Efficient Estimation of Population Median Using Supplementary Variable

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Abstract

In this paper, we propose an exponential cum ratio-product type class of estimators for population median under simple random sampling scheme using the supplementary variable. Expressions for bias and mean square error (MSE) are obtained up to first order of approximation. The proposed class of estimators is more efficient as compared to all considered estimators under certain conditions. Four real data sets and simulation studies are carried out to observe the performances of the estimators. Both numerical and simulation studies show that the proposed class of estimators performs better as compared to all considered estimators.

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1. Introduction

When the supplementary variable is correlated with the study variable, then in such situation, the use of the appropriate supplementary variable may results in considerable reduction in the mean squared error *(MSE)* of the regression, ratio and product estimators. A primary objective in many statistical analysis is to estimate the location parameter for the desired distribution, which describes the data in good manners. When the entire data is homogeneous, the sample mean performs better. In survey sampling, there are situations when the study variable follows extremely skewed distribution, such as expenditure, taxes, income, production and consumption or in data, if some of the observations lie very far from

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rest of the observations, then in such situation, the sample mean performs very poorly and the results are either under or over estimated. To handle this kind of situation, median is used as an alternative measure of location, which performs better in case of skewed variable and gives efficient estimate for population parameter. A large amount of literature is available to estimate the mean and total, but comparatively less efforts have been devoted to propose efficient estimators for population median (M_y). Gross [1] was the first who proposed a weighted estimator without the supplementary variable for population median (M_y) and derived its asymptotic distribution. Kuk and Mak [2] suggested the classical ratio method for population median using the supplementary information. Singh et al. [3] suggested a class of generalized ratio type estimators using the supplementary variable for population median. Singh et al. [4, 5] suggested certain estimators for population median using known parameters of the supplementary variables. Singh et al. [6] suggested chain ratio and regression estimators for population median. Gupta et al. [7] proposed number of difference types estimators for population median using the supplementary variables. Other important contributions in this area include, Aladag and Cingi [8], Singh and Solanki [9], Jhajj et al. [10], Sharma and Singh [11], Aladag and Cingi [12], Shabbir and Gupta [13, 14], Solanki and Singh [15], Enang et al. [16], Jhajj et al. [17], Koyuncu [18], Irfan et al. [19], Yadav et al. [20], Bandyopadhyay et el. [21], Javed et al. [22] and Baig et al. [23]. These researchers suggested large number of estimators for estimation of population parameters under different situations using the supplementary variables.

In this article, we propose a new class of exponential cum ratio-product type estimators for population median (M_y) under simple random sampling *(SRS)* scheme using the supplementary variable. The main aim of this research is to propose an efficient class of estimators to increase precision of the median estimators.

Consider a finite population $U = \{U_1, U_2, \ldots U_N\}$ consisting of N units. Let (y_i, x_i) be the ith values of the survey variable y and the supplementary variable x respectively. From this population, a sample of size n is selected using simple random sample without replacement *(SRSWOR)* scheme. Let $\widehat{M}_y$ and $\widehat{M}_x$ be the sample medians corresponding to population medians M_y and M_x respectively. Let $f_y(M_y)$ and $f_x(M_x)$ be the probability density functions of M_y and M_x respectively. Let $\rho_{(\widehat{M}_y,\widehat{M}_x)} = \rho_{yx} = 4P_{11}(y, x) - 1$ be the coefficient of correlation between $\widehat{M}_y$ and $\widehat{M}_x$, where P_{11} indicates the fraction of values in the population with $Y \leq M_Y$ and $X \leq M_X$. It is assumed that (y, x) follow a continuous distribution with $f_y(M_y)$ and $f_x(M_x)$ being their marginal densities.

To get the bias and *MSEs* of the estimators, we use the following relative error terms.
Let $\zeta_0 = \left(\frac{\widehat{M}_y - M_y}{M_y}\right)$ and $\zeta_1 = \left(\frac{\widehat{M}_x - M_x}{M_x}\right)$ such that $E(\zeta_j) = 0, (j = 0, 1)$,
$E(\zeta_0^2) = \lambda\psi_{M_y}^2$, $E(\zeta_1^2) = \lambda\psi_{M_x}^2$, $E(\zeta_0\zeta_1) = \lambda\psi_{M_{yx}}$, where $\psi_{M_y} = \frac{1}{[M_y f_y(M_y)]}$, $\psi_{M_x} = \frac{1}{[M_x f_x(M_x)]}$, $\psi_{M_{yx}} = \rho_{yx}\psi_{M_y}\psi_{M_x}$ and $\lambda = \frac{1}{4}(\frac{1}{n} - \frac{1}{N})$.

2. Estimators in Literature

We discuss the following usual median per unit estimator and some existing estimators of population median M_y.

(i). The most common estimator of median M_y is sample median estimator $\widehat{M}_y$ whose variance, is given by

$$V(\widehat{M}_y) = \lambda M_y^2 \psi_{M_y}^2. \tag{1}$$

(ii). The usual ratio estimator of median suggested by Kuk and Mak [2] as:

$$\widehat{M}_R = \widehat{M}_y \left(\frac{M_x}{\widehat{M}_x}\right). \tag{2}$$

The bias and *MSE* of $\widehat{M}_R$, are given by

$$B(\widehat{M}_R) \cong \lambda M_y(\psi_{M_x}^2 - \psi_{M_{yx}}) \tag{3}$$

and

$$MSE(\widehat{M}_R) \cong \lambda M_y^2(\psi_{M_y}^2 + \psi_{M_x}^2 - 2\psi_{M_{yx}}). \tag{4}$$

(iii). The usual exponential estimator is:

$$\widehat{M}_{Exp} = \widehat{M}_y \exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right). \tag{5}$$

The bias and *MSE* of $\widehat{M}_{Exp}$, are given by

$$B(\widehat{M}_{Exp}) \cong \lambda M_y \left(\frac{3}{8}\psi_{M_x}^2 - \frac{1}{2}\psi_{M_{yx}}\right) \tag{6}$$

and

$$MSE(\widehat{M}_{Exp}) \cong \lambda M_y^2 \left(\psi_{M_y}^2 + \frac{1}{4}\psi_{M_x}^2 - \psi_{M_{yx}}\right). \tag{7}$$

(iv). Singh and Solanki [9] proposed the following estimators for M_y as:

$$\widehat{M}_{SS1} = k_1\widehat{M}_y + (1-k_1)(M_x - \widehat{M}_x), \tag{8}$$

$$\widehat{M}_{SS2} = k_2\widehat{M}_y + k_3(M_x - \widehat{M}_x), \tag{9}$$

where k_1, k_2 and k_3 are constants.
The biases and minimum *MSEs* of $\widehat{M}_{SSi}(i = 1, 2)$, to first degree of approximation, are given by

$$B(\widehat{M}_{SS1}) \cong (k_1 - 1)M_y, \tag{10}$$

$$B(\widehat{M}_{SS2}) \cong (k_2 - 1)M_y, \tag{11}$$

$$MSE(\widehat{M}_{SS1})_{min} \cong M_y^2\left[1 + \frac{M_x^2}{M_y^2}\lambda\psi^2_{M_x} - \frac{(1 + \frac{M_x^2}{M_y^2}\lambda\psi^2_{M_x} + \frac{M_x}{M_y}\lambda\psi_{M_{yx}})^2}{1 + \lambda\psi^2_{M_y} + \frac{M_x^2}{M_y^2}\lambda\psi^2_{M_x} + 2\frac{M_x}{M_y}\lambda\psi_{M_{yx}}}\right], \tag{12}$$

$$MSE(\widehat{M}_{SS2})_{min} \cong \frac{\lambda M_y^2\psi^2_{M_y}(1-\rho^2_{yx})}{1 + \lambda\psi^2_{M_y}(1-\rho^2_{yx})}. \tag{13}$$

The optimal values of $k_j(j = 1, 2, 3)$ are:

$k_{1(opt)} = \frac{M_y^2 + M_x^2\lambda\psi^2_{M_x} - M_yM_x\lambda\psi_{M_{yx}}}{M_y^2 + M_y^2\lambda\psi^2_{M_y} + M_x^2\lambda\psi^2_{M_x}}$, $k_{2(opt)} = \frac{1}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})}$, $k_{3(opt)} = \frac{M_y}{M_x}\left[\frac{\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}}}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})}\right]$.

(v). The difference estimator of median as:

$$\widehat{M}_D = \widehat{M}_y - k_4(\widehat{M}_x - M_X), \tag{14}$$

where k_4 is unknown constant, whose value is to be find.
The minimum *MSE* of $\widehat{M}_D$, is given by

$$MSE(\widehat{M}_D)_{min} \cong \lambda M_y^2\psi^2_{M_y}\left(1-\rho^2_{yx}\right). \tag{15}$$

The optimum value of k_4 is $k_{4(opt)} = \frac{f_x(M_x)}{f_y(M_y)}\rho_{yx}$.

(vi). Solanki and Singh [15] proposed a generalized class of estimators for median as:

$$\widehat{M}_{SS3} = \widehat{M}_y\left[k_5\left(\frac{cM_X + d}{c\widehat{M}_x + d}\right) + k_6\left(\frac{c\widehat{M}_x + d}{cM_x + d}\right)\right], \tag{16}$$

where k_5 and k_6 are unknown constants and c, d are known population parameters of the supplementary variable.
The bias and minimum MSE of $\widehat{M}_{SS3}$, at $(c, d) = (1, 0)$, are given by

$$B(\widehat{M}_{SS3}) \cong M_y[k_5(1 - \lambda\psi_{M_{yx}} + \lambda\psi^2_{M_x}) + k_6(1 + \lambda\psi_{M_{yx}}) - 1] \tag{17}$$

and

$$MSE(\widehat{M}_{SS3})_{min} \cong \frac{\lambda M_y^2 \begin{pmatrix} \lambda\psi^6_{M_x} + \lambda\psi^2_{M_y}\psi^4_{M_x} + 2\lambda\psi^4_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x}\psi_{M_{yx}} \\ - 4\lambda\psi^2_{M_x}\psi^2_{M_{yx}} + 4\lambda\psi^2_{M_y}\psi^2_{M_{yx}} - 4\psi^2_{M_y}\psi^2_{M_x} + 4\psi^2_{M_{yx}} \end{pmatrix}}{16\lambda\psi^2_{M_{yx}} - 4\psi^2_{M_x} - 8\lambda\psi^2_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x} - 3\lambda\psi^4_{M_x}}. \tag{18}$$

The optimum values of k_5 and k_6 are

$$k_{5(opt)} = \left(\frac{\lambda\psi^2_{M_y}\psi^2_{M_x} - 3\lambda\psi^2_{M_x}\psi_{M_{yx}} - 2\lambda\psi^2_{M_y}\psi_{M_{yx}} + 4\lambda\psi^2_{M_{yx}} - 2\psi^2_{M_x} + 2\psi_{M_{yx}}}{16\lambda\psi^2_{M_{yx}} - 4\psi^2_{M_x} - 8\lambda\psi^2_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x} - 3\lambda\psi^4_{M_x}}\right)$$

and

$$k_{6(opt)} = \left(\frac{\lambda\psi^4_{M_x} + \lambda\psi^2_{M_y}\psi^2_{M_x} + 3\lambda\psi^2_{M_x}\psi_{M_{yx}} - 2\lambda\psi^2_{M_y}\psi_{M_{yx}} - 4\lambda\psi^2_{M_{yx}} + 2\psi^2_{M_x} + 2\psi_{M_{yx}}}{16\lambda\psi^2_{M_{yx}} - 4\psi^2_{M_x} - 8\lambda\psi^2_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x} - 3\lambda\psi^4_{M_x}}\right).$$

(vii). Shabbir and Gupta [14] suggested the following difference type estimators of median as:

$$\widehat{M}_{D1} = [k_7\widehat{M}_y + k_8(M_x - \widehat{M}_x)]\left(\frac{M_x}{\widehat{M}_x}\right), \tag{19}$$

$$\widehat{M}_{D2} = [k_9\widehat{M}_y + k_{10}(M_x - \widehat{M}_x)]\exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right), \tag{20}$$

where $k_j (j = 7, 8, \ldots, 10)$ are constants.
The biases and minimum $MSEs$ of $\widehat{M}_{Di}(i = 1, 2)$, are given by

$$B(\widehat{M}_{D1}) \cong (k_7 - 1)M_y + k_8\lambda M_y(\psi^2_{M_x} - \psi_{M_{yx}}) + k_8\lambda M_x\psi^2_{M_x}, \tag{21}$$

$$B(\widehat{M}_{D2}) \cong (k_9 - 1)M_y + k_9\lambda M_y\left(\frac{3}{8}\psi^2_{M_x} - \frac{1}{2}\psi_{M_{yx}}\right) + \frac{1}{2}k_{10}\lambda M_x\psi^2_{M_x}, \tag{22}$$

$$MSE(\widehat{M}_{D1})_{min} \cong M_y^2\left[\frac{(1 - \lambda\psi^2_{M_x})\lambda\psi^2_{M_y}(1 - \rho^2_{yx})}{(1 - \lambda\psi^2_{M_x}) + \lambda\psi^2_{M_y}(1 - \rho^2_{yx})}\right], \tag{23}$$

$$MSE(\widehat{M}_{D2})_{min} \cong M_y^2\left[1 - \frac{1}{4}\lambda\psi_{M_x}^2 - \frac{(1 - \frac{1}{8}\lambda\psi_{M_x}^2)^2}{1 + \lambda\psi_{M_y}^2(1 - \rho_{yx}^2)}\right]. \quad (24)$$

The optimal values of $k_j (j = 7, 8, 9, 10)$ are:
$k_{7(opt)} = \frac{(1-\lambda\psi_{M_x}^2)}{1-\lambda\psi_{M_x}^2+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}$, $k_{8(opt)} = \frac{M_y}{M_x}\left[1 + k_{7(opt)}\left(\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}} - 2\right)\right]$, $k_{9(opt)} = \frac{1-\frac{1}{8}\lambda\psi_{M_x}^2}{1+\lambda\psi_{M_y}^2(1-\rho_{yx}^2)}$ and $k_{10(opt)} = \frac{M_y}{M_x}\left[\frac{1}{2} + k_{9(opt)}\left(\frac{\rho_{yx}\psi_{M_y}}{\psi_{M_x}} - 1\right)\right]$.

3. The Proposed Estimator

Motivated by Solanki and Singh [15] and Muneer et al. [24], we propose the following class of exponential cum ratio-product type estimators of population median M_y as

$$\widehat{M}_{Prop} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right) + k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right]\left[\alpha\exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right) + (1-\alpha)\exp\left(\frac{\widehat{M}_x - M_x}{M_x + \widehat{M}_x}\right)\right], \quad (25)$$

where $(k_j, j = 11, 12)$ are unknown constants whose values are to be determined and α is known constant which takes values $(0, 1)$. Putting the values of α in Equation (25), we get the following two estimators.

(i) Put $\alpha = 0$, the proposed estimator is reduced to:

$$\widehat{M}_{Prop_{(\alpha=0)}} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right) + k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right]\exp\left(\frac{\widehat{M}_x - M_x}{M_x + \widehat{M}_x}\right). \quad (26)$$

(ii) Put $\alpha = 1$, the proposed estimator is reduced to:

$$\widehat{M}_{Prop_{(\alpha=1)}} = \widehat{M}_y\left[k_{11}\left(\frac{M_x}{\widehat{M}_x}\right) + k_{12}\left(\frac{\widehat{M}_x}{M_x}\right)\right]\exp\left(\frac{M_x - \widehat{M}_x}{M_x + \widehat{M}_x}\right). \quad (27)$$

Expressing $\widehat{M}_{Prop}$ in Equation (25) in terms of $(\zeta_j, j = 0, 1)$ upto first order of approximation, we get

$$\widehat{M}_{Prop} \cong M_y(1+\zeta_0)\left[k_{11}(1 - \zeta_1 + \zeta_1^2) + k_{12}(1 + \zeta_1)\right]\left[1 + (1-2\alpha)\frac{\zeta_1}{2} - (1-4\alpha)\frac{\zeta_1^2}{8}\right]. \quad (28)$$

Keeping $\zeta' s$ upto power two and expanding Equation (28), we have

$$\widehat{M}_{Prop} - M_y \cong M_y \left[\begin{array}{l} k_{11}\left\{1+\zeta_0-(\alpha+\frac{1}{2})\zeta_1-(\alpha+\frac{1}{2})\zeta_0\zeta_1+(\frac{3}{8}+\frac{3\alpha}{2})\zeta_1^2\right\} \\ \quad + k_{12}\left\{1+\zeta_0+(\frac{3}{2}-\alpha)\zeta_1+(\frac{3}{2}-\alpha)\zeta_0\zeta_1+(\frac{3}{8}-\frac{\alpha}{2})\zeta_1^2\right\}-1 \end{array} \right]. \tag{29}$$

Solving Equation (29), the bias and MSE of $\widehat{M}_{Prop}$ are:

$$Bias(\widehat{M}_{Prop}) \cong M_y \left[\begin{array}{l} (k_{11}+k_{12}-1)+k_{11}\lambda\left\{\left(\frac{3}{8}+\frac{3\alpha}{2}\right)\psi_{M_x}^2-\left(\alpha+\frac{1}{2}\right)\psi_{M_{yx}}\right\} \\ \quad + k_{12}\lambda\left\{\left(\frac{3}{2}-\alpha\right)\psi_{M_{yx}}+\left(\frac{3}{8}-\frac{\alpha}{2}\right)\psi_{M_x}^2\right\} \end{array} \right] \tag{30}$$

and

$$\begin{aligned} MSE(\widehat{M}_{Prop}) \cong M_y^2[1 &+ k_{11}^2\left\{1+\lambda\psi_{M_y}^2+\left(\alpha^2+4\alpha+1\right)\lambda\psi_{M_x}^2-(4\alpha+2)\lambda\psi_{M_{yx}}\right\} \\ &+k_{12}^2\left\{1+\lambda\psi_{M_y}^2+\left(\alpha^2-4\alpha+3\right)\lambda\psi_{M_x}^2+(6-4\alpha)\lambda\psi_{M_{yx}}\right\} \\ &+2k_{11}k_{12}\left\{1+\lambda\psi_{M_y}^2+(2-4\alpha)\lambda\psi_{M_{yx}}+\alpha^2\lambda\psi_{M_x}^2\right\} \\ &-2k_{11}\left\{1+\left(\frac{3}{8}+\frac{3\alpha}{2}\right)\lambda\psi_{M_x}^2-\left(\alpha+\frac{1}{2}\right)\lambda\psi_{M_{yx}}\right\} \\ &-2k_{12}\left\{1+\left(\frac{3}{8}-\frac{\alpha}{2}\right)\lambda\psi_{M_x}^2+\left(\frac{3}{2}-\alpha\right)\lambda\psi_{M_{yx}}\right\}] \end{aligned}$$

or

$$MSE(\widehat{M}_{Prop}) \cong M_y^2\left[1+k_{11}^2A_1+k_{12}^2A_2+2k_{11}k_{12}A_3-2k_{11}A_4-2k_{12}A_5\right], \tag{31}$$

where

$$A_1 = 1+\lambda\psi_{M_y}^2+\left(\alpha^2+4\alpha+1\right)\lambda\psi_{M_x}^2-(4\alpha+2)\lambda\psi_{M_{yx}},$$

$$A_2 = 1+\lambda\psi_{M_y}^2+\left(\alpha^2-4\alpha+3\right)\lambda\psi_{M_x}^2+(6-4\alpha)\lambda\psi_{M_{yx}},$$

$$A_3 = 1+\lambda\psi_{M_y}^2+(2-4\alpha)\lambda\psi_{M_{yx}}+\alpha^2\lambda\psi_{M_x}^2,$$

$$A_4 = 1+\left(\frac{3}{8}+\frac{3\alpha}{2}\right)\lambda\psi_{M_x}^2-\left(\alpha+\frac{1}{2}\right)\lambda\psi_{M_{yx}},$$

$$A_5 = 1+\left(\frac{3}{8}-\frac{\alpha}{2}\right)\lambda\psi_{M_x}^2+\left(\frac{3}{2}-\alpha\right)\lambda\psi_{M_{yx}}.$$

Differentiate Equation (31) with respect to k_{11} and k_{12}, we obtain the optimal values of k_{11} and k_{12} as:

$$k_{11(opt)} = \left(\frac{A_2A_4-A_3A_5}{A_1A_2-A_3^2}\right)$$

and

$$k_{12(opt)} = \left(\frac{A_1A_5 - A_3A_4}{A_1A_2 - A_3^2}\right).$$

Substitute the optimal values of k_{11} and k_{12} in Equation (31), we get the minimum *MSE* of $\widehat{M}_{Prop}$, is given by:

$$MSE(\widehat{M}_{Prop})_{min} \cong M_y^2\left(1 - \frac{A_1A_5^2 - 2A_3A_4A_5 + A_2A_4^2}{A_1A_2 - A_3^2}\right). \quad (32)$$

4. Theoretical Comparison

Now we compare the *MSE* of the proposed estimator $\hat{M}_{Prop}$ with the *MSEs* of considered estimators and derive the optimal conditions.

Condition (i): By (1) and (32),

$$V(\widehat{M}_y) - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left((\lambda\psi_{M_y}^2 - 1)\theta_2 + \theta_1\right) > 0,$$

where $\theta_1 = A_1A_5^2 - 2A_3A_4A_5 + A_2A_4^2$ and $\theta_2 = A_1A_2 - A_3^2$.

Condition (ii): By (4) and (32),

$$MSE(\widehat{M}_R) - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left((\lambda\psi_{M_y}^2 + \lambda\psi_{M_x}^2 - 2\lambda\psi_{M_{yx}} - 1)\theta_2 + \theta_1\right) > 0.$$

Condition (iii): By (7) and (32),

$$MSE(\widehat{M}_{Exp}) - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left((\lambda\psi_{M_y}^2 + \frac{1}{4}\lambda\psi_{M_x}^2 - \lambda\psi_{M_{yx}} - 1)\theta_2 + \theta_1\right) > 0.$$

Condition (iv): By (12) and (32),

$$MSE(\widehat{M}_{SS1})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left(\frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 - \frac{(1 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + \frac{M_x}{M_y}\lambda\psi_{M_{yx}})^2}{1 + \lambda\psi_{M_y}^2 + \frac{M_x^2}{M_y^2}\lambda\psi_{M_x}^2 + 2\frac{M_x}{M_y}\lambda\psi_{M_{yx}}} + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (v): By (13) and (32),

$$MSE(\widehat{M}_{SS2})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left(\frac{\lambda\psi^2_{M_y}(1-\rho^2_{yx})}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})} - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (vi): By (15) and (32),

$$MSE(\widehat{M}_{D})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left(\lambda\psi^2_{M_y}(1-\rho^2_{yx}) - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (vii): By (18) and (32),

$$MSE(\widehat{M}_{SS3})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left[\frac{\lambda\left(\begin{array}{c}\lambda\psi^6_{M_x} + \lambda\psi^2_{M_y}\psi^4_{M_x} + 2\lambda\psi^4_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x}\psi_{M_{yx}} \\ -4\lambda\psi^2_{M_x}\psi^2_{M_{yx}} + 4\lambda\psi^2_{M_y}\psi^2_{M_{yx}} - 4\psi^2_{M_y}\psi^2_{M_x} + 4\psi^2_{M_{yx}}\end{array}\right)}{16\lambda\psi^2_{M_{yx}} - 4\psi^2_{M_x} - 8\lambda\psi^2_{M_x}\psi_{M_{yx}} - 4\lambda\psi^2_{M_y}\psi^2_{M_x} - 3\lambda\psi^4_{M_x}} - 1 + \frac{\theta_1}{\theta_2}\right] > 0.$$

Condition (viii): By (23) and (32),

$$MSE(\widehat{M}_{D1})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left(\frac{(1-\lambda\psi^2_{M_x})\lambda\psi^2_{M_y}(1-\rho^2_{yx})}{(1-\lambda\psi^2_{M_x}) + \lambda\psi^2_{M_y}(1-\rho^2_{yx})} - 1 + \frac{\theta_1}{\theta_2}\right) > 0.$$

Condition (ix): By (24) and (32),

$$MSE(\widehat{M}_{D2})_{min} - MSE(\widehat{M}_{Prop})_{min} > 0, \text{ if}$$

$$M_y^2\left(\frac{\theta_1}{\theta_2} - \frac{1}{4}\lambda\psi^2_{M_x} - \frac{(1-\frac{1}{8}\lambda\psi^2_{M_x})^2}{1+\lambda\psi^2_{M_y}(1-\rho^2_{yx})}\right) > 0.$$

Note: The proposed estimator $\hat{M}_{Prop}$ performs better, when the conditions from (i)-(ix) are satisfied.

5. Numerical Comparisons

To examine performances of the estimators, we use the following data sets for comparisons.

- Population 1: [Source: PDS [25], Pages 114-116]

  Y: Number of teaching staff,

  X: Number of students in 4 different types of schools under 36 districts in Punjab province of Pakistan.

- Population 2: [Source: Aladag and Cingi [12]]

  Y: Number of Teachers,

  X: Number of students in 340 districts having elementary schools in 2007.

- Population 3: [Source: Bandyopadhyay et al. [21]]

  Y: Graduate or higher degrees in 2007 of United States,

  X: Graduate or higher degrees in 2006 of United States.

- Population 4: [Source: Amir et al. [26]]

  Y: Oil prices in current week from 1996 to 2017 ,

  X: Oil prices in previous week from 1996 to 2017.

The descriptive statistics of all populations are given in Table 1.

We use the following expression to obtain the percent relative efficiencies *(PREs)* with respect to Gross [1] unbiased per unit estimator:

$$PRE = \frac{V(\widehat{M}_y)}{MSE(\widehat{M}_j)\ or\ MSE(\widehat{M}_j)_{min}} \times 100,$$

where $j = y, R, Exp, SS1, SS2, D, D1, D2, SS3, Prop_{(\alpha=0)}, Prop_{(\alpha=1)}$.

$MSEs$ and $PREs$ values based on four populations are given in Table 2.

From Table 2, it is clear that the MSE of the proposed class of estimators in all four populations have minimum as compared to considered estimators. It is also observed that using different correlations, the proposed estimators are more efficient and stable.

6. Simulation Study

The simulation study is carried out to observe the performances of the estimators both empirically and theoretically. We consider three finite populations of size 1500 generated from a skewed multivariate normal distribution with same theoretical means of $[Y, X]$ as $\mu = [5, 5]$ and using different variance covariance matrices as given below.

Population 1

$$\Sigma = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$$

Population 2

$$\Sigma = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}$$

Population 3

$$\Sigma = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}$$

For each population, we consider sample sizes $n = 50, 100, 150, 300$ and probability density functions $f_y(M_y)$ and $f_x(M_x)$ are calculated using normal distribution. In Table 3, the empirical and theoretical mean squared errors and percent relative efficiencies are computed. We estimate the empirical *MSE* using 5000 samples of various sizes selected from each population.

Analyzing the simulation results of all three artificial populations in Table 3, it is clear that the proposed class of estimators performs more efficiently and have less MSE as compared to all considered estimators in all populations. It is noted that the empirical MSE of $\hat{M}_{SS3}$ is very high in all populations and performs very poorly.

7. Conclusion

In this article, we have proposed a new class of exponential cum ratio-product type estimators for the population median (M_y) under simple random sampling using the supplementary variable. The bias and *MSE* expressions are obtained upto first order of approximation. Theoretical conditions are derived under which the proposed class of estimators performs better. In Tables 2 and 3, the proposed class of estimators $\widehat{M}_{Prop_{(\alpha=0)}}$ and $\widehat{M}_{Prop_{(\alpha=1)}}$ are compared with unbiased estimator of median and other considered estimators both theoretically and empirically in terms of *MSEs* and *PREs* using different sample sizes. It is observed that the proposed class of estimators has minimum MSE and performs better in all four populations. A simulation study reflects the same behavior in all populations. So we recommend the practical use of proposed estimators $\widehat{M}_{Prop_{(\alpha=0)}}$ and $\widehat{M}_{Prop_{(\alpha=1)}}$ in various fields of engineering and data analysis.

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Table Captions

Table 1: Descriptive statistics of all four populations.

Table 2: *MSE* and *PRE* values of all considered and proposed estimators.

Table 3: Empirical and Theoretical *MSE* and *PRE* values of estimators.

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	Population 1	Population 2	Population 3	Population 4
N	144	340	51	1134
n	10	49	11	210
λ	0.02327	0.00436	0.01782	0.00097
M_y	2023	178	25.80	48.55
M_x	64659	3526	25.60	48.49
$f_y(M_y)$	0.00024	0.00182	0.0728	0.0075349
$f_x(M_x)$	0.00001	0.00800	0.1080	0.0075437
ρ_{yx}	0.86110	0.92000	0.9956	0.9953
ψ_{M_y}	2.05965	3.08680	0.53242	2.73359
ψ_{M_x}	1.54658	0.03545	0.36168	2.73378
$\psi_{M_{yx}}$	2.74295	0.10067	0.19172	7.43791

Table 1: Descriptive statistics of all populations

Pop.		$\widehat{M}_y$	$\widehat{M}_R$	$\widehat{M}_{Exp}$	$\widehat{M}_{SS1}$	$\widehat{M}_{SS2}$	$\widehat{M}_D$	$\widehat{M}_{D1}$	$\widehat{M}_{D2}$	$\widehat{M}_{SS3}$	$\widehat{M}_{Prop_{(\alpha=0)}}$	$\widehat{M}_{Prop_{(\alpha=1)}}$
1	MSE	403886.959	109312.950	199667.997	102203.340	174090.410	181824.980	173655.141	171479.031	103796.950	**100940.420**	**99054.183**
	PRE	100.000	369.478	202.279	395.179	231.999	222.129	232.579	235.532	389.113	**400.124**	**407.744**
2	MSE	1318.303	1290.618	1304.417	1242.712	201.205	202.492	201.205	201.205	173.7406	**173.066**	**172.717**
	PRE	100.000	102.145	101.065	106.083	655.203	651.042	655.203	655.203	758.776	**761.732**	**763.273**
3	MSE	3.36337	0.36594	1.47661	3.31657	0.02953	0.02953	0.02953	0.02946	0.03060	**0.01815**	**0.02924**
	PRE	100.000	919.111	227.777	101.107	11389.197	11388.691	11389.198	11417.774	11440.852	**18528.773**	**11501.591**
4	MSE	17.08538	0.1606138	4.35105	16.60673	0.16022	0.16023	0.16021	0.15798	0.16019	**0.11400**	**0.141623**
	PRE	100.000	10637.548	392.672	102.883	10664.082	10663.357	10664.087	10814.411	10665.607	**14986.947**	**12064.049**

Table 2: *MSE* and *PRE* values of all considered and proposed estimators.

		Population 1			Population 2			Population 3		
		MSE			MSE			MSE		
n	Estimator	Empirical	Theoretical	PRE	Empirical	Theoretical	PRE	Empirical	Theoretical	PRE
	$\hat{M}_y$	0.306148	0.323727	100.00	0.183689	0.194236	100.00	0.275533	0.291354	100.00
	$\hat{M}_R$	0.236473	0.244817	132.23	0.097730	0.107067	181.41	0.327556	0.341346	85.35
	$\hat{M}_{Exp}$	0.255370	0.267997	120.79	0.123801	0.134467	144.45	0.269551	0.283504	102.77
	$\hat{M}_{SS1}$	0.318258	0.318130	101.76	0.188586	0.191566	101.39	0.285143	0.287387	101.38
	$\hat{M}_{SS2}$	0.242741	0.241913	133.82	0.099727	0.104712	185.50	0.274293	0.275908	105.60
50	$\hat{M}_D$	0.238789	0.244085	132.63	0.098889	0.105124	184.77	0.267575	0.278749	104.52
	$\hat{M}_{D1}$	0.302483	0.241908	133.82	0.161117	0.104711	185.50	0.390574	0.275894	105.60
	$\hat{M}_{D2}$	0.258115	0.241766	133.90	0.114109	0.104646	185.61	0.302462	0.275563	105.73
	$\hat{M}_{SS3}$	0.296073	0.242102	133.71	0.131809	0.104845	185.26	0.411307	0.275573	105.73
	$\hat{M}_{Prop_{(\alpha=0)}}$	0.330339	0.241733	133.92	0.194626	0.104634	185.63	0.413497	0.275120	105.90
	$\hat{M}_{Prop_{(\alpha=1)}}$	0.252095	0.241607	133.99	0.108878	0.104620	185.66	0.277523	0.274128	106.28
	$\hat{M}_y$	0.136593	0.156282	100.00	0.081956	0.093769	100.00	0.122933	0.140654	100.00
	$\hat{M}_R$	0.105485	0.118188	132.23	0.043630	0.051688	181.41	0.145648	0.164788	85.35
	$\hat{M}_{Exp}$	0.113646	0.129378	120.79	0.054098	0.064915	144.45	0.119849	0.136864	102.77
	$\hat{M}_{SS1}$	0.138803	0.154965	100.85	0.082791	0.093142	100.67	0.124701	0.139721	100.67
	$\hat{M}_{SS2}$	0.106430	0.117325	133.20	0.045065	0.050953	184.03	0.119171	0.133903	105.04
100	$\hat{M}_D$	0.106030	0.117834	132.63	0.044935	0.050750	184.77	0.118115	0.134568	104.52
	$\hat{M}_{D1}$	0.132564	0.117325	133.20	0.070592	0.050853	184.39	0.173543	0.133901	105.04
	$\hat{M}_{D2}$	0.112452	0.117291	133.24	0.049209	0.050838	184.45	0.132336	0.133822	105.10
	$\hat{M}_{SS3}$	0.122492	0.117370	133.15	0.099777	0.050795	184.60	0.134908	0.133784	105.13
	$\hat{M}_{Prop_{(\alpha=0)}}$	0.143380	0.117368	133.16	0.092437	0.050735	184.82	0.179402	0.133717	105.19
	$\hat{M}_{Prop_{(\alpha=1)}}$	0.108088	0.117254	133.29	0.046540	0.050632	185.20	0.119465	0.133484	105.37
	$\hat{M}_y$	0.087658	0.100467	100.00	0.052595	0.060280	100.00	0.078892	0.090420	100.00
	$\hat{M}_R$	0.068742	0.075978	132.23	0.028247	0.033228	181.41	0.093848	0.105935	85.35
	$\hat{M}_{Exp}$	0.073758	0.083172	120.79	0.034606	0.041731	144.45	0.077276	0.087984	102.77
	$\hat{M}_{SS1}$	0.088259	0.099921	100.55	0.052769	0.060020	100.43	0.079355	0.090033	100.43
	$\hat{M}_{SS2}$	0.068951	0.075540	133.00	0.029302	0.032585	184.99	0.076545	0.086233	104.86
150	$\hat{M}_D$	0.069058	0.075750	132.63	0.029298	0.032625	184.77	0.076309	0.086508	104.52
	$\hat{M}_{D1}$	0.085043	0.075540	133.00	0.045133	0.032585	184.99	0.110684	0.086232	104.86
	$\hat{M}_{D2}$	0.072697	0.075526	133.02	0.031426	0.032579	185.00	0.084669	0.086199	104.90
	$\hat{M}_{SS3}$	0.107018	0.075559	132.97	0.048483	0.032598	184.92	0.089474	0.086899	104.06
	$\hat{M}_{Prop_{(\alpha=0)}}$	0.089938	0.075510	133.05	0.061190	0.032576	185.04	0.114622	0.086156	104.95
	$\hat{M}_{Prop_{(\alpha=1)}}$	0.069556	0.075001	133.95	0.029932	0.032147	187.51	0.076447	0.086059	105.07
	$\hat{M}_y$	0.037044	0.044652	100.00	0.022226	0.026791	100.00	0.033340	0.040187	100.00
	$\hat{M}_R$	0.029300	0.033768	132.23	0.011986	0.014768	181.41	0.040547	0.047082	85.35
	$\hat{M}_{Exp}$	0.031229	0.036965	120.79	0.014392	0.018547	144.45	0.032560	0.039104	102.77
	$\hat{M}_{SS1}$	0.037163	0.044544	100.24	0.022263	0.026740	100.19	0.033432	0.040110	100.19
	$\hat{M}_{SS2}$	0.029297	0.033625	132.79	0.012613	0.014492	184.87	0.032058	0.038394	104.67
300	$\hat{M}_D$	0.029387	0.033667	132.63	0.012624	0.014500	184.77	0.032029	0.038448	104.52
	$\hat{M}_{D1}$	0.035808	0.033625	132.79	0.018890	0.014492	184.87	0.048751	0.038394	104.67
	$\hat{M}_{D2}$	0.030652	0.033622	132.80	0.013044	0.014491	184.88	0.036065	0.038387	104.69
	$\hat{M}_{SS3}$	0.039510	0.033629	132.78	0.027763	0.014495	184.84	0.041918	0.038381	104.70
	$\hat{M}_{Prop_{(\alpha=0)}}$	0.038825	0.033642	132.73	0.028340	0.014490	184.89	0.050344	0.038378	104.71
	$\hat{M}_{Prop_{(\alpha=1)}}$	0.029498	0.033619	132.82	0.012789	0.014423	185.75	0.032083	0.038359	104.76

Table 3: Empirical and Theoretical MSE and PRE values of different estimators with respect to $\hat{M}_y$.