Three-valued soft set and its multi-criteria group decision making via TOPSIS and ELECTRE

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Abstract

The purpose of this paper is to introduce a generalization of Molodtsov’s approach to soft sets obtained by passing from the classical two-valued logic underlying those sets to a three-valued logic, where the third truth value can usually be interpreted as either non-determined (i.e., between true and false) or unknown. This extension of soft set approach allows for more intuitive and clearer representation of various decision making problems involving incomplete or uncertain information. In other words, it is a useful way to analyze soft set based multi-criteria group decision making problems under the lack of information resulting from the inability to determine the data. In this paper, we introduce the concept of three-valued soft set and its some operations and products. We propose the formulas to calculate all possible choice values for each object in the (weighted) three-valued soft sets, and thus calculate their respective decision values. By modifying the TOPSIS and ELECTRE methods to deal with multi-criteria group decision problems, three-valued soft set based decision making algorithms are constructed. To demonstrate the practicality of these algorithms, we address the examples adapted from the decision problems in real-life. Lastly, some aspects of the efficiency of the proposed algorithms are discussed with computational experiments.

Key words: Soft set, three-valued soft set, choice value, decision making, TOPSIS, ELECTRE.

1 Introduction

Classically, a logic is two-valued (Boolean) if every proposition is either false (“0”) or true (“1”). In 1930, Łukasiewicz initiated three-valued logic, a natural extension of the two-valued logic, in which there are three truth (logical) values indicating false (“0”), true (“1”) and some indeterminate third value (“1/2” as something in the middle between true and false). Thus, he pioneered the conceptual form and basic ideas of three-valued logic. By interpreting

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the intuition of Łukasiewicz three-valued logic from different perspectives, two-valued logic were extended to three-valued logic in various ways [2][3]. They differ by their choice of basic connectives, therefore also from a syntactic and proof-theoretic point of view. Although the idea of fuzzification of logic was envisaged by several researchers in the years following 1920, the concept of fuzzy logic in which the truth values of variables may be any real number between 0 and 1 both inclusive was explicitly and crisply proposed by Zadeh [6] in 1965. Fuzzy logic is based on the investigation and observation that people make decisions based on uncertain and non-numerical information. Fuzzy set, which is a generalization of the crisp set based on the two-valued logic, is mathematical mean of representing vagueness and imprecise information. This set is described by a membership function that assigns to each object a degree of membership ranging between 0 and 1. Pawlak [7] defined rough set, which can be considered as a new area of uncertainty mathematics closely related to set theory. This set is a formal approximation of a crisp set in terms of a pair of sets which give the lower and the upper approximations of the original set. The approximation spaces of rough set theory are sets with multiple memberships, while fuzzy sets are concerned with partial memberships. These sets are combined to derive the different variations such as the fuzzy rough set and the rough fuzzy set. In 1999, Molodtsov [8] described the soft set, based on two-valued logic, as a mathematical tool that deals with parametric data which are imprecise or uncertain in nature. In 2003, Maji et al. [9] published a study on the operations of soft sets, and in the later years the operational laws of the soft sets were derived [10][14]. Also, many authors described and discussed various types of soft sets such as bijective soft set [15], exclusive disjunctive soft set [16], bipolar soft set [17], inverse soft set [18][19] and so on.

The decision making, which is one of the issues involving uncertainty, is a problem that is frequently encountered in many commercial and scientific fields, even in every stage of daily life. For the deterministic modeling of decision making problems, many mathematical techniques such as TOPSIS, AHP, VIKOR, ELECTRE and PROMETHEE were developed. These mathematical techniques were adapted for decision making based the fuzzy set and its some extensions [20][26]. In addition to the fuzzy modelling of TOPSIS, ELECTRE and etc., many algorithmic solutions were proposed for decision making under fuzzy environment [27][31]. In 2002, Maji et al. [32] put forward that the soft sets can be used to solve decision making problems involving parametric data based on two-valued logic. In the following years, many soft decision algorithms were created and their applications for problems in real life were included [33][37]. Moreover, Eraslan [38] proposed a decision making procedure via classical method of TOPSIS on the soft sets. Thus, he pointed out that classical decision making techniques may be successfully applied to the soft set based decision making.

In 2008, Avron and Konikowska [39] explored the idea of describing Pawlak’s rough set utilizing three-valued logic. This paradigm presents a different perspective in the interpretation of issues containing indetermined or unknown data in many fields. As with the rough set, three-valued logic emerges in many real world scenes that are included in the scope of the soft set and it is difficult to deal with such broadly scoped issues whose third truth value is “undetermined”. In this study, we discuss a completely designed approach to soft set using three-valued logic. Thus, in practice, we aim to overcome the difficulties which include a third truth value caused by uncertain or unknown origin in addition to the truth values of two-valued logic. The present paper is motivated by the idea of proposing the notion of three-valued soft set that to try to describe soft set using three-valued logic. The fusion of three-valued logic into the soft set provides a clearer and intuitive way to explain various issues under incomplete or uncertain information. Therefore, we focus on the decision making based on this type of soft set and elaborate various algorithmic solutions.

The remainder of this paper is set out as follows. Section 2 gives an outline of soft set theory. Section 3 is devoted to the motivation to reinterpret soft sets using three-valued logic. In Section 4, we endeavor to calculate the choice value of an object in (weighted) three-valued soft sets, and accordingly propose two algorithms. In Section 5, we
create a multi-criteria group decision making algorithm based the modified TOPSIS on three-valued soft sets. In Section 6, we propose a three-valued soft decision making algorithm via the modified ELECTRE, which is based on three fundamental objectives that choosing, sorting and ranking. Also, the examples are given to analyze the performance of the algorithms emerging in these two sections. In Section 7, the matching numerical examples are solved to compare the results of the proposed algorithms and thus show that they are convincing. Final section is dedicated to conclusions and plans for further research.

2 Preliminaries

In this section, as preparation for introducing new concepts, some relevant arguments of soft set, two-valued logic and three-valued logic are given.

Let’s first talk about the soft set theory. In 1999, Molodtsov [8] introduced soft set theory as a useful way of classifying objects according to parametric data. In 2011, Çağman and Enginoğlu [40] recreated the soft sets to make their operations more practical in some cases. Maji et al. [32] put forward that the soft set could be represented in a tabular form. Thus, they showed that the soft sets are the parametric sets created using two-valued logic (i.e., 0 as false, 1 as true). Now, let us recall the definition of soft set.

Definition 2.1. (8, 40) Assume \( X^{(0,1)} \) denotes the set of all functions from \( X \) to \( \{0,1\} \). A pair \( (f_Q, \mathcal{P}) = F_Q \) is called a soft set over \( X \) when the mapping \( f_Q \) is defined by \( f_Q : \mathcal{P} \rightarrow X^{(0,1)} \), where for all \( p_j \in \mathcal{P} \)

\[
T_P = \{(p_1, \{x_1^{(0)}\}, \{x_2^{(1)}\}, \{x_3^{(0)}\}, \{x_4^{(1)}\}, \{x_5^{(0)}\}), (p_2, \{x_1^{(1)}\}, \{x_2^{(0)}\}, \{x_3^{(0)}\}, \{x_4^{(1)}\}, \{x_5^{(0)}\}), (p_3, \{x_1^{(1)}\}, \{x_2^{(1)}\}, \{x_3^{(0)}\}, \{x_4^{(1)}\}, \{x_5^{(0)}\}), (p_4, \{x_1^{(0)}\}, \{x_2^{(0)}\}, \{x_3^{(1)}\}, \{x_4^{(0)}\}, \{x_5^{(1)}\})\}.
\]

For the first pair in this soft set, we can interpret that the prices of hotels \( x_2, x_3 \) are suitable for us but the prices of hotels \( x_1, x_4, x_5 \) is not suitable. Other pairs can be interpreted similarly.

In 1930, Łukasiewicz [1] put forward (Łukasiewicz) three-valued logic extending two-valued logic (Boolean logic). Immediately after, many authors interested in the idea of three-valued logic and its operations 2, 4, 39, 41, 42. They argued that since two-valued logic (Boolean logic) is so far all kinds of scientific investigations, three-valued logic may be useful and as basic for a number of useful reasoning tasks. Boolean connectives can be extended to three-valued logic in various ways. That is, there are various ways of explicating third truth value that is different from true and false. Ciucci and Dubois [43] listed these ways as follows:
- **Possible**: this explanation was proposed by Lukasiewicz [1,2], the pioneer of three-valued logic. A proposition is “Possible” if its truth value will be only known in the future.

- **Unknown**: this explanation was proposed by Kleene [4] in the year 1952. A proposition is “Unknown” if its truth value cannot be computed for some reason (for instance, it is too time-consuming to do it).

- **Inconsistent**: the third value stands for a proposition which is both true and false, and also it is the dual of “unknown” in some sense.

- **Half-true**: this is typical of fuzzy logic [3]. The intuition is that for some propositions, truth is a importance degree. For instance, Shadowed set in [44,45] is based on the idea of turning fuzzy set into three-valued set.

- **Undefined**: this is another explanation of Kleene. The undefined state corresponds to the selection of the argument of the function outside its definition domain. A proposition is “Undefined” if its truth value involves undefined atoms.

- **Irrelevant**: the idea is that propositions are not applicable in some possible worlds.

In 1960, Skolem [5] initiated a set theory based on a certain three-valued logic. In this set theory, the variables such as \( p, q, r, \ldots \) take three values 0, \( \frac{1}{2} \), 1. We may interpret 0 as “false”, 1 as “true” and \( \frac{1}{2} \) as something in the middle between true and false, say “undetermined”. Moreover, Skolem presented a set of truth tables showing tree-valued logic operations like negation, disjunction and conjunction (for a detailed review, see [5]). In the literature, there are many truth tables illustrating tree-valued logic operations. However, this study will focus on the truth tables proposed by Skolem [5].

Three-valued logic emerges in several scenes of the real world. Three samples within the atmosphere of “undetermined” are presented in Figures 1, 2 and 3. In these figures, “?” symbolizes “undetermined”, i.e., \( \frac{1}{2} \).

Since the truth value of “undetermined” in Figures 1 and 2 will be precisely known in the future, this can be considered as “Possible”. Figure 3 can also be considered as an example for “Unknown”.

### 3 Three-valued soft sets

In this part, we discuss three-valued soft set which is the generalization of soft set obtained by passing from two-valued logic to three-valued logic.

The soft set is a set approach proposed using two-valued logic (true and false). In daily life, while evaluating alternatives according to parameters in decision making process, sometimes “undetermined” (neither true nor false, or both true and false) arises. In such decision making processes, the soft sets are insufficient. To overcome this shortcoming, the notion of three-valued soft set, which is a soft set using three-valued logic (i.e., 0 as false, 1 as true and \( \frac{1}{2} \) as “undetermined”) is created.
3.1 Three-valued soft set

Definition 3.1. Assume $\mathcal{X}^{(0, \frac{1}{2}, 1)}$ denotes the set of all functions from $\mathcal{X}$ to $\{0, \frac{1}{2}, 1\}$. A pair $(t_Q, \mathcal{P}) = T_Q$ is called a three-valued soft set over $\mathcal{X}$ when the mapping $t_Q$ is defined by $t_Q : \mathcal{P} \rightarrow \mathcal{X}^{(0, \frac{1}{2}, 1)}$, where for all $p_j \in \mathcal{P}$

$$t_Q(p_j) = \begin{cases} \{x_i^{(j)} : x_i \in \mathcal{X} \text{ and } \lambda_{t_Q(p_j)}^i \in \{0, \frac{1}{2}, 1\}\}, & \text{if } p_j \in Q \\ \{x_i^{(0)} : \forall x_i \in \mathcal{X}\}, & \text{if } p_j \notin \mathcal{P} - Q \end{cases}$$

Notation: For the parameter set $\mathcal{P}$, the set of all three-valued soft sets over $\mathcal{X}$ is denoted by $TVSS(\mathcal{X}, \mathcal{P})$.

Example 3.2. Assume $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$ is a set of multimedia features (attributes) that may be available on phones. Through the website "www.91mobiles.com" (date: 17.10.2019), we can interpret that: $\mathcal{P} = \{p_1 = \text{allergy}, p_2 = \text{fetal damage}, p_3 = \text{pharmacological effect}\}$ denotes some side effects of the drugs. One obtains the following table (Table 1) by examining the prospectus of drugs.

![Insert Table 1]

According to Table 1, the following three-valued soft set is constructed.

$$T_Q = \{(p_1, \{x_1^{(1)}, x_2^{(0)}, x_3^{(1)}, x_4^{(0)}, x_5^{(0)}\}), (p_2, \{x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(1)}, x_5^{(0)}\}), (p_3, \{x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, x_4^{(1)}, x_5^{(0)}\})\}.$$

Example 3.3. Let $\mathcal{X} = \{x_1 = \text{Vivo V17 Pro}, x_2 = \text{OnePlus 7}, x_3 = \text{Xiaomi Redmi K20 Pro}, x_4 = \text{Apple IPhone 11}\}$ be a set of four mobiles and $\mathcal{P} = \{p_1 = \text{FM Radio}, p_2 = \text{Stereo Speakers}, p_3 = \text{Loudspeaker}\}$ be a set of multimedia features (attributes) that may be available on phones. Considering the information presented on this website, as well as the pros and cons of each mobile presented on the website "www.91mobiles.com" (date: 17.10.2019), we can interpret that:

![Insert Figure 4]

According to Figure 4, we create the following three-valued soft set.

$$T_Q = \{(p_1, \{x_1^{(1)}, x_2^{(0)}, x_3^{(\frac{1}{2})}, x_4^{(0)}\}), (p_2, \{x_1^{(\frac{1}{2})}, x_2^{(1)}, x_3^{(0)}, x_4^{(1)}\}), (p_3, \{x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\})\}.$$
Example 3.4. Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$ be a set of four investments at the disposal of the investor to invest some money and $\mathcal{P} = \{p_1 = \text{riskless}, p_2 = \text{security}, p_3 = \text{tax free}, p_4 = \text{short period}\}$ be a set of parameters. For parameter subset $Q_1 = \{p_1, p_2, p_3\}$, an investor can create the following three-valued soft set over $\mathfrak{X}$.

$$T_{Q_1} = \{(p_1, \{x_1(1), x_2(0), x_3(0), x_4(1)\}), (p_2, \{x_1(0), x_2(1), x_3(1), x_4(0)\}), (p_3, \{x_1(0), x_2(0), x_3(1), x_4(1)\})\}.$$ 

The element $(p_1, \{x_1(1), x_2(0), x_3(0), x_4(1)\})$ in $T_{Q_1}$ means that
- the investments $x_1$ and $x_3$ are risk-free,
- the investment $x_2$ is risky,
- the investment $x_4$ is “undetermined” in terms of risk.

As shown in the above example, if $p_j \in \mathcal{P}\backslash Q_1$ then the pair $(p_j, t_{Q_1}(p_j))$ does not need to be displayed in the structure of three-valued soft set $T_{Q_1}$. However, it is known that $(p_j, \{x_i(0) : \forall x_i \in \mathfrak{X}\})$.

Each three-valued soft set can be represented in the form of binary table. This representation provides the usefulness of three-valued soft sets in various computer program languages and the practicality of calculations.

The binary tabular form of three-valued soft set $T_{Q_1}$ in Example 3.4 can be given as in Table 2.

[Insert Table 2]

In Table 2 each component $x_{ij}$ represents the truth value $\lambda_{iQ(p_j)}$ of alternative $x_i$ with respect to the parameter $p_j$.

That is, in Table 2 and onward,
- $x_{ij} = 1$ means that $x_i$ belongs to the subset of $\mathfrak{X}$ approximated by the parameter $p_j$,
- $x_{ij} = 0$ means that $x_i$ does not belong to the subset of $\mathfrak{X}$ approximated by the parameter $p_j$,
- $x_{ij} = \frac{1}{2}$ means that it is undetermined whether $x_i$ belongs to the subset of $\mathfrak{X}$ approximated by the parameter $p_j$.

From now on, in the examples, three-valued soft sets will be represented by the binary tables.

Definition 3.5. Let $T_{Q} \in \mathcal{T}_{\mathcal{V}SS}(\mathfrak{X}, \mathcal{P})$. It is called

a) an empty three-valued soft set when $t_{Q}(p_j) = \{x_i(0) : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in \mathcal{P}$, and it is denoted by $T_{\emptyset}$.

b) a $Q$-mid three-valued soft set when $t_{Q}(p_j) = \{x_i(\frac{1}{2}) : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\overline{Q}}$. If $Q = \mathcal{P}$ then the $Q$-mid three-valued soft set is called a mid three-valued soft set, and it is denoted by $T_{\overline{\mathcal{P}}}$.

c) a $Q$-universal three-valued soft set when $t_{Q}(p_j) = \{x_i(1) : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\widehat{Q}}$. If $Q = \mathcal{P}$ then the $Q$-universal three-valued soft set is called a universal three-valued soft set, and it is denoted by $T_{\widehat{\mathcal{P}}}$.

Definition 3.6. Let $T_{Q_1}, T_{Q_2} \in \mathcal{T}_{\mathcal{V}SS}(\mathfrak{X}, \mathcal{P})$. Then

a) $T_{Q_1}$ is a three-valued soft subset of $T_{Q_2}$ when $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for all $p_j \in \mathcal{P}$, and it is denoted by $T_{Q_1} \subseteq T_{Q_2}$.

Here, $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means $\lambda_{iQ_1(p_j)} \leq \lambda_{iQ_2(p_j)}$ for each $x_i \in \mathfrak{X}$.

b) $T_{Q_1}$ and $T_{Q_2}$ are equal three-valued soft sets when $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for all $p_j \in \mathcal{P}$, and it is denoted by $T_{Q_1} = T_{Q_2}$.

Here, $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means $\lambda_{iQ_1(p_j)} = \lambda_{iQ_2(p_j)}$ for each $x_i \in \mathfrak{X}$.
Example 3.7. Let us consider three-valued soft set $T_{Q_1}$ given in Table 2 of Example 3.4. Also, we take three-valued soft set $T_{Q_2}$ as in Table 3 respectively.

Then, it is seen that $T_{Q_2} \subseteq T_{Q_1}$.

Proposition 3.8. Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in TVSS(X, \mathcal{P})$.

i) $T_{\emptyset} \subseteq T_{\mathcal{P}} \subseteq T_{\overline{\mathcal{P}}}$.

ii) $T_{\emptyset} \subseteq T_{Q_k}$ for each $k$.

iii) $T_{Q_k} \subseteq T_{\overline{\mathcal{P}}}$.

iv) $T_{Q_k} \subseteq T_{Q_k}$ for each $k$.

v) $T_{Q_1} \subseteq T_{Q_2}$ and $T_{Q_2} \subseteq T_{Q_3} \Rightarrow T_{Q_1} \subseteq T_{Q_3}$.

Proof. The proofs are obvious from Definitions 3.5 and 3.6, therefore omitted.

3.2 Operations and products on three-valued soft sets

Definition 3.9. Let $T_Q \in TVSS(X, \mathcal{P})$. Then the complement of $T_Q$, denoted by $\overline{T_Q}$, is defined by the mapping $\overline{T_Q} : \mathcal{P} \rightarrow X^{\{0, \frac{1}{2}, 1\}}$ such that $\overline{T_Q}(p_j) = \{x_i \in X : \lambda_{\overline{T_Q}(p_j)}^i = \{0, \frac{1}{2}, 1\} \}$ for all $p_j \in \mathcal{P}$ where

$$\lambda_{\overline{T_Q}(p_j)}^i = 1 - \lambda_{T_Q(p_j)}^i.$$  \hfill (3.1)

Note: This definition clarifies why the truth value for “undetermined” is $\frac{1}{2}$. The negation of “undetermined” must also have the same truth value, because there is no gauge of “undetermined”. Accordingly, we say that “undetermined” implies something in the middle between true and false, i.e., its truth vale is $\frac{1}{2}$.

Example 3.10. We consider the universal set $X$ and the parameter set $\mathcal{P}$ in Example 3.4. Furthermore, we generate three-valued soft set $T_{Q_2}$ given in Table 4.

The complement of $T_{Q_2}$ is obtained as in Table 5.

Proposition 3.11. Let $T_Q \in TVSS(X, \mathcal{P})$.

i) $(T_Q) = T_Q$.

ii) $T_{\emptyset} = T_{\mathcal{P}}$.

iii) $T_{\overline{\mathcal{P}}} = T_{\overline{\mathcal{P}}}$. 
Proof. (i) We consider three-valued soft set \( T_Q = (t_Q, \mathcal{P}) \) over \( \mathcal{X} \). Then, we have the mapping \( t_Q : \mathcal{P} \rightarrow \mathcal{X}^\left(0, \frac{1}{2}, 1\right) \) such that for all \( p_j \in \mathcal{P} \)

\[
t_Q(p_j) = \begin{cases} 
\{ x_i^{(\lambda^i_{t_{Q_i}(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda^i_{t_{Q_i}(p_j)} \in \{0, \frac{1}{2}, 1\} \}, & \text{if } p_j \in Q \\
\{ x_i^{(0)} : \forall x_i \in \mathcal{X} \}, & \text{if } p_j \in \mathcal{P} - Q
\end{cases}
\]

By Definition 3.9 we can write for all \( p_j \in \mathcal{P} \)

\[
\overline{t_Q}(p_j) = \begin{cases} 
\{ x_i^{(1-\lambda^i_{t_{Q_i}(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda^i_{t_{Q_i}(p_j)} \in \{0, \frac{1}{2}, 1\} \}, & \text{if } p_j \in Q \\
\{ x_i^{(1-0)} : \forall x_i \in \mathcal{X} \}, & \text{if } p_j \in \mathcal{P} - Q
\end{cases}
\]

When proceeding in a similar manner, it is seen that for all \( p_j \in \mathcal{P} \)

\[
\overline{\overline{t_Q}}(p_j) = \begin{cases} 
\{ x_i^{(1-(1-\lambda^i_{t_{Q_i}(p_j)})}) : x_i \in \mathcal{X} \text{ and } \lambda^i_{t_{Q_i}(p_j)} \in \{0, \frac{1}{2}, 1\} \}, & \text{if } p_j \in Q \\
\{ x_i^{(1(1-0))} : \forall x_i \in \mathcal{X} \}, & \text{if } p_j \in \mathcal{P} - Q
\end{cases}
\]

Thus, we have \( \overline{\overline{t_Q}}(p_j) = t_Q(p_j) \) for all \( p_j \in \mathcal{P} \). So, \( \overline{\overline{T_Q}} = T_Q \).

The proofs of (ii) and (iii) are obvious, hence omitted.

\[\square\]

**Definition 3.12.** Let \( T_{Q_1}, T_{Q_2} \in \mathcal{T}VSS(\mathcal{X}, \mathcal{P}) \). Then, the intersection of \( T_{Q_1} \) and \( T_{Q_2} \), denoted by \( T_{Q_1} \cap T_{Q_2} \), is defined by the mapping \( t_{Q_1 \cap Q_2} : \mathcal{P} \rightarrow \mathcal{X}^\left(0, \frac{1}{2}, 1\right) \) such that \( t_{Q_1 \cap Q_2}(p_j) = \{ x_i^{(\lambda^i_{t_{Q_1 \cap Q_2}(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda^i_{t_{Q_1 \cap Q_2}(p_j)} \in \{0, \frac{1}{2}, 1\} \} \) for all \( p_j \in \mathcal{P} \), where

\[
\lambda^i_{t_{Q_1 \cap Q_2}(p_j)} = \min\{\lambda^i_{t_{Q_1}(p_j)}, \lambda^i_{t_{Q_2}(p_j)}\}.
\]

**Proposition 3.13.** Let \( T_Q \in \mathcal{T}VSS(\mathcal{X}, \mathcal{P}) \).

i) \( T_Q \cap T_{\emptyset} = T_{\emptyset} \).

ii) \( T_Q \cap T_{\overline{Q}} = T_Q \).

iii) \( T_Q \cap T_{\overline{\overline{Q}}} \subseteq T_Q \).

**Proof.** It is clear from Definitions 3.5, 3.6, and 3.12. \[\square\]

**Definition 3.14.** Let \( T_{Q_1}, T_{Q_2} \in \mathcal{T}VSS(\mathcal{X}, \mathcal{P}) \). Then, the union of \( T_{Q_1} \) and \( T_{Q_2} \), denoted by \( T_{Q_1} \sqcup T_{Q_2} \), is defined by the mapping \( t_{Q_1 \sqcup Q_2} : \mathcal{P} \rightarrow \mathcal{X}^\left(0, \frac{1}{2}, 1\right) \) such that \( t_{Q_1 \sqcup Q_2}(p_j) = \{ x_i^{(\lambda^i_{t_{Q_1 \sqcup Q_2}(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda^i_{t_{Q_1 \sqcup Q_2}(p_j)} \in \{0, \frac{1}{2}, 1\} \} \) for all \( p_j \in \mathcal{P} \), where

\[
\lambda^i_{t_{Q_1 \sqcup Q_2}(p_j)} = \max\{\lambda^i_{t_{Q_1}(p_j)}, \lambda^i_{t_{Q_2}(p_j)}\}.
\]

**Proposition 3.15.** Let \( T_Q \in \mathcal{T}VSS(\mathcal{X}, \mathcal{P}) \).
i) $T_Q \cup T_{Q} = T_Q$.

ii) $T_Q \cup T_{Q} = T_Q$.

iii) $T_Q \cup T_{\bar{Q}} = T_Q$.

Proof. It is clear from Definitions 3.5, 3.6 and 3.14.

Proposition 3.16. Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{T} \forall SS(X, \mathcal{P})$. For all $*, \circ \in \{\cap, \cup\}$,

i) $T_{Q_1} * T_{Q_2} = T_{Q_3}$.

ii) $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3}$.

iii) $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$.

Proof. The proof of (i) and (ii) is similar to that of (iii), so it is omitted.

(iii) Let us prove that $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$ for $* = \cap$ and $\circ = \cup$.

We consider $T_{Q_1} \cap (T_{Q_2} \cup T_{Q_3})$, where for all $p_j \in \mathcal{P}$

$$t_R(p_j) = t_{Q_2 \cup Q_3}(p_j) = \{x_i^{(\lambda^i_{t_R(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda_{t_R(p_j)}^i = \lambda_{t_{Q_2 \cap Q_3}(p_j)}^i \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda_{t_R(p_j)}^i = \lambda_{t_{Q_2 \cap Q_3}(p_j)}^i = \max\{\lambda_{t_{Q_2}(p_j)}^i, \lambda_{t_{Q_3}(p_j)}^i\}. \quad (3.4)$$

And assume that $T_{Q_1} \cap T_R = T_S$, where for all $p_j \in \mathcal{P}$

$$t_S(p_j) = t_{Q_1 \cap R}(p_j) = \{x_i^{(\lambda^i_{t_S(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda_{t_S(p_j)}^i = \lambda_{t_{Q_1 \cap \bar{Q}}(p_j)}^i \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda_{t_S(p_j)}^i = \lambda_{t_{Q_1 \cap \bar{Q}}(p_j)}^i = \min\{\lambda_{t_{Q_1}(p_j)}^i, \lambda_{t_{R}(p_j)}^i\} = \min\{\lambda_{t_{Q_1}(p_j)}, \max\{\lambda_{t_{Q_2}(p_j)}, \lambda_{t_{Q_3}(p_j)}\}\}. \quad (3.5)$$

Now, we consider $(T_{Q_1} \cap T_{Q_2}) \cup (T_{Q_1} \cap T_{Q_3})$. Assume that $T_{Q_1} \cap T_{Q_2} = T_U$, where for all $p_j \in \mathcal{P}$

$$t_U(p_j) = t_{Q_1 \cap Q_2}(p_j) = \{x_i^{(\lambda^i_{t_U(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda_{t_U(p_j)}^i = \lambda_{t_{Q_1 \cap Q_2}(p_j)}^i \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda_{t_U(p_j)}^i = \lambda_{t_{Q_1 \cap Q_2}(p_j)}^i = \min\{\lambda_{t_{Q_1}(p_j)}, \lambda_{t_{Q_2}(p_j)}\}. \quad (3.6)$$

And assume that $T_{Q_1} \cap T_{Q_3} = T_V$, where for all $p_j \in \mathcal{P}$

$$t_V(p_j) = t_{Q_1 \cap Q_3}(p_j) = \{x_i^{(\lambda^i_{t_V(p_j)})} : x_i \in \mathcal{X} \text{ and } \lambda_{t_V(p_j)}^i = \lambda_{t_{Q_1 \cap Q_3}(p_j)}^i \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda_{t_V(p_j)}^i = \lambda_{t_{Q_1 \cap Q_3}(p_j)}^i = \min\{\lambda_{t_{Q_1}(p_j)}, \lambda_{t_{Q_3}(p_j)}\}. \quad (3.7)$$
Suppose that $T_U \sqcup T_V = T_W$, where for all $p_j \in P$

$$t_W(p_j) = t_{U \sqcup V}(p_j) = \{x_i^{(\lambda_j^{i}(p_j)}) : x_i \in \mathcal{X} \text{ and } \lambda_j^{i}(p_j) = \lambda_j^{i}(p_j) = \{0, \frac{1}{2}, 1\}$$

such that

$$\lambda_j^{i}(p_j) = \lambda_j^{i}(p_j) = \max\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\} = \max\{\min\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\}, \min\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\}\}.$$

(3.8)

Since $\lambda_j^{i}(p_j), \lambda_j^{i}(p_j), \lambda_j^{i}(p_j) \in \{0, \frac{1}{2}, 1\}$, we obtain that $\lambda_j^{i}(p_j) = \lambda_j^{i}(p_j)$ for all $p_j \in P$ (by Eq. [3.5] and Eq. (3.8)). Thereby, we say that $T_S$ and $T_W$ are indeed the same set-valued mapping, so $T_Q_1 \cap (T_Q_2 \cup T_Q_3) = (T_Q_1 \cap T_Q_2) \cup (T_Q_1 \cap T_Q_3)$.

Other cases can be proved in a similar way.

**Proposition 3.17.** Let $T_Q_1, T_Q_2 \in \mathcal{TVSS}(\mathcal{X}, P)$. Then, the following De Morgan’s rules are hold.

i) $(T_Q_1 \cap T_Q_2) = T_Q_1 \cup T_Q_2$.

ii) $(T_Q_1 \cup T_Q_2) = T_Q_1 \cap T_Q_2$.

**Proof.** (i) Since $1 - \min\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\} = \max\{1 - \lambda_j^{i}(p_j), 1 - \lambda_j^{i}(p_j)\}$ for all $p_j \in P$, we can say that $(T_Q_1 \cap T_Q_2) = T_Q_1 \cup T_Q_2$.

(ii) It is similar to the proof of (i). □

**Definition 3.18.** Let $T_Q_1, T_Q_2 \in \mathcal{TVSS}(\mathcal{X}, P)$. Then, the difference of $T_Q_1$ and $T_Q_2$, denoted by $T_Q_1 \setminus T_Q_2$, is defined by the mapping $t_{Q_1 \setminus Q_2} : P \to \mathcal{X}^{(0, \frac{1}{2})}$ such that $t_{Q_1 \setminus Q_2}(p_j) = \{x_i^{(\lambda_j^{i}(p_j))} : x_i \in \mathcal{X} \text{ and } \lambda_j^{i}(p_j) \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in P$, where

$$\lambda_j^{i}(p_j) = \min\{\lambda_j^{i}(p_j), 1 - \lambda_j^{i}(p_j)\}. \quad (3.9)$$

**Definition 3.19.** Let $T_Q_1, T_Q_2 \in \mathcal{TVSS}(\mathcal{X}, P)$. Then, the symmetric difference of $T_Q_1$ and $T_Q_2$, denoted by $T_Q_1 \Delta T_Q_2$, is defined by the mapping $t_{Q_1 \Delta Q_2} : P \to \mathcal{X}^{(0, \frac{1}{2})}$ such that $t_{Q_1 \Delta Q_2}(p_j) = \{x_i^{(\lambda_j^{i}(p_j))} : x_i \in \mathcal{X} \text{ and } \lambda_j^{i}(p_j) \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in P$, where

$$\lambda_j^{i}(p_j) = \min\{\max\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\}, 1 - \min\{\lambda_j^{i}(p_j), \lambda_j^{i}(p_j)\}\}. \quad (3.10)$$

**Example 3.20.** Consider three-valued soft sets $T_{Q_1}$ in Table 2 and $T_{Q_2}$ in Table 4. Then, we obtain the difference and symmetric difference of $T_{Q_1}$ and $T_{Q_2}$ as in Tables 6 and 7 respectively.

[Insert Table 6]

[Insert Table 7]

**Proposition 3.21.** Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathcal{X}, P)$.

i) $T_{Q_1 \setminus Q_2} = T_{Q_1} \cap T_{Q_2}$.
ii) \( T_{Q_1} \triangle T_{Q_2} = (T_{Q_1} \searrow T_{Q_2}) \cup (T_{Q_2} \searrow T_{Q_1}) \). 

Proof. (i) Assume \( T_{Q_1} \searrow T_{Q_2} = T_R \). Then, by Eq. 3.9 we have, for all \( p_j \in P \),

\[
\lambda_{i,(p_j)}^{t_i} = \min\{\lambda_{i,Q_1(p_j)}^{t_i}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\}. \tag{3.11}
\]

On the other hand, assume that \( T_{Q_1} \triangledown T_{Q_2} = T_S \). Then, from Definition 3.12 we obtain that

\[
\lambda_{i,(p_j)}^{t_i} = \min\{\lambda_{i,Q_1(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\} = \min\{\lambda_{i,Q_1(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\}. \tag{3.12}
\]

By Eqs. 3.11 and 3.12 we prove that the above equality is achieved.

(ii) Assume \( T_{Q_1} \triangledown T_{Q_2} = T_U \). Then, by Eq. 3.10 we have, for all \( p_j \in P \),

\[
\lambda_{i,(p_j)}^{t_i} = \min\{\lambda_{i,Q_1(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\} \tag{3.13}
\]

Considering the right side of equality, let \( T_{Q_1} \triangledown T_{Q_2} = T_V \) and \( T_{Q_2} \triangledown T_{Q_1} = T_W \), where

\[
\lambda_{i,(p_j)}^{t_i} = \min\{\lambda_{i,Q_2(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\}. \tag{3.14}
\]

and

\[
\lambda_{i,(p_j)}^{t_i} = \min\{\lambda_{i,Q_2(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\}. \tag{3.15}
\]

For \( T_Z = T_V \triangledown T_W \), from Definition 3.14 we obtain that

\[
\lambda_{i,(p_j)}^{t_i} = \max\{\lambda_{i,Q_1(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\} = \max\{\min\{\lambda_{i,Q_1(p_j)}, 1 - \lambda_{i,Q_2(p_j)}^{t_i}\}, \min\{\lambda_{i,Q_2(p_j)}, 1 - \lambda_{i,Q_1(p_j)}^{t_i}\}\}. \tag{3.16}
\]

We know that \( \lambda_{i,Q_1(p_j)}, \lambda_{i,Q_2(p_j)} \in \{0, \frac{1}{2}, 1\} \) for all \( p_j \in P \), and so \( \lambda_{i,(p_j)}^{t_i} = \lambda_{i,(p_j)}^{t_i} \). This completes the proof. \( \square \)

Definition 3.22. Let \( T_{Q_1}, T_{Q_2} \in TVSS(\mathcal{X}, P) \), then, the And-product of \( T_{Q_1} \) and \( T_{Q_2} \), denoted by \( T_{Q_1} \triangledown T_{Q_2} \), is defined by the mapping \( t_{Q_1 \triangledown Q_2} : P \times P \to \mathcal{X} \{0, \frac{1}{2}, 1\} \) such that

\[
t_{Q_1 \triangledown Q_2}(p_j, p_k)(x_i) = \lambda_{i,Q_1 \triangledown Q_2}(p_j, p_k) = \min\{\lambda_{i,Q_1(p_j)}, \lambda_{i,Q_2(p_k)}\}. \tag{3.17}
\]

Definition 3.23. Let \( T_{Q_1}, T_{Q_2} \in TVSS(\mathcal{X}, P) \), then, the Or-product of \( T_{Q_1} \) and \( T_{Q_2} \), denoted by \( T_{Q_1} \triangleleft T_{Q_2} \), is defined by the mapping \( t_{Q_1 \triangleleft Q_2} : P \times P \to \mathcal{X} \{0, \frac{1}{2}, 1\} \) such that

\[
t_{Q_1 \triangleleft Q_2}(p_j, p_k)(x_i) = \lambda_{i,Q_1 \triangleleft Q_2}(p_j, p_k) = \max\{\lambda_{i,Q_1(p_j)}, \lambda_{i,Q_2(p_k)}\}. \tag{3.18}
\]

Proposition 3.24. Let \( T_{Q_1}, T_{Q_2}, T_{Q_3} \in TVSS(\mathcal{X}, P) \). For all \( * \in \{\triangledown, \triangleleft\} \),

i) \( T_{Q_1} * (T_{Q_2} \triangledown T_{Q_3}) = (T_{Q_1} \triangledown T_{Q_2}) * T_{Q_3} \).

ii) \( T_{Q_1} * (T_{Q_2} \triangleleft T_{Q_3}) = (T_{Q_1} \triangleleft T_{Q_2}) \circ (T_{Q_1} \triangledown T_{Q_2}) \).

iii) \( (T_{Q_1} * T_{Q_2}) \circ T_{Q_3} = (T_{Q_1} \circ T_{Q_2}) * (T_{Q_2} \circ T_{Q_3}) \).
Proof. They can be shown in a similar way to the proofs of Proposition 3.16.

**Proposition 3.25.** Let $T_{Q_1}, T_{Q_2} \in TVSS(\mathcal{X}, \mathcal{P})$. Then, the following De Morgan’s laws are hold.

i) $(T_{Q_1} \bar{\lor} T_{Q_2}) = \overline{T_{Q_1}} \lor T_{Q_2}$.

ii) $(T_{Q_1} \lor T_{Q_2}) = T_{Q_1} \bar{\lor} T_{Q_2}$.

**Proof.** The proofs are similar to those of Proposition 3.17.

4 Choice value of an object in three-valued soft set(s)

In 2002, Maji et al. [32] defined the choice value of an object in soft set and thereby put forward that this indication can be used to prioritize the objects in the soft set during decision making. We also adapt the idea of choice value initiated by Maji et al. for the three-valued soft sets.

In this part, it is taken $J = \{1, 2, \ldots, |\mathcal{P}|\}$ where $|\mathcal{P}|$ denotes the cardinality of the parameter set $\mathcal{P}$.

**Definition 4.1.** Let $\mathcal{X}$ be a set of alternatives (objects).

1. The choice value of an object $x_i \in \mathcal{X}$ in the three-valued soft set $T_Q$ is defined and denoted by

\[ \alpha_i = \sum_{j \in J} (x_{ij})^\xi, \tag{4.1} \]

where $x_{ij}$ for all $i, j$ are the entries in the table of three-valued soft set $T_Q$. Also, the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of “undetermined” on the choice value.

2. The choice value of an object $x_i \in \mathcal{X}$ in the three-valued soft sets $T_{Q_k}$ for $k = 1, 2, \ldots, s$ is defined and denoted by

\[ \alpha_i = \frac{\sum_{j \in J} (x_{ij}^T + x_{ij}^L)^\xi}{2}, \tag{4.2} \]

where $x_{ij}^T$ and $x_{ij}^L$ for all $i, j$ are the entries in the tables of three-valued soft sets $\prod_{k=1}^s T_{Q_k}$ and $\bigcup_{k=1}^s T_{Q_k}$, respectively. Also, the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of “undetermined” on the choice value.

**Remark:** It is clear that for $x_{ij} = \frac{1}{2}$, $(x_{ij})^\xi \to 1$ when $\xi \to 0$ and $(x_{ij})^\xi \to 0$ when $\xi \to +\infty$. For instance, consider Figure 3. If the revision is minor then it is more appropriate to specify as $0 < \xi < 1$. If the revision is major then it is more appropriate to specify as $\xi \in (1, +\infty)$.

**Algorithm for Selection (Algorithm 1)**

Step 1. Choose feasible subsets $Q_k (k = 1, 2, \ldots, s)$ of the parameter set $\mathcal{P}$,

Step 2. Create the three-valued soft sets $T_{Q_k}$ for parameter subsets $Q_k (k = 1, 2, \ldots, s)$,

Step 3. Specify the overall impact coefficient of “undetermined” on the choice value, i.e., $\xi \in \mathbb{R}^+$.

Step 4.

- If $k > 1$ then obtain the intersection and union of three-valued soft sets $T_{Q_k} (k = 1, 2, \ldots, s)$,
- If $k = 1$ then skip to step 4,

Step 5. Calculate $\alpha_i$ for all $i$,
Step 6. Find \( l \), for which \( \alpha_l = \max \alpha_i \).

Then, \( x_l \) is the optimal choice object. If \( l \) has more than one value, then any one of them could be chosen.

**Example 4.2.** As an implementation of Algorithm 1, we try to solve our numerical problem in Example 3.4.

![Insert Table 8]

![Insert Table 9]

By Tables 8 and 9 we have \( \max \alpha_i = \alpha_4 \) for \( \xi = 1, \frac{1}{4}, \frac{3}{2}, \sqrt{13}, 10 \) and so the investment \( x_4 \) is optimal choice to invest some money.

**Example 4.3.** We consider three-valued soft sets \( T_{Q1} \) in Example 3.4 and \( T_{Q2} \) in Example 3.10. Let’s try to make a common decision by using the data in these two three-valued soft sets for \( \xi = 2 \). We obtain the intersection and union of \( T_{Q1} \) and \( T_{Q2} \) as in Tables 10 and 11, respectively.

![Insert Table 10]

![Insert Table 11]

By Tables 10 and 11 we have \( \max \alpha_i = \alpha_1 = \alpha_4 = 2.125 \). So, we interpret that any of the investments \( x_1 \) and \( x_4 \) is optimal choice to invest some money.

In the decision making process, all parameters of the parameter set may not be of equal importance. In such cases, weights can be imposed on the choice parameters, i.e., there is a weight \( \omega_j \in \mathbb{R}^+ \) corresponding to each parameter \( p_j \in Q \). If \( p_j \in P^* \) then we know that \( \omega_j = 0 \). Generally, total weight is \( \sum_j \omega_j = 1 \). Now, let us describe the weighted choice value of an object in the structures of (weighted) three-valued soft sets.

**Definition 4.4.** Let \( \mathcal{X} \) be a set of alternatives (objects).

1. The weighted choice value of an object \( x_i \in \mathcal{X} \) in the (weighted) three-valued soft set \( T_Q \) is defined and denoted by

   \[
   \alpha_i^\omega = \sum_{j \in J} \omega_j \times (x_{ij})^\xi,
   \]

   where the arbitrary number \( \xi \in \mathbb{R}^+ \) is the overall impact coefficient of “undetermined” on the choice value. Also, \( \omega_j \) denotes the weight corresponding to each parameter \( p_j \) in the structure of three-valued soft set \( T_Q \).

2. The weighted choice value of an object \( x_i \in \mathcal{X} \) in the (weighted) three-valued soft sets \( T_{Qk} \) for \( k = 1, 2, \ldots, s \) is defined and denoted by

   \[
   \alpha_i^\omega = \sum_{j \in J} \omega_j^{ort} \times \left( (x_{ij}^\xi)^\xi + (x_{ij}^{\xi^*})^\xi \right),
   \]

   where the arbitrary number \( \xi \in \mathbb{R}^+ \) is the overall impact coefficient of “undetermined” on the choice value. Also,

   \[
   \omega_j^{ort} = \frac{\sum_{k=1}^s \omega_j^k}{s},
   \]

   where \( \omega_j^k \) denotes the weight corresponding to each parameter \( p_j \) in the structure of three-valued soft set \( T_{Qk} \).
Algorithm for Selection by Imposing Weights on Parameters (Algorithm 2)

Step 1. Choose feasible subsets \( Q_k \) \((k = 1, 2, ..., s)\) of the parameter set \( P \) and determine the parameter’s weights (i.e., \( \omega^k \)) for each subsets \( Q_k \).

Step 2. Create the (weighted) three-valued soft sets \( T_{Q_k} \) for the parameter subsets \( Q_k \) \((k = 1, 2, ..., s)\).

Step 3. Specify the overall impact coefficient of “undetermined” on the choice value, i.e., \( \xi \in \mathbb{R}^+ \).

Step 4. • If \( k > 1 \) then obtain the intersection and union of three-valued soft sets \( T_{Q_k} \) \((k = 1, 2, ..., s)\),

• If \( k = 1 \) then skip to step 4.

Step 5. Calculate \( \alpha^i_\omega \) for all \( i \).

Step 6. Find \( l \), for which \( \alpha^l_\omega = \max \alpha^i_\omega \).

Then, \( x_l \) is the optimal choice object. If \( l \) has more than one value, then any one of them could be chosen.

Example 4.5. We consider the numerical problem proposed in Example 3.4. Also, let us determine the following weights for the parameters of \( Q_1 \): \( \omega^1_1 = 0.6 \) for the parameter \( p_1 = \text{high returns} \), \( \omega^1_2 = 0.3 \) for the parameter \( p_2 = \text{low risk} \) and \( \omega^1_3 = 0.1 \) for the parameter \( p_3 = \text{high security} \). Since \( p_4 \notin Q_1 \), the weight of parameter \( p_4 \in P \) can be considered “0”.

[Insert Table 12]

By Table 12 we obtain \( \max \alpha^i_\omega = \alpha^3_\omega \) (for \( \xi = 1, \frac{1}{10}, 3 \)). Then, the optimal choice is \( x_3 \).

Example 4.6. Let’s the weighted three-valued soft sets \( T_{Q_1} \) in Example 4.3. Also, we take following parameter weights for \( T_{Q_2} \) in Table 4, \( \omega^2_1 = \omega^2_3 = 0.2 \) and \( \omega^2_2 = \omega^2_3 = 0.3 \). For \( \xi = 1 \), we obtain weighted choice values of alternatives \( x_i \) \((i = 1, 2, 3, 4)\) as \( \alpha^1_\omega = 0.65, \alpha^2_\omega = 0.2, \alpha^3_\omega = 0.6, \alpha^4_\omega = 0.6 \). Since \( \max \alpha^i_\omega = \alpha^1_\omega = 0.65 \), the optimal choice is \( x_1 \).

In Algorithms 1 and 2, for each decision maker, the overall impact coefficient (\( \xi \)) of “undetermined” on the choice value is taken the same. These two algorithms cannot be used if each decision maker selects the impact coefficient of “undetermined” differently. To address these shortcomings, we will create new decision making algorithms.

5 Three-valued soft decision making model based on TOPSIS

In this section, we focus on the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), which produces satisfactory results during decision making. We rebuild this technique to deal with the multi-criteria group decision making problems based on three-valued soft sets, and thus construct a novel decision making model.

The TOPSIS is an approach proposed to solve the multi-criteria decision making problems based on a decision maker. In 2007, Shih et al. extended this method for group decision making. The operations in the process of TOPSIS include decision matrix normalization, distance measures and aggregation operators. Generally, a decision matrix is required prior to the beginning of the process. As a results of this process, the output data are interpreted and thus the ranking order of alternatives is obtained. In summary, the TOPSIS approach is a practical and useful method for ranking and selection of a number of externally determined alternatives through distance measures. The main procedure of TOPSIS is given in a series of steps (see 48[51]).
Now, we propose a multi-criteria group decision making model using the TOPSIS on three-valued soft sets.

**Algorithm of TOPSIS based three-valued soft sets (Algorithm 3)**

**Step 1.** Describe the multi-criteria group decision making problem.

In this step, the decision makers (experts), the alternatives and the choice parameters are determined. Suppose that $DM = \{E_k : k \in I_s = \{1, 2, ..., s\}\}$ is a set of decision makers (experts) and $E_k$ denotes $k$th decision maker (expert). Also, $x_i (i \in I_m = \{1, 2, ..., m\})$ denotes $i$th alternative and $p_j (j \in I_n = \{1, 2, ..., n\})$ represents $j$th parameter (criterion or attribute).

Considering these data, each decision maker $E_k (k \in I_s)$ create three-valued soft set $T_{Q_k}$ and determine the weights of parameters as $\omega^k_j (j \in I_n)$ satisfying the condition $\sum_{j=1}^n \omega^k_j = 1$.

Also, each decision maker $E_k$ specifies the impact coefficient of “undetermined” in the decision making, i.e., $\xi_k$.

**Step 2.** For each decision maker $E_k$; the decision matrix $D^k$ is constructed and represented as follows:

$$D^k = \begin{bmatrix} d_{11}^k & d_{12}^k & \cdots & d_{1n}^k \\ d_{21}^k & d_{22}^k & \cdots & d_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1}^k & d_{m2}^k & \cdots & d_{mn}^k \end{bmatrix} = [d_{ij}^k]_{m \times n}$$

where $d_{ij}^k = (x_{ij}^k)^{\xi_k}$ that $x_{ij}^k$ for all $i, j$ are the entries in the table of three-valued soft set $T_{Q_k}$.

**Step 3.** After constructing the decision matrices, these are normalized (standardized).

For each decision matrix $D^k$, the normalized decision decision matrix $R^k$ is constructed and expressed as follows:

$$R^k = \begin{bmatrix} r_{11}^k & r_{12}^k & \cdots & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdots & r_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1}^k & r_{m2}^k & \cdots & r_{mn}^k \end{bmatrix} = [r_{ij}^k]_{m \times n}$$

where

$$r_{ij}^k = \begin{cases} \frac{d_{ij}^k}{\sqrt{\sum_{l=1}^{n} (d_{il}^k)^2}}, & \text{if } d_{ij}^k \neq 0 \\ 0, & \text{if } d_{ij}^k = 0 \end{cases} \quad (5.1)$$

for all $k \in I_s$, $i \in I_m$ and $j \in I_n$.

**Step 4.** Considering the different weights of parameters for each decision maker, the weighted normalized decision matrix is calculated by multiply the weights of evaluation parameters and values in the normalized decision matrix.
For each normalized decision matrix $R^k$, the weighted normalized decision matrix $V^k$ is created as below:

$$V^k = \begin{bmatrix} v^k_{11} & v^k_{12} & \cdots & v^k_{1n} \\ v^k_{21} & v^k_{22} & \cdots & v^k_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ v^k_{m1} & v^k_{m2} & \cdots & v^k_{mn} \end{bmatrix} = [v_{ij}]_{m \times n}$$

where

$$v^k_{ij} = w^k_j \times r^k_{ij} \quad (5.2)$$

for all $k \in I_s$, $i \in I_m$ and $j \in I_n$.

**Step 5.** Combining the weighted normalized decision matrices $V^k \ (k \in I_s)$, the average weighted normalized decision matrix $V$ is created.

The structure of matrix $V$ can be expressed as below:

$$V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ v_{m1} & v_{m2} & \cdots & v_{mn} \end{bmatrix} = [v_{ij}]_{m \times n}$$

where for all $i \in I_m$ and $j \in I_n$

$$v_{ij} = v^1_{ij} \oplus v^2_{ij} \oplus \ldots \oplus v^s_{ij}. \quad (5.3)$$

In other words, the value $v_{ij}$ is obtained by combining the weighted normalized decision value $v^k_{ij}$ through an operation $\oplus$. Here, the operation $\oplus$ can offer many choices: arithmetic mean, geometric mean, harmonic mean and their modification.

**Note:** In this study, we will take the arithmetic mean of all individual measures.

**Step 6.** The positive and negative ideal solutions $V^+\up{7}$ and $V^-\up{8}$ are determined by using the average weighted normalized decision matrix $V$.

In the TOPSIS approach, the parameters (criteria or attributes) are evaluated from aspect of benefit (cf. Example 3.4) and cost (cf. Example 3.2). Suppose that $J_1$ and $J_2$ are respectively the sets of benefit parameters and cost parameters, where $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = \{1, 2, \ldots, n\}$.

- $V^+$ is the set showing the most suitable alternative for each parameter may be preferred (PIS). This set is given by the following

$$V^+ = \{v^+_1, v^+_2, \ldots, v^+_j, \ldots, v^+_n\} = \{(\max_i v^i_{1j} : j \in J_1), (\min_i v^i_{1j} : j \in J_2), \ i \in I_m\}. \quad (5.4)$$

- $V^-$ is the set showing at least preferable alternative for each parameter (NIS). This set is given by the following

$$V^- = \{v^-_1, v^-_2, \ldots, v^-_j, \ldots, v^-_n\} = \{(\min_i v^i_{1j} : j \in J_1), (\max_i v^i_{1j} : j \in J_2), \ i \in I_m\}. \quad (5.5)$$
Step 7. The separation measurements of alternatives to the ideal solutions are obtained with the Euclidean distance formula.

The separation measurement of each alternative \( x_i \) to the positive ideal solution \( V^T \) is calculated as

\[
S_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_{ij}^T)^2}. \tag{5.6}
\]

The separation measurement of each alternative \( x_i \) to the negative ideal solution \( V^L \) is calculated as

\[
S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_{ij}^L)^2}. \tag{5.7}
\]

Here, \( S_i^+ \) and \( S_i^- \) represent the distance of alternative \( x_i \) from PIS and NIS, respectively.

Step 8. The relative closeness of each alternative to the ideal solutions is calculated.

The relative closeness \( C_i^\dagger \) of the alternatives \( x_i \) with respect to the ideal solutions can be expressed as

\[
C_i^\dagger = \frac{S_i^-}{S_i^+ + S_i^-}, \quad \forall i \in \mathcal{I}_m \quad (0 \leq C_i^\dagger \leq 1). \tag{5.8}
\]

Step 9. Ranking the preference order.

A set of alternatives \( x_i \) can be ranked according to the descending order of the values \( C_i^\dagger \).

To illustrate the potential of this proposed approach, we present a real life practice adapted from Figure 4.

**Example 5.1.** Assume that two experts want to determine best mobile phone brand by examining the new model mobile phones presented by six different mobile phone brands. The first expert \( E_1 \) will review each brand’s mobile phone with memory of 128 GB and the second expert \( E_2 \) will review each brand’s mobile phone with memory of 64 GB. Let \( \mathcal{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\} \) be the set of six different mobile phone brands. Also, the set of parameters will be used to determine the brand having the best of mobile phones is given as \( \mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\} \) where \( p_1= \) optical image stabilisation, \( p_2= \) quickly charging, \( p_3= \) expandable memory, \( p_4= \) water proof, \( p_5= \) autofocus, \( p_6= \) cheap, \( p_7= \) fingerprint sensor. Each of experts proceeds to the decision making stage after reviewing comparisons on mobile phone comparison-focused website (such as “www.91mobiles.com” and “www.smartprix.com”).

**Step 1.** The experts \( E_1 \) and \( E_2 \) determine the parameter sets as \( Q_1 = Q_2 = \mathcal{P} \), respectively. Also, they determine weights of their choice parameters as \( \omega_i^k \) for all \( j = 1, 2, \ldots, 7 \) and \( k = 1, 2 \). (see: Table 13)

[Insert Table 13]

The experts collect data about each brand’s mobile phone \( x_i \) (\( i = 1, 2, \ldots, 6 \)) for each attribute \( p_j \) and create three-valued soft sets in Table ??.

[Insert Table 14]

**Step 2.** According to Table 14 the decision matrices \( \mathcal{D}_k^i \) (\( k = 1, 2 \)) are constructed as follows:

\[
\mathcal{D}_1^1 = \begin{bmatrix}
1 & 0 & 0 & 0.3535 & 0.3535 & 1 & 0 \\
1 & 0.3535 & 0.3535 & 1 & 1 & 1 & 0.3535 \\
0.3535 & 0 & 0.3535 & 1 & 0.3535 & 0 & 1 \\
0.3535 & 0.3535 & 1 & 0.3535 & 0.3535 & 1 & 1 \\
0.3535 & 0.3535 & 1 & 0.3535 & 0 & 1 & 1 \\
0.3535 & 0.3535 & 1 & 1 & 0 & 1 & 1 \\
\end{bmatrix} = [d_{ij}^1]_{6 \times 7}, \quad \mathcal{D}_2^2 = \begin{bmatrix}
0.25 & 0.25 & 1 & 0 & 0.25 & 1 & 0 \\
1 & 0.25 & 1 & 1 & 1 & 1 & 1 \\
0.25 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0.25 & 1 & 0 & 0 & 1 & 0.25 \\
0.25 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{bmatrix} = [d_{ij}^2]_{6 \times 7}.
\]
Step 3. For each decision matrix $\mathbf{D}^k$ ($k = 1, 2$), the normalized decision decision matrix $\mathbf{R}^k$ ($k = 1, 2$) is constructed as follows:

$$
\mathbf{R}^1 = \begin{bmatrix}
0.6489 & 0 & 0 & 0.1714 & 0.2886 & 0.5 & 0 \\
0.6489 & 0.5001 & 0.2293 & 0.485 & 0.8166 & 0.5 & 0.174 \\
0.2293 & 0 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
0 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
0.2293 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
0 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\
\end{bmatrix}
$$

$$
\text{and } \mathbf{R}^2 = \begin{bmatrix}
0.169 & 0.2357 & 0.5714 & 0 & 0.0174 & 0.4472 & 0 \\
0.6761 & 0 & 0.1428 & 0.5 & 0.6963 & 0.4472 & 0.4961 \\
0.169 & 0 & 0 & 0.5 & 0.6963 & 0 & 0.4961 \\
0 & 0.2357 & 0.5714 & 0 & 0 & 0.4472 & 0.124 \\
0.169 & 0 & 0 & 0.5 & 0 & 0.4472 & 0.4961 \\
0.6761 & 0.9428 & 0.5714 & 0.5 & 0 & 0.4472 & 0.4961 \\
\end{bmatrix}
$$

Step 4. For each normalized decision decision matrix $\mathbf{R}^k$ ($k = 1, 2$), the weighted normalized decision decision matrix $V^k$ ($k = 1, 2$) is created as below:

$$
\mathbf{V}^1 = \begin{bmatrix}
0.1427 & 0 & 0 & 0.0257 & 0.0375 & 0.055 & 0 \\
0.1427 & 0.075 & 0.0298 & 0.0727 & 0.1061 & 0.055 & 0.0191 \\
0.0504 & 0 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\
0.0504 & 0.075 & 0.0843 & 0.0257 & 0.0375 & 0.055 & 0.0541 \\
0 & 0.075 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\
0.0504 & 0.075 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\
\end{bmatrix}
$$

$$
\mathbf{V}^2 = \begin{bmatrix}
0.0405 & 0.0282 & 0.0685 & 0 & 0.0243 & 0.0447 & 0 \\
0.1622 & 0 & 0.0171 & 0.08 & 0.0974 & 0.0447 & 0.0595 \\
0.0405 & 0 & 0 & 0.08 & 0.0974 & 0 & 0.0595 \\
0 & 0.0282 & 0.0685 & 0 & 0 & 0.0447 & 0.0148 \\
0.0405 & 0 & 0 & 0.08 & 0 & 0.0447 & 0.0595 \\
0.1622 & 0.1131 & 0.0685 & 0.08 & 0 & 0.0447 & 0.0595 \\
\end{bmatrix}
$$

Step 5. Then, the average weighted normalized decision matrix is

$$
\mathbf{V} = \mathbf{V}^1 \oplus \mathbf{V}^2 = \begin{bmatrix}
0.0916 & 0.0141 & 0.0342 & 0.0128 & 0.0309 & 0.0498 & 0 \\
0.1524 & 0.0375 & 0.0234 & 0.0763 & 0.1017 & 0.0498 & 0.0393 \\
0.0454 & 0 & 0.0149 & 0.0763 & 0.0674 & 0 & 0.0568 \\
0.0252 & 0.0516 & 0.0764 & 0.0128 & 0.0187 & 0.0498 & 0.0344 \\
0.0202 & 0.0375 & 0.0149 & 0.0763 & 0.0187 & 0.0223 & 0.0568 \\
0.1063 & 0.094 & 0.0491 & 0.0763 & 0 & 0.0498 & 0.0568 \\
\end{bmatrix}
$$

where the operation $\oplus$ represents the arithmetic mean.

Step 6. The positive and negative ideal solutions $\mathbf{V}^+$ and $\mathbf{V}^-$ are determined as follows:

$$
\mathbf{V}^+ = \begin{bmatrix}
v_1^+ = 0.1524, v_2^+ = 0.094, v_3^+ = 0.0764, v_4^+ = 0.0763, v_5^+ = 0.1017, v_6^+ = 0.0498, v_7^+ = 0.0568
\end{bmatrix}
$$

and

$$
\mathbf{V}^- = \begin{bmatrix}
v_1^- = 0.0202, v_2^- = 0, v_3^- = 0.0149, v_4^- = 0.0128, v_5^- = 0, v_6^- = 0, v_7^- = 0
\end{bmatrix}.
$$

Step 7. The separation measurements $S_i^+$ and $S_i^-$ of each alternative $x_i$ to the ideal solutions are given in Table 15

[Insert Table 15]

Step 8. The relative closeness $C_i^+$ of each alternative to the ideal solutions is calculated as follows:

$$
C_1^+ = 0.3767, C_2^+ = 0.7095, C_3^+ = 0.399, C_4^+ = 0.3706, C_5^+ = 0.3513, C_6^+ = 0.589.
$$

Step 9. According to the descending order of the values $C_i^+$, the ranking order of alternatives is obtained as below

$$
x_2 > x_6 > x_3 > x_1 > x_4 > x_5.
$$

Then, it can be said that $x_2$ is the best mobile phone brand according to the data presented by experts
6 Three-valued soft decision making model based on ELECTRE

In this section, we introduce a modified version of the ELECTRE technique (“ELimination Et Choix Traduisant la Réalité” or “Elimination and Choice Expressing Reality”), which are generally intended to output choosing, sorting and ranking, to deal with the multi-criteria group decision making problems based on three-valued soft sets.

As it was first applied in 1965, the ELECTRE method was to choose the best alternative(s) from the given set of alternatives, but it was soon applied three fundamental problems:

Choosing: Selecting a restricted number of the most interesting potential alternatives, as small as possible which will justify to eliminate all others.

Sorting: Assigning each potential alternative to one of the categories a family previously described; the categories are ordered from the worst to the best one.

Ranking: Ordering of alternatives from the best to the worst with the possibility of ties.

The main procedure of ELECTRE is described in a series of the steps (see [52–56]).

Now, we construct a multi-criteria group decision making model on three-valued soft sets via the modified ELECTRE technique.

Algorithm of ELECTRE based three-valued soft sets (Algorithm 4)

Step 1. Describe the multi-criteria group decision making problem (same as Step 1 in Algorithm of TOPSIS based three-valued soft sets).

Step 2. For each decision maker $E_k$; the decision matrix $D_k$ is constructed (same as Step 2 in Algorithm of TOPSIS based three-valued soft sets).

Step 3. For each decision matrix $D_k$, the normalized decision decision matrix $R_k$ is constructed (same as Step 3 in Algorithm of TOPSIS based three-valued soft sets).

Step 4. For each normalized decision matrix $R_k$, the weighted normalized decision matrix $V_k$ is created (same as Step 4 in Algorithm of TOPSIS based three-valued soft sets).

Step 5. Combining the weighted normalized decision matrices $V_k$ ($k \in I$), the average weighted normalized decision matrix $V$ is created (same as Step 5 in Algorithm of TOPSIS based three-valued soft sets).

Step 6. The concordance sets and discordance sets are determined.

The concordance set is composed of index of all parameters for which the alternative $x_\tau$ is preferred to the alternative $x_\kappa$. This set can be described as follows:

For $\tau, \kappa \in I_m$ and $\tau \neq \kappa$ (note that an alternative is not compared to itself),

$$J_{\tau\kappa}^+ = \{j : v_{\tau j} \geq v_{\kappa j}\},$$

(6.1)

The discordance set contains the index of all parameters for which the alternative $x_\tau$ is worse than the alternative $x_\kappa$. This set can be described as follows:

For $\tau, \kappa \in I_m$ and $\tau \neq \kappa$,

$$J_{\tau\kappa}^- = \{j : v_{\tau j} < v_{\kappa j}\}.$$

(6.2)

In the other words, this set can be considered as the complement of the concordance set $J_{\tau\kappa}^+$, i.e., $J_{\tau\kappa}^- = J \setminus J_{\tau\kappa}^+$, where $J = \{j : p_j \in P\}$. 


Step 7. The concordance matrix and discordance matrix are generated by employing the sets of concordance and discordance, respectively.

The concordance matrix can be expressed as follows:

\[
\mathbf{A} = \begin{bmatrix}
- & a_{1\kappa} & \cdots & a_{1m} \\
. & . & \ddots & . \\
. & . & \ddots & . \\
. & . & \ddots & . \\
a_{\tau 1} & a_{\tau \kappa} & \cdots & a_{\tau m} \\
. & . & \ddots & . \\
. & . & \ddots & . \\
. & . & \ddots & . \\
a_{m 1} & a_{m \kappa} & \cdots & -
\end{bmatrix} = [a_{\tau \kappa}]_{m \times m}
\]

where

\[
a_{\tau \kappa} = \frac{\sum_{j \in \mathcal{J}^+} \sum_{k=1}^{s} \omega_j^k \omega_j}{\sum_{j \in \mathcal{J}} \sum_{k=1}^{s} \omega_j^k}, \quad (0 \leq a_{\tau \kappa} \leq 1)
\]  

for all \( \tau, \kappa \in \mathcal{I}_m \).

That is, each component of concordance matrix is found as summation of the (standardized) weights of all parameters corresponding the indices in the concordance set \( \mathcal{J}^+ \).

The discordance matrix can be expressed as follows:

\[
\mathbf{B} = \begin{bmatrix}
- & b_{1\kappa} & \cdots & b_{1m} \\
. & . & \ddots & . \\
. & . & \ddots & . \\
. & . & \ddots & . \\
b_{\tau 1} & b_{\tau \kappa} & \cdots & b_{\tau m} \\
. & . & \ddots & . \\
. & . & \ddots & . \\
. & . & \ddots & . \\
b_{m 1} & b_{m \kappa} & \cdots & -
\end{bmatrix} = [b_{\tau \kappa}]_{m \times m}
\]

where

\[
b_{\tau \kappa} = \frac{\sum_{j \in \mathcal{J}^-} \sum_{k=1}^{s} \omega_j^k (v_{\tau j} - v_{\kappa j})}{\sum_{j \in \mathcal{J}} \sum_{k=1}^{s} \omega_j^k}, \quad (0 \leq b_{\tau \kappa} \leq 1)
\]  

for all \( \tau, \kappa \in \mathcal{I}_m \).

Step 8. The concordance threshold \( \mathbf{A} \) and discordance threshold \( \mathbf{B} \) are found.

The concordance threshold is calculated as

\[
\mathbf{A} = \frac{\sum_{\tau=1}^{m} \sum_{\kappa=1}^{m} a_{\tau \kappa}}{m(m-1)}, \quad (0 \leq \mathbf{A} \leq 1)
\]  

and the discordance threshold is calculated as

\[
\mathbf{B} = \frac{\sum_{\tau=1}^{m} \sum_{\kappa=1}^{m} b_{\tau \kappa}}{m(m-1)}, \quad (0 \leq \mathbf{B} \leq 1).
\]
Step 9. The effective concordance matrix $\mathcal{F}$ and the effective discordance matrix $\mathcal{G}$ are created.

The effective concordance matrix $\mathcal{F}$ is measured according to the concordance threshold $A$ and it is expressed as follows:

$$
\mathcal{F} = \begin{bmatrix}
- & \cdots & f_{1k} & \cdots & f_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
f_{m1} & \cdots & f_{mk} & \cdots & -
\end{bmatrix} = [f_{\tau\kappa}]_{m \times m}
$$

where

$$
f_{\tau\kappa} = \begin{cases} 
1 & \text{if } a_{\tau\kappa} \geq A \\
0 & \text{if } a_{\tau\kappa} < A 
\end{cases}. \quad (6.7)
$$

The effective discordance matrix $\mathcal{G}$ is measured according to the discordance threshold $B$ and it is expressed as follows:

$$
\mathcal{G} = \begin{bmatrix}
- & \cdots & g_{1k} & \cdots & g_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
g_{m1} & \cdots & g_{mk} & \cdots & -
\end{bmatrix} = [g_{\tau\kappa}]_{m \times m}
$$

where

$$
g_{\tau\kappa} = \begin{cases} 
0 & \text{if } b_{\tau\kappa} > B \\
1 & \text{if } b_{\tau\kappa} \leq B 
\end{cases}. \quad (6.8)
$$

Step 10. The aggregated outranking matrix $\mathcal{H}$ is constructed.

The aggregated outranking matrix $\mathcal{H}$ is found by merging the effective concordance information and the effective discordance information. Relatedly, the matrix $\mathcal{H}$ can be described as follows:

$$
\mathcal{H} = \begin{bmatrix}
- & \cdots & h_{1k} & \cdots & h_{1m} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
h_{m1} & \cdots & h_{mk} & \cdots & -
\end{bmatrix} = [h_{\tau\kappa}]_{m \times m}
$$
where
\[ h_{\tau \kappa} = f_{\tau \kappa} \times g_{\tau \kappa} \]  
(6.9)

for all \( \tau, \kappa \in I_m \).

Step 11. Ranking the preference order.

The components in the aggregated outranking matrix \( H \) present the dominance of any alternative over another alternative. Thus, we can mention a choice priority among the alternatives and can rank the alternatives according to this priority.

Considering the aggregated outranking matrix \( H \), the binary relations among the alternatives may occur as one of the following three situations:

(a) \( x_{\tau} > x_{\kappa} \) (i.e., \( x_{\tau} \) is strictly preferred to \( x_{\kappa} \) or \( x_{\tau} \) is dominance to \( x_{\kappa} \)) if \( h_{\tau \kappa} = 1 \) and \( h_{\kappa \tau} = 0 \),

(b) \( x_{\tau} \approx x_{\kappa} \) (i.e., \( x_{\tau} \) is indifferent to \( x_{\kappa} \)) if \( h_{\tau \kappa} = 1 \) and \( h_{\kappa \tau} = 1 \),

(c) \( x_{\tau} \equiv x_{\kappa} \) (i.e., \( x_{\tau} \) and \( x_{\kappa} \) are incomparability) if \( h_{\tau \kappa} = 0 \) and \( h_{\kappa \tau} = 0 \).

Thus, the ranking order of alternatives can be interpreted.

As an implementation of this model, we try to offer a solution for the following multi-criteria group decision making problem.

**Example 6.1.** Supplier selection is among the most important issues in the supply chain management area. Now, we consider a numerical example of a supplier selection problem adapted from [57,58]. A high technology company that manufactures electronic products aims to evaluate and choose a materials supplier. Assume \( X = \{ x_1, x_2, x_3, x_4 \} \) is a set of four suppliers that chosen as candidates (alternatives). A single decision maker may not be able to accurately consider all relevant aspects during the decision making. Therefore, the company’s leader decides to put together a decision committee to determine a suitable supplier. A committee of three decision makers (experts) is established: Financial expert \((E_1)\), which evaluates alternatives in terms of cost and finance, Quality control expert \((E_2)\), which evaluates alternatives in terms of quality and safety, Engineering expert \((E_3)\), which evaluates alternatives in terms of engineering and technical aspects.

Also, six evaluation parameters are considered: performance \((p_1)\), cost control \((p_2)\), management audit \((p_3)\), service \((p_4)\), company reputation \((p_5)\) and quality \((p_6)\).

**Step 1.** The decision makers (experts) \( E_1, E_2 \) and \( E_3 \) make their decisions based on the parameter subsets \( Q_1 = \{ p_1, p_2, p_3, p_5 \} \), \( Q_2 = \{ p_1, p_2, p_3, p_5, p_6 \} \) and \( Q_3 = \{ p_1, p_2, p_4, p_5, p_6 \} \), respectively. Also, they determine weights of their choice parameters as in Table 16.

[Insert Table 16]

The decision makers (experts) respectively present their opinion about the truth of alternative \( x_i \) under the parameter \( p_j \) and construct Table 17.

[Insert Table 17]

**Step 2.** The (three-valued) decision matrices \( D^k \) \((k = 1, 2, 3)\) are constructed as follows:
The concordance threshold and discordance threshold are calculated as
\[ \oplus \]
where the operation represents the arithmetic mean.

Steps 3-4. For the decision matrices \( D^k \) (\( k = 1, 2, 3 \)), the normalized decision decision matrices \( R^k \) (\( k = 1, 2, 3 \)) and then the weighted normalized decision decision matrices \( V^k \) (\( k = 1, 2, 3 \)) are constructed in a similar manner to Steps 3 and 4 of Example 5.1 so it is omitted.

Step 5. Then, the average weighted normalized decision matrix is
\[ V = V^1 \oplus V^2 \oplus V^3 = \begin{bmatrix} 0.0744 & 0.0189 & 0.0586 & 0 & 0.0917 & 0 \\ 0.0509 & 0.0504 & 0.116 & 0 & 0.0333 & 0.0589 \\ 0.0816 & 0.117 & 0.1747 & 0 & 0.0656 & 0.1333 \\ 0.0665 & 0.0732 & 0.0363 & 0.0333 & 0.0681 & 0.0589 \end{bmatrix} = [v_{ij}]_{4 \times 6}, \]
where the operation \( \oplus \) represents the arithmetic mean.

Step 6. Considering the average weighted normalized decision matrix \( V \), the concordance set and discordance set are determined in Table 18.

[Insert Table 18]

Step 7. Then, the concordance matrix and discordance matrix are respectively generated as below:
\[ A = \begin{bmatrix} -0.3333 & 0.15 & 0.55 \\ 0.7 & -0.3333 & 0.4666 \\ 0.8833 & 1 & -0.85 \\ 0.45 & 0.75 & 0.15 \end{bmatrix} = [a_{rk}]_{4 \times 4} \text{ and } B = \begin{bmatrix} -0.6434 & 0.9314 & 0.7314 \\ 0.3565 & -1 & 0.5719 \\ 0.0685 & 0 & 0.1164 \\ 0.2685 & 0.428 & 0.8718 \end{bmatrix} = [b_{rk}]_{4 \times 4}. \]

Step 8. The concordance threshold and discordance threshold are calculated as \( A = 0.5263 \) and \( B = 0.4989 \) respectively.

Step 9. The effective concordance matrix \( F \) and the effective discordance matrix \( G \) are created as follows:
\[ F = \begin{bmatrix} -0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [f_{rk}]_{4 \times 4} \text{ and } G = \begin{bmatrix} -0 & 0 & 0 & 0 \\ 1 & -0 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} = [g_{rk}]_{4 \times 4}. \]

Step 10. Then, the aggregated outranking matrix \( H \) is
\[ H = \begin{bmatrix} -0 & 0 & 0 & 0 \\ 1 & -0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} = [h_{rk}]_{4 \times 4}. \]

Step 11. Considering the aggregated outranking matrix \( H \), we obtain the following binary relations:

- \( h_{21} = 1 \) and \( h_{12} = 0 \) \( \Rightarrow \) \( x_2 > x_1 \),
- \( h_{31} = 1 \) and \( h_{13} = 0 \) \( \Rightarrow \) \( x_3 > x_1 \),
- \( h_{32} = 1 \) and \( h_{23} = 0 \) \( \Rightarrow \) \( x_3 > x_2 \),
- \( h_{34} = 1 \) and \( h_{43} = 0 \) \( \Rightarrow \) \( x_3 > x_4 \),
- \( h_{42} = 1 \) and \( h_{24} = 0 \) \( \Rightarrow \) \( x_4 > x_2 \).

Thus, the ranking order of alternatives is found as \( x_3 > x_4 > x_2 > x_1 \).
7 Comparison and Discussion

Algorithm 2 is the more general than Algorithm 1, namely, it is the version that takes into account parameter weights. Therefore, in this part, we will discuss the performances of Algorithms 2, 3 and 4. All of these algorithms can be used to deal with multi-criteria group decision problems involving incomplete information. However each of them has a different operating philosophy, while they have one common goal which is to combine the evaluations of multiple decision makers and then to propose an optimal choice. They can also offer a choice according to the assessment of only one decision maker. While Algorithm 2 can be applied if the decision makers specify the same impact coefficient for “undetermined”, there is no such limitation for Algorithms 3 and 4. The computational performance of each of our algorithms is critically analyzed by the experimental studies, and hence creating the following Table 19.

[Insert Table 19]

As shown in Table 19, the outputs of Algorithms 3 and 4 are identical. For many decision making problems, the results obtained by using these two algorithms are either identical or cover each other. Moreover, all of these algorithms can be used for Examples 4.5 and 4.6 and produce same results. For the problems in Examples 5.1 and 6.1, the outputs of the algorithms coincide. These results support the efficiency and usefulness of the proposed algorithms.

On the other hand, since three-valued soft set is an extension of soft set, the emerging algorithms can be applied to decision making problems based on the soft set(s). So, we endeavor to compare the results of our algorithms with some of the existing soft decision making algorithms. The details supporting this argument are presented in Table 20.

[Insert Table 20]

In Table 20, the weights of parameters in each of problems are considered equally when making calculations in Algorithms 2, 3 and 4 (for instance, in Application (Sect. 5) in [38], the parameter set is $X = \{x_1, x_2, x_3\}$ and so we specify $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$). Also, $\xi$ can be arbitrarily chosen in accordance with the comments on the selection of impact coefficient mentioned above. It is clear that the arbitrary selection of $\xi$ will not change the result(s).

As shown in Table 20, the results of our algorithms coincide with those of existing soft decision making algorithms. For Example 5.17 in [35] and Example 3.3 in [36], the optimal choice by the algorithms proposed in [35, 36] is \{h_1, h_2, h_3\} while the optimal choice by Algorithm 3 is $h_1$ (where $h_1 > h_3 > h_2$). This is not a contradiction, and this is the effect of normalizing the decision matrices in the model of TOPSIS (Algorithm 3). Considering the result of Algorithm 4 for the same problems, we say that $h_1$ and $h_2$ are incomparability, and $h_3$ and $h_2$ are incomparability. This does not contradict the result that the optimal choice is \{h_1, h_2, h_3\}. Because what’s certain that $h_1 > h_3?h_5$, $h_2 > h_4?h_5$ and $h_3 > h_4?h_5$. Consequently, the applicability of our algorithms in decision making based on both soft set and three-valued soft set demonstrates their performance range and advantages.

8 Conclusion

In this paper, we have defined three-valued soft set which is a generalization of the soft set and its set-theoretic operations like intersection, union, difference and symmetric difference. Moreover, we have described the basic relationships on three-valued soft sets, highlighted the corresponding generalization of the operations on soft sets to these sets and provided some examples for them. We have created the algorithms supporting multi-criteria decision making for three-valued soft set based on TOPSIS and ELECTRE techniques and compared their outputs. Thus,
we have pointed out that these algorithms show the applicability and efficiency of three-valued soft sets in handling the multi-criteria decision making involving uncertain or incomplete information. We hope that this work will contribute to the decision making under uncertain and incomplete information in the context of soft sets and also will provide new ideas for future studies related to soft sets. Also, it will motivate researchers for the use of three-valued logic stems in many practical applications such as the data mining, data selection, data integration, data analysis, control of production processes and pattern evaluation. In the near future, we intend to explore new operations on three-valued soft sets and their practical applications in the fields such as science, social science, medical science, environmental science, economics and so on.

References


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Biographies

Eyüp Akçetin is an Associate Professor at the Department of Accounting and Financial Management, Seydikemer School of Applied Sciences of Muğla Sıtkı Koçman University in Turkey. He received his MSc degrees in Logistics and Maritime Transport from Dokuz Eylül University in 2007 and also he received his another MSc degrees in Informatics and Computer Engineering from Sakarya University in 2014 Turkey. He received his PhD degrees in Maritime Economy and Maritime Business Management from Istanbul University in 2012 Turkey. His research interests include multi criteria decision making, game theory, business intelligence and data mining, gamification, systems dynamics in business. He has contributed with many articles, books and book chapters on these scientific topics.

Hüseyin Kanacı is a Research Assistant at the Mathematics Department in the Science and Arts Faculty of Yozgat Bozok University, Turkey. He received his MSc and PhD degrees in Mathematics from Bozok University, Yozgat, Turkey in 2014 and 2018, respectively. His research interests include mathematical logic, set theory, operational research, computational intelligence, decision making. He has many valuable publications on these issues in different scientific journals.
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Figure 2: Three-valued logic for reviewer recommendation in journal
Figure 3: Three-valued logic for Cilia-Related Lesions in Hydrocephalic Mice (see 46)
Figure 4: Comparison for some multimedia features of four mobiles

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Table 3: The tabular form of $T_{Q_2}$ for $Q_2 = \mathcal{P}$.
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Table 7: The tabular form of $T_{Q_1} \setminus T_{Q_2}$.
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Table 20: The comparison results of Algorithms 2, 3 and 4 with some existing soft decision making algorithms.
Figure 1: Three-valued logic for classifying mail in a mailbox

Figure 2: Three-valued logic for reviewer recommendation in journal
<table>
<thead>
<tr>
<th>Gene Allele Symbol</th>
<th>Rhinosinusitis</th>
<th>Otitis Media</th>
<th>Situs Inversus</th>
<th>Infertility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ulk4^milkex</td>
<td>Yes</td>
<td>Yes</td>
<td>Undetermined</td>
<td>Undetermined</td>
</tr>
<tr>
<td>Nme5^O(OST3R08)lEx</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Nme7^G(OST3R08)lEx</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Kif27^G(OST3R08)lEx</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Undetermined</td>
</tr>
<tr>
<td>Stk36^milkex</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Dpcd^G(OST3R08)lEx, PolI^G(OST3R08)lEx</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Ak7^G(OST3R08)lEx</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Ak8^G(OST3R08)lEx</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>493044A02Rik^G(OST2442093)lEx</td>
<td>No</td>
<td>No</td>
<td>Undetermined</td>
<td>Yes</td>
</tr>
<tr>
<td>Celar2^milkex</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Mboat2^milkex</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Tg(FZD3)lEx</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

Figure 3: Three-valued logic for Cilia-Related Lesions in Hydrocephalic Mice (see [46])

Figure 4: Comparison for some multimedia features of four mobiles
List of Tables:

Table 1: The side effects specified in the prospectus of drugs.

<table>
<thead>
<tr>
<th></th>
<th>$p_1 = \text{allergy}$</th>
<th>$p_2 = \text{fetal damage}$</th>
<th>$p_3 = \text{pharmacological effect}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>✓</td>
<td>×</td>
<td>--</td>
</tr>
<tr>
<td>$x_2$</td>
<td>×</td>
<td>--</td>
<td>✓</td>
</tr>
<tr>
<td>$x_3$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>$x_4$</td>
<td>×</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>$x_5$</td>
<td>×</td>
<td>✓</td>
<td>×</td>
</tr>
</tbody>
</table>

In the table, the symbols ✓, × and -- represent Yes, No and Undetermined, respectively.

Table 2: The tabular form of three-valued soft set $T_{Q_1}$ in Example 3.4.

<table>
<thead>
<tr>
<th>$X/P$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: The tabular form of $T_{Q_2}$ for $Q_2 = P$.

<table>
<thead>
<tr>
<th>$X/P$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 4: The tabular form of $T_{Q_2}$ for $Q_2 = \mathcal{P}$.

<table>
<thead>
<tr>
<th>$\chi / \mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5: The tabular form of $\widetilde{T_{Q_2}}$.

<table>
<thead>
<tr>
<th>$\chi / \mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6: The tabular form of $T_{Q_1 \triangle T_{Q_2}}$.

<table>
<thead>
<tr>
<th>$\chi / \mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7: The tabular form of $T_{Q_1 \bigcap T_{Q_2}}$.

<table>
<thead>
<tr>
<th>$\chi / \mathcal{P}$</th>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_3$</th>
<th>$p_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 8: The choice value for three-valued soft set \( T_{Q_1} \) in Example 3.4

<table>
<thead>
<tr>
<th>( \mathcal{X}/\mathcal{P} )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( \xi = 1 )</th>
<th>( \xi = \frac{1}{3} )</th>
<th>( \xi = \frac{2}{3} )</th>
<th>( \xi = \sqrt{13} )</th>
<th>( \xi = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>1.5</td>
<td>1.8408</td>
<td>1.3535</td>
<td>1.0824</td>
<td>1.0009</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>2</td>
<td>2.6816</td>
<td>1.707</td>
<td>1.1648</td>
<td>1.0018</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>2.5</td>
<td>2.8408</td>
<td>2.3535</td>
<td>2.0824</td>
<td>2.0009</td>
</tr>
</tbody>
</table>

Table 9: The ranking preference order of objects for three-valued soft set \( T_{Q_1} \) in Example 3.4

<table>
<thead>
<tr>
<th>( \xi = 1 )</th>
<th>( \alpha_4 &gt; \alpha_1 = \alpha_3 &gt; \alpha_2 )</th>
<th>( x_4 &gt; x_1 \neq x_3 &gt; x_2 )</th>
<th>( \alpha_4 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi = \frac{1}{3} )</td>
<td>( \alpha_4 &gt; \alpha_3 &gt; \alpha_1 &gt; \alpha_2 )</td>
<td>( x_4 &gt; x_3 &gt; x_1 &gt; x_2 )</td>
<td>( \alpha_4 )</td>
<td>( x_4 )</td>
</tr>
<tr>
<td>( \xi = \frac{2}{3} )</td>
<td>( \alpha_4 &gt; \alpha_1 &gt; \alpha_3 &gt; \alpha_2 )</td>
<td>( x_4 &gt; x_1 &gt; x_3 &gt; x_2 )</td>
<td>( \alpha_4 )</td>
<td>( x_4 )</td>
</tr>
<tr>
<td>( \xi = \sqrt{13} )</td>
<td>( \alpha_4 &gt; \alpha_1 &gt; \alpha_3 &gt; \alpha_2 )</td>
<td>( x_4 &gt; x_1 &gt; x_3 &gt; x_2 )</td>
<td>( \alpha_4 )</td>
<td>( x_4 )</td>
</tr>
<tr>
<td>( \xi = 10 )</td>
<td>( \alpha_4 &gt; \alpha_1 &gt; \alpha_3 &gt; \alpha_2 )</td>
<td>( x_4 &gt; x_1 &gt; x_3 &gt; x_2 )</td>
<td>( \alpha_4 )</td>
<td>( x_4 )</td>
</tr>
</tbody>
</table>

Table 10: The table of three-valued soft set \( T_{Q_1 \cap T_{Q_2}} \)

<table>
<thead>
<tr>
<th>( \mathcal{X}/\mathcal{P} )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( \sum_j(x_{ij}^\xi)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1.25</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>0.75</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Table 11: The table of three-valued soft set \( T_{Q_1 \cup T_{Q_2}} \)

<table>
<thead>
<tr>
<th>( \mathcal{X}/\mathcal{P} )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( \sum_j(x_{ij}^\xi)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>0.25</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>2.25</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 12: The weighted choice value for three-valued soft set \( T_{Q_1} \) in Example 3.4

<table>
<thead>
<tr>
<th>( \mathcal{X}/\mathcal{P} )</th>
<th>( p_1(\omega_1 = 0.6) )</th>
<th>( p_2(\omega_2 = 0.3) )</th>
<th>( p_3(\omega_3 = 0.1) )</th>
<th>( p_4(\omega_4 = 0) )</th>
<th>( \alpha_i^\omega = \sum_j \omega_j^1 \times (x_{ij})^\xi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 )</td>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>0.7</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>1</td>
<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>0.35</td>
</tr>
<tr>
<td>( x_3 )</td>
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<td>( \frac{1}{7} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>0.8</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( \frac{1}{7} )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Table 13: The weights of choice parameters of experts \( E_1 \) and \( E_2 \).

<table>
<thead>
<tr>
<th></th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
<th>( \omega_5 )</th>
<th>( \omega_6 )</th>
<th>( \omega_7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Expert ( (E_1) )</td>
<td>0.22</td>
<td>0.15</td>
<td>0.13</td>
<td>0.15</td>
<td>0.13</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td>Second Expert ( (E_2) )</td>
<td>0.24</td>
<td>0.12</td>
<td>0.12</td>
<td>0.16</td>
<td>0.14</td>
<td>0.1</td>
<td>0.12</td>
</tr>
</tbody>
</table>

\( \sum_j \omega_j^1 = 1 \)
Table 14: The table of three-valued soft sets of experts $E_1$ and $E_2$.

<table>
<thead>
<tr>
<th>Decision makers</th>
<th>First Expert ($E_1$)</th>
<th>Second Expert ($E_2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X/P$</td>
<td>$p_1$ $p_2$ $p_3$ $p_4$ $p_5$ $p_6$ $p_7$</td>
<td>$p_1$ $p_2$ $p_3$ $p_4$ $p_5$ $p_6$ $p_7$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1 0 0 $\frac{1}{2}$ $\frac{1}{2}$ 1 0</td>
<td>$\frac{1}{2}$ $\frac{1}{2}$ 1 0 $\frac{1}{2}$ 1 0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1 $\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{1}{2}$ $\frac{1}{2}$ 1</td>
<td>1 $\frac{1}{2}$ 1 1 $\frac{1}{2}$ 1 1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{2}$ 0 $\frac{1}{2}$ 1 $\frac{1}{2}$ 0 1</td>
<td>$\frac{1}{2}$ 0 0 1 1 0 1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{1}{2}$ 1 1</td>
<td>0 $\frac{1}{2}$ 1 0 0 1 $\frac{1}{2}$</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0 $\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{1}{2}$ 0 1</td>
<td>$\frac{1}{2}$ 0 0 1 0 1 1</td>
</tr>
<tr>
<td>$x_6$</td>
<td>$\frac{1}{2}$ $\frac{1}{2}$ 1 1 0 1 1</td>
<td>1 1 1 0 1 1</td>
</tr>
</tbody>
</table>

Impact coefficient $\xi_k$: $\xi_1 = 1.5$, $\xi_2 = 2$

Table 15: The table of separation measurements $S^7_i$ and $S^4_i$.

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^7_i$</td>
<td>0.1542</td>
<td>0.0787</td>
<td>0.167</td>
<td>0.1705</td>
<td>0.178</td>
</tr>
<tr>
<td>$S^4_i$</td>
<td>0.0932</td>
<td>0.1923</td>
<td>0.1109</td>
<td>0.1004</td>
<td>0.0964</td>
</tr>
</tbody>
</table>

Table 16: The weights of the decision maker’s choice parameters.

<table>
<thead>
<tr>
<th>Decision makers/Weights</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
<th>$\omega_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engineering expert ($E_1$)</td>
<td>0.2</td>
<td>0.15</td>
<td>0.5</td>
<td>0</td>
<td>0.15</td>
<td>0</td>
</tr>
<tr>
<td>Financial expert ($E_2$)</td>
<td>0.15</td>
<td>0.25</td>
<td>0.25</td>
<td>0</td>
<td>0.1</td>
<td>0.25</td>
</tr>
<tr>
<td>Quality control expert ($E_3$)</td>
<td>0.2</td>
<td>0.2</td>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Table 17: The table of the expert’s three-valued soft sets $T_{Q_1}T_{Q_2}T_{Q_3}$.

<table>
<thead>
<tr>
<th>Decision makers</th>
<th>Financial expert ($E_1$)</th>
<th>Quality control expert ($E_2$)</th>
<th>Engineering expert ($E_3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X/P$</td>
<td>$p_1$ $p_2$ $p_3$ $p_4$ $p_5$ $p_6$</td>
<td>$p_1$ $p_2$ $p_3$ $p_4$ $p_5$ $p_6$</td>
<td>$p_1$ $p_2$ $p_3$ $p_4$ $p_5$ $p_6$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1 $\frac{1}{2}$ 0 0 1 0</td>
<td>0 $\frac{1}{2}$ 1 0 $\frac{1}{2}$ 0</td>
<td>$\frac{1}{2}$ 0 0 0 $\frac{1}{2}$ 0</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\frac{1}{2}$ 1 1 0 0 0</td>
<td>1 0 0 0 0 1</td>
<td>0 $\frac{1}{2}$ 0 0 $\frac{1}{2}$ 0</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0 1 1 0 $\frac{1}{2}$ 0</td>
<td>1 1 1 0 $\frac{1}{2}$ 0</td>
<td>1 0 0 0 $\frac{1}{2}$ 1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$\frac{1}{2}$ $\frac{1}{2}$ 0 1 0</td>
<td>$\frac{1}{2}$ 0 1 0 1</td>
<td>1 1 0 $\frac{1}{2}$ 0</td>
</tr>
</tbody>
</table>

Impact coefficient $\xi_k$: $\xi_1 = 2$, $\xi_2 = 3$, $\xi_3 = 2$

Table 18: The table of sets of concordance and discordance.

<table>
<thead>
<tr>
<th>Concordance set ($J^+$)</th>
<th>Discardance set ($J^-$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>-</td>
<td>{1,4,5}</td>
</tr>
<tr>
<td>{2,3,4,6}</td>
<td>-</td>
</tr>
<tr>
<td>{1,2,3,4,6}</td>
<td>{1,2,3,4,5,6}</td>
</tr>
<tr>
<td>{2,4,6}</td>
<td>{1,2,4,5,6}</td>
</tr>
</tbody>
</table>
Table 19: Table of comparing and matching the results of Algorithms 2, 3 and 4.

<table>
<thead>
<tr>
<th>Problems</th>
<th>Impact coefficients</th>
<th>Algorithm 2</th>
<th>Algorithm 3</th>
<th>Algorithm 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 4.5</td>
<td>$\xi = 1$ (i.e., $\xi_1 = 1, \xi_2 = 1$)</td>
<td>$x_3$</td>
<td>$x_3$</td>
<td>$x_3$</td>
</tr>
<tr>
<td>Example 4.6</td>
<td>$\xi = 1$ (i.e., $\xi_1 = 1, \xi_2 = 1$)</td>
<td>$x_1$</td>
<td>$x_1$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>Example 5.1</td>
<td>$\xi_1 = 1.5, \xi_2 = 2$</td>
<td>$-$</td>
<td>$x_2$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>Example 6.1</td>
<td>$\xi_1 = 2, \xi_2 = 3, \xi_3 = 2$</td>
<td>$-$</td>
<td>$x_3$</td>
<td>$x_3$</td>
</tr>
</tbody>
</table>

"-" means that the algorithm is not applicable to this problem.

Table 20: The comparison results of Algorithms 2, 3 and 4 with some existing soft decision making algorithms.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Problem in the paper</th>
<th>Algorithm in the paper</th>
<th>Algorithm 2</th>
<th>Algorithm 3</th>
<th>Algorithm 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>38</td>
<td>Application (Sect. 5) in 38</td>
<td>$u_1$</td>
<td>$u_1$</td>
<td>$u_1$</td>
<td>$u_1$</td>
</tr>
<tr>
<td>35, 36</td>
<td>Example 5.17 in 35</td>
<td>$h_1, h_2, h_3$</td>
<td>$h_1, h_2, h_3$</td>
<td>$h_1$</td>
<td>$h_1 &gt; h_3 &gt; h_4 ? h_5 &amp;$</td>
</tr>
<tr>
<td>59</td>
<td>Example 31 in 59</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
</tr>
<tr>
<td>32</td>
<td>Table 2 (Sect. 3.4) in 32</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
<td>$h_1, h_6$</td>
</tr>
</tbody>
</table>

For Algorithms 2, 3 and 4, the weights of parameters in decision problem are taken equally and their sum is 1.