Three-valued soft set and its multi-criteria group decision making via TOPSIS and ELECTRE

E. Akçetin\(^a\) and H. Kamacı\(^b,\)\(^,*\)

\(^a\) Department of Accounting and Financial Management, Seydi kemere School of Applied Sciences, Muğla Sıtkı Koçman University, Muğla, Turkey.
\(^b\) Department of Mathematics, Faculty of Science and Arts, Yozgat Bozok University, Yozgat, Turkey.

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- Decision making;
- TOPSIS;
- ELECTRE.

Abstract. The main objective of the present study is to point to the generalization of Molodtsov’s approach to soft sets obtained by passing from the classical two-valued logic underlying those sets to a three-valued logic where the third truth value can usually be interpreted as either non-determined (i.e., between true and false) or unknown. This extension of soft set approach allows a more intuitive and clearer representation of various decision-making problems involving incomplete or uncertain information. In other words, it is a viable approach to analyze soft-set-based multi-criteria group decision-making problems in the absence of adequate information resulting from the inability to determine the data.

In this regard, this study introduced the concept of three-valued soft set and some of its operations and products. In addition, the formulas required to calculate all possible choice values were proposed for each object in the (weighted) three-valued soft sets and their respective decision values were calculated. Both Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and Elimination and Choice Translating Reality (ELECTRE) methods were modified to deal with multi-criteria group decision problems and then, three-valued soft-set-based decision-making algorithms were constructed. To demonstrate the practicability of these algorithms, the examples adopted from the decision problems in real life were addressed. Lastly, some aspects of the efficiency of the proposed algorithms were discussed using computational experiments.

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1. Introduction

Classically, a logic is two-valued (Boolean) if every proposition is either false ("\(\emptyset\)") or true ("\(1\)"). In 1920, Łukasiewicz [1] initiated a three-valued logic, i.e., a natural extension of the two-valued logic with three truth (logical) values indicating false ("\(\emptyset\)"), true ("\(1\)"), and some indeterminate third values ("\(\frac{1}{2}\) as something in the middle between true and false"). Further, he pioneered the conceptual form and basic ideas of three-valued logic. By interpreting the intuition of Łukasiewicz three-valued logic from different perspectives, two-valued logic was extended to three-valued logic in different ways [2–5]. Based on their choice of basic connectives, they were different from a syntactic and proof-theoretic point of view. Although the idea of fuzzification of logic was envisaged by several researchers in the years following 1920, the concept of fuzzy logic in which the truth values of

\(^*\) Corresponding author. Tel.: +90 354 242 10 21; Fax: +90 354 242 10 22
E-mail addresses: eyup.akcetin@mu.edu.tr (E. Akçetin);
huseyin.kamaci@hotmail.com, huseyin.kamaci@bozok.edu.tr (H. Kamacı)
variables might be any real number between 0 and 1, both inclusive, was explicitly and crisply proposed by Zadeh [6] in 1965. Fuzzy logic is based on the investigation and observation that people take into account while making decisions based on uncertain and non-numerical information. Fuzzy set, which is a generalization of the crisp set based on the two-valued logic, is the mathematical means of representing vagueness and imprecise information. This set is described by a membership function that assigns to each object a degree of membership ranging between 0 and 1. Pawlak [7] defined a rough set that can be considered as a new area of uncertainty mathematics closely related to set theory. This set is a formal approximation of a crisp set in a pair of sets which give the lower and upper approximations of the original set. The approximation spaces of rough set theory are sets with multiple memberships, while fuzzy sets are concerned with partial memberships. These sets are combined to derive different variations such as the fuzzy rough set and rough fuzzy set. In 1999, Molodtsov [8] described the soft set based on two-valued logic as a mathematical tool dealing with parametric data which were imprecise or uncertain in nature. In 2003, Maji et al. [9] published a study on the operations of soft sets. Later, the operational laws of the soft sets were derived [10–14]. In addition, many authors have described and discussed different types of soft sets such as bijective soft set [15], exclusive disjunctive soft set [16], bipolar soft set [17], inverse soft set [18,19], etc.

Decision making, which is one of the issues that triggers uncertainty, is a frequently encountered problem in many commercial and scientific fields, even in every stage of daily life. For deterministic modeling of decision-making problems, many mathematical techniques such as Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), AHP, VIKOR, Elimination Et Choice Translating (ELECTRE), and PROMETHEE have been developed. These mathematical techniques were adopted for decision-making based on the fuzzy set and some of its extensions [20–26]. In addition to the fuzzy modelling of TOPSIS, ELECTRE, etc., many algorithmic solutions were proposed for decision-making in a fuzzy environment [27–31]. In 2002, Maji et al. [32] reported that soft sets could be used to solve decision-making problems involving parametric data based on two-valued logic. In the following years, many soft decision algorithms were created and their applications to problems in real life were specified [33–37]. Moreover, Eraslav [38] proposed a decision-making procedure through the classical method of TOPSIS on the soft sets. In this regard, he pointed out that classical decision-making techniques could be successfully applied to the soft-set-based decision-making.

In 2008, Avron and Konikowska [39] explored the idea of describing Pawlak's rough set using three-valued logic. This paradigm presents a different perspective in the interpretation of issues involving indeterminate or unknown data in many fields. As with the rough set, three-valued logic emerges in many real-world scenarios that are included in the scope of the soft set, and it is difficult to deal with such broadly scoped issues whose third truth value is "undetermined". In this study, a completely designed approach to soft set was discussed using three-valued logic. Thus, in practice, we aim to overcome the difficulties that include a third truth value caused by uncertain or unknown origin, in addition to the truth values of a two-valued logic. The present study was conducted based on the idea of proposing the notion of three-valued soft set to describe a soft set using three-valued logic. The fusion of three-valued logic into the soft set suggests a clearer and intuitive way to explain different issues under incomplete or uncertain information. In this regard, the main focus was put on decision-making based on this type of soft set and offering different algorithmic solutions.

The rest of this paper is organized as follows. Section 2 gives an outline of soft set theory. Section 3 elaborates the motivation to reinterpret soft sets using three-valued logic. Section 4 calculates the choice value of an object in (weighted) three-valued soft sets and accordingly, proposes two algorithms. Section 5 creates a multi-criteria group decision-making algorithm based on the modified TOPSIS on three-valued soft sets. Section 6 proposes a three-valued soft decision-making algorithm via the modified ELECTRE, which is based on three fundamental objectives called choosing, sorting, and ranking. In addition, some examples were given to analyze the performance of the algorithms emerging in these two sections. Section 7 solves the matching numerical examples to compare the results of the proposed algorithms, thus showing that they are convincing. The last section presents the concluding remarks and suggests plans for further research.

2. Preliminaries

As a preparatory opening for new concepts, this section elaborates some relevant arguments of soft set, two-valued logic, and three-valued logic.

Consider the soft set theory first. In 1999, Molodtsov [8] introduced soft set theory as a useful way of classifying objects based on parametric data. In 2010, Çağman and Enginoğlu [40] recreated soft sets to make their operations more practical in some cases. Maji et al. [32] put forward that the soft set could be represented in a tabular form. They also demonstrated that soft sets were the parametric sets created based on two-valued logic (i.e., 0 as false, 1 as true). Now, recall the definition of soft set.
From now on, $\mathcal{X}$ is a universal set, $\mathcal{P}$ is a parameter set, and $Q \subseteq \mathcal{P}$.

**Definition 2.1** [8,40]. Assume that $\mathcal{X}^{(0,1)}$ denotes the set of all functions from $\mathcal{X}$ to $\{0,1\}$. A pair $(f_Q, \mathcal{P}) = F_Q$ is called a soft set over $\mathcal{X}$ when the mapping $f_Q$ is defined by $f_Q : \mathcal{P} \to \mathcal{X}^{(0,1)}$, where for all $p_j \in \mathcal{P}$, the approximate function $f_Q(p_j)$ is shown in Box I.

**Example 2.1.** In daily life, many operations such as shopping, booking, money transfer, etc., can be done through online websites. Recently, some websites have made it feasible to reserve hotel rooms for trips and holidays; take Booking.com and Expedia.com as examples. Now, consider the following problem that one may encounter when making a hotel room reservation.

Let $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five hotels that are available for booking a room. Let $\mathcal{P} = \{p_1 = \text{price}, p_2 = \text{location}, p_3 = \text{amenities}, p_4 = \text{satisfaction}\}$ be a set of the choice parameters. Then, we can create the following soft set over $\mathcal{X}$:

$$T_p = \{(p_1, \{x_1^{(0)}, x_2^{(1)}, x_3^{(1)}, x_4^{(0)}, x_5^{(0)}\}),$$

$$\quad(p_2, \{x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(1)}\}),$$

$$\quad(p_3, \{x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}\}),$$

$$\quad(p_4, \{x_1^{(0)}, x_2^{(1)}, x_3^{(1)}, x_4^{(0)}, x_5^{(1)}\})\}.$$

For the first pair in this soft set, it can be interpreted that the prices of hotels $x_2$ and $x_3$ are suitable for us, while the prices of hotels $x_1$, $x_4$, and $x_5$ are not suitable. Other pairs can be interpreted similarly.

In 1930, Łukasiewicz [1] put forward (Łukasiewicz) three-valued logic by extending two-valued logic called Boolean logic. Immediately after the above author, many authors have exhibited their interest in the idea of three-valued logic and its applications [2,4,39,41,42]. They argued that since two-valued logic (Boolean logic) could cover all kinds of scientific investigations, three-valued logic might be useful as a basis for a number of useful reasoning tasks. Boolean connectives can be extended to three-valued logic in different ways. In other words, the third truth value can be explained in different ways that are different from true and false. Ciucci and Dubois [43] listed these ways as follows:

- **Possible**: this explanation was proposed by Łukasiewicz [1] and Borowski [2] the pioneer of three-valued logic. A proposition is regarded as “Possible” if its truth value will be known only in the future;
- **Unknown**: This explanation was proposed by Kleene [4] in 1952. A proposition is “Unknown” if its truth value cannot be computed for some reasons (for instance, it is too time-consuming to compute);
- **Inconsistent**: The third value stands for a proposition that is both true and false, and also it is the dual of “Unknown” in some sense;
- **Half-true**: This is the typical of fuzzy logic [3]. The intuition is that for some propositions, truth is a importance degree. For instance, Shadowed set in [44,45] is based on the idea of turning fuzzy set into three-valued one;
- **Undefined**: This is another explanation of Kleene [4]. The undefined state corresponds to the selection of the argument of the function outside its definition domain. A proposition is “Undefined” if its truth value involves undefined atoms;
- **Irrelevant**: The idea behind it is that propositions are not applicable in some possible worlds.

In 1960, Skolem [5] initiated a set theory based on a certain three-valued logic. In this set theory, the variables such as $p, q, r, ..., \ldots$ take three values: 0, $\frac{1}{2}$, 1. We may interpret $0$ as "false", 1 as "true", and $\frac{1}{2}$ as something in the middle between true and false, say "undetermined". Moreover, Skolem presented a set of truth tables showing tree-valued logic operations like negation, disjunction and conjunction (for a detailed review, see [5]). In the literature, there are many truth tables illustrating tree-valued logic operations. However, this study focused on the truth tables proposed by Skolem [5].

Three-valued logic emerges in several real-world scenes. Three samples in the atmosphere of "undetermined" are presented in Figures 1–3. In these figures, "?", symbolizes "undetermined", i.e., $\frac{1}{2}$.

Since the truth value of "undetermined" in Figures 1 and 2 is precisely known in the future, it can

$$f_Q(p_j) = \begin{cases} 
\{\{x_i \mid \lambda_{f_Q(p_j)}(x_i) \in \mathcal{X} \text{ and } \lambda_{f_Q(p_j)}(x_i) \in \{0,1\}\} 
\text{, if } p_j \in Q \\
\{x_i \mid \forall x_i \in \mathcal{X}\}, 
\text{ if } p_j \in \mathcal{P} \setminus Q
\end{cases}$$

Box I
be considered as “Possible”. Figure 3 can also be considered as an example for “Unknown”.

### 3. Three-valued soft sets

This section discusses three-valued soft set which is a generalization of soft set obtained by passing from two-valued logic to three-valued one.

The soft set is a set approach proposed by two-valued logic (true and false). In daily life, while evaluating alternatives according to parameters in the decision-making process, “undetermined” mode (neither true nor false, or both true and false) sometimes arises. In such decision-making processes, the soft sets are insufficient. To overcome this shortcoming, the notion of three-valued soft set, which is a soft set based on three-valued logic (i.e., 0 as false, 1 as true, and $\frac{1}{2}$ as “undetermined”) is formed.

#### 3.1. Three-valued soft set

**Definition 3.1.** Assume that $X^{(0,\frac{1}{2},1)}$ denotes the set of all functions from $X$ to $\{0, \frac{1}{2}, 1\}$. A pair $(t_Q, \mathcal{P}) = T_Q$ is called a three-valued soft set over $X$ when the mapping $t_Q$ is defined by $t_Q : \mathcal{P} \rightarrow X^{(0,\frac{1}{2},1)}$, where for all $p_j \in \mathcal{P}$ the equation shown in Box II is obtained.

**Notation:** For the parameter set $\mathcal{P}$, the set of all three-valued soft sets over $X$ is denoted by $T\forall SS(X, \mathcal{P})$.

**Example 3.1.** Assume that $X = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of drugs that can be taken by pregnant

![Figure 1. Three-valued logic for classifying mail in a mailbox.](image1)

![Figure 2. Three-valued logic for reviewers’ recommendation in journal.](image2)

![Figure 3. Three-valued logic for cilia-related lesions in hydrocephalic mice (see [46]).](image3)
women and those of childbearing age. The drugs that can be taken by these women, can be observed without increase in the frequency of malformation or other direct or indirect harmful effects on the human fetus. Therefore, these women should carefully check their side effects while using the drug. Suppose that $\mathcal{P} = \{p_1 = \text{allergy}, p_2 = \text{fetal damage}, p_3 = \text{pharmacological effect}\}$ denotes some side effects of the drugs. Here, the following table (Table 1) can be obtained by examining the prospectus of drugs.

Based on Table 1, the following three-valued soft set is constructed:

$$ T_\mathcal{P} = \{(p_1, \{x_1^{(1)}, x_2^{(0)}, x_3^{(1)}, x_4^{(0)}, x_5^{(0)}\}), \) 
\{(p_2, \{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(1)}\}), \) 
\{(p_3, \{x_1^{(0)}, x_2^{(0)}, x_3^{(1)}, x_4^{(0)}, x_5^{(0)}\}) \). 

Example 3.2. Let $\mathcal{X} = \{x_1 = \text{Vivo V17 Pro}, x_2 = \text{OnePlus 7}, x_3 = \text{Xiaomi Redmi K20 Pro}, x_4 = \text{Apple IPhone 11}\}$ be a set of four mobiles, and $\mathcal{P} = \{p_1 = \text{FM Radio}, p_2 = \text{Stereo Speakers}, p_3 = \text{Loudspeaker}\}$ a set of multimedia features (attributes) that may be available on phones. Through the website “www.91mobiles.com” (date: 17.10.2019), we have Figure 4. Based on this figure, the following three-valued soft set can be created:

$$ T_\mathcal{P} = \{(p_1, \{x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}\}), \) 
\{(p_2, \{x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, x_5^{(0)}\}), \) 
\{(p_3, \{x_1^{(0)}, x_2^{(0)}, x_3^{(1)}, x_4^{(0)}, x_5^{(0)}\}) \). 

Based on the information presented on this website, we can interpret that for the multimedia feature $p_1$ (FM Radio):
- The mobile $x_1$ (Vivo V17 Pro) has an FM Radio;
- The mobiles $x_2$ and $x_4$ (OnePlus 7 and Apple IPhone 11) do not have an FM Radio; and
- It remains “undetermined” whether the mobile $x_3$ (Xiaomi Redmi K20 Pro) has an FM Radio.

Figure 4. Comparison of some multi-media features of four mobiles.
Comment: The truth value \( \frac{1}{2} \) in the three-valued logic differs from the membership degree 0.5 in the fuzzy logic. The truth value \( \frac{1}{2} \) in the three-valued logic represents “undetermined”, not membership. Why is this truth value between 0 and 1, not 0? 0 indicates that the given object does not certainly have the desired property, while \( \frac{1}{2} \) means that the object might possibly have this property. To be specific, assume that one wants to buy any item from the mobiles \( x_2 \) and \( x_3 \) (presented in Figure 4). As a criterion for purchasing, the requirement to have only FM Radio is determined. In such situations, it is more convenient to select the mobile \( x_3 \). Since it is not known (undetermined) whether or not the mobile \( x_3 \) has an FM Radio and the mobile \( x_2 \) has no FM Radio, there is also the possibility that the mobile \( x_3 \) has an FM Radio. Indeed, we reexamine through the website “www.smartprix.com” that the mobile \( x_3 \) (Xiaomi Redmi K20 Pro) has an FM Radio.

**Example 3.3.** Let \( X = \{ x_1, x_2, x_3, x_4 \} \) be a set of four investments at the disposal of the investor to invest some money and \( P = \{ p_1 = \text{riskless}, p_2 = \text{security}, p_3 = \text{tax free}, p_4 = \text{short period} \} \) a set of parameters. For parameter subset \( Q_1 = \{ p_1, p_2, p_3 \} \), an investor can create the following three-valued soft set over \( X \):

\[
T_{Q_1} = \{ (p_1, \{ x_1(1), x_2(0), x_3(1), x_4(1) \}), (p_2, \{ x_1(0), x_2(1), x_3(0), x_4(1) \}), (p_3, \{ x_1(1), x_2(0), x_3(1), x_4(0) \}) \}.
\]

The element \( (p_1, \{ x_1(1), x_2(0), x_3(1), x_4(1) \}) \) in \( T_{Q_1} \), means that:

- The investments \( x_1 \) and \( x_3 \) are risk-free;
- The investment \( x_2 \) is risky; and
- The investment \( x_4 \) is “undetermined” in terms of risk.

As shown in the above example, if \( p_j \in P \setminus Q_1 \), the pair \( (p_j, t_{Q_1}(p_j)) \) does not need to be displayed in the structure of the three-valued soft set \( T_{Q_1} \). However, it is known that \( (p_j, \{ x_i(0) : \forall x_i \in X \}) \).

Each three-valued soft set can be represented in the form of a binary table. This representation makes three-valued soft sets useful in different computer program languages as well as the practicality of calculations.

The binary tabular form of three-valued soft set \( T_{Q_1} \) in Example 3.3 is presented in Table 2.

<table>
<thead>
<tr>
<th>( X / P )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( \frac{1}{7} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

- \( x_{ij} = 1 \) means that \( x_i \) belongs to the subset of \( X \) approximated by the parameter \( p_j \);
- \( x_{ij} = 0 \) means that \( x_i \) does not belong to the subset of \( X \) approximated by the parameter \( p_j \); and
- \( x_{ij} = \frac{1}{7} \) means that it is undetermined whether \( x_i \) belongs to the subset of \( X \) approximated by the parameter \( p_j \).

From now on, in the examples, the three-valued soft sets will be represented by the binary tables.

**Definition 3.2.** Let \( T_Q \in TVSS(X, P) \). It is called:

a) An empty three-valued soft set when \( t_Q(p_j) = \{ x_i(0) : \forall x_i \in X \} \) for all \( p_j \in P \) and it is denoted by \( T_\emptyset \);

b) A \( Q \)-mid three-valued soft set when \( t_Q(p_j) = \{ x_i(\frac{1}{2}) : \forall x_i \in X \} \) for all \( p_j \in Q \), and it is denoted by \( T_Q \). If \( Q = P \), the \( Q \)-mid three-valued soft set is called a mid three-valued soft set and it is denoted by \( T_P \);

c) A \( Q \)-universal three-valued soft set when \( t_Q(p_j) = \{ x_i(1) : \forall x_i \in X \} \) for all \( p_j \in Q \), and it is denoted by \( T_{Q^*} \). If \( Q = P \), the \( Q \)-universal three-valued soft set is called a universal three-valued soft set, and it is denoted by \( T_P \).

**Definition 3.3.** Let \( T_{Q_1}, T_{Q_2} \in TVSS(X, P) \), then, we have:

a) \( T_{Q_1} \) is a three-valued soft subset of \( T_{Q_2} \) when \( t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j) \) for all \( p_j \in P \) and it is denoted by \( T_{Q_1} \subseteq T_{Q_2} \). Here, \( t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j) \) for \( p_j \in P \) means \( \lambda_t^{i_{Q_1}(p_j)} \leq \lambda_t^{i_{Q_2}(p_j)} \) for each \( x_i \in X \).

b) \( T_{Q_1} \) and \( T_{Q_2} \) are both equal three-valued soft sets when \( t_{Q_1}(p_j) = t_{Q_2}(p_j) \) for all \( p_j \in P \), denoted by \( T_{Q_1} = T_{Q_2} \). Here, \( t_{Q_1}(p_j) = t_{Q_2}(p_j) \) for \( p_j \in P \) means that \( \lambda_t^{i_{Q_1}(p_j)} = \lambda_t^{i_{Q_2}(p_j)} \) for each \( x_i \in X \).

**Example 3.4.** Let us consider the three-valued soft set \( T_{Q_1} \) given in Table 2 of Example 3.3. In addition, the three-valued soft set \( T_{Q_1} \) is shown in Table 3. Then, \( T_{Q_1} \subseteq T_{Q_1} \).

**Proposition 3.1.** Let \( T_{Q_1}, T_{Q_2}, T_{Q_3} \in TVSS(X, P) \).
Table 3. The tabular form of $T_{Q_1}$ for $Q_2 = \mathcal{P}$.

<table>
<thead>
<tr>
<th>$\mathcal{X}/\mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

i. $T_\emptyset \subseteq \overline{T_\mathcal{P}} \subseteq \overline{\mathcal{P}}$;

ii. $T_\emptyset \subseteq T_{Q_k}$ for each $k$;

iii. $T_{Q_k} \subseteq \overline{\mathcal{P}}$;

iv. $T_{Q_k} \subseteq T_{Q_k}$ for each $k$;

v. $T_{Q_1} \subseteq T_\emptyset$, and $T_{Q_1} \subseteq T_{Q_3} \Rightarrow T_{Q_1} \subseteq T_{Q_3}$.

**Proof.** The proofs are obvious according to Definitions 3.2 and 3.3, hence they are omitted.\[\square\]

3.2. Operations and products on three-valued soft sets

**Definition 3.4.** Let $T_Q \in \mathcal{T}\mathcal{V}\mathcal{S}(\mathcal{X}, \mathcal{P})$. Then the complement of $T_Q$, denoted by $\overline{T_Q}$, is defined by the mapping $\overline{t_Q} : \mathcal{P} \rightarrow \mathcal{X}^{\{0, \frac{1}{2}, 1\}}$ such that:

$$\overline{t_Q}(p_j) = \left\{ \begin{array}{ll}
\{ \lambda_{Q(p_j)}^i : x_i \in \mathcal{X} \text{ and } \\
\lambda_{Q(p_j)}^i \in \{0, \frac{1}{2}, 1\} \}
\end{array} \right.,
$$

for all $p_j \in \mathcal{P}$ where:

$$\lambda_{Q(p_j)}^i = 1 - \lambda_{Q(p_j)}^i. \tag{1}$$

**Note:** This definition clarifies why the truth value for “undetermined” is $\frac{1}{2}$. The negation of “undetermined” must also have the same truth value because there is no gauge of “undetermined”. Accordingly, we argue that “undetermined” implies something in the middle between true and false, whose truth value is $\frac{1}{2}$.

**Example 3.5.** We consider the universal set $\mathcal{X}$ and parameter set $\mathcal{P}$ in Example 3.3. Furthermore, we generated the three-valued soft set $T_{Q_1}$, given in Table 4.

The complement of $T_{Q_1}$ is obtained and shown in Table 5.

**Proposition 3.2.** Let $T_Q \in \mathcal{T}\mathcal{V}\mathcal{S}(\mathcal{X}, \mathcal{P})$.

i. $\overline{(T_Q)} = T_Q$;

ii. $\overline{T_\emptyset} = \overline{\mathcal{P}}$;

iii. $\overline{T_\mathcal{P}} = \overline{\mathcal{P}}$.

**Proof.** (i) We consider three-valued soft set $T_Q = (t_Q, \mathcal{P})$ over $\mathcal{X}$. Then, we have the mapping $t_Q : \mathcal{P} \rightarrow \mathcal{X}^{\{0, \frac{1}{2}, 1\}}$ so that for all $p_j \in \mathcal{P}$ we obtain the equation shown in Box III. Based on Definition 3.4, we can write for all $p_j \in \mathcal{P}$ the equation shown in Box IV. When proceeding in a similar manner, for all $p_j \in \mathcal{P}$ we...
obtained the equation shown in Box V. Therefore, we have \( T Q(p_j) = t_Q(p_j) \) for all \( p_j \in P \). So, \( T Q = T Q \).

Proofs (ii) and (iii) are obvious, hence they are omitted. \( \square \)

**Definition 3.5.** Let \( T Q_1, T Q_2 \in TVSS(\mathcal{X}, P) \). Then, the intersection of \( T Q_2 \) and \( T Q_2 \), denoted by \( T Q_1 \cap T Q_2 \), is defined by the mapping \( t_{Q_1, Q_2} : P \to \mathcal{X}^{(0,1)} \) such that:

\[
t_{Q_1, Q_2}(p_j) = \begin{cases} 
(\lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j) : x_i \in \mathcal{X} \\
\lambda_{Q_1}^{(i)} p_j \in \{0, \frac{1}{2}, 1\}
\end{cases},
\]

for all \( p_j \in P \), where:

\[
\lambda_{Q_1, Q_2}^{(i)} p_j = \min \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\}.
\]

**Proposition 3.3.** Let \( T Q \in TVSS(\mathcal{X}, P) \).

i. \( T Q \cap T O = T O \);

ii. \( T Q \cap T Q = T Q \);

iii. \( T Q \cap T Q = T Q \).

**Proof.** It is clear from Definitions 3.2, 3.3, and 3.5. \( \square \)

**Definition 3.6.** Let \( T Q_1, T Q_2 \in TVSS(\mathcal{X}, P) \). Then, the union of \( T Q_1 \) and \( T Q_2 \), denoted by \( T Q_1 \cup T Q_2 \), is defined by the mapping \( t_{Q_1 \cup Q_2} : P \to \mathcal{X}^{(0,1)} \) such that:

\[
t_{Q_1 \cup Q_2}(p_j) = \begin{cases} 
(\lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j) : x_i \in \mathcal{X} \\
\lambda_{Q_1 \cup Q_2}^{(i)} p_j \in \{0, \frac{1}{2}, 1\}
\end{cases},
\]

for all \( p_j \in P \), where:

\[
\lambda_{Q_1 \cup Q_2}^{(i)} p_j = \max \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\}.
\]

**Proposition 3.4.** Let \( T Q \in TVSS(\mathcal{X}, P) \).

i. \( T Q \cup T O = T Q \);

ii. \( T Q \cup T Q = T Q \);

iii. \( T Q \cup T Q = T Q \).

**Proof.** It is clear from Definitions 3.2, 3.3, and 3.6. \( \square \)

**Proposition 3.5.** Let \( T Q_1, T Q_2, T Q_3 \in TVSS(\mathcal{X}, P) \). For all \( s \in \{\cap, \cup\} \),

i. \( T Q_1 \cap T Q_2 = T Q_1 \cap T Q_2 \);

ii. \( T Q_1 \cap T Q_2 = T Q_3 \);

iii. \( T Q_1 \cap T Q_2 = T Q_3 \).

**Proof.** Proofs (i) and (ii) are similar to (iii), hence omitted. (iii) Let us prove that \( T Q_1 \cap (T Q_2 \cap T Q_3) = (T Q_1 \cap T Q_2) \cap T Q_3 \) for \( s = \cap \) and \( s = \cup \).

We consider \( T Q_1 \cap T Q_2 = T R \), where for all \( p_j \in P \):

\[
t_R(p_j) = t_{Q_1 \cap Q_2}(p_j) = \begin{cases} 
(\lambda_{Q_1}^{(i)} p_j) : x_i \in \mathcal{X} \\
\lambda_{Q_1 \cap Q_2}^{(i)} p_j \in \{0, \frac{1}{2}, 1\}
\end{cases},
\]

such that:

\[
\lambda_{Q_1 \cap Q_2}^{(i)} p_j = \lambda_{Q_1}^{(i)} p_j \lambda_{Q_2}^{(i)} p_j = \max \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\}.
\]

Assume that \( T Q_1 \cap T R = T S \), where for all \( p_j \in P \):

\[
t_S(p_j) = t_{Q_1 \cap R}(p_j) = \begin{cases} 
(\lambda_{Q_1}^{(i)} p_j) : x_i \in \mathcal{X} \\
\lambda_{Q_1 \cap R}^{(i)} p_j \in \{0, \frac{1}{2}, 1\}
\end{cases},
\]

such that:

\[
\lambda_{Q_1 \cap R}^{(i)} p_j = \min \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\}.
\]

Assume that \( \lambda_{Q_1 \cap R}^{(i)} p_j = \min \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\} \),

\[
\lambda_{Q_1 \cap R}^{(i)} p_j = \min \left\{ \lambda_{Q_1}^{(i)} p_j, \lambda_{Q_2}^{(i)} p_j \right\}.
\]
Now, we consider $(T_{Q_1} \sqcap T_{Q_2}) \cup (T_{Q_3} \sqcup T_{Q_4})$. Assume that $T_{Q_1} \sqcap T_{Q_2} = T_U$, where for all $p_j \in \mathcal{P}$:

$$t_U(p_j) = t_{Q_1} \rho_{Q_2}(p_j) = \left\{ x_i^{(\lambda_{t_U(p_j)}^i)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_U(p_j)}^i = \lambda_{t_{Q_1}, \rho_{Q_2}}^i \in \left\{ 0, \frac{1}{2}, 1 \right\} \right\},$$

such that:

$$\lambda_{t_U(p_j)}^i = \lambda_{t_{Q_1}, \rho_{Q_2}}^i = \min \left\{ \lambda_{t_{Q_1}, \rho_{Q_2}}^i, \lambda_{t_{Q_2}, \rho_{Q_2}}^i \right\}. \quad (6)$$

Assume that $T_{Q_1} \sqcap T_{Q_4} = T_V$, where for all $p_j \in \mathcal{P}$:

$$t_V(p_j) = t_{Q_1} \rho_{Q_4}(p_j) = \left\{ x_i^{(\lambda_{t_V(p_j)}^i)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_V(p_j)}^i = \lambda_{t_{Q_1}, \rho_{Q_4}}^i \in \left\{ 0, \frac{1}{2}, 1 \right\} \right\},$$

such that:

$$\lambda_{t_V(p_j)}^i = \lambda_{t_{Q_1}, \rho_{Q_4}}^i = \min \left\{ \lambda_{t_{Q_1}, \rho_{Q_4}}^i, \lambda_{t_{Q_2}, \rho_{Q_4}}^i \right\}. \quad (7)$$

Suppose that $T_U \sqcup T_V = T_W$, where for all $p_j \in \mathcal{P}$:

$$t_W(p_j) = t_{U \cup V}(p_j) = \left\{ x_i^{(\lambda_{t_W(p_j)}^i)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_W(p_j)}^i = \lambda_{t_{U \cup V}(p_j)}^i \in \left\{ 0, \frac{1}{2}, 1 \right\} \right\},$$

such that:

$$\lambda_{t_W(p_j)}^i = \lambda_{t_{U \cup V}(p_j)}^i = \max \left\{ \lambda_{t_{U \cup V}(p_j)}^i, \lambda_{t_{W}(p_j)}^i \right\} = \max\left\{ \min\left\{ \lambda_{t_{Q_1}, \rho_{Q_2}}^i, \lambda_{t_{Q_2}, \rho_{Q_2}}^i \right\}, \min\left\{ \lambda_{t_{Q_1}, \rho_{Q_4}}^i, \lambda_{t_{Q_2}, \rho_{Q_4}}^i \right\} \right\}. \quad (8)$$

Since $\lambda_{t_{Q_1}, \rho_{Q_2}}^i, \lambda_{t_{Q_2}, \rho_{Q_2}}^i, \lambda_{t_{Q_1}, \rho_{Q_4}}^i, \lambda_{t_{Q_2}, \rho_{Q_4}}^i \in \left\{ 0, \frac{1}{2}, 1 \right\}$, we have:

$$\lambda_{t_{U \cup V}(p_j)}^i = \lambda_{t_{U \cup V}(p_j)}^i,$$

for all $p_j \in \mathcal{P}$ (by Eqs. (5) and (8)). Therefore, it can be concluded that $T_S$ and $T_W$ are indeed the same set-valued mappings, and $T_{Q_1} \sqcap T_{Q_2} = (T_{Q_2} \sqcup T_{Q_4}) \cup (T_{Q_2} \sqcup T_{Q_3})$.

Other cases can be proved in a similar way. □

**Proposition 3.6.** Let $T_{Q_1}, T_{Q_2} \in T^3 S(\mathcal{X}, \mathcal{P})$. Then, the following De Morgan’s rules are held:

i. $(T_{Q_1} \sqcap T_{Q_2}) = T_{Q_1} \cup T_{Q_2}$

ii. $(T_{Q_1} \cup T_{Q_2}) = T_{Q_1} \sqcap T_{Q_2}$

**Proof.** (i) Since $1 - \min\left\{ \lambda_{t_{Q_1}, \rho_{Q_2}}^i, \lambda_{t_{Q_2}, \rho_{Q_2}}^i \right\} = \max\left\{ 1 - \lambda_{t_{Q_2}, \rho_{Q_2}}^i, 1 - \lambda_{t_{Q_1}, \rho_{Q_2}}^i \right\}$ for $\lambda_{t_{Q_1}, \rho_{Q_2}}^i, \lambda_{t_{Q_2}, \rho_{Q_2}}^i \in \left\{ 0, \frac{1}{2}, 1 \right\}$, we can say that $(T_{Q_1} \sqcap T_{Q_2}) = T_{Q_1} \cup T_{Q_2}$. (ii) It is similar to the proof of (i). □

**Definition 3.7.** Let $T_{Q_1}, T_{Q_2} \in T^3 S(\mathcal{X}, \mathcal{P})$. Then, the difference between $T_{Q_1}$ and $T_{Q_2}$, denoted by $T_{Q_1} \setminus T_{Q_2}$, is defined by the mapping $t_{Q_1} \setminus T_{Q_2} : \mathcal{P} \rightarrow \mathcal{X}((0, \frac{1}{2}, 1))$ such that:

$$t_{Q_1 \setminus Q_2}(p_j) = \left\{ x_i^{(\lambda_{t_{Q_1 \setminus Q_2}(p_j)}^i)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_{Q_1 \setminus Q_2}(p_j)}^i = \lambda_{t_{Q_1 \setminus Q_2}(p_j)}^i \in \left\{ 0, \frac{1}{2}, 1 \right\} \right\},$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda_{t_{Q_1 \setminus Q_2}(p_j)}^i = \min\left\{ \lambda_{t_{Q_1 \setminus Q_2}(p_j)}^i, 1 - \lambda_{t_{Q_2}(p_j)}^i \right\}. \quad (9)$$

**Definition 3.8.** Let $T_{Q_1}, T_{Q_2} \in T^3 S(\mathcal{X}, \mathcal{P})$. Then, the symmetric difference between $T_{Q_1}$ and $T_{Q_2}$, denoted by $T_{Q_1} \triangle T_{Q_2}$, is defined by the mapping $t_{Q_1 \triangle Q_2} : \mathcal{P} \rightarrow \mathcal{X}((0, \frac{1}{2}, 1))$ such that:

$$t_{Q_1 \triangle Q_2}(p_j) = \left\{ x_i^{(\lambda_{t_{Q_1 \triangle Q_2}(p_j)}^i)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_{Q_1 \triangle Q_2}(p_j)}^i = \lambda_{t_{Q_1 \triangle Q_2}(p_j)}^i \in \left\{ 0, \frac{1}{2}, 1 \right\} \right\},$$

for all $p_j \in \mathcal{P}$, where:

$$\lambda_{t_{Q_1 \triangle Q_2}(p_j)}^i = \min\left\{ \lambda_{t_{Q_1 \triangle Q_2}(p_j)}^i, \lambda_{t_{Q_2}(p_j)}^i \right\}.$$ 

**Example 3.6.** Consider three-valued soft sets $T_{Q_1}$ and $T_{Q_2}$ in Tables 2 and 4, respectively. Therefore, the difference and symmetric difference between $T_{Q_1}$ and $T_{Q_2}$ can be measured as shown in Tables 6 and 7, respectively.

**Proposition 3.7.** Let $T_{Q_1}, T_{Q_2} \in T^3 S(\mathcal{X}, \mathcal{P})$.

i. $T_{Q_1 \setminus T_{Q_2}} = T_{Q_1} \sqcap T_{Q_2}$

ii. $T_{Q_1 \triangle T_{Q_2}} = (T_{Q_1 \setminus T_{Q_2}}) \cup (T_{Q_1} \setminus T_{Q_2})$

**Proof.**
Table 6. The tabular form of $T_{Q_1 \setminus T_{Q_2}}$.

<table>
<thead>
<tr>
<th>$\mathcal{X}$ / $\mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7. The tabular form of $T_{Q_1 \triangle T_{Q_2}}$.

<table>
<thead>
<tr>
<th>$\mathcal{X}$ / $\mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(i) Assume that $T_{Q_1 \setminus T_{Q_1}} = T_R$. According to Eq. (9), for all $p_j \in \mathcal{P}$, we have:

$$\lambda^i_{n, p_j} = \min \left\{ \lambda^i_{Q_1(p_j)}, 1 - \lambda^i_{Q_1(p_j)} \right\}. \quad (11)$$

On the contrary, assume that $T_{Q_1 \triangle T_{Q_2}} = T_S$. Then, based on Definition 3.5, we have:

$$\lambda^i_{l, p_j} = \min \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_j)} \right\} = \min \left\{ \lambda^i_{Q_1(p_j)}, 1 - \lambda^i_{Q_1(p_j)} \right\}. \quad (12)$$

By Eqs. (11) and (12), we prove that the above equality is achieved.

(ii) Assume $T_{Q_1 \triangle T_{Q_1}} = T_U$. Then, by Eq. (10), we have for all $p_j \in \mathcal{P}$:

$$\lambda^i_{m, p_j} = \min \left\{ \max \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_j)} \right\}, 1 - \min \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_j)} \right\} \right\}. \quad (13)$$

With the consideration of the right side of equality, we have $T_{Q_1 \setminus T_{Q_1}} = T_V$ and $T_{Q_1 \setminus T_{Q_1}} = T_W$, where:

$$\lambda^i_{n, p_j} = \min \left\{ \lambda^i_{Q_1(p_j)}, 1 - \lambda^i_{Q_1(p_j)} \right\}. \quad (14)$$

and:

$$\lambda^i_{m, p_j} = \min \left\{ \lambda^i_{Q_2(p_j)}, 1 - \lambda^i_{Q_2(p_j)} \right\}. \quad (15)$$

For $T_Z = T_V \cap T_W$, from Definition 3.6, we obtain that:

$$\lambda^i_{z, p_j} = \max \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_j)} \right\} = \max \left\{ \min \left\{ \lambda^i_{Q_1(p_j)}, 1 - \lambda^i_{Q_1(p_j)} \right\}, \min \left\{ \lambda^i_{Q_2(p_j)}, 1 - \lambda^i_{Q_2(p_j)} \right\} \right\}. \quad (16)$$

We know that $\lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_j)} \in \{0, \frac{1}{2}, 1\}$ for all $p_j \in \mathcal{P}$, and $\lambda^i_{z, p_j} = \lambda^i_{z, p_j}$. This completes the proof $\square$

**Definition 3.9.** Let $T_{Q_1}, T_{Q_2} \in \mathcal{T}_{\mathcal{VSS}(\mathcal{X}, \mathcal{P})}$. Then, the And-product of $T_{Q_1}$ and $T_{Q_2}$, denoted by $T_{Q_1 \wedge T_{Q_2}}$, is defined by the mapping $t_{Q_1 \wedge T_{Q_2}} : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{X}^{(0, \frac{1}{2}, 1)}$ such that:

$$t_{Q_1 \wedge T_{Q_2}}(p_j, p_k) = \left\{ \begin{array}{ll}
\lambda^i_{Q_1(p_j), Q_2(p_k)} & : x_i \in \mathcal{X} \\
\lambda^i_{Q_1(p_j), Q_2(p_k)} & \in \left\{0, \frac{1}{2}, 1\right\}
\end{array} \right. \text{ for all } (p_j, p_k) \in \mathcal{P} \times \mathcal{P},$$

where:

$$\lambda^i_{Q_1(p_j), Q_2(p_k)} = \min \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_k)} \right\}. \quad (17)$$

**Definition 3.10.** Let $T_{Q_1}, T_{Q_2} \in \mathcal{T}_{\mathcal{VSS}(\mathcal{X}, \mathcal{P})}$. Then, the Or-product of $T_{Q_1}$ and $T_{Q_2}$, denoted by $T_{Q_1 \vee T_{Q_2}}$, is defined by the mapping $t_{Q_1 \vee T_{Q_2}} \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{X}^{(0, \frac{1}{2}, 1)}$ such that:

$$t_{Q_1 \wedge T_{Q_2}}(p_j, p_k) = \left\{ \begin{array}{ll}
\lambda^i_{Q_1(p_j), Q_2(p_k)} & : x_i \in \mathcal{X} \\
\lambda^i_{Q_1(p_j), Q_2(p_k)} & \in \left\{0, \frac{1}{2}, 1\right\}
\end{array} \right. \text{ for all } (p_j, p_k) \in \mathcal{P} \times \mathcal{P},$$

where:

$$\lambda^i_{Q_1(p_j), Q_2(p_k)} = \max \left\{ \lambda^i_{Q_1(p_j)}, \lambda^i_{Q_2(p_k)} \right\}. \quad (18)$$

**Proposition 3.8.** Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{T}_{\mathcal{VSS}(\mathcal{X}, \mathcal{P})}$. For all $*, \circ \in \{\wedge, \vee\}$:

i. $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3};$

ii. $T_{Q_1} \circ (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} \circ T_{Q_2}) \circ (T_{Q_2} \circ T_{Q_3});$

iii. $(T_{Q_1} \star T_{Q_2}) \circ T_{Q_3} = (T_{Q_1} \star T_{Q_1}) \star (T_{Q_1} \circ T_{Q_3}).$

**Proof.** They can be shown in a similar way to the proofs of Proposition 3.5 $\square$

**Proposition 3.9.** Let $T_{Q_1}, T_{Q_2} \in \mathcal{T}_{\mathcal{VSS}(\mathcal{X}, \mathcal{P})}$. Then, the following De Morgan’s laws are held:
i. \( (T_Q, \wedge T_{Q_i}) = T_Q \vee T_{Q_i} \);

ii. \( (T_Q, \vee T_{Q_i}) = T_Q \wedge T_{Q_i} \).

**Proof.** The proofs are similar to those of Proposition 3.6.

4. Choice value of an object in three-valued soft set(s)

In 2002, Maji et al. [32] defined the choice value of an object in a soft set and concluded that this indication could be used to prioritize the objects in the soft set during the decision-making process. In addition, the idea of choice value initiated by Maji et al. was used for the three-valued soft sets.

In this part, it is taken \( J = \{1, 2, ..., |\mathcal{P}|\} \) where \(|\mathcal{P}|\) denotes the cardinality of the parameter set \( \mathcal{P} \).

**Definition 4.1.** Let \( \mathcal{X} \) be a set of alternatives (objects).

1. The choice value of an object \( x_i \in \mathcal{X} \) in the three-valued soft set \( T_Q \) is defined and denoted by:

\[
\alpha_i = \sum_{j \in J} (x_{ij})^\xi,
\]

(19)

where \( x_{ij} \) for all \( i, j \) are the entries in the table of the three-valued soft set \( T_Q \). Further, the arbitrary number \( \xi \in \mathbb{R}^+ \) represents the overall impact coefficient of “undetermined” on the choice value.

2. The choice value of an object \( x_i \in \mathcal{X} \) in the three-valued soft sets \( T_{Q_k} \) for \( k = 1, 2, ..., s \) is defined and denoted by:

\[
\alpha_i = \frac{\sum_{j \in J} (x_{ij}^0)^\xi + (x_{ij}^1)^\xi}{2},
\]

(20)

where \( x_{ij}^0 \) and \( x_{ij}^1 \) for all \( i, j \) are the entries in the tables of three-valued soft sets \( \prod_{k=1}^{s} T_{Q_k} \) and \( \bigcup_{k=1}^{s} T_{Q_k} \), respectively. Also, the arbitrary number \( \xi \in \mathbb{R}^+ \) is the overall impact coefficient of “undetermined” on the choice value.

**Remark:** It is clear that for \( x_{ij} = \frac{1}{2} \), \( (x_{ij})^\xi \to 1 \) when \( \xi \to 0 \) and \( (x_{ij})^\xi \to 0 \) when \( \xi \to +\infty \). Take Figure 2 as an example. In case of minor modification, it is more appropriate to consider \( 0 < \xi < 1 \). However, if the modification is major, and should consider \( \xi \in (1, +\infty) \).

**Algorithm 1: Selection**

**Step 1.** Choose feasible subsets \( Q_k \) \( (k = 1, 2, ..., s) \) of the parameter set \( \mathcal{P} \);

**Step 2.** Create the three-valued soft sets \( T_{Q_k} \) for parameter subsets \( Q_k \) \( (k = 1, 2, ..., s) \);

**Step 3.** Specify the overall impact coefficient of “undetermined” on the choice value, i.e., \( \xi \in \mathbb{R}^+ \);

**Step 4.**

- If \( k > 1 \), obtain the intersection and union of three-valued soft sets \( T_{Q_k} \) \( (k = 1, 2, ..., s) \);
- If \( k = 1 \), skip to Step 4.

**Step 5.** Calculate \( \alpha_i \) for all \( i \)’s;

**Step 6.** Find \( l \), for which \( \alpha_l = \max \alpha_i \).

Then, \( x_l \) is the optimal choice object. If \( l \) has more than one value, any one of them could be chosen.

**Example 4.1.** As an implementation of Algorithm 1, we attempt to solve our numerical problem in Example 3.3. According to Tables 8 and 9, we have max \( \alpha_i = \alpha_4 \) for \( \xi = 1, \frac{1}{2}, \frac{3}{2}, \sqrt{13}, 10 \); then, the investment \( x_4 \) is an

| Table 8. The choice value for three valued soft set \( T_{Q_1} \) in Example 3.3. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( x / \mathcal{P} \) | \( p_1 \) | \( p_2 \) | \( p_3 \) | \( p_4 \) | \( \xi = 1 \) | \( \xi = \frac{1}{2} \) | \( \xi = \frac{3}{2} \) | \( \xi = \sqrt{13} \) | \( \xi = 10 \) |
| \( x_1 \) | 1 | 0 | 1 | 0 | 1 | 2 | 2 | 2 | 2 |
| \( x_2 \) | 0 | 1 | 1 | 1 | 0.5 | 1.8108 | 1.3335 | 1.0824 | 1.0009 |
| \( x_3 \) | 1 | 1 | 0 | 1 | 2 | 2.0816 | 1.7077 | 1.1648 | 1.0018 |
| \( x_4 \) | 1 | 0 | 1 | 1 | 2.5 | 2.8108 | 2.3335 | 2.0824 | 2.0009 |

| Table 9. The ranking preference order of objects for three valued soft set \( T_{Q_1} \) in Example 3.3. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Ranking order of | Ranking preference | max \( \alpha_i \) | Optimal choice |
| choice value \( \alpha_i \) | order of \( x_i \) | object |
| \( \xi = 1 \) | \( \alpha_4 > \alpha_1 = \alpha_3 > \alpha_2 \) | \( x_4 > x_1 \approx x_3 > x_2 \) | \( \alpha_4 \) | \( x_4 \) |
| \( \xi = \frac{1}{2} \) | \( \alpha_4 > \alpha_1 > \alpha_3 > \alpha_2 \) | \( x_4 > x_1 > x_3 > x_2 \) | \( \alpha_4 \) | \( x_4 \) |
| \( \xi = \frac{3}{2} \) | \( \alpha_4 > \alpha_1 > \alpha_3 > \alpha_2 \) | \( x_4 > x_1 > x_3 > x_2 \) | \( \alpha_4 \) | \( x_4 \) |
| \( \xi = \sqrt{13} \) | \( \alpha_4 > \alpha_1 > \alpha_3 > \alpha_2 \) | \( x_4 > x_1 > x_3 > x_2 \) | \( \alpha_4 \) | \( x_4 \) |
| \( \xi = 10 \) | \( \alpha_4 > \alpha_1 > \alpha_3 > \alpha_2 \) | \( x_4 > x_1 > x_3 > x_2 \) | \( \alpha_4 \) | \( x_4 \) |
optimal choice to invest some money.

**Example 4.2.** We consider three-valued soft sets $T_{Q_1}$ in Example 3.3 and $T_{Q_1}$ in Example 3.5. Then, we should make a common decision based on the data in these two three-valued soft sets for $\xi = 2$. We obtain the intersection and union of $T_{Q_1}$ and $T_{Q_1}$, as in Tables 10 and 11, respectively.

According to Tables 10 and 11, we have $\max \alpha_i = \alpha_1 = \alpha_4 = 2.125$. It can be concluded that any of the investments $x_1$ and $x_4$ can be an optimal choice to invest some money.

In the decision-making process, all parameters of a parameter set may not be of equal importance. In such cases, weights can be imposed on the choice parameters; in other words, there is a weight $\omega_j \in (0, 1]$ corresponding to each parameter $p_j \in Q$. If $p_j \in \mathcal{P} - Q$, we know that $\omega_j = 0$. Generally, the total weight is $\sum_j \omega_j = 1$. Now, let us describe the weighted choice value of an object in the structures of (weighted) three-valued soft sets.

**Definition 4.2.** Let $\mathcal{X}$ be a set of alternatives (objects). Also, $T_{Q_1}$ is a (weighted) three-valued soft set over $\mathcal{X}$:

1. The weighted choice value of an object $x_i \in \mathcal{X}$ in the (weighted) three-valued soft set $T_{Q_1}$ is defined and denoted by:

$$
\alpha_i^w = \sum_{j \in J} \omega_j \times (x_{ij})^\xi,
$$

(21)

where the arbitrary number $\xi \in \mathbb{R}^+$ represents the overall impact coefficient of “undetermined” on the choice value. Also, $\omega_j$ denotes the weight corresponding to each parameter $p_j$ in the structure of three-valued soft set $T_{Q_1}$.

2. The weighted choice value of an object $x_i \in \mathcal{X}$ in the (weighted) three-valued soft sets $T_{Q_k}$ for $k = 1, 2, \ldots, s$ is defined and denoted by:

$$
\alpha_i^w = \sum_{j \in \mathcal{J}} \omega_j \times (x_{ij})^\xi,
$$

(22)

where the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of “undetermined” on the choice value. Also:

$$
\omega_{j}^{\mathcal{P}} = \sum_{k=1}^{s} \omega_{j}^{k},
$$

(23)

where $\omega_{j}^{k}$ indicates the weight corresponding to each parameter $p_j$ in the structure of three-valued soft set $T_{Q_k}$.

**Algorithm 2: Selection by imposing weights on parameters**

**Step 1.** Choose the feasible subsets $Q_k$ ($k = 1, 2, \ldots, s$) of the parameter set $\mathcal{P}$ and determine its weights (i.e., $\omega_k^\mathcal{P}$) for each subsets $Q_k$.

**Step 2.** Create the (weighted) three-valued soft sets $T_{Q_k}$ for the parameter subsets $Q_k$ ($k = 1, 2, \ldots, s$).

**Step 3.** Specify the overall impact coefficient of “undetermined” on the choice value, i.e., $\xi \in \mathbb{R}^+$.

**Step 4.**
- If $k > 1$, obtain the intersection and union of three-valued soft sets $T_{Q_k}$ ($k = 1, 2, \ldots, s$), and
- If $k = 1$, skip to Step 4.

**Step 5.** Calculate $\alpha_i^w$ for all $i$’s;

**Step 6.** Find $l$ for which $\alpha_i^w = \max \alpha_i^w$. Then, $x_l$ is the optimal choice object. If $l$ has more than one value, any one of them could be chosen.

**Example 4.3.** The numerical problem proposed in Example 3.3 was taken into consideration. Also, the following weights were measured for the parameters of $Q_1$: $\omega_1^1 = 0.6$ for the parameter $p_1 = \text{high returns}$, $\omega_2^1 = 0.3$ for the parameter $p_2 = \text{low risk}$ and $\omega_3^1 = 0.1$ for the parameter $p_3 = \text{high security}$. Since $p_4 \notin Q_1$, the weight of parameter $p_4 \in \mathcal{P}$ can be considered ‘U’.

According to Table 12, we have $\max \alpha_i^w = \alpha_3^w$ (for $\xi = 1, \frac{1}{2}$, 3). Then, the optimal choice is $x_3$.

**Example 4.4.** Consider the weighted three-valued soft sets $T_{Q_1}$ in Example 4.3. Also, we take the following parameter weights for $T_{Q_1}$ in Table 4: $\omega_1^1 = \omega_4^1 = 0.2$ and $\omega_2^1 = \omega_3^1 = 0.3$. For $\xi = 1$, we obtain weighted choice values of alternatives $x_i$ ($i = 1, 2, 3, 4$) as $\alpha_i^1 = 0.65$, $\alpha_i^2 = 0.2$, $\alpha_i^3 = 0.6$, $\alpha_i^4 = 0.6$. Since max $\alpha_i^w = \alpha_3^w = 0.65$, the optimal choice is $x_3$.

In Algorithms 1 and 2, for each decision-maker, the overall impact coefficient ($\xi$) of “undetermined” on the choice value is taken the same. These two algorithms cannot be used if each decision-maker selects the impact coefficient of “undetermined” differently. To address these shortcomings, we will create new decision-making algorithms.
5. Three-valued soft decision-making model based on TOPSIS

This section primarily focuses on the TOPSIS, which produces satisfactory results during decision-making. Here, this technique was rebuilt to deal with the multi-criteria-group decision-making problems based on three-valued soft sets, thus constructing a novel decision-making model.

TOPSIS is an approach to solving multi-criteria decision-making problems based on a decision maker. In 2007, Shih et al. [47] extended this method for group decision-making. The operations in the process of TOPSIS include decision matrix normalization, distance measures, and aggregation operators [47]. Generally, a decision matrix is required prior to the beginning of the process. As a result of this process, the output data are interpreted so that the ranking order of alternatives can be obtained. In summary, the TOPSIS approach is a practical and useful method for ranking and selecting a number of externally determined alternatives through distance measures. The main procedure of TOPSIS is given in a series of steps (see [48-51]).

Now, a multi-criteria group decision-making model is proposed using the TOPSIS on three-valued soft sets.

Algorithm 3: TOPSIS based three-valued soft sets

**Step 1.** The multi-criteria group decision-making problem is identified. In this step, decision makers (experts), alternatives, and choice parameters are determined. Suppose that $DM = \{E_k : k \in I_s = \{1, 2, \ldots, s\}\}$ is a set of decision makers (experts) and $E_k$ denotes the $k$th decision maker (expert). Also, $x_i$ ($i \in I_m = \{1, 2, \ldots, m\}$) denotes the $i$th alternative, and $p_j$ ($j \in I_n = \{1, 2, \ldots, n\}$) represents the $j$th parameter (criterion or attribute).

Considering these data, each decision maker $E_k$ ($k \in I_s$) create three-valued soft set $T_{Q_k}$ and measures the weights of parameters as $\omega_j^k$ ($j \in I_n$) satisfying the condition $\sum_{j=1}^{n} \omega_j^k = 1$.

Moreover, each decision maker $E_k$ specifies the impact coefficient of “undetermined” in decision-making, i.e., $\xi_k$.

**Step 2.** For each decision maker $E_k$, the decision matrix $D^k$ is constructed and represented as follows:

$$\begin{align*}
D^k &= \begin{bmatrix}
    d_{11}^k & d_{12}^k & \cdots & d_{1n}^k \\
    d_{21}^k & d_{22}^k & \cdots & d_{2n}^k \\
    \vdots & \vdots & \ddots & \vdots \\
    d_{m1}^k & d_{m2}^k & \cdots & d_{mn}^k
\end{bmatrix} = [d_{ij}^k]_{m \times n},
\end{align*}$$

where $d_{ij}^k = (x_{ij}^k)^{\xi_k}$ that $x_{ij}^k$ for all $i, j$ are the entries in the table of three-valued soft set $T_{Q_k}$.

**Step 3.** After constructing the decision matrices, these are normalized (standardized).

For each decision matrix $D^k$, the normalized decision matrix $R^k$ is constructed and expressed as follows:

$$\begin{align*}
R^k &= \begin{bmatrix}
    r_{11}^k & r_{12}^k & \cdots & r_{1n}^k \\
    r_{21}^k & r_{22}^k & \cdots & r_{2n}^k \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{m1}^k & r_{m2}^k & \cdots & r_{mn}^k
\end{bmatrix} = [r_{ij}^k]_{m \times n},
\end{align*}$$

where:

$$r_{ij}^k = \begin{cases}
    \frac{d_{ij}^k}{\sqrt{\sum_{j=1}^{n} d_{ij}^k}}, & \text{if } d_{ij}^k \neq 0 \\
    0, & \text{if } d_{ij}^k = 0
\end{cases}$$

for all $k \in I_s, i \in I_m$, and $j \in I_n$.

**Step 4.** Given different weights of parameters for each decision maker, the weighted normalized decision matrix is calculated by multiplying the weights of evaluation parameters by values in the normalized decision matrix.

For each normalized decision matrix $R^k$, the weighted normalized decision matrix $V^k$ is created as follows:

$$\begin{align*}
V^k &= \begin{bmatrix}
    v_{11}^k & v_{12}^k & \cdots & v_{1n}^k \\
    v_{21}^k & v_{22}^k & \cdots & v_{2n}^k \\
    \vdots & \vdots & \ddots & \vdots \\
    v_{m1}^k & v_{m2}^k & \cdots & v_{mn}^k
\end{bmatrix} = [v_{ij}^k]_{m \times n},
\end{align*}$$

**Table 12.** The weighted choice value for three valued soft set $T_{Q_1}$ in Example 3.3

<table>
<thead>
<tr>
<th>$\chi/P$</th>
<th>$p_1(\omega_1^1 = 0.6)$</th>
<th>$p_2(\omega_2^1 = 0.3)$</th>
<th>$p_3(\omega_3^1 = 0.1)$</th>
<th>$p_4(\omega_4^1 = 0)$</th>
<th>$\alpha_i^2 = \sum_{j=1}^{n} \omega_j^1 \times (x_{ij}^1)^{\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{7}$</td>
<td>0</td>
<td>0.35</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>$\frac{1}{7}$</td>
<td>1</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{7}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.7</td>
</tr>
</tbody>
</table>
where:
\[ v_{ij}^k = w_i^k \times r_{ij}^k, \]
for all \( k \in \mathcal{I}_k, i \in \mathcal{I}_m \) and \( j \in \mathcal{I}_n. \)

**Step 5.** By combining the weighted normalized decision matrices \( V^k (k \in \mathcal{I}_k) \), the average weighted normalized decision matrix \( V \) can be obtained.

The structure of the matrix \( V \) is expressed as follows:
\[
V = \begin{bmatrix}
  v_{11} & v_{12} & \cdots & v_{1n} \\
  v_{21} & v_{22} & \cdots & v_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  v_{m1} & v_{m2} & \cdots & v_{mn}
\end{bmatrix} = [v_{ij}]_{m \times n},
\]
where for all \( i \in \mathcal{I}_m \) and \( j \in \mathcal{I}_n.\)
\[
v_{ij} = v_{ij}^1 \oplus v_{ij}^2 \oplus \cdots \oplus v_{ij}^k.
\]
In other words, the value \( v_{ij} \) is obtained by combining the weighted normalized decision value \( v_{ij}^k \) through an operation \( \oplus. \) Here, the operation \( \oplus \) can offer many choices: arithmetic mean, geometric mean, harmonic mean and their modification.

**Note:** In this study, we will take the arithmetic mean of all individual measures.

**Step 6.** The positive and negative ideal solutions \( V^\top \) and \( V^\perp \) are determined using the average weighted normalized decision matrix \( V \).

In the TOPSIS approach, the parameters (criteria or attributes) are evaluated in terms of benefit (cf. Example 3.3) and cost (cf. Example 3.1). Suppose that \( \mathcal{J}_1 \) and \( \mathcal{J}_2 \) are the sets of benefit and cost parameters, respectively, where \( \mathcal{J}_1 \cap \mathcal{J}_2 = \phi \) and \( \mathcal{J}_1 \cup \mathcal{J}_2 = \{1, 2, \ldots, n\}. \) \( V^\top \) and \( V^\perp \) are described as follows:

- \( V^\top \) is the set which shows that the most suitable alternative for each parameter may be preferred (PIS). This set is obtained and shown in the following:
\[
V^\top = \{v^\top_1, v^\top_2, \ldots, v^\top_j, \ldots, v^\top_n\} = \{(\max_i v^\top_{ij} : j \in \mathcal{J}_1), (\min_i v^\top_{ij} : j \in \mathcal{J}_2) : i \in \mathcal{I}_m\}.
\]

- \( V^\perp \) is the set showing the least preferable alternative for each parameter (NIS). This set is obtained as follows:
\[
V^\perp = \{v^\perp_1, v^\perp_2, \ldots, v^\perp_j, \ldots, v^\perp_n\} = \{(\min_i v^\perp_{ij} : j \in \mathcal{J}_1), (\max_i v^\perp_{ij} : j \in \mathcal{J}_2) : i \in \mathcal{I}_m\}.
\]

**Step 7.** The separation measurements of alternatives to the ideal solutions are obtained through the Euclidean distance formula.

The separation measurement of each alternative \( x_i \) to the positive ideal solution \( V^\top \) is calculated as follows:
\[
S_i^\top = \sqrt{\sum_{j=1}^{n} (v_{ij} - v^\top_j)^2}.
\]

The separation measurement of each alternative \( x_i \) to the negative ideal solution \( V^\perp \) is calculated as follows:
\[
S_i^\perp = \sqrt{\sum_{j=1}^{n} (v_{ij} - v^\perp_j)^2}.
\]

Here, \( S_i^\top \) and \( S_i^\perp \) represent the distance of the alternative \( x_i \) from PIS and NIS, respectively.

**Step 8.** The relative closeness of each alternative to the ideal solutions is also calculated.

The relative closeness \( C_i \) of the alternatives \( x_i \) with respect to the ideal solutions can be expressed as:
\[
C_i = \frac{S_i^\perp}{S_i^\top + S_i^\perp}, \quad \forall i \in \mathcal{I}_m \quad (0 \leq C_i \leq 1).
\]

**Step 9.** The alternatives (objects) are ranked in order of preference.

A set of alternatives \( x_i \) can be ranked according to the descending order of the values \( C_i \).

To show the potential of the proposed approach, a real-life practice was suggested, adopted from Figure 4.

**Example 5.1.** Assume that two experts are about to determine the best mobile brand by examining the new model mobile phones presented by six different mobile phone brands. The first expert \( E_1 \) reviews each brand’s mobile phone with memory of 128 GB, and the second expert \( E_2 \) reviews each brand’s mobile phone with memory of 64 GB. Let \( X = \{x_1, x_2, x_3, x_4, x_5, x_6\} \) be a set of six different mobile phone brands. Also, the set of parameters is employed to determine the brand with the best mobile phones, which is given as \( \mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} \) where \( p_1 = \text{optical image stabilization}, p_2 = \text{quick charging}, p_3 = \text{expandable memory}, p_4 = \text{waterproof}, p_5 = \text{autofocus}, p_6 = \text{cheap}, \) and \( p_7 = \text{fingerprint sensor}. \) Each of these experts proceeds to the decision making stage after reviewing comparisons on mobile phones on a comparison-focused website (such as “www.9mobiles.com” and “www.smartprix.com”).

To deal with this problem, the steps of Algorithm 3 are followed:

**Step 1.** The experts \( E_1 \) and \( E_2 \) determine the
Table 13. The weights of choice parameters of experts $E_1$ and $E_2$.

<table>
<thead>
<tr>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
<th>$\omega_6$</th>
<th>$\omega_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>First expert ($E_1$)</td>
<td>0.22</td>
<td>0.15</td>
<td>0.13</td>
<td>0.15</td>
<td>0.13</td>
<td>0.11</td>
</tr>
<tr>
<td>Second expert ($E_2$)</td>
<td>0.24</td>
<td>0.12</td>
<td>0.12</td>
<td>0.16</td>
<td>0.14</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 14. Three valued soft sets of experts $E_1$ and $E_2$.

<table>
<thead>
<tr>
<th>Decision makers:</th>
<th>First expert ($E_1$)</th>
<th>Second expert ($E_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X/P$</td>
<td>$p_1$</td>
<td>$p_2$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_6$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Impact coefficient $\xi_k$: $\xi_1 = 1.5$, $\xi_2 = 2$

Step 2. According to Table 14, the decision matrices $D^k (k = 1, 2)$ are constructed in the equations shown in Box VI.

Step 3. For each decision matrix $D^k (k = 1, 2)$, the normalized decision matrix $N^k (k = 1, 2)$ is constructed in the equations shown in Box VII.

Step 4. For each normalized decision matrix $N^k (k = 1, 2)$, the weighted normalized decision matrix $V^k (k = 1, 2)$ is formed by the equations shown in Box VIII.

Step 5. Then, the average weighted normalized decision matrix is constructed by the equation shown in Box IX, where the operation $\oplus$ represents the arithmetic mean.

Step 6. The positive and negative ideal solutions $V^+$ and $V^-$ are determined as follows:

$$V^+ = \{v^+_1 = 0.1524, v^+_2 = 0.094, v^+_3 = 0.0764, v^+_4 = 0.0763, v^+_5 = 0.1017, v^+_6 = 0.0498, v^+_7 = 0.0568\}.$$
\[
\mathbf{g}^1 = \begin{bmatrix}
0.6489 & 0 & 0 & 0.1714 & 0.2886 & 0.5 & 0 \\
0.6489 & 0.5001 & 0.2293 & 0.485 & 0.8166 & 0.5 & 0.174 \\
0.2293 & 0 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
0.2293 & 0.5001 & 0.6489 & 0.1714 & 0.2886 & 0.5 & 0.4923 \\
0 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
0.2293 & 0.5001 & 0.2293 & 0.485 & 0 & 0.5 & 0.4923
\end{bmatrix}
\]

\[
\mathbf{g}^2 = \begin{bmatrix}
0.169 & 0.2357 & 0.5714 & 0 & 0.0174 & 0.4472 & 0 \\
0.6761 & 0 & 0.1428 & 0.5 & 0.6963 & 0.4472 & 0.4961 \\
0.169 & 0 & 0 & 0.5 & 0.6963 & 0 & 0.4961 \\
0 & 0.2357 & 0.5714 & 0 & 0 & 0.4472 & 0.124 \\
0.169 & 0 & 0 & 0.5 & 0 & 0.4472 & 0.4961 \\
0.6761 & 0.9428 & 0.5714 & 0.5 & 0 & 0.4472 & 0.4961
\end{bmatrix}
\]

**Box VII**

\[
\mathbf{v}^1 = \begin{bmatrix}
0.1427 & 0 & 0 & 0.0257 & 0.0375 & 0.055 & 0 \\
0.1427 & 0.075 & 0.0298 & 0.0727 & 0.1061 & 0.055 & 0.0191 \\
0.0504 & 0 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\
0.0504 & 0.075 & 0.0843 & 0.0257 & 0.0375 & 0.055 & 0.0541 \\
0 & 0.075 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\
0.0504 & 0.075 & 0.0298 & 0.0727 & 0 & 0.055 & 0.0541
\end{bmatrix}
\]

\[
\mathbf{v}^2 = \begin{bmatrix}
0.0405 & 0.0282 & 0.0685 & 0 & 0.0243 & 0.0447 & 0 \\
0.1622 & 0 & 0.0171 & 0.08 & 0.0974 & 0.0447 & 0.0595 \\
0.0405 & 0 & 0 & 0.08 & 0.0974 & 0 & 0.0595 \\
0 & 0.0282 & 0.0685 & 0 & 0 & 0.0447 & 0.0148 \\
0.0405 & 0 & 0 & 0.08 & 0 & 0.0447 & 0.0595 \\
0.1622 & 0.1131 & 0.0685 & 0.08 & 0 & 0.0447 & 0.0595
\end{bmatrix}
\]

**Box VIII**

and:

\[
\mathbf{v}^3 = \{v_1^3 = 0.0202, v_2^3 = 0, v_3^3 = 0.0149, v_4^3 = 0.0128, v_5^3 = 0, v_6^3 = 0, v_7^3 = 0\}.
\]

**Step 7.** The separation measurements \(s_i^a\) and \(s_i^l\) of each alternative \(x_i\) to the ideal solutions are given in Table 15.

**Step 8.** The relative closeness \(c_i^a\) of each alternative to the ideal solutions is calculated as follows:

\[
c_1^a = 0.3767, \quad c_2^a = 0.7005, \quad c_3^a = 0.399, \quad c_4^a = 0.3706, \quad c_5^a = 0.3513, \quad c_6^a = 0.589.
\]

**Step 9.** According to the descending order of

\[
\mathbf{v} = \mathbf{v}^1 \oplus \mathbf{v}^2 = \begin{bmatrix}
0.0916 & 0.0141 & 0.0342 & 0.0128 & 0.0399 & 0.0498 & 0 \\
0.1524 & 0.0375 & 0.0234 & 0.0763 & 0.1017 & 0.0498 & 0.0393 \\
0.0454 & 0 & 0.0149 & 0.0763 & 0.0674 & 0 & 0.0568 \\
0.0252 & 0.0516 & 0.0764 & 0.0128 & 0.0187 & 0.0498 & 0.0344 \\
0.0202 & 0.0375 & 0.0149 & 0.0763 & 0.0187 & 0.0223 & 0.0568 \\
0.1063 & 0.094 & 0.0491 & 0.0763 & 0 & 0.0498 & 0.0568
\end{bmatrix}
\]

\(=[v_{ij}]_{6 \times 7}\).

**Box IX**
the values $c_i^j$, the ranking order of alternatives is obtained below:

$$x_2 > x_6 > x_3 > x_1 > x_4 > x_5.$$  

Then, it can be argued that $x_2$ is the best mobile phone brand according to the data presented by experts.

6. Three-valued soft decision-making model based on ELECTRE

This section introduces a modified version of ELECTRE technique ("ELimination Et Choix Traduisant la REALité" or "Elimination and Choice Expressing Reality"). which is generally intended to output choosing, sorting, and ranking, to deal with the multi-criteria group decision-making problems based on three-valued soft sets. 

As was first applied in 1965, the ELECTRE method was employed to choose the best alternative(s) from a given set of alternatives and it was applied to three fundamental problems:

Choosing: Selecting a restricted number of the most interesting potential alternatives, as small as possible which will justify elimination of the others.

Sorting: Assigning each potential alternative to one of the categories a family previously described; the categories are ordered from the worst to best.

Ranking: Ordering alternatives from the best to worst with the possibility of ties.

The main procedure of ELECTRE is described in a series of steps (see [52–56].

Now, a multi-criteria group decision making model is constructed on three-valued soft sets using the modified ELECTRE technique.

Algorithm 4: ELECTRE based three-valued soft sets

Step 1. Describe the multi-criteria group-decision-making problem (the same as Step 1 in Algorithm 3).

Step 2. For each decision maker $E_k$, the decision matrix $D^k$ is constructed (the same as Step 2 in Algorithm 3).

Step 3. For each decision matrix $D^k$, the normalized decision matrix $R^k$ is constructed (the same as Step 3 in Algorithm 3).

Step 4. For each normalized decision matrix $R^k$, the weighted normalized decision matrix $V^k$ is created (the same as Step 4 in Algorithm 3).

Step 5. After combining the weighted normalized decision matrices $V^k$ ($k \in I_k$), the average weighted normalized decision matrix $V$ is formed (the same as Step 5 in Algorithm 3).

Step 6. The concordance sets and discordance sets are determined. The concordance set is composed of the index of all parameters for which the alternative $x_\tau$ is preferred over the alternative $x_\kappa$. This set can be described as follows:

$$J_{\tau \kappa}^+ = \{ j : v_{\tau j} > v_{\kappa j} \}.  \tag{32}$$

The discordance set contains the index of all parameters for which the alternative $x_\tau$ is worse than the alternative $x_\kappa$. This set can be described as follows:

$$J_{\tau \kappa}^- = \{ j : v_{\tau j} < v_{\kappa j} \}.  \tag{33}$$

In other words, this set can be considered as the complement of the concordance set $J_{\tau \kappa}^+$, i.e.,

$$J_{\tau \kappa}^- = J \setminus J_{\tau \kappa}^+,$$

where $J = \{ j : p_j \in P \}$.

Step 7. The concordance matrix and discordance matrix are generated by employing the sets of concordance and discordance, respectively.

The concordance matrix can be expressed as follows:

$$A = \begin{bmatrix}
- & \ldots & a_{1 \kappa} & \ldots & a_{1 m} \\
. & \ddots & \vdots & \ddots & \vdots \\
. & \ddots & \ddots & \ddots & \vdots \\
. & \ddots & \ddots & \ddots & \vdots \\
. & \ddots & \ddots & \ddots & a_{m \kappa} \\
. & \ddots & \ddots & \ddots & a_{m m} \\
\end{bmatrix}$$

$$= [a_{\tau \kappa}]_{m \times m},$$

where:
\[ a_{\tau\kappa} = \frac{\sum_{j \in J^+} \sum_{\kappa = 1}^{m} w_j^k}{\sum_{j \in J} \sum_{\kappa = 1}^{m} w_j^k}, \quad (0 \leq a_{\tau\kappa} \leq 1). \] (34)

for all \( \tau, \kappa \in I_m. \) In other words, each component of the concordance matrix is found as a summation of the (standardized) weights of all parameters corresponding to the indices in the concordance set \( J^+. \)

The discorandance matrix can be expressed as follows:

\[
B = \begin{bmatrix}
- & \cdots & b_{1\kappa} & \cdots & b_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
& \cdots & b_{\tau1} & \cdots & b_{\tau m} \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & - \\
\end{bmatrix}
\]

\[ = [b_{\tau\kappa}]_{m \times m}, \]

where:

\[ b_{\tau\kappa} = \frac{\sum_{j \in J^+} [v_{\tau j} - v_{\kappa j}]}{\sum_{j \in J} [v_{\tau j} - v_{\kappa j}]}, \quad (0 \leq b_{\tau\kappa} \leq 1). \] (35)

for all \( \tau, \kappa \in I_m. \)

**Step 8.** The concordance threshold \( A \) and discordance threshold \( B \) are found.

The concordance threshold is calculated as follows:

\[ A = \frac{\sum_{\tau = 1}^{m} \sum_{\kappa = 1}^{m} a_{\tau\kappa}}{m(m - 1)}, \quad (0 \leq A \leq 1). \] (36)

and the discordance threshold is calculated below:

\[ B = \frac{\sum_{\tau = 1}^{m} \sum_{\kappa = 1}^{m} b_{\tau\kappa}}{m(m - 1)}, \quad (0 \leq B \leq 1). \] (37)

**Step 9.** The effective concordance matrix \( F \) and effective discorandance matrix \( G \) are created.

The effective concordance matrix \( F \) is measured based on the concordance threshold \( A \) as expressed in the following:

\[
F = \begin{bmatrix}
- & \cdots & f_{1\kappa} & \cdots & f_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
f_{\tau1} & \cdots & f_{\tau\kappa} & \cdots & f_{\tau m} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
f_{m1} & \cdots & f_{m\kappa} & \cdots & - \\
\end{bmatrix}
\]

\[ = [f_{\tau\kappa}]_{m \times m}, \]

where:

\[ f_{\tau\kappa} = \begin{cases} 1, & \text{if } a_{\tau\kappa} \geq A \\ 0, & \text{if } a_{\tau\kappa} < A \end{cases} \] (38)

The effective discorandance matrix \( G \) is measured based on the discordance threshold \( B \), as expressed in the following:

\[
G = \begin{bmatrix}
- & \cdots & g_{1\kappa} & \cdots & g_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
g_{\tau1} & \cdots & g_{\tau\kappa} & \cdots & g_{\tau m} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
g_{m1} & \cdots & g_{m\kappa} & \cdots & - \\
\end{bmatrix}
\]

\[ = [g_{\tau\kappa}]_{m \times m}, \]

where:

\[ g_{\tau\kappa} = \begin{cases} 0, & \text{if } b_{\tau\kappa} > B \\ 1, & \text{if } b_{\tau\kappa} \leq B \end{cases} \] (39)

**Step 10.** The aggregated outranking matrix \( H \) is constructed. Then, the aggregated outranking matrix \( H \) is established by merging the effective concordance information with effective discorandance information.

The matrix \( H \) can be described as follows:

\[
H = \begin{bmatrix}
- & \cdots & h_{1\kappa} & \cdots & h_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
h_{\tau1} & \cdots & h_{\tau\kappa} & \cdots & h_{\tau m} \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
h_{m1} & \cdots & h_{m\kappa} & \cdots & - \\
\end{bmatrix}
\]

\[ = [h_{\tau\kappa}]_{m \times m}, \]

where:

\[ h_{\tau\kappa} = f_{\tau\kappa} \times g_{\tau\kappa}, \] (40)

for all \( \tau, \kappa \in I_m. \)

**Step 11.** The alternatives (objects) are ranked in order of preference. The components in the aggregated outranking matrix \( H \) are indicative of the dominance of any alternative over the other. Therefore, according to this priority, a choice priority among the alternatives is considered to rank the alternatives.

Given the aggregated outranking matrix \( H, \) the binary relations among the alternatives may take place as one of the following three situations:

(a) \( x_\tau \succ x_\kappa \) (i.e., \( x_\tau \) is strictly preferred over \( x_\kappa \) or \( x_\tau \) is dominant over \( x_\kappa \)) if \( h_{\tau\kappa} = 1 \) and \( h_{\kappa\tau} = 0; \)
(b) \( x_r \approx x_e \) (i.e., \( x_r \) is indifferent to \( x_e \)) if \( h_{r,e} = 1 \) and \( h_{e,r} = 1 \);
(c) \( x_r \approx x_e \) (i.e., \( x_r \) and \( x_e \) are incomparable) if \( h_{r,e} = 0 \) and \( h_{e,r} = 0 \).

Therefore, the ranking order of alternatives can be interpreted.

For the implementation of this model, a solution that follows the multi-criteria group-decision-making problem is offered.

**Example 6.1.** Supplier selection is among the most important issues in the supply chain management area. In this regard, a numerical example of a supplier selection problem adopted from [57,58] was taken into account. A high-technology company that manufactures electronic products aims to evaluate and choose a materials supplier. Assume that \( \mathcal{X} = \{x_1, x_2, x_3, x_4\} \) is a set of four suppliers chosen as candidates (alternatives). A single decision maker may not be able to accurately consider all relevant aspects during decision-making. Therefore, the company’s leader decides to put together a decision committee to determine a suitable supplier. A committee of three decision makers (experts) is established containing: 1) Financial expert (\( E_1 \)), who evaluates alternatives in terms of cost and finance; 2) Quality control expert (\( E_2 \)), who evaluates alternatives in terms of quality and safety; and 3) Engineering expert (\( E_3 \)), who evaluates alternatives in terms of engineering and technical aspects.

Six, evaluation parameters are also considered: 1) performance (\( p_1 \)), 2) cost control (\( p_2 \)), 3) management audit (\( p_3 \)), 4) service (\( p_4 \)), 5) company reputation (\( p_5 \)), and 6) quality (\( p_6 \)). To deal with this problem, the steps of Algorithm 4 are followed:

**Step 1.** Decision makers (experts) \( E_1, E_2, \) and \( E_3 \) make their decisions based on the parameter subsets \( Q_1 = \{p_1, p_2, p_3, p_6\} \), \( Q_2 = \{p_1, p_2, p_3, p_5, p_6\} \), and \( Q_3 = \{p_1, p_2, p_4, p_5, p_6\} \), respectively. In addition, they measure the weights of their choice parameters, as shown in Table 16.

Decision makers (experts) present their opinion about the truth of alternative \( x_i \) under the parameter \( p_j \) and construct Table 17.

**Step 2.** The (three-valued) decision matrices \( \mathcal{D}^k \) \((k = 1, 2, 3)\) are constructed as follows:

\[
\mathcal{D}^1 = \begin{bmatrix}
1 & 0.25 & 0 & 0 & 1 & 0 \\
0.25 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0.25 & 0 \\
0.25 & 0.25 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\mathcal{D}^2 = \begin{bmatrix}
0 & 0.25 & 1 & 0 & 0.125 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0.125 & 0 \\
0.125 & 0.125 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\mathcal{D}^3 = \begin{bmatrix}
0.25 & 0 & 0 & 0 & 0.25 & 0 \\
0 & 0.25 & 0 & 0 & 0.25 & 0 \\
1 & 0 & 0 & 0 & 0.25 & 1 \\
0 & 1 & 0 & 1 & 0.25 & 0 \\
\end{bmatrix}
\]

**Steps 3 and 4.** For the decision matrices \( \mathcal{D}^k \) \((k = 1, 2, 3)\), the normalized decision matrices \( \mathcal{N}^k \) \((k = 1, 2, 3)\) and weighted normalized decision matrices \( \mathcal{V}^k \) \((k = 1, 2, 3)\) are constructed similar to that in Steps 3 and 4 of Example 5; hence, it is omitted.

<table>
<thead>
<tr>
<th>Table 16. The weights of the decision maker’s choice parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision makers/weights</td>
</tr>
<tr>
<td>Engineering expert (( E_1 ))</td>
</tr>
<tr>
<td>Financial expert (( E_2 ))</td>
</tr>
<tr>
<td>Quality control expert (( E_3 ))</td>
</tr>
</tbody>
</table>

\[ \sum_{j=1}^{6} \omega^j_1 = 1 \]

<table>
<thead>
<tr>
<th>Table 17. The expert’s three-valued soft sets ( T_{Q_1}, T_{Q_2}, ) and ( T_{Q_3}. )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decision makers:</td>
</tr>
<tr>
<td>( \mathcal{X}_P )</td>
</tr>
<tr>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_2 )</td>
</tr>
<tr>
<td>( x_3 )</td>
</tr>
<tr>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

Impact coefficient \( \xi_k \): \( \xi_1 = 2 \), \( \xi_2 = 3 \), \( \xi_3 = 2 \).
Table 18. Sets of concordance and discordance.

<table>
<thead>
<tr>
<th></th>
<th>Concordance set ((J^+))</th>
<th></th>
<th>Discordance set ((J))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x_1)</td>
<td>(x_2)</td>
<td>(x_3)</td>
</tr>
<tr>
<td></td>
<td>{-}</td>
<td>{1,4,5}</td>
<td>{4,5}</td>
</tr>
<tr>
<td></td>
<td>{2,3,4,6}</td>
<td></td>
<td>{-}</td>
</tr>
<tr>
<td></td>
<td>{1,2,3,4,6}</td>
<td>{1,2,3,4,6}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>{2,4,6}</td>
<td>{1,2,4,5,6}</td>
<td>{4,5}</td>
</tr>
</tbody>
</table>

Step 5. Then, the average weighted normalized decision matrix is:
\[
V = V^1 \oplus V^2 \oplus V^3 = \\
\begin{bmatrix}
0.0744 & 0.0189 & 0.0586 & 0 & 0.0917 & 0 \\
0.0509 & 0.0504 & 0.116 & 0 & 0.0333 & 0.0589 \\
0.0816 & 0.1174 & 0.1747 & 0 & 0.0656 & 0.1333 \\
0.0065 & 0.0732 & 0.0363 & 0.0333 & 0.0681 & 0.0589 \\
\end{bmatrix} = [v_{ij}]_{4 \times 6},
\]
where the operation \(\oplus\) represents the arithmetic mean.

Step 6. With the consideration of the average weighted normalized decision matrix \(V\), the concordance set and discordance set are determined and presented in Table 18.

Step 7. Then, the concordance matrix and discordance matrix are respectively generated as follows:
\[
A = \begin{bmatrix}
- & 0.3333 & 0.15 & 0.55 \\
0.7 & - & 0.0333 & 0.4666 \\
0.8833 & 1 & - & 0.85 \\
0.45 & 0.75 & 0.15 & - \\
\end{bmatrix} = [a_{ij}]_{4 \times 4}, \text{ and}
\]
\[
B = \begin{bmatrix}
- & 0.6434 & 0.9314 & 0.7314 \\
0.3565 & - & 1 & 0.5719 \\
0.0685 & 0 & - & 0.1164 \\
0.2685 & 0.428 & 0.8718 & - \\
\end{bmatrix} = [b_{ij}]_{4 \times 4},
\]

Step 8. The concordance threshold and discordance threshold are calculated as \(A = 0.5263\) and \(B = 0.4989\), respectively.

Step 9. The effective concordance matrix \(\mathcal{F}\) and effective discordance matrix \(\mathcal{G}\) are created as follows:
\[
\mathcal{F} = \begin{bmatrix}
- & 0 & 0 & 1 \\
1 & - & 0 & 0 \\
1 & 1 & - & 1 \\
0 & 1 & 0 & - \\
\end{bmatrix} = [f_{ij}]_{4 \times 4}, \text{ and}
\]
\[
\mathcal{G} = \begin{bmatrix}
- & 0 & 0 & 0 \\
1 & - & 0 & 0 \\
1 & 1 & - & 1 \\
1 & 1 & 0 & - \\
\end{bmatrix} = [g_{ij}]_{4 \times 4},
\]

Step 10. Then, the aggregated outranking matrix \(\mathcal{H}\) is
\[
\mathcal{H} = \begin{bmatrix}
- & 0 & 0 & 0 \\
1 & - & 0 & 0 \\
1 & 1 & - & 1 \\
0 & 1 & 0 & - \\
\end{bmatrix} = [h_{ij}]_{4 \times 4}.
\]

Step 11. Considering the aggregated outranking matrix \(\mathcal{H}\), we obtain the following binary relations as:
- \(h_{21} = 1\) and \(h_{12} = 0 \Rightarrow x_2 \succ x_1\),
- \(h_{31} = 1\) and \(h_{13} = 0 \Rightarrow x_3 \succ x_1\),
- \(h_{22} = 1\) and \(h_{32} = 0 \Rightarrow x_3 \succ x_2\),
- \(h_{33} = 1\) and \(h_{43} = 0 \Rightarrow x_3 \succ x_4\),
- \(h_{41} = 1\) and \(h_{24} = 0 \Rightarrow x_4 \succ x_2\).

Therefore, the ranking order of alternatives is found as \(x_3 \succ x_4 \succ x_2 \succ x_1\).

7. Comparison and discussion

Algorithm 2 is more general than Algorithm 1; in other words, it is the version that takes into account parameter weights. In this section, the performances of Algorithms 2, 3, and 4 are explained and evaluated. All of these algorithms can be used to deal with multi-criteria group decision problems involving incomplete information. While each of them has a different operating philosophy, they also have one goal in common, that is, to combine the evaluations of multiple decision-makers and to propose an optimal choice. They can also offer a choice according to the assessment of only one decision-maker. While Algorithm 2 can be applied if the decision-makers determine the same impact coefficient for “undetermined”, there is no such limitation for Algorithms 3 and 4. The computational performance of each of our algorithms is critically analyzed by the experimental studies; hence, we have Table 19.

As shown in Table 19, the outputs of Algorithms 3 and 4 are the same. For many decision-making problems, the results obtained from these two algorithms either are identical or overlap each other. Moreover, all of these algorithms can be used for Examples 4.3 and 4.4, and produce the same results. For the problems in
Examples 5.1 and 6.1, the outputs of the algorithms coincide. These results support the efficiency and usefulness of the proposed algorithms.

Since the three-valued soft set is an extension of soft set, the emerging algorithms can be applied to decision-making problems based on the soft set(s). In this respect, the results of the proposed algorithms with those of some of the existing soft decision-making algorithms were compared. The details supporting this argument are presented in Table 20.

In this table, the weights of parameters in each of these problems were equally considered when making calculations in Algorithms 2, 3, and 4 (for instance, in application (Section 5) in [38], the parameter set is \( X = \{x_1, x_2, x_3\} \) and so we specify \( \omega_1 = \omega_2 = \omega_3 = \frac{1}{3} \). Also, \( \xi \) can be arbitrarily chosen in accordance with the comments on the selection of the impact coefficients mentioned above. It is clear that the arbitrary selection of \( \xi \) will not change the result(s).

As shown in Table 20, the results of our algorithms coincide with those of existing soft decision-making algorithms. For Example 5.17 in [35] and Example 3.3 in [36], the optimal choice by the algorithms proposed in [35, 36] is \( \{h_1, h_2, h_3\} \), while the optimal choice by Algorithm 3 is \( h_1 \) (where \( h_1 \succ h_2 \)). This is not a contradiction, and this is the effect of normalizing the decision matrices in the model of TOPSIS (Algorithm 3). Considering the result of Algorithm 4 for the same problems, we say that \( h_1 \) and \( h_2 \) are incomparable, while \( h_3 \) and \( h_2 \) are incomparable. This does not contradict the result that the optimal choice is \( \{h_1, h_2, h_3\} \), because what is certain is that \( h_1 \succ h_4,h_5 \), \( h_2 \succ h_4,h_5 \), \( h_3 \succ h_4 \). Consequently, the applicability of our algorithms to decision making based on both soft set and three-valued soft set demonstrates their performance range and advantages.

8. Conclusion

This study defined a three-valued soft set as a generalization of the soft set and its set-theoretic operations like intersection, union, difference, and symmetric difference. Moreover, the basic relationships concerning three-valued soft sets were described and the corresponding generalization of the operations on soft sets to these sets was highlighted. In this regard, some examples for them were provided. The algorithms supporting multi-criteria decision making for the three-valued soft set based on Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and ELimitation Ex Choice Translating (ELECTRE) techniques were formed and their outputs were compared. Thus, it was pointed out that these algorithms exhibited the applicability and efficiency of three-valued soft sets in handling the multi-criteria decision making involving uncertain or incomplete information.

We hope that this work will contribute to decision-making under uncertain and incomplete information in the context of soft sets and also will provide new ideas for future studies related to soft sets. Also, this study will motivate researchers to use three-
valued logic stems in many practical applications such as data mining, data selection, data integration, data analysis, control of production processes, and pattern evaluation. In near future, we intend to explore new operations on three-valued soft sets and their practical applications in the fields such as science, social science, medical science, environmental science, economics, and so on.

References

Biographies

Eyüp Akçetin is an Associate Professor at the Department of Accounting and Financial Management, Seydikemer School of Applied Sciences of Mugla Sıtkı Koçman University in Turkey. He received his MSc degrees in Logistics and Maritime Transport from Dokuz Eylül University in 2007 and received another MSc degree in Informatics and Computer Engineering from Sakarya University in 2014 Turkey. He received his PhD degrees in Maritime Economy and Maritime Business Management from Istanbul University in 2012 Turkey. His research interests include multi-criteria decision-making, game theory, business intelligence and data mining, gamification, system dynamics in business. He has contributed to these scientific topics with many articles, books, and book chapters.

Hişeyin Kamacı is a Research Assistant at the Mathematics Department in the Science and Arts Faculty of Yozgat Bozok University, Turkey. He received his MSc and PhD degrees in Mathematics from Bozok University, Yozgat, Turkey in 2014 and 2018, respectively. His research interests include mathematical logic, set theory, operational research, computational intelligence, and decision-making. He has many valuable publications on these issues in different scientific journals.