Three-valued soft set and its multi-criteria group decision making via TOPSIS and ELECTRE

Eyüp Akçetin[1] , Hüseyin Kamacı[2,*]

[1] Department of Accounting and Financial Management, Seydikemer School of Applied Sciences,
Muğla Sıtkı Koçman University, Muğla, Turkey
E-mail: eyup.akcetin@mu.edu.tr

[2] Department of Mathematics, Faculty of Science and Arts,
Yozgat Bozok University, Yozgat, Turkey
E-mails: huseyin.kamaci@hotmail.com, huseyin.kamaci@bozok.edu.tr
Tel:+90 354 242 10 21, Fax: +90 354 242 10 22

Abstract

The purpose of this paper is to introduce a generalization of Molodtsov's approach to soft sets obtained by passing from the classical two-valued logic underlying those sets to a three-valued logic, where the third truth value can usually be interpreted as either non-determined (i.e., between true and false) or unknown. This extension of soft set approach allows for more intuitive and clearer representation of various decision making problems involving incomplete or uncertain information. In other words, it is a useful way to analyze soft set based multi-criteria group decision making problems under the lack of information resulting from the inability to determine the data. In this paper, we introduce the concept of three-valued soft set and its some operations and products. We propose the formulas to calculate all possible choice values for each object in the (weighted) three-valued soft sets, and thus calculate their respective decision values. By modifying the TOPSIS and ELECTRE methods to deal with multi-criteria group decision problems, three-valued soft set based decision making algorithms are constructed. To demonstrate the practicality of these algorithms, we address the examples adapted from the decision problems in real-life. Lastly, some aspects of the efficiency of the proposed algorithms are discussed with computational experiments.

Key words: Soft set, three-valued soft set, choice value, decision making, TOPSIS, ELECTRE.

1 Introduction

Classically, a logic is two-valued (Boolean) if every proposition is either false ("0") or true ("1"). In 1930, Łukasiewicz [1] initiated three-valued logic, a natural extension of the two-valued logic, in which there are three truth (logical) values indicating false ("0"), true ("1") and some indeterminate third value ("$\frac{1}{2}$" as something in the middle between true and false). Thus, he pioneered the conceptual form and basic ideas of three-valued logic. By interpreting

*Corresponding author.

the intuition of Łukasiewicz three-valued logic from different perspectives, two-valued logic were extended to three-valued logic in various ways [2–5]. They differ by their choice of basic connectives, therefore also from a syntactic and proof-theoretic point of view. Although the idea of fuzzification of logic was envisaged by several researchers in the years following 1920, the concept of fuzzy logic in which the truth values of variables may be any real number between 0 and 1 both inclusive was explicitly and crisply proposed by Zadeh [6] in 1965. Fuzzy logic is based on the investigation and observation that people make decisions based on uncertain and non-numerical information. Fuzzy set, which is a generalization of the crisp set based on the two-valued logic, is mathematical mean of representing vagueness and imprecise information. This set is described by a membership function that assigns to each object a degree of membership ranging between 0 and 1. Pawlak [7] defined rough set, which can be considered as a new area of uncertainty mathematics closely related to set theory. This set is a formal approximation of a crisp set in terms of a pair of sets which give the lower and the upper approximations of the original set. The approximation spaces of rough set theory are sets with multiple memberships, while fuzzy sets are concerned with partial memberships. These sets are combined to derive the different variations such as the fuzzy rough set and the rough fuzzy set. In 1999, Molodtsov [8] described the soft set, based on two-valued logic, as a mathematical tool that deals with parametric data which are imprecise or uncertain in nature. In 2003, Maji et al. [9] published a study on the operations of soft sets, and in the later years the operational laws of the soft sets were derived [10–14]. Also, many authors described and discussed various types of soft sets such as bijective soft set [15], exclusive disjunctive soft set [16], bipolar soft set [17], inverse soft set [18, 19] and so on.

The decision making, which is one of the issues involving uncertainty, is a problem that is frequently encountered in many commercial and scientific fields, even in every stage of daily life. For the deterministic modeling of decision making problems, many mathematical techniques such as TOPSIS, AHP, VIKOR, ELECTRE and PROMETHEE were developed. These mathematical techniques were adapted for decision making based the fuzzy set and its some extensions [20–26]. In addition to the fuzzy modelling of TOPSIS, ELECETRE and etc., many algorithmic solutions were proposed for decision making under fuzzy environment [27–31]. In 2002, Maji et al. [32] put forward that the soft sets can be used to solve decision making problems involving parametric data based on two-valued logic. In the following years, many soft decision algorithms were created and their applications for problems in real life were included [33–37]. Moreover, Eraslan [38] proposed a decision making procedure via classical method of TOPSIS on the soft sets. Thus, he pointed out that classical decision making techniques may be successfully applied to the soft set based decision making.

In 2008, Avron and Konikowska [39] explored the idea of describing Pawlak's rough set utilizing three-valued logic. This paradigm presents a different perspective in the interpretation of issues containing indetermined or unknown data in many fields. As with the rough set, three-valued logic emerges in many real world scenes that are included in the scope of the soft set and it is difficult to deal with such broadly scoped issues whose third truth value is "undetermined". In this study, we discuss a completely designed approach to soft set using three-valued logic. Thus, in practice, we aim to overcome the difficulties which include a third truth value caused by uncertain or unknown origin in addition to the truth values of two-valued logic. The present paper is motivated by the idea of proposing the notion of three-valued soft set that to try to describe soft set using three-valued logic. The fusion of three-valued logic into the soft set provides a clearer and intuitive way to explain various issues under incomplete or uncertain information. Therefore, we focus on the decision making based on this type of soft set and elaborate various algorithmic solutions.

The remainder of this paper is set out as follows. Section 2 gives an outline of soft set theory. Section 3 is devoted to the motivation to reinterpret soft sets using three-valued logic. In Section 4, we endeavor to calculate the choice value of an object in (weighted) three-valued soft sets, and accordingly propose two algorithms. In Section 5, we create a multi-criteria group decision making algorithm based the modified TOPSIS on three-valued soft sets. In Section 6, we propose a three-valued soft decision making algorithm via the modified ELECTRE, which is based on three fundamental objectives that choosing, sorting and ranking. Also, the examples are given to analyze the performance of the algorithms emerging in these two sections. In Section 7, the matching numerical examples are solved to compare the results of the proposed algorithms and thus show that they are convincing. Final section is dedicated to conclusions and plans for further research.

2 Preliminaries

In this section, as preparation for introducing new concepts, some relevant arguments of soft set, two-valued logic and three-valued logic are given.

Let's first talk about the soft set theory. In 1999, Molodtsov [8] introduced soft set theory as a useful way of classifying objects according to parametric data. In 2011, Çağman and Enginoğlu [40] recreated the soft sets to make their operations more practical in some cases. Maji et al. [32] put forward that the soft set could be represented in a tabular form. Thus, they showed that the soft sets are the parametric sets created using two-valued logic (i.e., 0 as false, 1 as true). Now, let us recall the definition of soft set.

From now on, $\mathfrak{X}$ is a universal set, $\mathcal{P}$ is a parameter set and $Q \subseteq \mathcal{P}$.

Definition 2.1. ([8,40]) Assume $\mathfrak{X}^{(\{0,1\})}$ denotes the set of all functions from $\mathfrak{X}$ to $\{0,1\}$. A pair $(f_Q, \mathcal{P}) = F_Q$ is called a soft set over $\mathfrak{X}$ when the mapping f_Q is defined by $f_Q : \mathcal{P} \to \mathfrak{X}^{(\{0,1\})}$, where for all $p_j \in \mathcal{P}$

$$f_Q(p_j) = \begin{cases} \{x_i^{\langle \lambda^i_{f_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{f_{Q(p_j)}} \in \{0,1\}\}, & \text{if } p_j \in Q \\ \{x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X}\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

Example 2.2. In daily life, many operations like shopping, booking, money transfer can be done through online websites. Recently, hotel room reservations for trips and holidays are made through many websites such as Booking.com and Expedia.com. Now let's consider the following problem to make a hotel room reservation.
Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five hotels that are available for booking a room. Let $\mathcal{P} = \{p_1 = price, p_2 = location, p_3 = amenities, p_4 = satisfaction\}$ be a set of the choice parameters. Then, we can create the following soft set over $\mathfrak{X}$.

$$\begin{aligned} T_{\mathcal{P}} = \{&(p_1, \{x_1^{\langle 0 \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 1 \rangle}, x_4^{\langle 0 \rangle}, x_5^{\langle 0 \rangle}\}), (p_2, \{x_1^{\langle 1 \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 0 \rangle}, x_4^{\langle 0 \rangle}, x_5^{\langle 1 \rangle}\}), \\ &(p_3, \{x_1^{\langle 1 \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 0 \rangle}, x_4^{\langle 1 \rangle}, x_5^{\langle 0 \rangle}\}), (p_4, \{x_1^{\langle 0 \rangle}, x_2^{\langle 1 \rangle}, x_3^{\langle 1 \rangle}, x_4^{\langle 0 \rangle}, x_5^{\langle 1 \rangle}\})\}. \end{aligned}$$

For the first pair in this soft set, we can interpret that the prices of hotels x_2, x_3 are suitable for us but the prices of hotels x_1, x_4, x_5 is not suitable. Other pairs can be interpreted similarly.

In 1930, Łukasiewicz [1] put forward (Lukasiewicz) three-valued logic extending two-valued logic (Boolean logic). Immediately after, many authors interested in the idea of three-valued logic and its operations [2,4,39,41,42]. They argued that since two-valued logic (Boolean logic) is so far all kinds of scientific investigations, three-valued logic may be useful and as basic for a number of useful reasoning tasks. Boolean connectives can be extended to three-valued logic in various ways. That is, there are various ways of explicating third truth value that is different from true and false. Ciucci and Dubois [43] listed these ways as follows:

- *Possible*: this explanation was proposed by Łukasiewicz [1,2], the pioneer of three-valued logic. A proposition is "Possible" if its truth value will be only known in the future.

- *Unknown*: this explanation was proposed by Kleene [4] in the year 1952. A proposition is "Unknown" if its truth value cannot be computed for some reason (for instance, it is too time-consuming to do it).

- *Inconsistent*: the third value stands for a proposition which is both true and false, and also it is the dual of "unknown" in some sense.

- *Half-true*: this is typical of fuzzy logic [3]. The intuition is that for some propositions, truth is a importance degree. For instance, Shadowed set in [44,45] is based on the idea of turning fuzzy set into three-valued set.

- *Undefined*: this is another explanation of Kleene. The undefined state corresponds to the selection of the argument of the function outside its definition domain. A proposition is "Undefined" if its truth value involves undefined atoms.

- *Irrelevant*: the idea is that propositions are not applicable in some possible worlds.

In 1960, Skolem [5] initiated a set theory based on a certain three-valued logic. In this set theory, the variables such as $p, q, r, ...$ take three values $0, \frac{1}{2}, 1$. We may interpret 0 as "false", 1 as "true" and $\frac{1}{2}$ as something in the middle between true and false, say "undetermined". Moreover, Skolem presented a set of truth tables showing tree-valued logic operations like negation, disjunction and conjunction (for a detailed review, see [5]). In the literature, there are many truth tables illustrating tree-valued logic operations. However, this study will focus on the truth tables proposed by Skolem [5].

Three-valued logic emerges in several scenes of the real world. Three samples within the atmosphere of "undetermined" are presented in Figures 1, 2 and 3. In these figures, "?" symbolizes "undetermined", i.e., $\frac{1}{2}$.

[Insert Figure 1]

[Insert Figure 2]

[Insert Figure 3]

Since the truth value of "undetermined" in Figures 1 and 2 will be precisely known in the future, this can be considered as "Possible". Figure 3 can also be considered as an example for "Unknown".

3 Three-valued soft sets

In this part, we discuss three-valued soft set which is the generalization of soft set obtained by passing from two-valued logic to three-valued logic.

The soft set is a set approach proposed using two-valued logic (true and false). In daily life, while evaluating alternatives according to parameters in decision making process, sometimes "undetermined" (neither true nor false, or both true and false) arises. In such decision making processes, the soft sets are insufficient. To overcome this shortcoming, the notion of three-valued soft set, which is a soft set using three-valued logic (i.e., 0 as false, 1 as true and $\frac{1}{2}$ as "undetermined") is created.

3.1 Three-valued soft set

Definition 3.1. Assume $\mathfrak{X}^{\langle\{0,\frac{1}{2},1\}\rangle}$ denotes the set of all functions from $\mathfrak{X}$ to $\{0,\frac{1}{2},1\}$. A pair $(t_Q,\mathcal{P})=T_Q$ is called a three-valued soft set over $\mathfrak{X}$ when the mapping t_Q is defined by $t_Q:\mathcal{P}\to\mathfrak{X}^{\langle\{0,\frac{1}{2},1\}\rangle}$, where for all $p_j\in\mathcal{P}$

$$t_Q(p_j) \quad = \quad \begin{cases} \{x_i^{\langle\lambda^i_{t_{Q(p_j)}}\rangle} : x_i\in\mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}}\in\{0,\frac{1}{2},1\}\}, & \text{if } \ p_j\in Q \\ \{x_i^{\langle 0\rangle}:\forall x_i\in\mathfrak{X}\}, & \text{if } \ p_j\in\mathcal{P}-Q \end{cases}$$

Notation: For the parameter set $\mathcal{P}$, the set of all three-valued soft sets over $\mathfrak{X}$ is denoted by $\mathcal{TVSS}(\mathfrak{X},\mathcal{P})$.

Example 3.2. Assume $\mathfrak{X}=\{x_1,x_2,x_3,x_4,x_5\}$ is a set of drugs that can be taken by pregnant women and women of childbearing age. The drugs that can be taken by these women, without an increase in the frequency of malformation or other direct or indirect harmful effects on the human fetus can be observed. Therefore, these women should carefully check their side effects while using the drug. Suppose that $\mathcal{P}=\{p_1=\textit{allergy}, p_2=\textit{fetal damage}, p_3=\textit{pharmacological effect}\}$ denotes some side effects of the drugs. One obtains the following table (Table 1) by examining the prospectus of drugs.

[Insert Table 1]

According to Table 1, the following three-valued soft set is constructed.

$$T_{\mathcal{P}}=\{(p_1,\{x_1^{\langle 1\rangle},x_2^{\langle 0\rangle},x_3^{\langle 1\rangle},x_4^{\langle 0\rangle},x_5^{\langle 0\rangle}\}),(p_2,\{x_1^{\langle 0\rangle},x_2^{\langle\frac{1}{2}\rangle},x_3^{\langle 0\rangle},x_4^{\langle\frac{1}{2}\rangle},x_5^{\langle 1\rangle}\}),(p_3,\{x_1^{\langle\frac{1}{2}\rangle},x_2^{\langle 1\rangle},x_3^{\langle 0\rangle},x_4^{\langle\frac{1}{2}\rangle},x_5^{\langle 0\rangle}\}).$$

Example 3.3. Let $\mathfrak{X}=\{x_1=\textit{Vivo V17 Pro}, x_2=\textit{OnePlus 7}, x_3=\textit{Xiaomi Redmi K20 Pro}, x_4=\textit{Apple IPhone 11}\}$ be a set of four mobiles and $\mathcal{P}=\{p_1=\textit{FM Radio}, p_2=\textit{Stereo Speakers}, p_3=\textit{Loudspeaker}\}$ be a set of multimedia features (attributes) that may be available on phones. Through the website "*www.91mobiles.com*" (date: 17.10.2019), we have Figure 4.

[Insert Figure 4]

According to Figure 4, we create the following three-valued soft set.

$$T_{\mathcal{P}}=\{(p_1,\{x_1^{\langle 1\rangle},x_2^{\langle 0\rangle},x_3^{\langle\frac{1}{2}\rangle},x_4^{\langle 0\rangle}\}),(p_2,\{x_1^{\langle\frac{1}{2}\rangle},x_2^{\langle 1\rangle},x_3^{\langle\frac{1}{2}\rangle},x_4^{\langle 1\rangle}\}),(p_3,\{x_1^{\langle 1\rangle},x_2^{\langle 1\rangle},x_3^{\langle 1\rangle},x_4^{\langle 1\rangle}\})\}.$$

Considering the information presented on this website, we can interpret that:
For the multimedia feature p_1 (FM Radio),
-the mobile x_1 (Vivo V17 Pro) has an FM Radio,
-the mobiles x_2 and x_4 (OnePlus 7 and Apple IPhone 11) does not have an FM Radio,
-it is "*undetermined*" whether the mobile x_3 (Xiaomi Redmi K20 Pro) has an FM Radio.

Comment: The truth value $\frac{1}{2}$ in the three-valued logic is different from the membership degree 0.5 in the fuzzy logic. The truth value $\frac{1}{2}$ in the three-valued logic represents "undetermined", not membership. Why is this truth value between 0 and 1, not 0? 0 means that given object certainly does't have the desired property, while $\frac{1}{2}$ means that the object might possible have this property. For instance: Assume that, one wants to buy any from the mobiles x_2 and x_3 (in Figure 4). As a criterion for the purchase, it determines the requirement to have only FM Radio. In such situations, it is more convenient to select the mobile x_3. Because it is not known (undetermined) whether the mobile x_3 has an FM Radio or not while the mobile x_2 has no FM Radio. So there is also the possibility that the mobile x_3 has an FM Radio. Indeed, we reconsider through the website "www.smartprix.com" that the mobile x_3 (Xiaomi Redmi K20 Pro) has an FM Radio.

Example 3.4. Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$ be a set of four investments at the disposal of the investor to invest some money and $\mathcal{P} = \{p_1 = riskless, p_2 = security, p_3 = tax\ free, p_4 = short\ period\}$ be a set of parameters. For parameter subset $Q_1 = \{p_1, p_2, p_3\}$, an investor can create the following three-valued soft set over $\mathfrak{X}$.

$$T_{Q_1} = \{(p_1, \{x_1^{\langle 1\rangle}, x_2^{\langle 0\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle \frac{1}{2}\rangle}\}), (p_2, \{x_1^{\langle 0\rangle}, x_2^{\langle 1\rangle}, x_3^{\langle \frac{1}{2}\rangle}, x_4^{\langle 1\rangle}\}), (p_3, \{x_1^{\langle 1\rangle}, x_2^{\langle \frac{1}{2}\rangle}, x_3^{\langle \frac{1}{2}\rangle}, x_4^{\langle 1\rangle}\})\}.$$

The element $(p_1, \{x_1^{\langle 1\rangle}, x_2^{\langle 0\rangle}, x_3^{\langle 1\rangle}, x_4^{\langle \frac{1}{2}\rangle}\})$ in T_{Q_1} means that
-the investments x_1 and x_3 are risk-free,
-the investment x_2 is risky,
-the investment x_4 is "*undetermined*" in terms of risk.

As shown in the above example, if $p_j \in \mathcal{P}\backslash Q_1$ then the pair $(p_j, t_{Q_1}(p_j))$ does not need to be displayed in the structure of three-valued soft set T_{Q_1}. However, it is known that $(p_j, \{x_i^{\langle 0\rangle} : \forall x_i \in \mathfrak{X}\})$.

Each three-valued soft set can be represented in the form of binary table. This representation provides the usefulness of three-valued soft sets in various computer program languages and the practicality of calculations.

The binary tabular form of three-valued soft set T_{Q_1} in Example 3.4 can be given as in Table 2.

[Insert Table 2]

In Table 2, each component x_{ij} represents the truth value $\lambda^i_{t_{Q(p_j)}}$ of alternative x_i with respect to the parameter p_j. That is, in Table 2 and onward,

- $x_{ij} = 1$ means that x_i belongs to the subset of $\mathfrak{X}$ approximated by the parameter p_j,

- $x_{ij} = 0$ means that x_i does not belong to the subset of $\mathfrak{X}$ approximated by the parameter p_j,

- $x_{ij} = \frac{1}{2}$ means that it is undetermined whether x_i belongs to the subset of $\mathfrak{X}$ approximated by the parameter p_j.

From now on, in the examples, three-valued soft sets will be represented by the binary tables.

Definition 3.5. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. It is called

a) an empty three-valued soft set when $t_Q(p_j) = \{x_i^{\langle 0\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in \mathcal{P}$, and it is denoted by T_Θ.

b) a Q-mid three-valued soft set when $t_Q(p_j) = \{x_i^{\langle \frac{1}{2}\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\widetilde{Q}}$. If $Q = \mathcal{P}$ then the Q-mid three-valued soft set is called a mid three-valued soft set, and it is denoted by $T_{\widetilde{\mathcal{P}}}$.

c) a Q-universal three-valued soft set when $t_Q(p_j) = \{x_i^{\langle 1\rangle} : \forall x_i \in \mathfrak{X}\}$ for all $p_j \in Q$, and it is denoted by $T_{\widehat{Q}}$. If $Q = \mathcal{P}$ then the Q-universal three-valued soft set is called a universal three-valued soft set, and it is denoted by $T_{\widehat{\mathcal{P}}}$.

Definition 3.6. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then

a) T_{Q_1} is a three-valued soft subset of T_{Q_2} when $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for all $p_j \in P$, and it is denoted by $T_{Q_1} \sqsubseteq T_{Q_2}$. Here, $t_{Q_1}(p_j) \subseteq t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means $\lambda^i_{t_{Q_1(p_j)}} \leq \lambda^i_{t_{Q_2(p_j)}}$ for each $x_i \in \mathfrak{X}$.

b) T_{Q_1} and T_{Q_2} are equal three-valued soft sets when $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for all $p_j \in P$, and it is denoted by $T_{Q_1} = T_{Q_2}$. Here, $t_{Q_1}(p_j) = t_{Q_2}(p_j)$ for $p_j \in \mathcal{P}$ means $\lambda^i_{t_{Q_1(p_j)}} = \lambda^i_{t_{Q_2(p_j)}}$ for each $x_i \in \mathfrak{X}$.

Example 3.7. Let us consider three-valued soft set T_{Q_1} given in Table 2 of Example 3.4. Also, we take three-valued soft set T_{Q_2} as in Table 3, respectively.

[Insert Table 3]

Then, it is seen that $T_{Q_2} \sqsubseteq T_{Q_1}$.

Proposition 3.8. *Let* $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i) $T_{\Theta} \sqsubseteq T_{\widetilde{\mathcal{P}}} \sqsubseteq T_{\widehat{\mathcal{P}}}$.

ii) $T_{\Theta} \sqsubseteq T_{Q_k}$ *for each* k.

iii) $T_{Q_k} \sqsubseteq T_{\widehat{\mathcal{P}}}$.

iv) $T_{Q_k} \sqsubseteq T_{Q_k}$ *for each* k..

v) $T_{Q_1} \sqsubseteq T_{Q_2}$ *and* $T_{Q_2} \sqsubseteq T_{Q_3} \Rightarrow T_{Q_1} \sqsubseteq T_{Q_3}$.

Proof. The proofs are obvious from Definitions 3.5 and 3.6, therefore omitted. □

3.2 Operations and products on three-valued soft sets

Definition 3.9. Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then the complement of T_Q, denoted by $\overline{T_Q}$, is defined by the mapping $\overline{t_Q} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that $\overline{t_Q}(p_j) = \{x_i^{\langle \lambda^i_{\overline{t_Q}(p_j)} \rangle} : x_i \in \mathfrak{X}$ and $\lambda^i_{\overline{t_Q}(p_j)} \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in \mathcal{P}$ where

$$\lambda^i_{\overline{t_Q}(p_j)} = 1 - \lambda^i_{t_Q(p_j)}. \tag{3.1}$$

Note: This definition clarifies why the truth value for "undetermined" is $\frac{1}{2}$. The negation of "undetermined" must also have the same truth value, because there is no gauge of "undetermined". Accordingly, we say that "undetermined" implies something in the middle between true and false, i.e., its truth vale is $\frac{1}{2}$.

Example 3.10. We consider the universal set $\mathfrak{X}$ and the parameter set $\mathcal{P}$ in Example 3.4. Furthermore, we generate three-valued soft set T_{Q_2} given in Table 4.

[Insert Table 4]

The complement of T_{Q_2} is obtained as in Table 5.

[Insert Table 5]

Proposition 3.11. *Let* $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.

i) $\overline{(\overline{T_Q})} = T_Q$.

ii) $\overline{T_{\Theta}} = T_{\widehat{\mathcal{P}}}$.

iii) $\overline{T_{\widetilde{\mathcal{P}}}} = T_{\widetilde{\mathcal{P}}}$.

Proof. (i) We consider three-valued soft set $T_Q = (t_Q, \mathcal{P})$ over $\mathfrak{X}$. Then, we have the mapping $t_Q : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that for all $p_j \in \mathcal{P}$

$$t_Q(p_j) = \begin{cases} \{x_i^{\langle \lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\}\}, & \text{if } p_j \in Q \\ \{x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X}\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

By Definition 3.9, we can write for all $p_j \in \mathcal{P}$

$$\overline{t_Q}(p_j) = \begin{cases} \{x_i^{\langle 1-\lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\}\}, & \text{if } p_j \in Q \\ \{x_i^{\langle 1-0 \rangle} : \forall x_i \in \mathfrak{X}\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases}$$

When proceeding in a similar manner, it is seen that for all $p_j \in \mathcal{P}$

$$\begin{aligned} \overline{\overline{t_Q}}(p_j) &= \begin{cases} \{x_i^{\langle 1-(1-\lambda^i_{t_{Q(p_j)}}) \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\}\}, & \text{if } p_j \in Q \\ \{x_i^{\langle 1-(1-0) \rangle} : \forall x_i \in \mathfrak{X}\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases} \\ &= \begin{cases} \{x_i^{\langle \lambda^i_{t_{Q(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{Q(p_j)}} \in \{0, \frac{1}{2}, 1\}\}, & \text{if } p_j \in Q \\ \{x_i^{\langle 0 \rangle} : \forall x_i \in \mathfrak{X}\}, & \text{if } p_j \in \mathcal{P} - Q \end{cases} \end{aligned}$$

Thus, we have $\overline{\overline{t_Q}}(p_j) = t_Q(p_j)$ for all $p_j \in \mathcal{P}$. So, $\overline{(\overline{T_Q})} = T_Q$.
The proofs of (ii) and (iii) are obvious, hence omitted. □

Definition 3.12. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the intersection of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \bar{\sqcap} T_{Q_2}$, is defined by the mapping $t_{Q_1 \bar{\sqcap} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that $t_{Q_1 \bar{\sqcap} Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X}$ and $\lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} = min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.2}$$

Proposition 3.13. *Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.*

i) $T_Q \bar{\sqcap} T_\Theta = T_\Theta$.

ii) $T_Q \bar{\sqcap} T_{\widehat{Q}} = T_Q$.

iii) $T_Q \bar{\sqcap} T_{\widetilde{Q}} \sqsubseteq T_Q$.

Proof. It is clear from Definitions 3.5, 3.6 and 3.12. □

Definition 3.14. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the union of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \bar{\sqcup} T_{Q_2}$, is defined by the mapping $t_{Q_1 \bar{\sqcup} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0,\frac{1}{2},1\})}$ such that $t_{Q_1 \bar{\sqcup} Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \bar{\sqcup} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X}$ and $\lambda^i_{t_{(Q_1 \bar{\sqcup} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \bar{\sqcup} Q_2)(p_j)}} = max\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.3}$$

Proposition 3.15. *Let $T_Q \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.*

i) $T_Q \sqcup T_\Theta = T_Q$.

ii) $T_Q \sqcup T_{\widehat{Q}} = T_{\widehat{Q}}$.

iii) $T_Q \sqcup T_{\widetilde{Q}} \sqsupseteq T_Q$.

Proof. It is clear from Definitions 3.5, 3.6 and 3.14. □

Proposition 3.16. *Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. For all $*, \circ \in \{\bar{\sqcap}, \sqcup\}$,*

i) $T_{Q_k} * T_{Q_k} = T_{Q_k}$.

ii) $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3}$.

iii) $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$.

Proof. The proof of (i) and (ii) is similar to that of (iii), so it is omitted.
(iii) Let us prove that $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$ for $* = \bar{\sqcap}$ and $\circ = \sqcup$.
We consider $T_{Q_1} \bar{\sqcap} (T_{Q_2} \sqcup T_{Q_3})$. Suppose that $T_{Q_2} \sqcup T_{Q_3} = T_R$, where for all $p_j \in \mathcal{P}$

$$t_R(p_j) = t_{Q_2 \sqcup Q_3}(p_j) = \{x_i^{\langle \lambda^i_{t_R(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_R(p_j)} = \lambda^i_{t_{(Q_2 \sqcup Q_3)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda^i_{t_{R(p_j)}} = \lambda^i_{t_{(Q_2 \sqcup Q_3)(p_j)}} = max\{\lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}}\}. \tag{3.4}$$

And assume that $T_{Q_1} \bar{\sqcap} T_R = T_S$, where for all $p_j \in \mathcal{P}$

$$t_S(p_j) = t_{Q_1 \bar{\sqcap} R}(p_j) = \{x_i^{\langle \lambda^i_{t_S(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_S(p_j)} = \lambda^i_{t_{(Q_1 \bar{\sqcap} R)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\begin{aligned} \lambda^i_{t_{S(p_j)}} = \lambda^i_{t_{(Q_1 \bar{\sqcap} R)(p_j)}} &= min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{R(p_j)}}\} \\ &= min\{\lambda^i_{t_{Q_1(p_j)}}, max\{\lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}}\}\}. \end{aligned} \tag{3.5}$$

Now, we consider $(T_{Q_1} \bar{\sqcap} T_{Q_2}) \sqcup (T_{Q_1} \bar{\sqcap} T_{Q_3})$. Assume that $T_{Q_1} \bar{\sqcap} T_{Q_2} = T_U$, where for all $p_j \in \mathcal{P}$

$$t_U(p_j) = t_{Q_1 \bar{\sqcap} Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_U(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_U(p_j)} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda^i_{t_{U(p_j)}} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} = min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.6}$$

And assume that $T_{Q_1} \bar{\sqcap} T_{Q_3} = T_V$, where for all $p_j \in \mathcal{P}$

$$t_V(p_j) = t_{Q_1 \bar{\sqcap} Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_V(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_V(p_j)} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\lambda^i_{t_{V(p_j)}} = \lambda^i_{t_{(Q_1 \bar{\sqcap} Q_2)(p_j)}} = min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.7}$$

Suppose that $T_U \sqcup T_V = T_W$, where for all $p_j \in \mathcal{P}$

$$t_W(p_j) = t_{U \sqcup V}(p_j) = \{x_i^{\langle \lambda^i_{t_W(p_j)} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_W(p_j)} = \lambda^i_{t_{(U \sqcup V)}(p_j)} \in \{0, \frac{1}{2}, 1\}\}$$

such that

$$\begin{aligned} \lambda^i_{t_{W(p_j)}} = \lambda^i_{t_{(U \sqcup V)(p_j)}} &= max\{\lambda^i_{t_{U(p_j)}}, \lambda^i_{t_{W(p_j)}}\} \\ &= max\{min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}, min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_3(p_j)}}\}\}. \end{aligned} \tag{3.8}$$

Since $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}, \lambda^i_{t_{Q_3(p_j)}} \in \{0, \frac{1}{2}, 1\}$, we obtain that $\lambda^i_{t_{S(p_j)}} = \lambda^i_{t_{W(p_j)}}$ for all $p_j \in \mathcal{P}$ (by Eq. (3.5) and Eq. (3.8)). Thereby, we say that T_S and T_W are indeed the same set-valued mapping, so $T_{Q_1} \sqcap (T_{Q_2} \sqcup T_{Q_3}) = (T_{Q_1} \sqcap T_{Q_2}) \sqcup (T_{Q_1} \sqcap T_{Q_3})$.
Other cases can be proved in a similar way. □

Proposition 3.17. *Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the following De Morgan's rules are hold.*

i) $\overline{(T_{Q_1} \sqcap T_{Q_2})} = \overline{T_{Q_1}} \sqcup \overline{T_{Q_2}}$.

ii) $\overline{(T_{Q_1} \sqcup T_{Q_2})} = \overline{T_{Q_1}} \sqcap \overline{T_{Q_2}}$.

Proof. (i) Since $1 - min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\} = max\{1 - \lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}$ for $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \in \{0, \frac{1}{2}, 1\}$, we can say that $\overline{(T_{Q_1} \sqcap T_{Q_2})} = \overline{T_{Q_1}} \sqcup \overline{T_{Q_2}}$.
(ii) It is similar to the proof of (i). □

Definition 3.18. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the difference of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \curlyeqsucc T_{Q_2}$, is defined by the mapping $t_{Q_1 \curlyeqsucc Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that $t_{Q_1 \curlyeqsucc Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \curlyeqsucc Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \curlyeqsucc Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \curlyeqsucc Q_2)(p_j)}} = min\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.9}$$

Definition 3.19. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the symmetric difference of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \underline{\triangle} T_{Q_2}$, is defined by the mapping $t_{Q_1 \underline{\triangle} Q_2} : \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that $t_{Q_1 \underline{\triangle} Q_2}(p_j) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} \rangle} : x_i \in \mathfrak{X} \text{ and } \lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $p_j \in \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \underline{\triangle} Q_2)(p_j)}} = min\{max\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}, 1 - min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}\}. \tag{3.10}$$

Example 3.20. Consider three-valued soft sets T_{Q_1} in Table 2 and T_{Q_2} in Table 4. Then, we obtain the difference and symmetric difference of T_{Q_1} and T_{Q_2} as in Tables 6 and 7, respectively.

[Insert Table 6]

[Insert Table 7]

Proposition 3.21. *Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$.*

i) $T_{Q_1} \curlyeqsucc T_{Q_2} = T_{Q_1} \sqcap \overline{T_{Q_2}}$.

ii) $T_{Q_1} \underline{\triangle} \, T_{Q_2} = (T_{Q_1} \underline{\succ} \, T_{Q_2}) \sqcup (T_{Q_2} \underline{\succ} \, T_{Q_1})$.

Proof. (i) Assume $T_{Q_1} \underline{\succ} \, T_{Q_2} = T_R$. Then, by Eq. 3.9, we have, for all $p_j \in \mathcal{P}$,

$$\lambda^i_{t_R}(p_j) = min\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.11}$$

On the other hand, assume that $T_{Q_1} \,\overline{\sqcap}\, \overline{T_{Q_2}} = T_S$. Then, from Definition 3.12, we obtain that

$$\lambda^i_{t_S}(p_j) = min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{\overline{t_{Q_2}}_{(p_j)}}\} = min\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.12}$$

By Eqs. 3.11 and 3.12, we prove that the above equality is achieved.
(ii) Assume $T_{Q_1} \underline{\triangle} \, T_{Q_2} = T_U$. Then, by Eq. 3.10, we have, for all $p_j \in \mathcal{P}$,

$$\lambda^i_{t_U}(p_j) = min\{max\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}, 1 - min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}}\}\} \tag{3.13}$$

Considering the right side of equality, let $T_{Q_1} \underline{\succ} \, T_{Q_2} = T_V$ and $T_{Q_2} \underline{\succ} \, T_{Q_1} = T_W$, where

$$\lambda^i_{t_V}(p_j) = min\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}. \tag{3.14}$$

and

$$\lambda^i_{t_W}(p_j) = min\{\lambda^i_{t_{Q_2(p_j)}}, 1 - \lambda^i_{t_{Q_1(p_j)}}\}. \tag{3.15}$$

For $T_Z = T_V \,\overline{\sqcap}\, T_W$, from Definition 3.14, we obtain that

$$\lambda^i_{t_Z}(p_j) = max\{\lambda^i_{t_{V(p_j)}}, \lambda^i_{t_{W(p_j)}}\} = max\{min\{\lambda^i_{t_{Q_1(p_j)}}, 1 - \lambda^i_{t_{Q_2(p_j)}}\}, min\{\lambda^i_{t_{Q_2(p_j)}}, 1 - \lambda^i_{t_{Q_1(p_j)}}\}\}. \tag{3.16}$$

We know that $\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_j)}} \in \{0, \frac{1}{2}, 1\}$ for all $p_j \in \mathcal{P}$, and so $\lambda^i_{t_U}(p_j) = \lambda^i_{t_Z}(p_j)$. This completes the proof. ◻

Definition 3.22. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the And-product of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \,\overline{\wedge}\, T_{Q_2}$, is defined by the mapping $t_{Q_1 \overline{\wedge} Q_2} : \mathcal{P} \times \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that
$t_{Q_1 \overline{\wedge} Q_2}(p_j, p_k) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j,p_k)}} \rangle} : x_i \in \mathfrak{X}$ and $\lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j,p_k)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $(p_j, p_k) \in \mathcal{P} \times \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \overline{\wedge} Q_2)(p_j,p_k)}} = min\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_k)}}\}. \tag{3.17}$$

Definition 3.23. Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the Or-product of T_{Q_1} and T_{Q_2}, denoted by $T_{Q_1} \,\underline{\vee}\, T_{Q_2}$, is defined by the mapping $t_{Q_1 \underline{\vee} Q_2} : \mathcal{P} \times \mathcal{P} \to \mathfrak{X}^{(\{0, \frac{1}{2}, 1\})}$ such that
$t_{Q_1 \underline{\vee} Q_2}(p_j, p_k) = \{x_i^{\langle \lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j,p_k)}} \rangle} : x_i \in \mathfrak{X}$ and $\lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j,p_k)}} \in \{0, \frac{1}{2}, 1\}\}$ for all $(p_j, p_k) \in \mathcal{P} \times \mathcal{P}$, where

$$\lambda^i_{t_{(Q_1 \underline{\vee} Q_2)(p_j,p_k)}} = max\{\lambda^i_{t_{Q_1(p_j)}}, \lambda^i_{t_{Q_2(p_k)}}\}. \tag{3.18}$$

Proposition 3.24. *Let $T_{Q_1}, T_{Q_2}, T_{Q_3} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. For all $*, \circ \in \{\overline{\wedge}, \underline{\vee}\}$,*

i) $T_{Q_1} * (T_{Q_2} * T_{Q_3}) = (T_{Q_1} * T_{Q_2}) * T_{Q_3}$.

ii) $T_{Q_1} * (T_{Q_2} \circ T_{Q_3}) = (T_{Q_1} * T_{Q_2}) \circ (T_{Q_1} * T_{Q_3})$.

iii) $(T_{Q_1} \star T_{Q_2}) \circ T_{Q_3} = (T_{Q_1} \circ T_{Q_3}) \star (T_{Q_2} \circ T_{Q_3})$.

Proof. They can be shown in a similar way to the proofs of Proposition 3.16. □

Proposition 3.25. *Let $T_{Q_1}, T_{Q_2} \in \mathcal{TVSS}(\mathfrak{X}, \mathcal{P})$. Then, the following De Morgan's laws are hold.*

i) $\overline{(T_{Q_1} \barwedge T_{Q_2})} = \overline{T_{Q_1}} \veebar \overline{T_{Q_2}}$.

ii) $\overline{(T_{Q_1} \veebar T_{Q_2})} = \overline{T_{Q_1}} \barwedge \overline{T_{Q_2}}$.

Proof. The proofs are similar to those of Proposition 3.17. □

4 Choice value of an object in three-valued soft set(s)

In 2002, Maji et al. [32] defined the choice value of an object in soft set and thereby put forward that this indication can be used to prioritize the objects in the soft set during decision making. We also adapt the idea of choice value initiated by Maji et al. for the three-valued soft sets.

In this part, it is taken $J = \{1, 2, ..., |\mathcal{P}|\}$ where $|\mathcal{P}|$ denotes the cardinality of the parameter set $\mathcal{P}$.

Definition 4.1. Let $\mathfrak{X}$ be a set of alternatives (objects).

1. The choice value of an object $x_i \in \mathfrak{X}$ in the three-valued soft set T_Q is defined and denoted by

$$\alpha_i = \sum_{j \in J} (x_{ij})^{\xi}, \tag{4.1}$$

   where x_{ij} for all i, j are the entries in the table of three-valued soft set T_Q. Also, the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value.

2. The choice value of an object $x_i \in \mathfrak{X}$ in the three-valued soft sets T_{Q_k} for $k = 1, 2, ..., s$ is defined and denoted by

$$\alpha_i = \frac{\sum_{j \in J}((x_{ij}^{\sqcap})^{\xi} + (x_{ij}^{\sqcup})^{\xi})}{2}, \tag{4.2}$$

   where $x_{ij}^{\sqcap}$ and $x_{ij}^{\sqcup}$ for all i, j are the entries in the tables of three-valued soft sets $\bar{\sqcap}_{k=1}^{s} T_{Q_k}$ and $\underline{\sqcup}_{k=1}^{s} T_{Q_k}$, respectively. Also, the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value.

   Remark: It is clear that for $x_{ij} = \frac{1}{2}$, $(x_{ij})^{\xi} \to 1$ when $\xi \to 0$ and $(x_{ij})^{\xi} \to 0$ when $\xi \to +\infty$. For instance, consider Figure 2. If the revision is minor then it is more appropriate to specify as $0 < \xi < 1$. If the revision is major then it is more appropriate to specify as $\xi \in (1, +\infty)$.

Algorithm for Selection (Algorithm 1)

Step 1. Choose feasible subsets Q_k $(k = 1, 2, ..., s)$ of the parameter set $\mathcal{P}$,

Step 2. Create the three-valued soft sets T_{Q_k} for parameter subsets Q_k $(k = 1, 2, ..., s)$,

Step 3. Specify the overall impact coefficient of "undetermined" on the choice value, i.e, $\xi \in \mathbb{R}^+$,

Step 4.
- If $k > 1$ then obtain the intersection and union of three-valued soft sets T_{Q_k} $(k = 1, 2, ..., s)$,
- If $k = 1$ then skip to step 4,

Step 5. Calculate α_i for all i, Step 6. Find l, for which $\alpha_l = max\ \alpha_i$.

Then, x_l is the optimal choice object. If l has more than one value, then any one of them could be chosen.

Example 4.2. As an implementation of Algorithm 1, we try to solve our numerical problem in Example 3.4.

[Insert Table 8]

[Insert Table 9]

By Tables 8 and 9, we have $max\ \alpha_i = \alpha_4$ for $\xi = 1, \frac{1}{4}, \frac{3}{2}, \sqrt{13}, 10$ and so the investment x_4 is optimal choice to invest some money.

Example 4.3. We consider three-valued soft sets T_{Q_1} in Example 3.4 and T_{Q_2} in Example 3.10. Let's try to make a common decision by using the data in these two three-valued soft sets for $\xi = 2$. We obtain the intersection and union of T_{Q_1} and T_{Q_2} as in Tables 10 and 11, respectively.

[Insert Table 10]

[Insert Table 11]

By Tables 10 and 11, we have $max\alpha_i = \alpha_1 = \alpha_4 = 2.125$. So, we interpret that any of the investments x_1 and x_4 is optimal choice to invest some money.

In the decision making process, all parameters of the parameter set may not be of equal importance. In such cases, weights can be imposed on the choice parameters, i.e., there is a weight $\omega_j \in (0, 1]$ corresponding to each parameter $p_j \in Q$. If $p_j \in \mathcal{P} - Q$ then we know that $\omega_j = 0$. Generally, total weight is $\sum_j \omega_j = 1$. Now, let us describe the weighted choice value of an object in the structures of (weighted) three-valued soft sets.

Definition 4.4. Let $\mathfrak{X}$ be a set of alternatives (objects).

1. The weighted choice value of an object $x_i \in \mathfrak{X}$ in the (weighted) three-valued soft set T_Q is defined and denoted by

$$\alpha_i^{\omega} = \sum_{j \in J} \omega_j \times (x_{ij})^{\xi}, \tag{4.3}$$

   where the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value. Also, ω_j denotes the weight corresponding to each parameter p_j in the structure of three-valued soft set T_Q.

2. The weighted choice value of an object $x_i \in \mathfrak{X}$ in the (weighted) three-valued soft sets T_{Q_k} for $k = 1, 2, ..., s$ is defined and denoted by

$$\alpha_i^{\omega} = \frac{\sum_{j \in J} \omega_j^{ort} \times \left((x_{ij}^{\sqcap})^{\xi} + (x_{ij}^{\sqcup})^{\xi} \right)}{2}, \tag{4.4}$$

   where the arbitrary number $\xi \in \mathbb{R}^+$ is the overall impact coefficient of "undetermined" on the choice value. Also,

$$\omega_j^{ort} = \frac{\sum_{k=1}^{s} \omega_j^k}{s}, \tag{4.5}$$

   where ω_j^k denotes the weight corresponding to each parameter p_j in the structure of three-valued soft set T_{Q_k}.

Algorithm for Selection by Imposing Weights on Parameters (Algorithm 2)

Step 1. Choose feasible subsets Q_k $(k = 1, 2, ..., s)$ of the parameter set $\mathcal{P}$ and determine the parameter's weights (i.e., ω^k) for each subsets Q_k,

Step 2. Create the (weighted) three-valued soft sets T_{Q_k} for the parameter subsets Q_k $(k = 1, 2, ..., s)$,

Step 3. Specify the overall impact coefficient of "undetermined" on the choice value, i.e, $\xi \in \mathbb{R}^+$,

Step 4.
- If $k > 1$ then obtain the intersection and union of three-valued soft sets T_{Q_k} $(k = 1, 2, ..., s)$,
- If $k = 1$ then skip to step 4,

Step 5. Calculate α_i^ω for all i,

Step 6. Find l, for which $\alpha_l^\omega = max\ \alpha_i^\omega$.

Then, x_l is the optimal choice object. If l has more than one value, then any one of them could be chosen.

Example 4.5. We consider the numerical problem proposed in Example 3.4. Also, let us determine the following weights for the parameters of Q_1: $\omega_1^1 = 0.6$ for the parameter $p_1 = high\ returns$, $\omega_2^1 = 0.3$ for the parameter $p_2 = low\ risk$ and $\omega_3^1 = 0.1$ for the parameter $p_3 = high\ security$. Since $p_4 \notin Q_1$, the weight of parameter $p_4 \in \mathcal{P}$ can be considered "0".

[Insert Table 12]

By Table 12, we obtain $max\ \alpha_i^\omega = \alpha_3^\omega$ (for $\xi = 1, \frac{1}{10}, 3$). Then, the optimal choice is x_3.

Example 4.6. Let's the weighted three-valued soft sets T_{Q_1} in Example 4.5. Also, we take following parameter weights for T_{Q_2} in Table 4: $\omega_1^2 = \omega_4^2 = 0.2$ and $\omega_2^2 = \omega_3^2 = 0.3$. For $\xi = 1$, we obtain weighted choice values of alternatives x_i $(i = 1, 2, 3, 4)$ as $\alpha_1^\omega = 0.65$, $\alpha_2^\omega = 0.2$, $\alpha_3^\omega = 0.6$, $\alpha_4^\omega = 0.6$. Since $max\ \alpha_i^\omega = \alpha_1^\omega = 0.65$, the optimal choice is x_1.

In Algorithms 1 and 2, for each decision maker, the overall impact coefficient (ξ) of "undetermined" on the choice value is taken the same. These two algorithms cannot be used if each decision maker selects the impact coefficient of "undetermined" differently. To address these shortcomings, we will create new decision making algorithms.

5 Three-valued soft decision making model based on TOPSIS

In this section, we focus on the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), which produces satisfactory results during decision making. We rebuild this technique to deal with the multi-criteria group decision making problems based on three-valued soft sets, and thus construct a novel decision making model.

The TOPSIS is an approach proposed to solve the multi-criteria decision making problems based on a decision maker. In 2007, Shih et al. [47] extended this method for group decision making. The operations in the process of TOPSIS include decision matrix normalization, distance measures and aggregation operators [47]. Generally, a decision matrix is required prior to the beginning of the process. As a results of this process, the output data are interpreted and thus the ranking order of alternatives is obtained. In summary, the TOPSIS approach is a practical and useful method for ranking and selection of a number of externally determined alternatives through distance measures. The main procedure of TOPSIS is given in a series of steps (see [48–51]).

Now, we propose a multi-criteria group decision making model using the TOPSIS on three-valued soft sets.

Algorithm of TOPSIS based three-valued soft sets (Algorithm 3)

Step 1. Describe the multi-criteria group decision making problem.
In this step, the decision makers (experts), the alternatives and the choice parameters are determined. Suppose that $DM = \{E_k : \ k \in I_s = \{1, 2, ..., s\}\}$ is a set of decision makers (experts) and E_k denotes kth decision maker (expert). Also, x_i $(i \in \mathcal{I}_m = \{1, 2, ..., m\})$ denotes ith alternative and p_j $(j \in \mathcal{I}_n = \{1, 2, ..., n\})$ represents jth parameter (criterion or attribute).
Considering these data, each decision maker E_k $(k \in I_s)$ create three-valued soft set T_{Q_k} and determine the weights of parameters as ω_j^k $(j \in \mathcal{I}_n)$ satisfying the condition $\sum_{j=1}^{n} \omega_j^k = 1$.
Also, each decision maker E_k specifies the impact coefficient of "undetermined" in the decision making, i.e., ξ_k.

Step 2. For each decision maker E_k; the decision matrix $\mathfrak{D}^k$ is constructed and represented as follows:

$$\mathfrak{D}^k = \begin{array}{c} \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{array} \begin{array}{c} \begin{array}{ccccccc} p_1 & p_2 & \cdot & \cdot & \cdot & p_n \end{array} \\ \begin{bmatrix} d_{11}^k & d_{12}^k & \cdot & \cdot & \cdot & d_{1n}^k \\ d_{21}^k & d_{22}^k & \cdot & \cdot & \cdot & d_{2n}^k \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ d_{m1}^k & d_{m2}^k & \cdot & \cdot & \cdot & d_{mn}^k \end{bmatrix} \end{array} = [d_{ij}^k]_{m \times n}$$

where $d_{ij}^k = (x_{ij}^k)^{\xi_k}$ that x_{ij}^k for all i, j are the entries in the table of three-valued soft set T_{Q_k}.

Step 3. After constructing the decision matrices, these are normalized (standardized).
For each decision matrix $\mathfrak{D}^k$, the normalized decision decision matrix $\mathfrak{R}^k$ is constructed and expressed as follows:

$$\mathfrak{R}^k = \begin{bmatrix} r_{11}^k & r_{12}^k & \cdot & \cdot & \cdot & r_{1n}^k \\ r_{21}^k & r_{22}^k & \cdot & \cdot & \cdot & r_{2n}^k \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ r_{m1}^k & r_{m2}^k & \cdot & \cdot & \cdot & r_{mn}^k \end{bmatrix} = [r_{ij}^k]_{m \times n}$$

where

$$r_{ij}^k = \begin{cases} \frac{d_{ij}^k}{\sqrt{\sum_{\ell=1}^{m} (d_{\ell j}^k)^2}}, & \text{if } d_{ij}^k \neq 0 \\ 0, & \text{if } d_{ij}^k = 0 \end{cases} \tag{5.1}$$

for all $k \in \mathcal{I}_s$, $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$.

Step 4. Considering the different weights of parameters for each decision maker, the weighted normalized decision matrix is calculated by multiply the weights of evaluation parameters and values in the normalized decision matrix.

For each normalized decision matrix $\mathfrak{R}^k$, the weighted normalized decision matrix V^k is created as below:

$$\mathrm{V}^k = \begin{bmatrix} \mathrm{v}_{11}^k & \mathrm{v}_{12}^k & . & . & . & \mathrm{v}_{1n}^k \\ \mathrm{v}_{21}^k & \mathrm{v}_{22}^k & . & . & . & v_{2n}^k \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ \mathrm{v}_{m1}^k & \mathrm{v}_{m2}^k & . & . & . & \mathrm{v}_{mn}^k \end{bmatrix} = [\mathrm{v}_{ij}]_{m\times n}$$

where

$$\mathrm{v}_{ij}^k = w_j^k \times r_{ij}^k \tag{5.2}$$

for all $k \in \mathcal{I}_s$, $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$.

Step 5. Combining the weighted normalized decision matrices V^k $(k \in \mathcal{I}_s)$, the average weighted normalized decision matrix $\mathcal{V}$ is created.
The structure of matrix $\mathcal{V}$ can be expressed as below:

$$\mathcal{V} = \begin{bmatrix} v_{11} & v_{12} & . & . & . & v_{1n} \\ v_{21} & v_{22} & . & . & . & v_{2n} \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ v_{m1} & v_{m2} & . & . & . & v_{mn} \end{bmatrix} = [v_{ij}]_{m\times n}$$

where for all $i \in \mathcal{I}_m$ and $j \in \mathcal{I}_n$

$$v_{ij} = \mathrm{v}_{ij}^1 \oplus \mathrm{v}_{ij}^2 \oplus ... \oplus \mathrm{v}_{ij}^s. \tag{5.3}$$

In the other words, the value v_{ij} is obtained by combining the weighted normalized decision value v_{ij}^k through an operation $\oplus$. Here, the operation $\oplus$ can offer many choices: arithmetic mean, geometric mean, harmonic mean and their modification.
Note: In this study, we will take the arithmetic mean of all individual measures.

Step 6. The positive and negative ideal solutions $\mathcal{V}^\top$ and $\mathcal{V}^\perp$ are determined by using the average weighted normalized decision matrix $\mathcal{V}$.
In the TOPSIS approach, the parameters (criteria or attributes) are evaluated from aspect of benefit (cf. Example 3.4) and cost (cf. Example 3.2). Suppose that $\mathcal{J}_1$ and $\mathcal{J}_2$ are respectively the sets of benefit parameters and cost parameters, where $\mathcal{J}_1 \cap \mathcal{J}_2 = \varnothing$ and $\mathcal{J}_1 \cup \mathcal{J}_2 = \{1, 2, ..., n\}$.

- $\mathcal{V}^\top$ is the set showing the most suitable alternative for each parameter may be preferred (PIS). This set is given by the following

$$\mathcal{V}^\top = \{v_1^\top, v_2^\top, ..., v_j^\top, ..., v_n^\top\} = \{(\max_i v_{ij}^\perp : j \in \mathcal{J}_1), (\min_i v_{ij}^\perp : j \in \mathcal{J}_2),\ i \in \mathcal{I}_m\}. \tag{5.4}$$

- $\mathcal{V}^\perp$ is the set showing at least preferable alternative for each parameter (NIS). This set is given by the following

$$\mathcal{V}^\perp = \{v_1^\perp, v_2^\perp, ..., v_j^\perp, ..., v_n^\perp\} = \{(\min_i v_{ij}^\perp : j \in \mathcal{J}_1), (\max_i v_{ij}^\perp : j \in \mathcal{J}_2),\ i \in \mathcal{I}_m\}. \tag{5.5}$$

Step 7. The separation measurements of alternatives to the ideal solutions are obtained with the Euclidean distance formula.

The separation measurement of each alternative x_i to the positive ideal solution $\mathcal{V}^{\top}$ is calculated as

$$\mathcal{S}_i^{\top} = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^{\top})^2}. \tag{5.6}$$

The separation measurement of each alternative x_i to the negative ideal solution $\mathcal{V}^{\perp}$ is calculated as

$$\mathcal{S}_i^{\perp} = \sqrt{\sum_{j=1}^{n}(v_{ij} - v_j^{\perp})^2}. \tag{5.7}$$

Here, $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$ represent the distance of alternative x_i from PIS and NIS, respectively.

Step 8. The relative closeness of each alternative to the ideal solutions is calculated.

The relative closeness $\mathcal{C}_i^{\dagger}$ of the alternatives x_i with respect to the ideal solutions can be expressed as

$$\mathcal{C}_i^{\dagger} = \frac{\mathcal{S}_i^{\perp}}{\mathcal{S}_i^{\perp} + \mathcal{S}_i^{\top}}, \ \forall i \in \mathcal{I}_m \quad (0 \leq \mathcal{C}_i^{\dagger} \leq 1). \tag{5.8}$$

Step 9. Ranking the preference order.

A set of alternatives x_i can be ranked according to the descending order of the values $\mathcal{C}_i^{\dagger}$.

To illustrate the potential of this proposed approach, we present a real life practice adapted from Figure 4.

Example 5.1. Assume that two experts want to determine best mobile phone brand by examining the new model mobile phones presented by six different mobile phone brand. The first expert E_1 will review each brand's mobile phone with memory of 128 GB and the second expert E_2 will review each brand's mobile phone with memory of 64 GB. Let $\mathfrak{X} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the set of six different mobile phone brands. Also, the set of parameters will be used to determine the brand having the best of mobile phones is given as $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$ where p_1=optical image stabilisation, p_2=quickly charging, p_3=expandable memory, p_4=water proof, p_5=autofocus, p_6=cheap, p_7=fingerprint sensor. Each of experts proceeds to the decision making stage after reviewing comparisons on mobile phone comparison-focused website (such as "www.91mobiles.com" and "www.smartprix.com").

Step 1. The experts E_1 and E_2 determine the parameter sets as $Q_1 = Q_2 = \mathcal{P}$, respectively. Also, they determine weights of their choice parameters as ω_j^k for all $j = 1, 2, ..., 7$ and $k = 1, 2$. (see: Table 13)

[Insert Table 13]

The experts collect data about each brand's mobile phone x_i $(i = 1, 2, ..., 6)$ for each attribute p_j and create three-valued soft sets in Table **??**.

[Insert Table 14]

Step 2. According to Table 14, the decision matrices $\mathfrak{D}^k$ $(k = 1, 2)$ are constructed as follows:

$$\mathfrak{D}^1 = \begin{bmatrix} 1 & 0 & 0 & 0.3535 & 0.3535 & 1 & 0 \\ 1 & 0.3535 & 0.3535 & 1 & 1 & 1 & 0.3535 \\ 0.3535 & 0 & 0.3535 & 1 & 0.3535 & 0 & 1 \\ 0.3535 & 0.3535 & 1 & 0.3535 & 0.3535 & 1 & 1 \\ 0 & 0.3535 & 0.3535 & 1 & 0.3535 & 0 & 1 \\ 0.3535 & 0.3535 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = [d_{ij}^1]_{6\times 7}, \ \mathfrak{D}^2 = \begin{bmatrix} 0.25 & 0.25 & 1 & 0 & 0.25 & 1 & 0 \\ 1 & 0 & 0.25 & 1 & 1 & 1 & 1 \\ 0.25 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0.25 & 1 & 0 & 0 & 1 & 0.25 \\ 0.25 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = [d_{ij}^2]_{6\times 7}.$$

Step 3. For each decision matrix $\mathfrak{D}^k$ $(k = 1, 2)$, the normalized decision decision matrix $\mathfrak{R}^k$ $(k = 1, 2)$ is constructed as follows:

$$\mathfrak{R}^1 = \begin{bmatrix} 0.6489 & 0 & 0 & 0.1714 & 0.2886 & 0.5 & 0 \\ 0.6489 & 0.5001 & 0.2293 & 0.485 & 0.8166 & 0.5 & 0.174 \\ 0.2293 & 0 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\ 0.2293 & 0.5001 & 0.6489 & 0.1714 & 0.2886 & 0.5 & 0.4923 \\ 0 & 0.5001 & 0.2293 & 0.485 & 0.2886 & 0 & 0.4923 \\ 0.2293 & 0.5001 & 0.2293 & 0.485 & 0 & 0.5 & 0.4923 \end{bmatrix} \text{ and } \mathfrak{R}^2 = \begin{bmatrix} 0.169 & 0.2357 & 0.5714 & 0 & 0.0174 & 0.4472 & 0 \\ 0.6761 & 0 & 0.1428 & 0.5 & 0.6963 & 0.4472 & 0.4961 \\ 0.169 & 0 & 0 & 0.5 & 0.6963 & 0 & 0.4961 \\ 0 & 0.2357 & 0.5714 & 0 & 0 & 0.4472 & 0.124 \\ 0.169 & 0 & 0 & 0.5 & 0 & 0.4472 & 0.4961 \\ 0.6761 & 0.9428 & 0.5714 & 0.5 & 0 & 0.4472 & 0.4961 \end{bmatrix}.$$

Step 4. For each normalized decision decision matrix $\mathfrak{R}^k$ $(k = 1, 2)$, the weighted normalized decision decision matrix V^k $(k = 1, 2)$ is created as below:

$$\mathrm{V}^1 = \begin{bmatrix} 0.1427 & 0 & 0 & 0.0257 & 0.0375 & 0.055 & 0 \\ 0.1427 & 0.075 & 0.0298 & 0.0727 & 0.1061 & 0.055 & 0.0191 \\ 0.0504 & 0 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\ 0.0504 & 0.075 & 0.0843 & 0.0257 & 0.0375 & 0.055 & 0.0541 \\ 0 & 0.075 & 0.0298 & 0.0727 & 0.0375 & 0 & 0.0541 \\ 0.0504 & 0.075 & 0.0298 & 0.0727 & 0 & 0.055 & 0.0541 \end{bmatrix}, \mathrm{V}^2 = \begin{bmatrix} 0.0405 & 0.0282 & 0.0685 & 0 & 0.0243 & 0.0447 & 0 \\ 0.1622 & 0 & 0.0171 & 0.08 & 0.0974 & 0.0447 & 0.0595 \\ 0.0405 & 0 & 0 & 0.08 & 0.0974 & 0 & 0.0595 \\ 0 & 0.0282 & 0.0685 & 0 & 0 & 0.0447 & 0.0148 \\ 0.0405 & 0 & 0 & 0.08 & 0 & 0.0447 & 0.0595 \\ 0.1622 & 0.1131 & 0.0685 & 0.08 & 0 & 0.0447 & 0.0595 \end{bmatrix}.$$

Step 5. Then, the average weighted normalized decision matrix is

$$\mathcal{V} = \mathrm{V}^1 \oplus \mathrm{V}^2 = \begin{bmatrix} 0.0916 & 0.0141 & 0.0342 & 0.0128 & 0.0309 & 0.0498 & 0 \\ 0.1524 & 0.0375 & 0.0234 & 0.0763 & 0.1017 & 0.0498 & 0.0393 \\ 0.0454 & 0 & 0.0149 & 0.0763 & 0.0674 & 0 & 0.0568 \\ 0.0252 & 0.0516 & 0.0764 & 0.0128 & 0.0187 & 0.0498 & 0.0344 \\ 0.0202 & 0.0375 & 0.0149 & 0.0763 & 0.0187 & 0.0223 & 0.0568 \\ 0.1063 & 0.094 & 0.0491 & 0.0763 & 0 & 0.0498 & 0.0568 \end{bmatrix} = [v_{ij}]_{6\times 7},$$

where the operation $\oplus$ represents the arithmetic mean.
Step 6. The positive and negative ideal solutions $\mathcal{V}^{\top}$ and $\mathcal{V}^{\perp}$ are determined as follows:

$$\mathcal{V}^{\top} = \{v_1^{\top} = 0.1524, v_2^{\top} = 0.094, v_3^{\top} = 0.0764, v_4^{\top} = 0.0763, v_5^{\top} = 0.1017, v_6^{\top} = 0.0498, v_7^{\top} = 0.0568\}$$

and

$$\mathcal{V}^{\perp} = \{v_1^{\perp} = 0.0202, v_2^{\perp} = 0, v_3^{\perp} = 0.0149, v_4^{\perp} = 0.0128, v_5^{\perp} = 0, v_6^{\perp} = 0, v_7^{\perp} = 0\}.$$

Step 7. The separation measurements $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$ of each alternative x_i to the ideal solutions are given in Table 15.

[Insert Table 15]

Step 8. The relative closeness $\mathcal{C}_i^{\dagger}$ of each alternative to the ideal solutions is calculated as follows:

$$\mathcal{C}_1^{\dagger} = 0.3767,\ \mathcal{C}_2^{\dagger} = 0.7095,\ \mathcal{C}_3^{\dagger} = 0.399,\ \mathcal{C}_4^{\dagger} = 0.3706,\ \mathcal{C}_5^{\dagger} = 0.3513,\ \mathcal{C}_6^{\dagger} = 0.589.$$

Step 9. According to the descending order of the values $\mathcal{C}_i^{\dagger}$, the ranking order of alternatives is obtained as below

$$x_2 \succ x_6 \succ x_3 \succ x_1 \succ x_4 \succ x_5.$$

Then, it can be said that x_2 is the best mobile phone brand according to the data presented by experts

6 Three-valued soft decision making model based on ELECTRE

In this section, , we introduce a modified version of the ELECTRE technique ("ELimination Et Choix Traduisant la REalité" or "Elimination and Choice Expressing Reality"), which are generally intended to output choosing, sorting and ranking, to deal with the multi-criteria group decision making problems based on three-valued soft sets.

As it was first applied in 1965, the ELECTRE method was to choose the best alternative(s) from the given set of alternatives, but it was soon applied three fundamental problems:
Choosing: Selecting a restricted number of the most interesting potential alternatives, as small as possible which will justify to eliminate all others.
Sorting: Assigning each potential alternative to one of the categories a family previously described; the categories are ordered from the worst to the best one.
Ranking: Ordering of alternatives from the best to the worst with the possibility of ties.
The main procedure of ELECTRE is described in a series of the steps (see [52–56]).

Now, we construct a multi-criteria group decision making model on three-valued soft sets via the modified ELECTRE technique.

Algorithm of ELECTRE based three-valued soft sets (Algorithm 4)

Step 1. Describe the multi-criteria group decision making problem (same as Step 1 in Algorithm of TOPSIS based three-valued soft sets).

Step 2. For each decision maker E_k; the decision matrix $\mathfrak{D}^k$ is constructed (same as Step 2 in Algorithm of TOPSIS based three-valued soft sets).

Step 3. For each decision matrix $\mathfrak{D}^k$, the normalized decision decision matrix $\mathfrak{R}^k$ is constructed (same as Step 3 in Algorithm of TOPSIS based three-valued soft sets).

Step 4. For each normalized decision matrix $\mathfrak{R}^k$, the weighted normalized decision matrix V^k is created (same as Step 4 in Algorithm of TOPSIS based three-valued soft sets).

Step 5. Combining the weighted normalized decision matrices V^k $(k \in \mathcal{I}_s)$, the average weighted normalized decision matrix $\mathcal{V}$ is created (same as Step 5 in Algorithm of TOPSIS based three-valued soft sets).

Step 6. The concordance sets and discordance sets are determined.
The concordance set is composed of index of all parameters for which the alternative x_τ is preferred to the alternative x_κ. This set can be described as follows:
For $\tau, \kappa \in \mathcal{I}_m$ and $\tau \neq \kappa$ (note that an alternative is not compared to itself),

$$\mathcal{J}^+_{\tau\kappa} = \{j : \ v_{\tau j} \geq v_{\kappa j}\}, \tag{6.1}$$

The discordance set contains the index of all parameters for which the alternative x_τ is worse than the alternative x_κ. This set can be described as follows:
For $\tau, \kappa \in \mathcal{I}_m$ and $\tau \neq \kappa$,

$$\mathcal{J}^-_{\tau\kappa} = \{j : \ v_{\tau j} < v_{\kappa j}\}. \tag{6.2}$$

In the other words, this set can be considered as the complement of the concordance set $\mathcal{J}^+_{\tau\kappa}$, i.e., $\mathcal{J}^-_{\tau\kappa} = \mathcal{J} \setminus \mathcal{J}^+_{\tau\kappa}$ where $\mathcal{J} = \{j : \ p_j \in \mathcal{P}\}$.

Step 7. The concordance matrix and discordance matrix are generated by employing the sets of concordance and discordance, respectively.

The concordance matrix can be expressed as follows:

$$\mathcal{A} = \begin{bmatrix} - & . & . & . & a_{1\kappa} & . & . & . & a_{1m} \\ . & . & & & . & . & & & . \\ . & & . & . & . & & . & & . \\ . & & & . & . & . & . & . & . \\ a_{\tau 1} & . & . & . & a_{\tau\kappa} & . & . & . & a_{\tau m} \\ . & . & & & . & . & & & . \\ . & & . & . & . & & . & & . \\ . & & & . & . & . & . & . & . \\ a_{m1} & . & . & . & a_{m\kappa} & . & . & . & - \end{bmatrix} = [a_{\tau\kappa}]_{m\times m}$$

where

$$a_{\tau\kappa} = \frac{\sum_{j\in\mathcal{J}^+_{\tau\kappa}} \sum_{k=1}^{s} \omega_j^k}{\sum_{j\in\mathcal{J}} \sum_{k=1}^{s} \omega_j^k}, \quad (0 \le a_{\tau\kappa} \le 1) \tag{6.3}$$

for all $\tau, \kappa \in \mathcal{I}_m$.

That is, each component of concordance matrix is found as summation of the (standardized) weights of all parameters corresponding the indices in the concordance set $\mathcal{J}^+$.

The discordance matrix can be expressed as follows:

$$\mathcal{B} = \begin{bmatrix} - & . & . & . & b_{1\kappa} & . & . & . & b_{1m} \\ . & . & & & . & . & & & . \\ . & & . & . & . & & . & & . \\ . & & & . & . & . & . & . & . \\ b_{\tau 1} & . & . & . & b_{\tau\kappa} & . & . & . & b_{\tau m} \\ . & . & & & . & . & & & . \\ . & & . & . & . & & . & & . \\ . & & & . & . & . & . & . & . \\ b_{m1} & . & . & . & b_{m\kappa} & . & . & . & - \end{bmatrix} = [b_{\tau\kappa}]_{m\times m}$$

where

$$b_{\tau\kappa} = \frac{\sum_{j\in\mathcal{J}^-_{\tau\kappa}} |\mathrm{v}_{\tau j} - \mathrm{v}_{\kappa j}|}{\sum_{j\in\mathcal{J}} |\mathrm{v}_{\tau j} - \mathrm{v}_{\kappa j}|}, \quad (0 \le b_{\tau\kappa} \le 1) \tag{6.4}$$

for all $\tau, \kappa \in \mathcal{I}_m$.

Step 8. The concordance threshold $\underline{\mathcal{A}}$ and discordance threshold $\underline{\mathcal{B}}$ are found.

The concordance threshold is calculated as

$$\underline{\mathcal{A}} = \frac{\sum\limits_{\tau=1}^{m} \sum\limits_{\kappa=1}^{m} a_{\tau\kappa}}{m(m-1)}, \quad (0 \le \underline{\mathcal{A}} \le 1) \tag{6.5}$$

and the discordance threshold is calculated as

$$\underline{\mathcal{B}} = \frac{\sum\limits_{\tau=1}^{m} \sum\limits_{\kappa=1}^{m} b_{\tau\kappa}}{m(m-1)}, \quad (0 \le \underline{\mathcal{B}} \le 1). \tag{6.6}$$

Step 9. The effective concordance matrix $\mathcal{F}$ and the effective discordance matrix $\mathcal{G}$ are created.

The effective concordance matrix $\mathcal{F}$ is measured according to the concordance threshold $\underline{\mathcal{A}}$ and it is expressed as follows:

$$\mathcal{F} = \begin{bmatrix} - & . & . & . & f_{1\kappa} & . & . & . & f_{1m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ f_{\tau 1} & . & . & . & f_{\tau\kappa} & . & . & . & f_{\tau m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ f_{m1} & . & . & . & f_{m\kappa} & . & . & . & - \end{bmatrix} = [f_{\tau\kappa}]_{m\times m}$$

where

$$f_{\tau\kappa} = \begin{cases} 1 & \text{, if } a_{\tau\kappa} \geq \underline{\mathcal{A}} \\ 0 & \text{, if } a_{\tau\kappa} < \underline{\mathcal{A}} \end{cases}. \tag{6.7}$$

The effective discordance matrix $\mathcal{G}$ is measured according to the discordance threshold $\underline{\mathcal{B}}$ and it is expressed as follows:

$$\mathcal{G} = \begin{bmatrix} - & . & . & . & g_{1\kappa} & . & . & . & g_{1m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ g_{\tau 1} & . & . & . & g_{\tau\kappa} & . & . & . & g_{\tau m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ g_{m1} & . & . & . & g_{m\kappa} & . & . & . & - \end{bmatrix} = [g_{\tau\kappa}]_{m\times m}$$

where

$$g_{\tau\kappa} = \begin{cases} 0 & \text{, if } b_{\tau\kappa} > \underline{\mathcal{B}} \\ 1 & \text{, if } b_{\tau\kappa} \leq \underline{\mathcal{B}} \end{cases}. \tag{6.8}$$

Step 10. The aggregated outranking matrix $\mathcal{H}$ is constructed

The the aggregated outranking matrix $\mathcal{H}$ is found by merging the effective concordance information and the effective discordance information. Relatedly, the matrix $\mathcal{H}$ can be described as follows:

$$\mathcal{H} = \begin{bmatrix} - & . & . & . & h_{1\kappa} & . & . & . & h_{1m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ h_{\tau 1} & . & . & . & h_{\tau\kappa} & . & . & . & h_{\tau m} \\ . & . & & & . & . & & & . \\ . & & . & & . & & . & & . \\ . & & & . & . & . & & . & . \\ h_{m1} & . & . & . & h_{m\kappa} & . & . & . & - \end{bmatrix} = [h_{\tau\kappa}]_{m\times m}$$

where

$$h_{\tau\kappa} = f_{\tau\kappa} \times g_{\tau\kappa} \tag{6.9}$$

for all $\tau, \kappa \in \mathcal{I}_m$.

Step 11. Ranking the preference order.

The components in the aggregated outranking matrix $\mathcal{H}$ present the dominance of any alternative over another alternative. Thus, we can mention a choice priority among the alternatives and can rank the alternatives according to this priority.

Considering the aggregated outranking matrix $\mathcal{H}$, the binary relations among the alternatives may occur as one of the following three situations:

(a) $x_\tau \succ x_\kappa$ (i.e., x_τ is strictly preferred to x_κ or x_τ is dominance to x_κ) if $h_{\tau\kappa} = 1$ and $h_{\kappa\tau} = 0$,

(b) $x_\tau \approx x_\kappa$ (i.e., x_τ is indifferent to x_κ) if $h_{\tau\kappa} = 1$ and $h_{\kappa\tau} = 1$,

(c) $x_\tau ? x_\kappa$ (i.e., x_τ and x_κ are incomparability) if $h_{\tau\kappa} = 0$ and $h_{\kappa\tau} = 0$.

Thus, the ranking order of alternatives can be interpreted.

As an implementation of this model, we try to offer a solution for the following multi-criteria group decision making problem.

Example 6.1. Supplier selection is among the most important issues in the supply chain management area. Now, we consider a numerical example of a supplier selection problem adapted from [57,58]. A high technology company that manufactures electronic products aims to evaluate and choose a materials supplier. Assume $\mathfrak{X} = \{x_1, x_2, x_3, x_4\}$ is a set of four suppliers that chosen as candidates (alternatives). A single decision maker may not be able to accurately consider all relevant aspects during the decision making. Therefore, the company's leader decides to put together a decision committee to determine a suitable supplier. A committee of three decision makers (experts) is established
Financial expert (E_1), which evaluates alternatives in terms of cost and finance,
Quality control expert (E_2), which evaluates alternatives in terms of quality and safety,
Engineering expert (E_3), which evaluates alternatives in terms of engineering and technical aspects.
Also, six evaluation parameters are considered: performance (p_1), cost control (p_2), management audit (p_3), service (p_4), company reputation (p_5) and quality (p_6).

Step 1. The decision makers (experts) E_1, E_2 and E_3 make their decisions based on the parameter subsets $Q_1 = \{p_1, p_2, p_3, p_5\}$, $Q_2 = \{p_1, p_2, p_3, p_5, p_6\}$ and $Q_3 = \{p_1, p_2, p_4, p_5, p_6\}$, respectively. Also, they determine weights of their choice parameters as in Table 16.

[Insert Table 16]

The decision makers (experts) respectively present their opinion about the truth of alternative x_i under the parameter p_j and construct Table 17.

[Insert Table 17]

Step 2. The (three-valued) decision matrices $\mathfrak{D}^k$ $(k = 1, 2, 3)$ are constructed as follows:

$$\mathfrak{D}^1 = \begin{bmatrix} 1 & 0.25 & 0 & 0 & 1 & 0 \\ 0.25 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0.25 & 0 \\ 0.25 & 0.25 & 0.25 & 0 & 1 & 0 \end{bmatrix}, \mathfrak{D}^2 = \begin{bmatrix} 0 & 0.125 & 1 & 0 & 0.125 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0.125 & 0 \\ 0.125 & 0 & 0.125 & 0 & 0 & 1 \end{bmatrix}, \mathfrak{D}^3 = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & 0 & 0.25 & 0 \\ 1 & 0 & 0 & 0 & 0.25 & 1 \\ 1 & 1 & 0 & 1 & 0.25 & 0 \end{bmatrix}.$$

Steps 3-4. For the decision matrices $\mathfrak{D}^k$ $(k = 1, 2, 3)$, the normalized decision decision matrices $\mathfrak{R}^k$ $(k = 1, 2, 3)$ and then the weighted normalized decision decision matrices V^k $(k = 1, 2, 3)$ are constructed in a similar manner to Steps 3 and 4 of Example 5.1, so it is omitted.

Step 5. Then, the average weighted normalized decision matrix is

$$\mathcal{V} = \mathrm{V}^1 \oplus \mathrm{V}^2 \oplus \mathrm{V}^3 = \begin{bmatrix} 0.0744 & 0.0189 & 0.0586 & 0 & 0.0917 & 0 \\ 0.0509 & 0.0504 & 0.116 & 0 & 0.0333 & 0.0589 \\ 0.0816 & 0.117 & 0.1747 & 0 & 0.0656 & 0.1333 \\ 0.0665 & 0.0732 & 0.0363 & 0.0333 & 0.0681 & 0.0589 \end{bmatrix} = [v_{ij}]_{4\times 6},$$

where the operation $\oplus$ represents the arithmetic mean.

Step 6. Considering the average weighted normalized decision matrix $\mathcal{V}$, the concordance set and discordance set are determined in Table 18.

[Insert Table 18]

Step 7. Then, the concordance matrix and discordance matrix are respectively generated as below:

$$\mathcal{A} = \begin{bmatrix} - & 0.3333 & 0.15 & 0.55 \\ 0.7 & - & 0.0333 & 0.4666 \\ 0.8833 & 1 & - & 0.85 \\ 0.45 & 0.75 & 0.15 & - \end{bmatrix} = [a_{\tau\kappa}]_{4\times 4} \text{ and } \mathcal{B} = \begin{bmatrix} - & 0.6434 & 0.9314 & 0.7314 \\ 0.3565 & - & 1 & 0.5719 \\ 0.0685 & 0 & - & 0.1164 \\ 0.2685 & 0.428 & 0.8718 & - \end{bmatrix} = [b_{\tau\kappa}]_{4\times 4}.$$

Step 8. The concordance threshold and discordance threshold are calculated as $\underline{\mathcal{A}} = 0.5263$ and $\underline{\mathcal{B}} = 0.4989$ respectively.

Step 9. The effective concordance matrix $\mathcal{F}$ and the effective discordance matrix $\mathcal{G}$ are created as follows:

$$\mathcal{F} = \begin{bmatrix} - & 0 & 0 & 1 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 0 & 1 & 0 & - \end{bmatrix} = [f_{\tau\kappa}]_{4\times 4} \text{ and } \mathcal{G} = \begin{bmatrix} - & 0 & 0 & 0 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 1 & 1 & 0 & - \end{bmatrix} = [g_{\tau\kappa}]_{4\times 4}.$$

Step 10. Then, the aggregated outranking matrix $\mathcal{H}$ is

$$\mathcal{H} = \begin{bmatrix} - & 0 & 0 & 0 \\ 1 & - & 0 & 0 \\ 1 & 1 & - & 1 \\ 0 & 1 & 0 & - \end{bmatrix} = [h_{\tau\kappa}]_{4\times 4}.$$

Step 11. Considering the aggregated outranking matrix $\mathcal{H}$, we obtain the following binary relations:

- $h_{21} = 1$ and $h_{12} = 0 \Rightarrow x_2 \succ x_1$,
- $h_{31} = 1$ and $h_{13} = 0 \Rightarrow x_3 \succ x_1$,
- $h_{32} = 1$ and $h_{23} = 0 \Rightarrow x_3 \succ x_2$,
- $h_{34} = 1$ and $h_{43} = 0 \Rightarrow x_3 \succ x_4$,
- $h_{42} = 1$ and $h_{24} = 0 \Rightarrow x_4 \succ x_2$.

Thus, the ranking order of alternatives is found as $x_3 \succ x_4 \succ x_2 \succ x_1$.

7 Comparison and Discussion

Algorithm 2 is the more general than Algorithm 1, namely, it is the version that takes into account parameter weights. Therefore, in this part, we will discuss the performances of Algorithms 2, 3 and 4. All of these algorithms can be used to deal with multi-criteria group decision problems involving incomplete information. However each of them has a different operating philosophy, while they have one common goal which is to combine the evaluations of multiple decision makers and then to propose an optimal choice. They can also offer a choice according to the assessment of only one decision maker. While Algorithm 2 can be applied if the decision makers specify the same impact coefficient for "undetermined", there is no such limitation for Algorithms 3 and 4. The computational performance of each of our algorithms is critically analyzed by the experimental studies, and thence creating the following Table 19.

[Insert Table 19]

As shown in Table 19, the outputs of Algorithms 3 and 4 are identical. For many decision making problems, the results obtained by using these two algorithms are either identical or cover each other. Moreover, all of these algorithms can be used for Examples 4.5 and 4.6, and produce same results. For the problems in Examples 5.1 and 6.1, the outputs of the algorithms coincide. These results support the efficiency and usefulness of the proposed algorithms.

On the other hand, since three-valued soft set is an extension of soft set, the emerging algorithms can be applied to decision making problems based on the soft set(s). So, we endeavor to compare the results of our algorithms with some of the existing soft decision making algorithms. The details supporting this argument are presented in Table 20.

[Insert Table 20]

In Table 20, the weights of parameters in each of problems are considered equally when making calculations in Algorithms 2, 3 and 4 (for instance, in Application (Sect. 5) in [38], the parameter set is $X = \{x_1, x_2, x_3\}$ and so we specify $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$). Also, ξ can be arbitrarily chosen in accordance with the comments on the selection of impact coefficient mentioned above. It is clear that the arbitrary selection of ξ will not change the result(s).
As shown in Table 20, the results of our algorithms coincide with those of existing soft decision making algorithms. For Example 5.17 in [35] and Example 3.3 in [36], the optimal choice by the algorithms proposed in [35, 36] is $\{h_1, h_2, h_3\}$ while the optimal choice by Algorithm 3 is h_1 (where $h_1 \succ h_3 \succ h_2$). This is not a contradiction, and this is the effect of normalizing the decision matrices in the model of TOPSIS (Algorithm 3). Considering the result of Algorithm 4 for the same problems, we say that h_1 and h_2 are incomparability, and h_3 and h_2 are incomparability. This does not contradict the result that the optimal choice is $\{h_1, h_2, h_3\}$. Because what's certain that $h_1 \succ h_4 ? h_5$, $h_2 \succ h_4 ? h_5$ and $h_3 \succ h_4 ? h_5$. Consequently, the applicability of our algorithms in decision making based on both soft set and three-valued soft set demonstrates their performance range and advantages.

8 Conclusion

In this paper, we have defined three-valued soft set which is a generalization of the soft set and its set-theoretic operations like intersection, union, difference and symmetric difference. Moreover, we have described the basic relationships on three-valued soft sets, highlighted the corresponding generalization of the operations on soft sets to these sets and provided some examples for them. We have created the algorithms supporting multi-criteria decision making for three-valued soft set based on TOPSIS and ELECTRE techniques and compared their outputs. Thus,

we have pointed out that these algorithms show the applicability and efficiency of three-valued soft sets in handling the multi-criteria decision making involving uncertain or incomplete information.
We hope that this work will contribute to the decision making under uncertain and incomplete information in the context of soft sets and also will provide new ideas for future studies related to soft sets. Also, it will motivate researchers for the use of three-valued logic stems in many practical applications such as the data mining, data selection, data integration, data analysis, control of production processes and pattern evaluation. In the near future, we intend to explore new operations on three-valued soft sets and their practical applications in the fields such as science, social science, medical science, environmental science, economics and so on.

References

[1] Łukasiewicz, J. "Philosophische Bemerkungen zu mehrwertigen systemen des Aussagenkalkülls", *Algebra C.R. Soc. Sci. Lett. Varsovie*, **23**, pp. 51-57 (1930).

[2] Borowski, L. Ed. "Selected works of J. Łukasiewicz", North Holland Publishing Company, Amsterdam (1970).

[3] Hájek, P. "Metamathematics of fuzzy logic", Kluwer, Dordrecht (1998).

[4] Kleene, S.C. "Introduction to metamathematics", North Holland Publishing Company, Amsterdam (1952).

[5] Skolem, T.H. "A set theory based on a certain 3-valued logic", *Mathematica Scandinavica*, **8**, pp. 127-136 (1960).

[6] Zadeh, L.A. "Fuzzy sets", *Information and Control*, **8**, pp. 338-353 (1965).

[7] Pawlak, Z., "Rough sets", *International Journal of Computer and Information Sciences*, **11**, pp. 341-356 (1982).

[8] Molodtsov, D. "Soft set theory-first results", *Computers and Mathematics with Applications*, **37**, pp. 19-31 (1999).

[9] Maji, P.K., Biswas, R. and Roy, A.R. "Soft set theory", *Computers and Mathematics with Applications*, **45**, pp. 555-562 (2003).

[10] Ali, M.I., Feng, F., Liu, X., et al. "On some new operations in soft set theory", *Computers and Mathematics with Applications*, **57**, pp. 1547-1553 (2009).

[11] Aygün, E. and Kamacı, H. "Some generalized operations in soft set theory and their role in similarity and decision making", *Journal of Intelligent and Fuzzy Systems*, **36**, pp. 6537-6547 (2019).

[12] Sezgin, A. and Atagün, A.O. "On operations of soft sets", *Computers and Mathematics with Applications*, **61**, pp. 1457-1467 (2011).

[13] Kamacı, H. "Similarity measure for soft matrices and its applications", *Journal of Intelligent and Fuzzy Systems*, **36**, pp. 3061-3072 (2019).

[14] Zhu, P. and Wen, Q. "Operations of soft set revisited", *Journal of Applied Mathematics*, **2013**, 7 pages (2013).

[15] Gong, K., Xiao, Z. and Zhang, X. "The bijective soft set with its operations", *Computers and Mathematics with Applications*, **60**, pp. 2270-2258 (2010).

[16] Xiao, Z., Gong, K., Xia, S.,et al. "Exclusive disjunctive soft sets", *Computers and Mathematics with Applications*, **59**, pp. 2128-2137 (2010).

[17] Karaaslan, F. and Karataş, S. "A new approach to bipolar soft sets and its applications", *Discrete Mathematics, Algorithms and Applications*, **7**(4), 1550054 (2015).

[18] Çetkin, V., Aygünoğlu, A. and Aygün, H. "A new approach in handling soft decision making problems", *Journal of Nonlinear Science and Applications*, **9**, pp. 231-239 (2016).

[19] Khalil, A.M. and Hassan, N. "Inverse fuzzy soft set and its application in decision making", *International Journal of Information and Decision Sciences*, **11**(1), pp. 73-92 (2019).

[20] Abbasimehr, H. and Tarokh, M.J. "A novel interval type-2 fuzzy AHP-TOPSIS approach for ranking reviewers in online communities", *Scientia Iranica E*, **23**, pp. 2355-2373 (2016).

[21] Adeel, A., Akram, M. and Koam, Ali N.A. "Group decision-making based on m-polar hesitant fuzzy linguistic TOPSIS method", *Symmetry*, **11**(6), pp. 735 (2019).

[22] Adeel, A., Akram, M. and Koam, Ali N.A. "Multi-criteria decision-making under mHF ELECTRE-I and HmF ELECTRE-I", *Energies*, **12**(9), pp. 1661 (2019).

[23] Akram, M. and Adeel, A. "TOPSIS approach for MAGDM based on interval-valued hesitant fuzzy N-soft environment", *International Journal of Fuzzy Systems*, **21**(3), pp. 993-1009 (2019).

[24] Akram, M., Adeel, A. and Alcantud J.C.R. "Multi-criteria group decision making using an m-polar hesitant fuzzy TOPSIS approach", *Symmetry*, **11**(6), pp. 795 (2019).

[25] Garg, H. and Kaur G. "Extended TOPSIS method for multi-criteria group decision-making problems under cubic intuitionistic fuzzy environment", *Scientia Iranica E*, **25**, pp. 1-18 (2018).

[26] Jasemi, M. and Ahmadi, E. "A new fuzzy ELECTRE based multiple criteria method for personnel selection", *Scientia Iranica E*, **25**, pp. 943-953 (2018).

[27] Garg, H., Agarwal, N. and Tripathi A. "Some improved interactive aggregation operators under interval-valued intuitionistic fuzzy environment and its application to decision making process", *Scientia Iranica E*, **24**, pp. 2581-2604 (2017).

[28] Garg, H. and Kumar K. "A novel correlation coefficient of intuitionistic fuzzy sets based on the connection number of set pair analysis and its application", *Scientia Iranica E*, **25**, pp. 2373-2388 (2018).

[29] Gitinavard, G., Mousavi, S.M., Vahdani, B., et al. "A new distance-based decision model in interval-valued hesitant fuzzy setting for industrial selection problems", *Scientia Iranica E*, **23**, pp. 1928-1940 (2016).

[30] Peng, X. "New operations for interval-valued Pythagorean fuzzy set", *Scientia Iranica E*, **26**, pp. 1049-1076 (2019).

[31] Tang, J. and Meng F. "An approach to interval-valued intuitionistic fuzzy decision making based on induced generalized symmetrical Choquet Shapley operator", *Scientia Iranica E*, **25**, pp. 1456-1470 (2018).

[32] Maji, P.K., Roy, A.R. and Biswas, R. "An application of soft sets in a decision making problem", *Computers and Mathematics with Applications*, **44**, pp. 1077-1083 (2002).

[33] Arora, D. and Garg, H. "Robust aggregation operators for multi-criteria decision-making with intuitionistic fuzzy soft environment", *Scientia Iranica E*, **25**(2), pp. 931-942 (2018).

[34] Arora, R. and Garg, H. "Prioritized averaging/geometric aggregation operators under the intuitionistic fuzzy soft set environment", *Scientia Iranica E*, **25**, pp. 466-482 (2018).

[35] Feng, F. and Li, Y. "Generalized uni-int decision making schemes based on choice value soft sets", *European Journal of Operational Research*, **220**, pp. 162-170 (2012).

[36] Han, B.-H. and Geng, S.-L. "Pruning method for optimal solutions of $int^m - int^n$ decision making", *European Journal of Operational Research*, **231**, pp. 779-783 (2013).

[37] Kamacı, H. "Selectivity analysis of parameters in soft set and its effect on decision making", *International Journal of Machine Learning and Cybernetics*, **11**, pp. 313-324 (2020).

[38] Eraslan, S. "A decision making method via TOPSIS on soft sets", *Journal of New Results in Science*, No:**8**, pp. 57-71 (2015).

[39] Avron, A. and Konikowska, B. "Rough sets and 3-valued logics", *Studia Logica*, **90**(1), pp. 69-92 (2008).

[40] Çağman, N. and Enginoğlu, S. "Soft set theory and uni-int decision making", *European Journal of Operational Research*, **207**, pp. 848-855, (2010).

[41] D'ottaviano, I.M.L. "The completeness and compactness of a three-valued first-order logic", *Revista Colombiana de Matemáticas*, **19**, pp. 77-94 (1985).

[42] Jacquette, D. "An internal determinacy metatheorem for Łukasiewicz's Aussagenkalküls", *Bulletin of the Section of Logic*, **29**(3), pp. 115-124 (2000).

[43] Ciucci, D. and Dubois, D. "A map of dependencies among three-valued logics", *Information Sciences*, **250**, 162-177 (2013).

[44] Cai, M., Li, Q. and Lang, G. "Shadowed sets of dynamic fuzzy sets", *Granular Computing*, **2**, pp. 85-94 (2017).

[45] Pedrycz, W. "From fuzzy sets to shadowed sets: Interpretation and computing", *International Journal of Intelligent Systems*, **24**(1), pp. 48-61 (2009).

[46] Vogel, P., Read, R.W., Hansen, G.M., et al. "Congenital hydrocephalus in genetically engineered mice", *Veterinary Pathology*, **49**(1), pp. 166-181 (2012).

[47] Shih, H.-S., Shyur, H.-J. and Lee, E.S. "An extension of TOPSIS for group decision making", *Mathematical and Computer Modelling*, **45**, pp. 801-813 (2007).

[48] Hwang, C.L. and Yoon, K. "Multiple attribute decision making: Methods and applications", New York: Springer-Verlag (1981).

[49] Milani, A.S., Shanian, A. and Madoliat, R. "The effect of normalization norms in multiple attribute decision making models: A case study in gear material selection", *Structural Multidisciplinary Optimization*, **29**(4), pp. 312-318 (2005).

[50] Shih, H.-S., Lin, W.-Y. and Lee, E.S. "Group decision making for TOPSIS", *Joint 9th IFSA World Congress and 20th NAFIPS International Conference, IFSA/NAFIPS*, Vancouver, Canada, pp. 2712-2717 (2001).

[51] Yoon, K.P. and Hwang, C.L. "Multiple attribute decision making: An introduction", Sage Publishing Thousand Oaks, CA (1995).

[52] Akram, M., Ilyas, F. and Garg, H. “Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information”, *Soft Computing*, in press, (2019). https://doi.org/10.1007/s00500-019-04105-0

[53] Alper, D. and Başdar, C. “A comparison of TOPSIS and ELECTRE methods: An application on the factoring industry”, *Business and Economics Research Journal*, **8**(3), pp. 627-646 (2017).

[54] Birgün, S. and Cihan, E., “Supplier selection process using ELECTRE method”, *IEEE 2010 International Conference on Intelligent Systems and Knowledge Engineering*, Hangzhou, China, pp. 634-639 (2010). DOI: 10.1109/ISKE.2010.5680767

[55] Chen, T.-Y. “An IVIF-ELECTRE outranking method for multiple criteria decision-making with interval-valued intuitionistic fuzzy sets”, *Technological and Economic Development of Economy*, **22**(3), pp. 416-452 (2016).

[56] Devi, K. and Yadav, S.P. “A multicriteria intuitionistic fuzzy group decision making for plant location selection with ELECTRE method”, *The International Journal of Advanced Manufacturing Technology*, **66**, pp. 1219-1229 (2013).

[57] Xu, J. and Shen, F. “A new outranking choice method for group decision making under Atanassov’s interval-valued intuitionistic fuzzy environment”, *Knowledge-Based Systems*, **70**, pp. 177-188 (2014).

[58] Zhong, L. and Yao, L. “An ELECTRE I-based multi-criteria group decision making methodwith interval type-2 fuzzy numbers and its application to supplier selection”, *Applied Soft Computing*, **57**, pp. 556-576 (2017).

[59] Kharal, A. “Soft approximations and uni-int decision making”, *The Scientific World Journal*, **2014**, 7 pages (2014).

Biographies

Eyüp Akçetin is an Associate Professor at the Department of Accounting and Financial Management, Seydikemer School of Applied Sciences of Muğla SıtkıKoçman University in Turkey. He received his MSc degrees in Logistics and Maritime Transport from Dokuz Eylül University in 2007 and also he received his another MSc degrees in Informatics and Computer Engineering from Sakarya University in 2014 Turkey. He received his PhD degrees in Maritime Economy and Maritime Business Management from Istanbul University in 2012 Turkey. His research interests include multi criteria decision making, game theory, business intelligence and data mining, gamification, systems dinamics in business. He has contributed with many articles, books and book chapters on these scientific topics.

Hüseyin Kamacı is a Research Assistant at the Mathematics Department in the Science and Arts Faculty of Yozgat Bozok University, Turkey. He received his MSc and PhD degrees in Mathematics from Bozok University, Yozgat, Turkey in 2014 and 2018, respectively. His research interests include mathematical logic, set theory, operational research, computational intelligence, decision making. He has many valuable publications on these issues in different scientific journals.

Figures and Tables

Figure Captions:

Figure 1: Three-valued logic for classifying mail in a mailbox
Figure 2: Three-valued logic for reviewer recommendation in journal
Figure 3: Three-valued logic for Cilia-Related Lesions in Hydrocephalic Mice (see [46])
Figure 4: Comparison for some multimedia features of four mobiles

Table Captions:

Table 1: The side effects specified in the prospectus of drugs.
Table 2: The tabular form of three-valued soft set T_{Q_1} in Example 3.4.
Table 3: The tabular form of T_{Q_2} for $Q_2 = \mathcal{P}$.
Table 4: The tabular form of $\underline{T_{Q_2}}$ for $Q_2 = \mathcal{P}$.
Table 5: The tabular form of $\overline{T_{Q_2}}$.
Table 6: The tabular form of $T_{Q_1} \underline{\setminus} T_{Q_2}$.
Table 7: The tabular form of $T_{Q_1} \underline{\triangle} T_{Q_2}$.
Table 8: The choice value for three-valued soft set T_{Q_1} in Example 3.4.
Table 9: The ranking preference order of objects for three-valued soft set T_{Q_1} in Example 3.4.
Table 10: The table of three-valued soft set $T_{Q_1} \overline{\sqcap} T_{Q_2}$.
Table 11: The table of three-valued soft set $T_{Q_1} \underline{\sqcup} T_{Q_2}$.
Table 12: The weighted choice value for three-valued soft set T_{Q_1} in Example 3.4.
Table 13: The weights of choice parameters of experts E_1 and E_2.
Table 14: The table of three-valued soft sets of experts E_1 and E_2.
Table 15: The table of separation measurements $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$.
Table 16: The weights of the decision maker's choice parameters.
Table 17: The table of the expert's three-valued soft sets T_{Q_1},T_{Q_2},T_{Q_3}.
Table 18: The table of sets of concordance and discordance.
Table 19: Table of comparing and matching the results of Algorithms 2, 3 and 4.
Table 20: The comparison results of Algorithms 2, 3 and 4 with some existing soft decision making algorithms.

List of Figures:

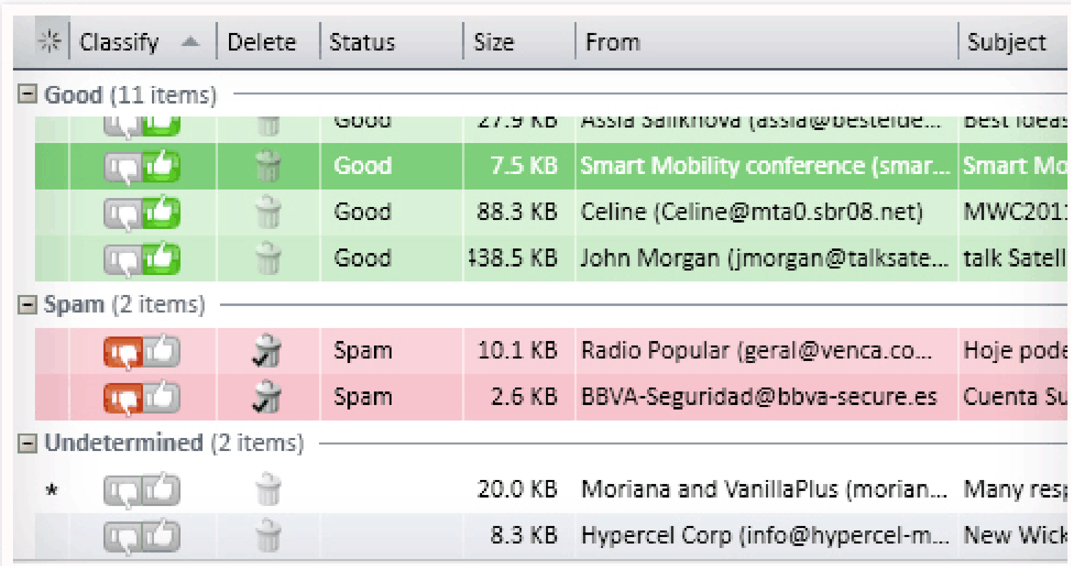

Figure 1: Three-valued logic for classifying mail in a mailbox

Figure 2: Three-valued logic for reviewer recommendation in journal

Gene Allele Symbol	Rhinosinusitis	Otitis Media	Situs Inversus	Infertifity
$Ulk4^{tmlLex}$	Yes	Yes	Undetermined	Undetermined
$Nme5^{\ tmlLex}$	Yes	No	No	Yes
$Nme7^{Gt(OST3II6)Lex}$	Yes	No	Yes	No
$Kif27^{Gt(OST4I9I5)Lex}$	Yes	Yes	No	Undetermined
$Stk36^{tmlLex}$	No	Yes	No	Yes
$Dpcd^{Gt(OST280355Lex}$, $Poll^{Gt(OST280355)Lex}$	Yes	No	Yes	Yes
$Ak7^{Gt(OST434404)Lex}$	Yes	No	No	Yes
$Ak8^{Gt(OSTI6378)Lex}$	No	No	No	No
$4930444A02Rik^{Gt(OST243203)Lex}$	No	No	Undetermined	Yes
$Celsr2^{tmlLex}$	No	No	No	No
$Mboat7^{tmlLex}$	No	No	No	No
Tg(FZD3)lLex	No	No	No	No

Figure 3: Three-valued logic for Cilia-Related Lesions in Hydrocephalic Mice (see [46])

Jump To	Vivo V17 Pro Rs. 27,990 89% SPEC SCORE Buy from	OnePlus 7 Rs. 31,975 96% SPEC SCORE Buy from	Xiaomi Redmi K20 Pro Rs. 23,175 94% SPEC SCORE Buy from	Apple iPhone 11 Rs. 62,900 96% SPEC SCORE Buy from
Multimedia				
FM Radio	✓	✗ No	✓	✗ No
Stereo Speakers		✓		✓
Loudspeaker	✓	✓	✓	✓

Figure 4: Comparison for some multimedia features of four mobiles

List of Tables:

Table 1: The side effects specified in the prospectus of drugs.

	$p_1 = allergy$	$p_2 = fetal\ damage$	$p_3 = pharmacological\ effect$
x_1	✓	×	−−
x_2	×	−−	✓
x_3	✓	×	×
x_4	×	−−	−−
x_5	×	✓	×

In the table, the symbols ✓, × and −− represent *Yes*, *No* and *Undetermined*, respectively.

Table 2: The tabular form of three-valued soft set T_{Q_1} in Example 3.4.

$\mathfrak{X}/P$	p_1	p_2	p_3	p_4
x_1	1	0	1	0
x_2	0	1	$\frac{1}{2}$	0
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0
x_4	$\frac{1}{2}$	1	1	0

Table 3: The tabular form of T_{Q_2} for $Q_2 = \mathcal{P}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	$\frac{1}{2}$	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	0	0	0
x_4	0	1	1	0

Table 4: The tabular form of T_{Q_2} for $Q_2 = \mathcal{P}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	1	1	0
x_2	0	0	0	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1	0
x_4	1	0	0	1

Table 5: The tabular form of $\overline{T_{Q_2}}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	0	1
x_2	1	1	1	1
x_3	$\frac{1}{2}$	$\frac{1}{2}$	0	1
x_4	0	1	1	0

Table 6: The tabular form of $T_{Q_1} \underline{\vee} T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	0	0	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	0	0
x_4	0	1	1	0

Table 7: The tabular form of $T_{Q_1} \underline{\triangle} T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4
x_1	$\frac{1}{2}$	1	0	0
x_2	0	1	$\frac{1}{2}$	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
x_4	$\frac{1}{2}$	1	1	1

Table 8: The choice value for three-valued soft set T_{Q_1} in Example 3.4.

					$\alpha_i = \sum_j (x_{ij})^\xi$				
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\xi = 1$	$\xi = \frac{1}{4}$	$\xi = \frac{3}{2}$	$\xi = \sqrt{13}$	$\xi = 10$
x_1	1	0	1	0	2	2	2	2	2
x_2	0	1	$\frac{1}{2}$	0	1.5	1.8408	1.3535	1.0824	1.0009
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	2.6816	1.707	1.1648	1.0018
x_4	$\frac{1}{2}$	1	1	0	2.5	2.8408	2.3535	2.0824	2.0009

Table 9: The ranking preference order of objects for three-valued soft set T_{Q_1} in Example 3.4.

	ranking order of choice value α_i	ranking preference order of x_i	$max\ \alpha_i$	optimal choice object
$\xi = 1$	$\alpha_4 > \alpha_1 = \alpha_3 > \alpha_2$	$x_4 \succ x_1 \approx x_3 \succ x_2$	α_4	x_4
$\xi = \frac{1}{4}$	$\alpha_4 > \alpha_3 > \alpha_1 > \alpha_2$	$x_4 \succ x_3 \succ x_1 \succ x_2$	α_4	x_4
$\xi = \frac{3}{2}$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4
$\xi = \sqrt{13}$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4
$\xi = 10$	$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$	$x_4 \succ x_1 \succ x_3 \succ x_2$	α_4	x_4

Table 10: The table of three-valued soft set $T_{Q_1} \bar{\sqcap}\, T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\sum_j (x_{ij}^{\sqcap})^2$
x_1	$\frac{1}{2}$	0	1	0	1.25
x_2	0	0	0	0	0
x_3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0.75
x_4	$\frac{1}{2}$	0	0	0	0.25

Table 11: The table of three-valued soft set $T_{Q_1} \underline{\sqcup}\, T_{Q_2}$.

$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	$\sum_j (x_{ij}^{\sqcup})^2$
x_1	1	1	1	0	3
x_2	0	1	$\frac{1}{2}$	0	1.25
x_3	1	$\frac{1}{2}$	1	0	2.25
x_4	1	1	1	1	4

Table 12: The weighted choice value for three-valued soft set T_{Q_1} in Example 3.4.

					$\alpha_i^\omega = \sum_j \omega_j^1 \times (x_{ij})^\xi$		
$\mathfrak{X}/\mathcal{P}$	$p_1(\omega_1^1 = 0.6)$	$p_2(\omega_2^1 = 0.3)$	$p_3(\omega_3^1 = 0.1)$	$p_4(\omega_4^1 = 0)$	$\xi = 1$	$\xi = \frac{1}{10}$	$\xi = 3$
x_1	1	0	1	0	0.7	0.7	0.7
x_2	0	1	$\frac{1}{2}$	0	0.35	0.3933	0.3125
x_3	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0.8	0.9732	0.65
x_4	$\frac{1}{2}$	1	1	0	0.7	0.9598	0.475

Table 13: The weights of choice parameters of experts E_1 and E_2.

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	
First Expert (E_1)	0.22	0.15	0.13	0.15	0.13	0.11	0.11	$\sum_{j=1}^7 \omega_j^1 = 1$
Second Expert (E_2)	0.24	0.12	0.12	0.16	0.14	0.1	0.12	$\sum_{j=1}^7 \omega_j^2 = 1$

Table 14: The table of three-valued soft sets of experts E_1 and E_2.

Decision makers:	First Expert (E_1)							Second Expert (E_2)						
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_1	p_2	p_3	p_4	p_5	p_6	p_7
x_1	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	0
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	1	1	1	1
x_3	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	1	1	0	1
x_4	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	$\frac{1}{2}$	1	0	0	1	$\frac{1}{2}$
x_5	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	1	0	1	1
x_6	$\frac{1}{2}$	$\frac{1}{2}$	1	1	0	1	1	1	1	1	1	0	1	1
impact coefficient ξ_k:	$\xi_1 = 1.5$							$\xi_2 = 2$						

Table 15: The table of separation measurements $\mathcal{S}_i^{\top}$ and $\mathcal{S}_i^{\perp}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\mathcal{S}_i^{\top}$	0.1542	0.0787	0.167	0.1705	0.178	0.1144
$\mathcal{S}_i^{\perp}$	0.0932	0.1923	0.1109	0.1004	0.0964	0.164

Table 16: The weights of the decision maker's choice parameters.

Decision makers/Weights	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	
Engineering expert (E_1)	0.2	0.15	0.5	0	0.15	0	$\sum_{j=1}^{6} \omega_j^1 = 1$
Financial expert (E_2)	0.15	0.25	0.25	0	0.1	0.25	$\sum_{j=1}^{6} \omega_j^2 = 1$
Quality control expert (E_3)	0.2	0.2	0	0.1	0.1	0.4	$\sum_{j=1}^{6} \omega_j^3 = 1$

Table 17: The table of the expert's three-valued soft sets T_{Q_1},T_{Q_2},T_{Q_3}.

Decision makers:	Financial expert (E_1)						Quality control expert (E_2)						Engineering expert (E_3)					
$\mathfrak{X}/\mathcal{P}$	p_1	p_2	p_3	p_4	p_5	p_6	p_1	p_2	p_3	p_4	p_5	p_6	p_1	p_2	p_3	p_4	p_5	p_6
x_1	1	$\frac{1}{2}$	0	0	1	0	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
x_2	$\frac{1}{2}$	1	1	0	0	0	1	0	0	0	0	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0
x_3	0	1	1	0	$\frac{1}{2}$	0	1	1	1	0	$\frac{1}{2}$	0	1	0	0	0	$\frac{1}{2}$	1
x_4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	1	1	0	1	$\frac{1}{2}$	0
impact coefficient ξ_k:	$\xi_1 = 2$						$\xi_2 = 3$						$\xi_3 = 2$					

Table 18: The table of sets of concordance and discordance.

	Concordance set ($\mathcal{J}^+$)				Discordance set ($\mathcal{J}^-$)			
	x_1	x_2	x_3	x_4	x_1	x_2	x_3	x_4
x_1	-	$\{1,4,5\}$	$\{4,5\}$	$\{1,3,5\}$	-	$\{2,3,6\}$	$\{1,2,3,6\}$	$\{2,4,6\}$
x_2	$\{2,3,4,6\}$	-	$\{4\}$	$\{3,6\}$	$\{1,5\}$	-	$\{1,2,3,5,6\}$	$\{1,2,4,5\}$
x_3	$\{1,2,3,4,6\}$	$\{1,2,3,4,5,6\}$	-	$\{1,2,3,6\}$	$\{5\}$	$\varnothing$	-	$\{4,5\}$
x_4	$\{2,4,6\}$	$\{1,2,4,5,6\}$	$\{4,5\}$	-	$\{1,3,5\}$	$\{3\}$	$\{1,2,3,6\}$	-

Table 19: Table of comparing and matching the results of Algorithms 2, 3 and 4.

Problems	Impact coefficients	Optimal Choice		
		Algorithm 2	Algorithm 3	Algorithm 4
Example 4.5	$\xi = 1$ (i.e., $\xi_1 = 1$, $\xi_2 = 1$)	x_3	x_3	x_3
Example 4.6	$\xi = 1$ (i.e., $\xi_1 = 1$, $\xi_2 = 1$)	x_1	x_1	x_1
Example 5.1	$\xi_1 = 1.5$, $\xi_2 = 2$	–	x_2	x_2
Example 6.1	$\xi_1 = 2$, $\xi_2 = 3$, $\xi_3 = 2$	–	x_3	x_3

"–" means that the algorithm is not applicable to this problem.

Table 20: The comparison results of Algorithms 2, 3 and 4 with some existing soft decision making algorithms.

Paper	Problem in the paper	Optimal Choice			
		Algorithm in the paper	Algorithm 2	Algorithm 3	Algorithm 4
[38]	Application (Sect. 5) in [38]	u_1	u_1	u_1	u_1
[35, 36]	(Example 5.17 in [35] Example 3.3 in [36])	h_1, h_2, h_3	h_1, h_2, h_3	h_1	$h_1 \succ h_3 \succ h_4 ? h_5$ & $h_2 \succ h_4 ? h_5$
[59]	Example 31 in [59]	u_3	u_3	u_3	u_3
[32]	Table 2 (Sect. 3.4) in [32]	h_1, h_6	h_1, h_6	h_1, h_6	h_1, h_6

For Algorithms 2, 3 and 4, the weights of parameters in decision problem are taken equally and their sum is 1.