 14 and 17 under LZMM operator, 14 and 17 under LZWMM operator, and 11 and 15 under LZDMM operator) have the same score values. Thus, they are ranked based on their accuracy values. Now, if we consider only the ESG criteria to calculate the score values and accuracy values of assets, then the ranking of assets according to their ESG performance in the descending order is as follows:

- LZMM operator
  \[ x_9 > x_7 > x_1 > x_{17} > x_{14} > x_{16} > x_2 > x_8 > x_{13} > x_4 > x_6 > x_9 > x_{11} > x_{15} > x_3 > x_{10} > x_{20} > x_{18} > x_5 > x_{12} \]

- LZWMM operator
  \[ x_9 > x_7 > x_1 > x_{17} > x_{14} > x_{16} > x_2 > x_8 > x_{13} > x_4 > x_6 > x_9 > x_{11} > x_{15} > x_3 > x_{10} > x_{20} > x_{18} > x_5 > x_{12} \]

- LZDMM operator
  \[ x_9 > x_7 > x_1 > x_{17} > x_{14} > x_{16} > x_2 > x_4 > x_8 > x_{13} > x_{19} > x_6 > x_{11} > x_{15} > x_3 > x_{10} > x_{20} > x_{18} > x_5 > x_{12} \]

- LZDWMM operator
  \[ x_9 > x_7 > x_1 > x_{16} > x_{14} > x_{17} > x_2 > x_4 > x_8 > x_{13} > x_{19} > x_6 > x_{15} > x_{11} > x_3 > x_{20} > x_{10} > x_{18} > x_5 > x_{12} \]

It is obvious that the ranking based on the financial criteria is absolutely different with the ranking based on the ESG criteria. For example, the assets 7 and 9 always have the best financial performance whilst their ESG performance is low. Therefore, we can use the main advantage of the proposed portfolio models and select the optimal assets based on a trade-off between the financial criteria and the ESG criteria according to investor’s preferences.

5.3. The third stage: portfolio selection based on aggregation operators under linguistic Z-number environment

In this step, the qualitative portfolio models are built based on Model (1), Model (2) and Model (3). These models can select the optimal combinations of assets by considering a trade-off among financial criteria and ESG criteria according to the preferences of investors. So, the parameters of the proposed models are considered as follows: \( h = 8 \), \( l_i = 0.05 \) and \( u_i = 0.5 \). It is clear that the proposed models are mixed integer programming models. Speranza [67] indicated that the computational complexity of the
mixed integer linear programming models (MILP) depends on the number of integer and binary variables. Also, he proved that obtaining the optimal solutions for MILP models in a rational time is impossible when the number of variables is greater than 15. In addition, Mansini and Speranza [68] proved that solving the portfolio selection model with round lots is NP-hard. Therefore, genetic algorithm (GA) is applied to solve Model (1), and the second version of non-dominated sorting genetic algorithm (NSGA-II) is used to solve Model (2) and Model (3). The parameters of GA and NSGA-II are adjusted as follows: \( POP_{size} \): 100; crossover rate: 0.8; mutation rate: 0.4; maximum iteration: 500. Also, GA and NSGA II are run 10 times for each case in MATLAB R2014a on a PC with Pentium(R) Dual core-CPU 2.0 GHz Processor and 2 GB of RAM memory.

- **Asset allocation using Model (1)**
  In this case, \( \gamma = 0.5 \) is applied to build Model (1) based on four proposed aggregation operators. Then, Model (1) is solved based on \( f^* (x_j) = LSF_1 \), \( g^* (x_j) = LSF_1 \), \( P_f = (0.25, 0.25, 0.25, 0.25) \), \( P_{ESG} = (0.33, 0.33, 0.33) \), \( w^f (0.3, 0.3, 0.25, 0.15) \) and \( w^{ESG} (0.4, 0.3, 0.3) \). The selected portfolios are reported in Table (12). Figure (2) indicates that the performance of asset 9 is better than other assets when LZMM operator and LZWMM operator are applied. Also, the budget is assigned more to the assets 1, 4 and 16 when LZZDM operator and LZDWMM operator are applied. All of these assets (1, 4, 9 and 16) have high financial score value, and their ESG score values are placed within admissible limits. Moreover, it can be seen that the portfolios selected based on four proposed aggregation operators are slightly different. This difference indicates an advantage of our proposed aggregation operators. In addition, more diversified portfolios can be constructed by changing the \( \gamma \)-value in the Model (1) according to investor’s preferences.

Table (12). The selected assets and their investment ratio in the portfolio by Model (1)

Figure (2). The selected assets and their investment ratio using Model (1)

- **Asset allocation using Model (2)**
  In this case, \( f^* (x_j) = LSF_1 \), \( g^* (x_j) = LSF_1 \), \( P_f = (0.25, 0.25, 0.25, 0.25) \), \( P_{ESG} = (0.33, 0.33, 0.33) \), \( w^f (0.3, 0.3, 0.25, 0.15) \) and \( w^{ESG} (0.4, 0.3, 0.3) \) are used to construct Model (2) based on four proposed aggregation operators. The selected portfolios are listed in Table (13). Figure (3) indicates that the performance of assets 1 and 9 is better than other assets when LZMM operator and LZWMM operator are applied. Also, the budget is assigned more to the assets 1, 9 and 16 when LZZDM operator and LZDWMM operator are applied. All of these assets are allocated to the optimal portfolios based on a trade-off between the financial goal and the ethical goal with consideration of expert’s reliability under the proposed aggregation operators. Moreover, we can find that the chosen assets in Model (2) based on four proposed aggregation operators are the same, however, their corresponding investment ratio are slightly different. This feature indicates an advantage of our proposed aggregation operators. The result reported in Table (13) is only one of the Pareto solutions, which is obtained by NSGA-II. It can be seen that the financial goals of the portfolios listed in Table (13) are greater than the ethical goals in all situations. Now, if investor prefers to choose the portfolios with higher ethical performance, then, he/she can take the assets listed in Table (14). Figure (4) indicates that the capital is devoted more to the assets 1, 4 and 16 under all proposed aggregation operators. Moreover, we can find
that some assets such as 5, 8, 12 and 20 selected in Table (14) have higher ESG score values in comparison with some assets such as 7, 9, 14 and 17 chosen in Table (13), contrariwise, the assets 7, 9, 14 and 17 chosen in Table (13) have higher financial score values in comparison with the assets 5, 8, 12 and 20 selected in Table (14). Therefore, this unique advantage shows that Model (2) can generate far more diversified portfolios according to investor’s preferences and examines the trade-off among the financial goal and the ethical goal.

Table (13). The selected assets and their investment ratio (Pareto solution 1) in the portfolio by Model (2)

Figure (3). The selected assets and their investment ratio (Pareto solution 1) using Model (2)

Table (14). The selected assets and their investment ratio (Pareto solution 2) in the portfolio by Model (2)

Figure (4). The selected assets and their investment ratio (Pareto solution 2) using Model (2)

- **Asset allocation using Model (3)**

  In this case, $\delta = 0.2$ and $\varphi = 0.4$ are employed to build Model (3) based on four proposed aggregation operators. The selected portfolios are listed in Table (15) and Figure (5) indicates that the performance of the assets 1 and 9 is better than the others when LZMM operator and LZWMM operator are used to fuse the evaluation information. Also, the budget is devoted more to the assets 2, 3 and 11 when LZDMM operator and LZDWMM operator are applied to aggregate the evaluation information. Moreover, financial score values and ESG score values of these selected assets (1, 2, 3, 9 and 11) are high, and their financial accuracy values and ESG accuracy values are placed within admissible bounds.

  If the diversification of the selected portfolios does not satisfy the investor, more portfolios can be constructed by changing the predefined $\delta$-value and $\varphi$-value in Model (3).

Table (15). The selected assets and their investment ratio in the portfolio by Model (3)

Figure (5). The selected assets and their investment ratio using Model (3)

5.4. **Discussion and sensitivity analysis**

It can be mentioned that the main feature of portfolio optimization problems emphasizes on the trade-off among two conflicting goals. In traditional portfolio theory, it is the conflict among return and risk. In this study, it is the conflict among the financial and ESG goals in addition to the conflict between the score values and accuracy values. These trade-offs are adequately considered in all the three proposed models to generate more diversified portfolios. In the following, the influence of critical parameters on the proposed qualitative portfolio models is analyzed.

5.4.1. **The influence of parameter vector $P$ on the asset allocation by Model (1)**
In this subsection, various parameter vector \( P \) are used to fuse the evaluation information and analyze the ranking orders with respect to only financial criteria based on LZMM operator and LZDMM operator. The results obtained by LZMM operator and LZDMM operator are shown in Tables (16) and (17), respectively. As obvious, it can be found that the ranking orders resulted by LZMM operator and LZDMM operator are slightly different. This difference shows an advantage of our proposed aggregation operators. Also, we find that the score values in LZMM operator become lower when the values of parameter vector \( P \) are greater; contrariwise, the score values in LZDMM operator become larger when the values of parameter vector \( P \) are greater. Consequently, the idea of decision-makers or investors can be indicated by the parameter vector \( P \). To further analyze, various parameter vector \( P \) are used to build Model (1) based on LZMM operator and LZDMM operator for \( \gamma = 0.3 \). Table (18) shows the financial goals and ESG goals for different values of \( P \). As shown in Figure (6), both financial performance and ESG performance of portfolios obtained based on LZMM operator are reduced when the values of vector \( P \) become greater. Contrariwise, Figure (7) indicates that both financial performance and ESG performance of portfolios obtained based on LZDMM operator are increased when the values of vector \( P \) become larger. Therefore, if an investor is a risk seeker, the parameter vector can generally be set as \( P = (1,0,\ldots,0) \) in LZMM operator or as \( P = (0,0,\ldots,1) \) in LZDMM operator; if an investor is a risk averter, the parameter vector can generally be set as \( P = (0,0,\ldots,1) \) in LZMM operator or as \( P = (1,0,\ldots,0) \) in LZDMM operator; and the parameter vector can generally be set as \( P = \left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right) \) for a neutral investor. Consequently, every investor can set the value of vector \( P \) according to his/her preferences.

Table (16). Ranking orders for various parameter vector \( P \) based on LZMM operator (Financial criteria)

Table (17). Ranking orders for various parameter vector \( P \) based on LZDMM operator (Financial criteria)

Table (18). Financial goal and ESG goal corresponding to different parameter vector \( P \) in Model (1) for \( \gamma = 0.3 \)

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Figure (7). The effect of different parameter vectors in Model (1) based on LZDMM operator for \( \gamma = 0.3 \)

5.4.2. The influence of desirable ESG level \( \gamma \) on the asset allocation by Model (1)

To investigate the role of the desirable ethical (ESG) level \( \gamma \) on the asset allocation by Model (1), the different \( \gamma \)-values are used to analyze the results, which are indicated in Table (19).

In this subsection, sensitivity analysis is implemented by changing the desirable ESG level of the portfolio \( \gamma \). As obvious in Figures (8) and (9), the attainment level of financial goal becomes lower in Model (1) based on both LZMM operator and LZDMM operator when the \( \gamma \)-value is increased. This is in line with trade-off among financial goal and ESG goal of the portfolio. Also, it is clear that Model (1)
selects the assets having better ESG performance when the desirable ESG level the portfolio $\gamma$ is increased. For example, the asset 9 that has high financial performance is not chosen when the $\gamma$-value is increased due to its low ethical performance. Moreover, we can mention that the cardinality constraint, and minimum and maximum fraction of the investment in each asset $(l_i, u_i)$ are applied to construct more diversified portfolios. Therefore, more diversified portfolios can be constructed by changing the maximum and minimum fraction $(l_i, u_i)$ in Model (1) according to investor’s preferences.

Table (19). Asset allocation corresponding to different predefined ESG level in Model (1)

Figure (8). Trade-off among financial goal and ESG level applying Model (1) based on LZMM operator

Figure (9). Trade-off among financial goal and ESG level applying Model (1) based on LZDMM operator

5.4.3. The influence of admissible financial accuracy level $\delta$ and admissible ESG accuracy level $\varphi$ on the asset allocation by Model (3)

In this case, sensitivity analysis is implemented by changing: (1) the minimum admissible accuracy level of financial goal $\delta$ and (2) the minimum admissible accuracy level of ESG goal $\varphi$ in the portfolio. Table (20) reports the computational results. It is obvious that the attainment level of both financial goal and ESG goal become lower when $\delta$-value and $\varphi$-value are increased. This issue is in line with trade-off among the score value and the accuracy value in the portfolio which can be seen in Figures (10)-(12). As mentioned in Section 4, the relation between the score and accuracy values are similar to the relation between the mean and variance in statistics. Hence, Model (3) can generate a comprehensive convergence between the financial score value and financial accuracy value and ESG score value and ESG accuracy value. Therefore, Model (3) is more proper for risky socially responsible investors as they wish to obtain both the maximum financial goal and the maximum ethical goal along with a limited accuracy level.

Table (20). Asset allocation by applying Model (3) based on LZMM operator

Figure (10). Trade-off among financial goal and ESG goal and financial accuracy level using Model (3) with ESG accuracy level $\varphi = 0.1$.

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5.5. Managerial results

As discussed above, our qualitative framework includes the application of aggregation methods to integrate the ethical criteria with asset allocation problems under linguistic Z-number environment. The three proposed models are developed for socially responsible investors. The first model (Model 1) is more suitable for general socially responsible investors as they want to obtain the maximum financial goal with the desirable ethical level. The second model (Model 2) is more suitable for general socially responsible investors as they wish to achieve both the maximum financial goal and the maximum ethical goal, simultaneously. Finally, the third model (Model 3) is more proper to risky socially responsible investors as they want to obtain both the maximum financial goal and the maximum ethical goal with a confined accuracy level as the portfolio risks. Furthermore, considering the predefined number of assets, which can be allocated to the portfolio along with lower and upper bounds of portion of capital that can be invested in each asset, guarantees the diversification of portfolios.

Based on above discussions, the advantages of our proposed approach can be highlighted as follows:

1. Our proposed approach is more holistic and more flexible than the other approaches because the linguistic Z-numbers is applied in the evaluation information modeling. The linguistic Z-numbers not only reduce the loss of information due to creating a more flexible, holistic and accurate structure but also capture the possibilistic and probabilistic constraints, simultaneously. So, they are more suitable for evaluating the information in financial markets.

2. Our proposed aggregation methods can consider interrelationship between all input arguments in the decision-making environment. Although there are some aggregation techniques to fuse the evaluation information in situations where the input arguments are dependent, they are unable to solve decision-making problems under linguistic Z-number environment. So, our proposed aggregation operators such as LZMM operator, LZWMM operator, LZDMM operator and LZDWMM operator are more general and more flexible to aggregate the assessment information. Our proposed aggregation operators not only consider interrelationship between all input arguments, but also capture the reliability of information, which can prevent the loss of information in the evaluation information aggregating.

3. Our proposed approach applies the max-score rule and the score-accuracy trade-off rule and develops three qualitative ethical-financial portfolio models under linguistic Z-number environment. Our proposed models are suitable for both the general socially responsible investors and the risky socially responsible investors. Moreover, our proposed models in comparison with traditional portfolio models not only can generate more diversified portfolios according to investor’s preferences, but also can examine the trade-off between financial goal and ethical goal in addition to the trade-off among the score value and accuracy value under different circumstances.

6. Conclusion

To choose the optimal combination of assets in a portfolio based on a trade-off between the financial and ethical criteria when the qualitative evaluation information is incomplete, vague and uncertain, this study has presented a comprehensive multi-stage methodology under linguistic Z-number environment in order to consider both investor’s preferences and expert’s reliability. The main stage of the methodology
extended here are: (1) the linguistic Z-number information has been aggregated using MM and DMM operators, and extended four new aggregation operators called LZMM operator, LZWMM operator, LZDMM operator and LZDWMM operator; (2) the score values and the accuracy values have separately been calculated for the financial and ethical criteria; (3) three qualitative portfolio optimization models have been developed to assist investors with different preferences. Two qualitative asset allocation models (Model 1 and 2), which are suitable for the general socially responsible investors, have been developed based on the max-score rule. Finally, the third qualitative portfolio model (Model 3), which is suitable for the risky socially responsible investors, has been designed based on the score-accuracy trade-off rule.

Then, we have applied the proposed qualitative models to select portfolios based on a real case and the results indicate that the proposed approach can construct more diversified portfolios based on a trade-off between the financial and ethical criteria and it considers investor’s preferences. Moreover, the proposed approach not only can consider the interrelationship among all input arguments, but also can capture expert’s reliability in the investment processes.

For future research, linguistic Z-numbers can be incorporated with many aggregation operators such as power aggregation operator, Heronian mean (HM) operator and etc. Moreover, our proposed approach can be developed by considering other assumptions, constraints and objectives such as entropy constraints, and transaction cost. Moreover, the proposed approach can be applied to model a multi-period portfolio selection problem.

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Table 1. Linguistic variables for evaluating social performance of firms ($T_i$)

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Table 2. Linguistic variables for evaluating environmental performance of firms (\( T_e \))

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Table 3. Linguistic variables for evaluating corporate governance performance of firms (\( T_g \))

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Table 4. Linguistic variables for evaluating financial performance of firms (\( T_f \))

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Table 5. Linguistic variables for evaluating expert’s reliability (\( T'' \))

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Table 6. The evaluation data of assets relative to financial criteria which is expressed by LZNs

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<td>( t_5, t_4 )</td>
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<td>( t_5, t_4 )</td>
<td>( t_1, t_4 )</td>
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</table>
Table (7). The evaluation data of assets relative to ESG criteria which is expressed by LZNs

<table>
<thead>
<tr>
<th>Asset ID</th>
<th>Social</th>
<th>Environmental</th>
<th>Governance</th>
<th>Asset ID</th>
<th>Social</th>
<th>Environmental</th>
<th>Governance</th>
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<tbody>
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<td>$t_2^s$, $t_3$</td>
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<td>$t_2^s$, $t_3$</td>
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<td>$t_4^s$, $t_5$</td>
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<td>$t_4^s$, $t_5$</td>
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<td>$t_5^s$, $t_3$</td>
<td>$t_3^s$, $t_3$</td>
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<td>$t_5^s$, $t_3$</td>
<td>$t_3^s$, $t_3$</td>
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<td>$t_2^s$, $t_3$</td>
<td>$t_4^s$, $t_4$</td>
<td>$t_3^s$, $t_3$</td>
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<td>$t_5^s$, $t_3$</td>
<td>$t_3^s$, $t_3$</td>
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<td>$t_3^s$, $t_3$</td>
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<td>$t_3^s$, $t_3$</td>
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<td>$t_3^s$, $t_3$</td>
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<td>$t_3^s$, $t_3$</td>
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<td>$t_5^s$, $t_3$</td>
<td>$t_3^s$, $t_3$</td>
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</table>

Table (8). The aggregated values and their corresponding score and accuracy values which resulted by LZMM operator for $f^*(x_t) = \text{LSF}^1$ and $g^*(x_t) = \text{LSF}^1$.

<table>
<thead>
<tr>
<th>Parameter vector</th>
<th>Asset ID</th>
<th>Financial criteria</th>
<th>ESG criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^{f} = (0.25, 0.25, 0.25, 0.25)$</td>
<td>1</td>
<td>$t_9^{0.898}, t_3^{4.646}$</td>
<td>(ESG $t_9^{0.158}, t_2^{2.884}$)</td>
</tr>
<tr>
<td>$P^{f} = (0.33, 0.33, 0.33)$</td>
<td>2</td>
<td>$t_7^{0.476}, t_3^{2.912}$</td>
<td>(ESG $t_9^{0.32}, t_2^{2.08}$)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>$t_5^{0.211}, t_1^{1.861}$</td>
<td>(ESG $t_4^{0.579}, t_3^{1.8}$)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$t_6^{0.928}, t_3^{4.646}$</td>
<td>(ESG $t_9^{1.588}, t_3^{3.301}$)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$t_5^{0.885}, t_1^{1.861}$</td>
<td>(ESG $t_7^{2.886}, t_2^{2.289}$)</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>$t_8^{0.458}, t_3^{3.632}$</td>
<td>(ESG $t_9^{0.32}, t_1^{1.817}$)</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>$t_{10.019}, t_3^{4.646}$</td>
<td>(ESG $t_5^{0.241}, t_2^{2.289}$)</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>$t_7^{3.325}, t_3^{4.646}$</td>
<td>(ESG $t_5^{5.04}, t_3^{6.34}$)</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>$t_{11.712}, t_3^{4.722}$</td>
<td>(ESG $t_4^{0.50}, t_2^{1.02}$)</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>$t_6^{9.211}, t_3^{4.646}$</td>
<td>(ESG $t_5^{5.241}, t_2^{2.289}$)</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>$t_9^{5.769}, t_3^{5.587}$</td>
<td>(ESG $t_5^{5.769}, t_3^{5.587}$)</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>$t_5^{5.885}, t_4^{1.565}$</td>
<td>(ESG $t_9^{0.158}, t_2^{2.62}$)</td>
</tr>
<tr>
<td>Parameter vector</td>
<td>Financial criteria</td>
<td>ESG criteria</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>--------------------</td>
<td>--------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Aggregated value</td>
<td>Score value</td>
<td>Accuracy value</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( f^* (x_i) = LSF 1 )</td>
<td>0.758 0.283</td>
<td>(ESG ( 5.769 ), f( 1 ), 1.817)</td>
</tr>
<tr>
<td>2</td>
<td>( f^* (x_i) = LSF 1 )</td>
<td>1.013 0.157</td>
<td>(ESG ( 6.32 ), f( 1 ), 1.587)</td>
</tr>
<tr>
<td>3</td>
<td>( LZWMM )</td>
<td>0.544 0.283</td>
<td>(ESG ( 6.604 ), f( 1 ), 2.08)</td>
</tr>
<tr>
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<td>( LZWMM )</td>
<td>0.980 0.152</td>
<td>(ESG ( 9.158 ), f( 2 ), 2.884)</td>
</tr>
<tr>
<td>5</td>
<td>( LZWMM )</td>
<td>1.013 0.282</td>
<td>(ESG ( 7.56 ), f( 1 ), 1.442)</td>
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<tr>
<td>6</td>
<td>( LZWMM )</td>
<td>0.437 0.298</td>
<td>(ESG ( 3.175 ), f( 2 ), 2.289)</td>
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<tr>
<td>7</td>
<td>( LZWMM )</td>
<td>0.655 0.341</td>
<td>(ESG ( 6.604 ), f( 1 ), 1.817)</td>
</tr>
<tr>
<td>8</td>
<td>( LZWMM )</td>
<td>0.463 0.173</td>
<td>(ESG ( 5.32 ), f( 2 ), 2.62)</td>
</tr>
</tbody>
</table>

Table (9). The aggregated values and their corresponding score and accuracy values which resulted by LZWM operator for \( f^* (x_i) = LSF 1 \), \( g^* (x_i) = LSF 1 \), \( w^f = (0.3, 0.3, 0.25, 0.15) \) and \( w^{ESG} = (0.4, 0.3, 0.3) \).
<table>
<thead>
<tr>
<th>Parameter vector</th>
<th>Asset ID</th>
<th>Financial criteria</th>
<th>ESG criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Aggregated value</td>
<td>Score value</td>
</tr>
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<td></td>
<td>(t_1^f, t_1^i)</td>
<td>0.299</td>
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<tr>
<td></td>
<td>1</td>
<td>(t_1^f, t_1^i)</td>
<td>0.227</td>
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<tr>
<td></td>
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<td>(t_1^f, t_1^i)</td>
<td>0.093</td>
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<tr>
<td></td>
<td>3</td>
<td>(t_1^f, t_1^i)</td>
<td>0.171</td>
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<td>(t_1^f, t_1^i)</td>
<td>0.129</td>
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<td>(t_1^f, t_1^i)</td>
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<td>(t_1^f, t_1^i)</td>
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<td>(t_1^f, t_1^i)</td>
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<td>19</td>
<td>(t_1^f, t_1^i)</td>
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</table>

Table (10). The aggregated values and their corresponding score and accuracy values which resulted by LZDMM operator for $f^f(x_i) = LSF, 1$ and $g^g(x_i) = LSF, 1$. 

$P^f = (0.25, 0.25, 0.25, 0.25)$, $P^ESG = (0.33, 0.33, 0.33)$.
Table (11). The aggregated values and their corresponding score and accuracy values which resulted by LZDWM operator for $f^\prime (x_t) = LSF \; 1$, $g^\prime (x_t) = LSF \; 1$, $w^f = (0.3, 0.3, 0.25, 0.15)$ and $w^{ESG} = (0.4, 0.3, 0.3)$.

<table>
<thead>
<tr>
<th>Parameter vector</th>
<th>Asset ID</th>
<th>Financial criteria</th>
<th>ESG criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Aggregated value</td>
<td>Score value</td>
</tr>
<tr>
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<td>($t_{3,204}$, $t_{5,02}$)</td>
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<tr>
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<td>($t_{3,134}$, $t_{2,387}$)</td>
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<td>($t_{2,836}$, $t_{1,196}$)</td>
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<td>($t_{1,566}$, $t_{1,194}$)</td>
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<td>($t_{1,341}$, $t_{9.05}$)</td>
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<td>($t_{3,478}$, $t_{9.25}$)</td>
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<td>($t_{1,85}$, $t_{2.908}$)</td>
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<td>($t_{2,764}$, $t_{1.442}$)</td>
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<td>($t_{1,566}$, $t_{1.145}$)</td>
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<tr>
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<td>($t_{2,002}$, $t_{1.957}$)</td>
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<td>($t_{2.874}$, $t_{5.219}$)</td>
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<td>($t_{1,889}$, $t_{2.116}$)</td>
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<td>($t_{2,915}$, $t_{5.231}$)</td>
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<td>($t_{3,406}$, $t_{5.267}$)</td>
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</table>

Table (12). The selected assets and their investment ratio in the portfolio by Model (1)

<p>| Operator | Financial | Selected assets and their investment ratio |</p>
<table>
<thead>
<tr>
<th>Operator</th>
<th>Financial goal</th>
<th>Ethical goal</th>
<th>Selected assets and their investment ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>LZMM</td>
<td>1.153</td>
<td></td>
<td>ID 1 2 7 8 9 14 16 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.079 0.073 0.079 0.079 0.460 0.077 0.077 0.077</td>
</tr>
<tr>
<td>LZWMM</td>
<td>1.114</td>
<td></td>
<td>ID 1 2 7 8 9 14 16 17</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td>Portions 0.078 0.077 0.077 0.076 0.469 0.070 0.077 0.076</td>
</tr>
<tr>
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<td></td>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.2503</td>
<td></td>
<td>ID 1 2 4 8 12 14 16 20</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.285 0.049 0.138 0.049 0.049 0.096 0.285 0.049</td>
</tr>
</tbody>
</table>

Table (13). The selected assets and their investment ratio (Pareto solution 1) in the portfolio by Model (2)

<table>
<thead>
<tr>
<th>Operator</th>
<th>Financial goal</th>
<th>Ethical goal</th>
<th>Selected assets and their investment ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>LZMM</td>
<td>1.123</td>
<td>0.925</td>
<td>ID 1 2 4 7 9 14 16 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.391 0.048 0.048 0.048 0.302 0.048 0.067 0.048</td>
</tr>
<tr>
<td>LZWMM</td>
<td>1.087</td>
<td>0.905</td>
<td>ID 1 2 4 7 9 14 16 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.396 0.048 0.048 0.048 0.313 0.048 0.048 0.048</td>
</tr>
<tr>
<td>LZDMM</td>
<td>0.364</td>
<td>0.249</td>
<td>ID 1 2 4 7 9 14 16 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.200 0.049 0.049 0.049 0.413 0.049 0.143 0.049</td>
</tr>
<tr>
<td>LZDWMM</td>
<td>0.361</td>
<td>0.259</td>
<td>ID 1 2 4 7 9 14 16 17</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Portions 0.202 0.049 0.051 0.049 0.397 0.049 0.154 0.049</td>
</tr>
</tbody>
</table>

Table (14). The selected assets and their investment ratio (Pareto solution 2) in the portfolio by Model (2)
Table (15). The selected assets and their investment ratio in the portfolio by Model (3)

<table>
<thead>
<tr>
<th>Operator</th>
<th>Financial goal</th>
<th>Ethical goal</th>
<th>Selected assets and their investment ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>LZMM</td>
<td>1.059</td>
<td>0.962</td>
<td>ID 1 2 3 7 9 14 16 17 Portions 0.396 0.066 0.064 0.067 0.229 0.058 0.055 0.066</td>
</tr>
<tr>
<td>LZWMM</td>
<td>1.012</td>
<td>0.911</td>
<td>ID 1 2 3 7 9 14 16 17 Portions 0.375 0.089 0.071 0.074 0.170 0.070 0.076 0.075</td>
</tr>
<tr>
<td>LZDMM</td>
<td>0.1403</td>
<td>0.1403</td>
<td>ID 2 3 6 11 12 13 17 19 Portions 0.155 0.276 0.089 0.179 0.069 0.073 0.079 0.080</td>
</tr>
<tr>
<td>LZDWMM</td>
<td>0.1435</td>
<td>0.045</td>
<td>ID 2 3 6 11 12 13 17 19 Portions 0.103 0.237 0.076 0.345 0.055 0.050 0.083 0.051</td>
</tr>
</tbody>
</table>

Table (16). Ranking orders for various parameter vector $P$ based on LZMM operator (Financial criteria)

<table>
<thead>
<tr>
<th>$P$ vector</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^{f}=(1,0,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_4 &gt; x_6 &gt; x_{13} &gt; x_{19} &gt; x_8 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^{f}=(2,0,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_{16} &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_2 &gt; x_4 &gt; x_{15} &gt; x_{19} &gt; x_{11} &gt; x_{13} &gt; x_6 &gt; x_8 &gt; x_{10} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^{f}=(1,1,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_8 &gt; x_4 &gt; x_{13} &gt; x_{19} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^{f}=(1,1,1,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_8 &gt; x_4 &gt; x_{13} &gt; x_{19} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^{f}=(1,1,1,1)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_8 &gt; x_4 &gt; x_{13} &gt; x_{19} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
</tbody>
</table>
Table (17). Ranking orders for various parameter vector $P$ based on LZDMM operator (Financial criteria)

<table>
<thead>
<tr>
<th>$P$ vector</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^f = (1,0,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_6 &gt; x_8 &gt; x_9 &gt; x_{11} &gt; x_{13} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^f = (2,0,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_6 &gt; x_8 &gt; x_9 &gt; x_{11} &gt; x_{13} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^f = (1,1,0,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_4 &gt; x_8 &gt; x_{13} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^f = (1,1,1,0)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_4 &gt; x_8 &gt; x_{13} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
<tr>
<td>$P^f = (1,1,1,1)$</td>
<td>$x_9 &gt; x_7 &gt; x_1 &gt; x_{17} &gt; x_{14} &gt; x_{16} &gt; x_2 &gt; x_4 &gt; x_8 &gt; x_{13} &gt; x_6 &gt; x_{11} &gt; x_{15} &gt; x_3 &gt; x_{10} &gt; x_{20} &gt; x_{18} &gt; x_5 &gt; x_{12}$</td>
</tr>
</tbody>
</table>

Table (18). Financial goal and ESG goal corresponding to different parameter vector $P$ in Model (1) for $\gamma = 0.3$

<table>
<thead>
<tr>
<th>Operator</th>
<th>Parameter vector</th>
<th>Financial goal</th>
<th>ESG level</th>
</tr>
</thead>
<tbody>
<tr>
<td>LZDMM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1: $P^f = (1,0,0,0)$</td>
<td>$P^{ESG} = (1,0,0)$</td>
<td>1.206</td>
<td>0.694</td>
</tr>
<tr>
<td>C2: $P^f = (2,0,0,0)$</td>
<td>$P^{ESG} = (2,0,0)$</td>
<td>0.8844</td>
<td>0.543</td>
</tr>
<tr>
<td>C3: $P^f = (1,1,0,0)$</td>
<td>$P^{ESG} = (1,1,0)$</td>
<td>0.8324</td>
<td>0.526</td>
</tr>
<tr>
<td>C4: $P^f = (1,1,1,0)$</td>
<td>$P^{ESG} = (1,1,0)$</td>
<td>0.7447</td>
<td>0.4786</td>
</tr>
<tr>
<td>C5: $P^f = (1,1,1,1)$</td>
<td>$P^{ESG} = (1,1,1)$</td>
<td>0.6817</td>
<td>0.421</td>
</tr>
<tr>
<td>LZDMM</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1: $P^f = (1,0,0,0)$</td>
<td>$P^{ESG} = (1,0,0)$</td>
<td>0.281</td>
<td>0.3</td>
</tr>
<tr>
<td>C2: $P^f = (2,0,0,0)$</td>
<td>$P^{ESG} = (2,0,0)$</td>
<td>0.6571</td>
<td>0.36</td>
</tr>
<tr>
<td>C3: $P^f = (1,1,0,0)$</td>
<td></td>
<td>0.7166</td>
<td>0.375</td>
</tr>
</tbody>
</table>
\[ P_{ESG} = (1,1,0) \]

\[ C4: \; P' = (1,1,1,0) \]
\[ P_{ESG} = (1,1,0) \]
\[
\begin{array}{c}
1.096 \\
0.393
\end{array}
\]

\[ C5: \; P' = (1,1,1,1) \]
\[ P_{ESG} = (1,1,1) \]
\[
\begin{array}{c}
1.483 \\
0.577
\end{array}
\]

**Table (19). Asset allocation corresponding to different predefined ESG level in Model (1)**

<table>
<thead>
<tr>
<th>Operator</th>
<th>Financial goal</th>
<th>Selected Assets and their investment ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>LZMM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>1.156</td>
<td>1.128</td>
</tr>
<tr>
<td>0.5</td>
<td>1.142</td>
<td>1.096</td>
</tr>
<tr>
<td>0.7</td>
<td>1.137</td>
<td>1.037</td>
</tr>
<tr>
<td>0.9</td>
<td>1.112</td>
<td>0.985</td>
</tr>
<tr>
<td>1.25</td>
<td>0.887</td>
<td>0.853</td>
</tr>
<tr>
<td>≥ 1.5</td>
<td>Infeasible</td>
<td></td>
</tr>
<tr>
<td>LZDMM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.367</td>
<td>0.393</td>
</tr>
<tr>
<td>0.2</td>
<td>0.365</td>
<td>0.393</td>
</tr>
<tr>
<td>0.3</td>
<td>0.323</td>
<td>0.393</td>
</tr>
<tr>
<td>0.4</td>
<td>0.271</td>
<td>0.393</td>
</tr>
<tr>
<td>0.45</td>
<td>0.2387</td>
<td>0.393</td>
</tr>
<tr>
<td>≥ 0.58</td>
<td>Infeasible</td>
<td></td>
</tr>
</tbody>
</table>

**Table (20). Asset allocation by applying Model (3) based on LZMM operator**

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \delta )</th>
<th>Financial goal</th>
<th>ESG goal</th>
<th>Selected assets and investment ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.128</td>
<td>0.911</td>
<td>1</td>
<td>0.393 0.048 0.048 0.048 0.317 0.048 0.048 0.048 0.048</td>
</tr>
<tr>
<td>0.3</td>
<td>0.9446</td>
<td>0.702</td>
<td>2</td>
<td>0.153 0.100 0.144 0.101 0.141 0.115 0.103 0.144</td>
</tr>
<tr>
<td>0.5</td>
<td>0.597</td>
<td>0.604</td>
<td>2</td>
<td>0.050 0.382 0.050 0.063 0.055 0.094 0.252 0.054</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7446</td>
<td>0.602</td>
<td>3</td>
<td>0.050 0.382 0.050 0.063 0.055 0.094 0.252 0.054</td>
</tr>
</tbody>
</table>

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Appendix A

Theorem 1. Let \( z_i = (A_{\theta(i)}, B_{\eta(i)}) \) (\( i = 1, \ldots, k \)) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then, the aggregated result acquired based on LZMM operator is a LZN and it is indicated as follows:

\[
LZMM^p(z_1, z_2, \ldots, z_k) = \left\{ x^* \left( \frac{1}{k^t} \sum_{i=1}^{k} \left( f^* \left( A_{\theta(i)} \right) \right)^{P_i} \right)^{\frac{1}{P_i}} \right\} \geq \left\{ \frac{\sum_{i=1}^{k} \left( f^* \left( A_{\theta(i)} \right) \right)^{P_i}}{\sum_{i=1}^{k} \left( g^* \left( B_{\eta(i)} \right) \right)^{P_i}} \right\}
\]

(A.1)

Proof. According to Definition 4, it is clear that the aggregated value is a LZN. Now, by applying the mathematical induction method, Eq. (A.1) is easily proven in the following.

Firstly, it is assumed that \( k = 2 \). Hence, if \( k = 2 \), then \( S_2 = \{ (\theta(1), \theta(2)) \} \). According to Definition 4, we have:

\[
\left( Z_{\theta(1)} \right)^{P_1} = \left\{ f^* \left( f^* \left( A_{\theta(1)} \right) \right)^{P_1} \right\}, g^* \left( g^* \left( B_{\eta(1)} \right) \right)^{P_1}
\]

\[
\left( Z_{\eta(1)} \right)^{P_1} = \left\{ f^* \left( A_{\theta(1)} \right)^{P_1} \right\}, g^* \left( g^* \left( B_{\eta(1)} \right) \right)^{P_1}
\]
\[
(Z_{g(2)})^p = \left( f^{-1} \left( f^* \left( A_{\omega(\theta(2))} \right) \right)^p \right) g^{-1} \left( g^* \left( B_{\phi(\theta(2))} \right) \right)^p
\]

\[
(Z_{g(2)})^p \times (Z_{g(1)})^p = \left( f^{-1} \left( f^* \left( A_{\omega(\theta(2))} \right) \right)^p \times f^{-1} \left( f^* \left( A_{\omega(\theta(1))} \right) \right)^p \right) g^{-1} \left( g^* \left( B_{\phi(\theta(2))} \right) \right)^p \times \left( g^* \left( B_{\phi(\theta(1))} \right) \right)^p
\]

\[
\frac{1}{2} \left( (Z_{g(2)})^p \times (Z_{g(1)})^p \right) = \left( f^{-1} \left( f^* \left( A_{\omega(\theta(2))} \right) \right)^p \times f^{-1} \left( f^* \left( A_{\omega(\theta(1))} \right) \right)^p \right) g^{-1} \left( g^* \left( B_{\phi(\theta(2))} \right) \right)^p \times \left( g^* \left( B_{\phi(\theta(1))} \right) \right)^p
\]

Obviously, Theorem 1 is true for \( k = 2 \). Now, it is assumed that this theorem be true for \( k = t \), therefore, we will have:

\[
\frac{1}{t!} \left( \prod_{\theta \in \mathbb{X}} \left( f^* \left( A_{\omega(\theta(t))} \right) \right) \right)^p = \sum_{\theta \in \mathbb{X}} \left( f^* \left( A_{\omega(\theta(t))} \right) \right)^p
\]

\[
LZMM^p (Z_1, Z_2 \ldots Z_t) = \left( \frac{1}{t!} \prod_{\theta \in \mathbb{X}} \left( f^* \left( A_{\omega(\theta(t))} \right) \right) \right)^p \prod_{\theta \in \mathbb{X}} \left( g^* \left( B_{\phi(\theta(t))} \right) \right)^p
\]

Consequently, for \( k = t + 1 \), we can obtain the following expression:
$$LZMM^p(Z_1,Z_2,...,Z_k) = \left\{ \begin{array}{l} f^{-1} \left( \left( \frac{1}{(t+1)!} \left( \sum_{\alpha \in S_{t+1}} \left( \prod_{i=1}^{t+1} (f^+ \left( A_{\varphi(\alpha(i))} \right)) \right)^{\nu_\alpha} \right) \sum_{\alpha \in S_{t+1}} \nu_\alpha \right), \right. \\
\left. g^{-1} \left( \sum_{\alpha \in S_{t+1}} \left( \prod_{i=1}^{t} (f^+ \left( A_{\varphi(\alpha(i))} \right)) \times \prod_{i=1}^{t} (g^+ \left( B_{\varphi(\alpha(i))} \right)) \right)^{\nu_\alpha} \sum_{\alpha \in S_{t+1}} \nu_\alpha \right) \right. \end{array} \right.$$

Since this theorem is true for \( k = t \), it will be also true for \( k = t + 1 \). Finally, according to the mathematical induction, Eq. (A.1) is true for all \( k \).

Appendix B.

**Theorem 5.** Let \( z_i = \left( A_{\varphi(i)}, B_{\varphi(i)} \right) (i = 1, \ldots, k) \) be a set of LZNs and \( P = (P_1, P_2, \ldots, P_k) \in \mathbb{R}^k \) be a parametric vector. Then, the aggregated result acquired based on LZDMM operator is a LZN and it is shown as follows:

$$LZDMM^p(Z_1,Z_2,...,Z_k) = \left\{ \begin{array}{l} f^{-1} \left( \left( \sum_{i=1}^{k} p_i \left( \prod_{\alpha \in S_1} \left( \sum_{i=1}^{k} \left( p_i f^+ \left( A_{\varphi(\alpha(i))} \right) \right) \right)^{\nu_\alpha} \right) \right)^{\nu_\alpha} \right), \\
\left. g^{-1} \left( \prod_{\alpha \in S_1} \left( \sum_{i=1}^{k} \left( p_i f^+ \left( A_{\varphi(\alpha(i))} \right) \times g^+ \left( B_{\varphi(\alpha(i))} \right) \right)^{\nu_\alpha} \right) \right) \right. \end{array} \right.$$

(B.1)

**Proof.** According to Definition 4, it is obvious that the aggregated value obtained by LZDMM operator is a LZN. Now, by applying the mathematical induction method, Eq. (B.1) is easily proven in the following.

Firstly, it is assumed \( k = 2 \). Hence, if \( k = 2 \), then \( S_2 = \{ (\theta(1), \theta(2)), (\theta(2), \theta(1)) \} \). According to Definition 4, we have:

\[
P_{Z(\theta(1))} = \left( f^{-1} \left( P_1 f^+ \left( A_{\varphi(\theta(1))} \right) \right) \right)_2 \]  

\[
P_{Z(\theta(2))} = \left( f^{-1} \left( P_2 f^+ \left( A_{\varphi(\theta(2))} \right) \right) \right)_2 \]  

\[
P_{Z(\theta(1))} = \left( f^{-1} \left( P_1 f^+ \left( A_{\varphi(\theta(1))} \right) \right) \right)_2 \]  

\[
P_{Z(\theta(2))} = \left( f^{-1} \left( P_2 f^+ \left( A_{\varphi(\theta(2))} \right) \right) \right)_2 \]  

\[
P_{Z(\theta(1))} + P_{Z(\theta(2))} = \left( \left( f^{-1} \left( P_1 f^+ \left( A_{\varphi(\theta(1))} \right) \right) + P_2 f^+ \left( A_{\varphi(\theta(2))} \right) \right) \right)_2 \]  

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\[
P_1 Z_{\varphi(2)} + P_2 Z_{\varphi(1)} = \left( f^{-1} \left( P_1 \varphi \left( A_{\varphi(2)} \right) + P_2 \varphi \left( A_{\varphi(1)} \right) \right) \right).
\]

\[
g^{-1} \left( \left( P_1 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( g \varphi \left( B_{\varphi(2)} \right) \right) \right) + \left( P_2 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( g \varphi \left( B_{\varphi(1)} \right) \right)
\]

\[
P_1 \varphi \left( A_{\varphi(1)} \right) + P_2 \varphi \left( A_{\varphi(1)} \right)
\]

\[
(P_1 Z_{\varphi(1)} + P_2 Z_{\varphi(2)}) \times \left( P_1 Z_{\varphi(2)} + P_2 Z_{\varphi(1)} \right) = \left( P_1 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( P_1 \varphi \left( A_{\varphi(2)} \right) \right) + \left( P_2 \varphi \left( A_{\varphi(2)} \right) \right) \times \left( P_2 \varphi \left( A_{\varphi(1)} \right) \right)
\]

\[
g^{-1} \left( \left( P_1 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( g \varphi \left( B_{\varphi(2)} \right) \right) \right) + \left( P_2 \varphi \left( A_{\varphi(2)} \right) \right) \times \left( g \varphi \left( B_{\varphi(1)} \right) \right)
\]

\[
P_1 \varphi \left( A_{\varphi(1)} \right) + P_2 \varphi \left( A_{\varphi(1)} \right)
\]

\[
\left( \left( P_1 Z_{\varphi(2)} \right) \times \left( P_1 Z_{\varphi(2)} + P_2 Z_{\varphi(1)} \right) \right)^{1/2T} = \left( P_1 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( P_1 \varphi \left( A_{\varphi(2)} \right) \right) + \left( P_2 \varphi \left( A_{\varphi(2)} \right) \right) \times \left( P_2 \varphi \left( A_{\varphi(1)} \right) \right)
\]

\[
g^{-1} \left( \left( P_1 \varphi \left( A_{\varphi(1)} \right) \right) \times \left( g \varphi \left( B_{\varphi(2)} \right) \right) \right) + \left( P_2 \varphi \left( A_{\varphi(2)} \right) \right) \times \left( g \varphi \left( B_{\varphi(1)} \right) \right)
\]

\[
P_1 \varphi \left( A_{\varphi(1)} \right) + P_2 \varphi \left( A_{\varphi(1)} \right)
\]

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\[
\frac{1}{p_1 + p_2} \left[ \left( P_1 Z_{\theta(1)} + P_2 Z_{\theta(2)} \right) \times \left( P_1 Z_{\theta(2)} + P_2 Z_{\theta(1)} \right) \right]^{\frac{1}{k+1}} = \left[ \left( P_1 f^* \left( A_{\varphi(\theta(1))} \right) + P_2 f^* \left( A_{\varphi(\theta(2))} \right) \right) \times \left( P_1 f^* \left( A_{\varphi(\theta(2))} \right) + P_2 f^* \left( A_{\varphi(\theta(1))} \right) \right) \right]^{\frac{1}{k+1}}
\]

\[
g^{-1} \left[ \left( P_1 f^* \left( A_{\varphi(\theta(1))} \right) \right) \times \left( g^* \left( B_{\varphi(\theta(1))} \right) \right) \right] + \left[ \left( P_2 f^* \left( A_{\varphi(\theta(2))} \right) \right) \times \left( g^* \left( B_{\varphi(\theta(2))} \right) \right) \right] \times \left( P_1 f^* \left( A_{\varphi(\theta(1))} \right) + P_2 f^* \left( A_{\varphi(\theta(2))} \right) \right)
\]

Obviously, Theorem 5 is true for \( k = 2 \). Now, it is assumed that this theorem be true for \( k = t \), therefore, we will have:

\[
LZDM^*_m (Z_1, Z_2, ..., Z_t) = f^{-1} \left[ \frac{1}{\sum_{i=1}^{t} p_i} \left( \prod_{i=1}^{t} \sum_{j=1}^{p} f^*(A_{\theta(i)}) \right)^{\frac{1}{t}} \right] \times g^{-1} \left[ \frac{\sum_{i=1}^{t} \left( p_i f^*(A_{\theta(i)}) \right) \times \left( B_{\varphi(i)} \right) \right]^{\frac{1}{t+1}}
\]

Consequently, for \( k = t + 1 \), we can obtain the following expression:

\[
LZDM^*_m (Z_1, Z_2, ..., Z_{t+1}) = f^{-1} \left[ \frac{1}{\sum_{i=1}^{t+1} p_i} \left( \prod_{i=1}^{t+1} \sum_{j=1}^{p} f^*(A_{\theta(i)}) \right)^{\frac{1}{t+1}} \right] \times g^{-1} \left[ \frac{\sum_{i=1}^{t+1} \left( p_i f^*(A_{\theta(i)}) \right) \times \left( B_{\varphi(i)} \right) \right]^{\frac{1}{t+1}}
\]

Since this theorem is true for \( k = t \), it will be also true for \( k = t + 1 \). Finally, according to the mathematical induction, Eq. (B.1) is true for all \( k \).

**Technical Biography:**

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