Maclaurin symmetric Means for Linguistic Z-numbers and Their Application to Multiple-Attribute Decision Making

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Abstract: Linguistic Z-numbers (LZNs), as a more rational extension of linguistic description, not only consider the fuzzy restriction of assessment information but also take the reliability of the information into account. Maclaurin symmetric mean (MSM) operator has the advantage which can take account of interrelationship of different attributes and there are a lot of research results on it. However, it has not been used to handle multi-attribute decision-making (MADM) problems expressed by LZNs. To sum up the advantages of LZNs and MSM, in this paper, we present the linguistic Z-Numbers MSM (LZMSM) and linguistic Z-Numbers weight MSM (LZWMSM) operators, respectively, and several characters and several special cases of them are explored. Moreover, we propose an approach to handle some MADM problems by using LZWMSM operator. In the end, an example is given to illustrate the effectiveness and superiority of this new presented approach by comparing with several existing approaches.

Keywords: Maclaurin symmetric mean operator; linguistic Z-numbers; multi-attribute decision making

I. INTRODUCTION

Zadeh [1] put forward the fuzzy sets (FSs) which laid the basis of fuzzy evaluation. However, the FSs only contain the membership function which is a crisp number in [0, 1]. So when the fuzzy information is complicated, it is difficult to describe by FSs. Later, Atanassov [2] introduced the intuitionistic FS (IFS) which includes not only membership degree but also non-membership degree to express more complex fuzzy information. Deschrijver and Kerre [3] made further exploration, and defined the intuitionistic fuzzy numbers (IFNs). However, we find that these information expression methods do not think about the confidence degree of decision making information. In fact, because everyone is a bounded rational there exists cultural differences, this will produce a significant difference in the evaluating the same things. To handle this problem, Zadeh [4] introduced the conception of Z-number firstly. A Z-number is an ordered pair of fuzzy numbers (FNs) that are expressed as \( Z = (A, B) \) with the meaning of constraint \( A \) and reliability \( B \). Since then, there are many research fruits based on Z-numbers [5-9]. Aliev et al. [10] investigated the operational rules of Z-numbers, but the operations are also complicated. Further, in order to avoid the complexity of calculating directed in Z-numbers, Kang et al. [11] presented a method which can transform the Z-numbers to traditional FNs. In the same way, Yaakob et.al [6] converted the Z-numbers to trapezoidal FNs and presented the interactive Z-TOPSIS approach to solve MADM problems. Peng et.al [12] presented the asymmetric normal Z-value (ANZ) and gave an innovative Multi-criteria game model. Further, Saravi et.al [13] presented the Z-numbers DEA for location optimization of agricultural residues-based biomass plant. However, in many cases, people are more inclined to give linguistic evaluation information, so it is also necessary to apply Z-numbers to a linguistic evaluation environment.

In real life, there are great deals of qualitative information for MADM problems, which are easily expressed by linguistic terms (LTs). For facilitating the processing of linguistic information, Zadeh [14-16] introduced the definition of the linguistic variable (LV) firstly. However, the original LVs proposed by Zadeh are discrete and they might lose a lot of original information during the calculation. To overcome the drawback, Xu [17] extended the discrete LTs set \( S \) to a continuous one. Based on this, LVs were extended to a series of types about fuzzy information, such as uncertain LVs [18,19], hesitant fuzzy linguistic information [20] and so on [21,22]. As an extension of quantitative research [23,24], based on the LVs and FSs, Chen and Liu [25] presented the linguistic IFNs (LIFNs) that can integrate the advantages of IFNs and LVs. Then, there are many researches based on LIFNs, such as the Partitioned Heronian means for LIFNs [26], preference relations for LIFNs [27]. In the same way, for combining the advantages of Z-number and LTs, Wang et al. [28] proposed the LZNs. At the same
time, they modified and defined the operational laws by using linguistic scale functions (LSFs) to overcome the limitation of losing original information by using traditional operations [11] [6].

Moreover, compared with the traditional operations of Z-numbers in [4,29], the operations proposed by Wang et al. [28] is simpler and more flexible, because different LSFs can be selected according to different situations. Besides, Wang et al. [28] proposed the extended TODIM method by using Choquet integral with Z-numbers for Multi-Criteria Decision-Making (MCDM) problems. However, we find that Wang’s methods cannot handle the variable interrelationship between different attributes. In real decision, many attributes are mutually influential. So, we will consider the aggregation operators to handle such problems in this paper.

Many traditional aggregation operators [30] just can aggregate a collection of crisp values into one. Now, for dealing with some special functions, the researchers have developed many extended aggregation operators. The BM operator [31] presented by Bonferroni can capture the relationship between any pair of different attributes $c_i$ and $c_j (i \neq j, i, j = 1,2,\ldots,n)$ in the decision-making problems. However, it ignores the correlation between $c_i$ and $c_j$ when $i = j$. Then, the Heronian mean (HM) [32] was proposed by Beliacov, which can overcome this drawback. Based on the two operators, a lot of researches are done. Such as Liang et al. [33] proposed BM operators for Pythagorean FNs (PFNs), Yang and Pang [34] proposed the partitioned BM operators for q-rung orthopair FNs (q-ROFNs) and Wei et al. [35] introduced HM operators for picture fuzzy numbers and so on. However, either HM operator or BM operator just can think about interrelationship among any two attributes at most. The MSM operator was first presented by Maclaurin [36], and developed by Detemple et al. [37], which considered the interrelationship between different numbers of attributes by adjusting the variable parameter. Further, there are many achievements to solve MADM problems by using MSM operator: Wei et al. [38] presented the MSM operators for q-ROFNs; Yang and Pang [39] proposed MSM operators for Pythagorean FNs (PFNs); Peng [40] presented the MSM operators for single-valued neutrosophic numbers and so on [41].

To sum up the advantages of LZNs and MSM, we extend MSM operator to LZNs, and develop the LZMSM operator and LZWMSM operator. The advantages of our presented operators are that they can not only take account of the merits of MSM by considering interrelationship among multi-attributes with a variable parameter, but also consider the reliability about the constraint $A$ of Z-number in qualitative environment. So the aims of our paper are given as follows: (1) investigate several new MSM operators for LZNs based on LSFs; (2) discuss the desired characters of the proposed operators and several special cases; (3) use our proposed operator to handle MADM problems under the circumstance of LZNs and to propose a novel decision method which can take interrelationship of multi-attributes into account; (4) demonstrate the merits of this new method by comparing with several existing approaches.

The remaining structure of the paper is constructed as: in Section 2, we give an outline of several basic notions of LSFs, MSM operator, LZNs and some new operational rules; in Section 3, we introduce LZMSM and LZWMSM operators and study some properties about them; in Section 4, we present a MADM approach according to our proposed LZWMSM operator; in Section 5, an illustrate example is shown to express the validity and superiority of our presented new approach by comparing with several existing other approaches; in Section 6, we given the conclusions about this paper.

II. PRELIMINARIES

In this part, we review some essential notions and basic theories of Z-numbers, LZNs, LSFs and MSM operator. Suppose LT set (LTS) $S = \{s_1, g = 0.1,\ldots,l\}$ is finite and totally ordered discrete LTS, and $s_i$ denotes a LV. $l$ is even number. In practice, $l$ can be set to 4, 6, 8, etc. For instance, when $l = 4$, it is represented as: $S = \{s_1,s_2,s_3,s_4\}$ = (very low, low, fair, high, very high).

2.1 LSFs

Definition 1 [28,42]. If $\rho_s \in [0,1]$ is a number, then the LSF $F$ conducts the mapping from $s_i$ to $\rho_s$ ($g = 0.1,\ldots,l$) which can be expressed as:

$$F: s_i \rightarrow \rho_s (g = 0.1,\ldots,l)$$

where $0 \leq \rho_1 < \rho_2 < \ldots < \rho_l$.

Based on the function, the symbols $\rho_i$ can express the LT $s_i \in S$ which reflects the assessment information, and the semantics of the LTs are denoted by the function or value. Next, we introduce four useful LSFs.

(1) $F_1(s_i) = \rho_s = \frac{g}{l} (g = 0.1,2,\ldots,l)$

(2) $F_2(s_i) = \rho_s = \left(\frac{g}{l}\right)^2 (g = 0.1,2,\ldots,l)$
In formula (5), \( a, b \in (0, 1] \), and if \( a = b = 1 \), then \( \rho_x = \frac{g}{l} \). In this paper, for easily calculating we let \( a = b = 0.5 \).

2.2 LZNs

Definition 2 [4]. A Z-number is an ordered pair of fuzzy numbers \((A, B)\) that is related to a real-valued uncertain variable \(x\), where \(A\) is a fuzzy restriction on the values that the variable \(x\) is allowed to take, and \(B\) is a measure of the certainty of \(A\). In general, \(A\) and \(B\) are described in natural language.

For easily understanding the concept of Z-numbers, a brief explanation will be made here. For instance, we can use the Z-number (about 1 hour and 45 minutes, sure) to express “I am sure that it takes about 1 hour and 45 minutes to travel from Jinan to Beijing by high-speed train.”, where “travel from Jinan to Beijing by high-speed train” is the uncertainty variable \(x\) in the Z-number. Zadeh noted that the underlying probability distribution in a Z-number is unknown, and the Z-number processing method he gave is complicated. Based on the idea of Z-number, Wang et al. [28] proposed a more understandable Z-number subclass - LZNs.

Definition 3 [28]. Suppose \(Y\) is a universe of discourse, \(S_1 = \{s_i \mid i = 0, 1, \ldots , T\} \) and \(S_2 = \{s_j \mid j = 0, 1, \ldots , L\} \) be two LTs, \(T\) and \(L\) be two even numbers. In practice, \(T\) and \(L\) can be set to 4, 6, 8, etc. For instance, when \(T = 4\), it can be expressed by \(S = (s_0, s_1, s_2, s_3, s_4) = \) (very bad, bad, general, good, very good). Further, let \(A_{i(1)} \in S_1\) and \(B_{i(1)} \in S_2\), then, the set of LZNs \(Z\) in \(Y\) is given by:

\[
Z = \left\{ (y, A_{i(1)}, B_{i(1)}) \mid y \in Y \right\},
\]

in which \(A_{i(1)}\) is the fuzzy linguistic measure of \(y\), and \(B_{i(1)}\) is the probability measure of \(A_{i(1)}\) that can measure the reliability. In general, the LTs \(S_1\) and \(S_2\) are different. To the given element \(x\), each pair of \((A_{i(1)}, B_{i(1)})\) in \(Z\) is referred to as a LZN. For convenience, \(z_i = (A_{i(1)}, B_{i(1)})\) is used to describe a LZN, which meets \(A_{i(1)} \in S_1\) and \(B_{i(1)} \in S_2\) are two LTs.

Example 1. Let \(Y = \{y_1, y_2, y_3\}\) be the universe of discourse, and LTs: \(A = \{A_1, A_2, \ldots , A_6\} = \) \{very bad, bad, almost bad, general, almost good, good, very good\} and \(B = \{B_1, B_2, B_3\} = \) \{rarely, occasionally, usual\}. A LZN can be expressed as \(z_i = (A_{i(1)}, B_{i(1)})\), where \(A_{i(1)} \in A\) is the evaluation information about the discourse given by the decision makers (DMs) and \(B_{i(1)} \in B\) is used to express the reliability measure of the LZN. Then, the LZN \(z_i = (A_{i(1)}, B_{i(1)})\) represents as \((fair, usually)\).

For reducing the information distortion caused by information loss during calculation process, it is very necessary to extend the original discrete LTs \(A\) and \(B\) to the continuous LTs \(\tilde{A} = \{A_i \mid i \in [0, T]\}\) and \(\tilde{B} = \{B_i \mid i \in [0, L]\}\)

like [17].

Definition 4 [28]. Let \(z_i = (A_{i(1)}, B_{i(1)})\) be a LZN, then the score function \(LS\) of \(z_i\) can be given as the following:

\[
LS(z_i) = \nu^*(A_{i(1)}) \times h^*(B_{i(1)})
\]

where \(\nu^*(A_{i(1)})\) and \(h^*(B_{i(1)})\) are any two LSFs in Definition 1.

Definition 5 [28]. Suppose \(z_i = (A_{i(1)}, B_{i(1)})\) and \(z_j = (A_{j(1)}, B_{j(1)})\) are any two LZNs, then according to the score function, the comparison approach of LZNs can be given as:
(1) if \(A_{i(0)} > A_{i(1)}\) and \(B_{i(0)} > B_{i(1)}\), then \(z_i\) is strictly better than \(z_j\), i.e., \(z_i > z_j\);

(2) if \(LS(z_i) > LS(z_j)\) or \(LS(z_i) = LS(z_j)\), then \(z_i > z_j\) or \(z_i = z_j\);

(3) if \(LS(z_i) < LS(z_j)\), then \(z_i < z_j\).

**Example 2.** Continue to Example 1, let \(z_i = (A_i, B_i)\) and \(z_j = (A_j, B_j)\) be two LZNs, and suppose \(\psi^*(s_j) = F_i(s_j)\) and \(h^*(s_j) = F_i(s_j)\), then we can get

\[
LS_i = \frac{3^{0.5} - 1^{0.5}}{2} \times \frac{2}{2} = 0.21, \quad LS_j = \frac{3^{0.5} - 0^{0.5}}{2} = 0.5
\]

So, \(LS_i < LS_j\), i.e., \(z_i < z_j\).

**Definition 6 [28].** Let \(z_i = (A_{i(0)}, B_{i(0)})\) and \(z_j = (A_{j(0)}, B_{j(0)})\) be any two LZNs; \(\psi^*\) and \(h^*\) be the possible functions of \(F_i(s_j), F_i(s_j), F_i(s_j)\), and \(F_i(s_j)\), \(\lambda \geq 0\), and the operational rules of LZNs is given as following formulas:

1) \(neg(z_i) = \left[\psi^{-1}(\psi^*(A_{i(0)}) - \psi^*(A_{j(0)})\right]. \]

\[
h^{-1}\left[h^*(B_{i(0)}) - h^*(B_{j(0)})\right]\]

(8)

2) \(z_i \oplus z_j = \left[\psi^{-1}(\psi^*(A_{i(0)}) + \psi^*(A_{j(0)})\right]. \]

\[
h^{-1}\left[\psi^*(A_{i(0)}) + \psi^*(A_{j(0)})\right]\]

(9)

3) \(\lambda z_i = \left[\psi^{-1}(\lambda \psi^*(A_{i(0)})\right] \oplus B_{j(0)}\]

(10)

4) \(z_i \odot z_j = \left[\psi^{-1}(\psi^*(A_{i(0)}) \psi^*(A_{j(0)})\right]. \]

\[
h^{-1}\left[h^*(B_{i(0)}) \oplus h^*(B_{j(0)})\right]\]

(11)

5) \(z_i = \left[\psi^{-1}(\psi^*(A_{i(0)})\right] \odot h^{-1}\left[h^*(B_{i(0)})\right]\)

\[
(1, 2)
\]

**Theorem 1 [28].** Suppose \(z_i = (A_{i(0)}, B_{i(0)})\) and \(z_j = (A_{j(0)}, B_{j(0)})\) are any two LZNs, and \(\lambda, \lambda_i, \lambda_j > 0\), then

(1) \(z_i \oplus z_j = z_i \oplus z_j\)

(2) \(z_i \odot z_j = z_i \odot z_j\)

(3) \(\lambda z_i \oplus z_j = \lambda z_i \oplus \lambda z_j\)

(4) \((\lambda_i + \lambda_j) z_i = \lambda_z z_i \oplus \lambda_z z_i\)

(5) \((z_i \odot z_j) = z_i \odot z_j\)

(6) \(z_i \odot z_j = z_i ^{(k + k_1)}\)

**Example 3.** Continue to Example 2, we can get

\[
z_j \oplus z_j = \psi^{-1}(0.21 + 0.5), h^{-1}\left[0.21 \times 1 + 0.5 \times 1\right]
\]

\[
= (A_{(0.712 + 0.5)}, B_{(0.21 + 0.5)}) = (A_{1(1)}, B_{1(1)})
\]
$$z_i \otimes z_j = (v^{-1}(0.21 \times 0.5), h^{-1}(1 \times 1)) =$$
$$(A_{n \times (0.21 \times 0.5)}, B_{n \times 1}) = (A_{11}, B_1).$$

### 2.3 The MSM Operator

MSM operator is a helpful technique that proposed by Maclaurin [36] firstly, which can take the interrelationship of different numbers of attributes into account.

**Definition 7 [36].** Let $q_i (i = 1, 2, \ldots, n)$ be a set of nonnegative real numbers, and $\delta = 1, 2, \ldots, n$, then MSM can be defined as:

$$MSM^{(\delta)}(q_1, q_2, \ldots, q_n) = \left( \sum_{i \in \{1, \ldots, \delta\}} \prod_{j \in \{i, \ldots, \delta\}} \frac{d_j}{C_{\delta}^i} \right)^{\|v^\delta\|} \left( \prod_{i = 1}^{n} \frac{1}{C_{\delta}^i} \right)^{\|v^\delta\|} = \left( \sum_{i \in \{1, \ldots, \delta\}} \prod_{j \in \{i, \ldots, \delta\}} \frac{d_j}{C_{\delta}^i} \right)^{\|v^\delta\|} \left( \prod_{i = 1}^{n} \frac{1}{C_{\delta}^i} \right)^{\|v^\delta\|}$$

(19)

where $C_{\delta}^i$ is the binomial coefficient and $(i, i_2, \ldots, i_\delta)$ traverses all the $\delta$-tuple combinations of $(1, 2, \ldots, n)$.

Apparently, MSM operator has several properties as follows:

1. $MSM^{(\delta)}(0, 0, \ldots, 0) = 0$, $MSM^{(\delta)}(q_1, q_2, \ldots, q_n) = q$;
2. $MSM^{(\delta)}(q_1, q_2, \ldots, q_n) \leq MSM^{(\delta)}(p_1, p_2, \ldots, p_n)$, if $q_i \leq p_i$ for all $i$;
3. $\min\{q_i\} \leq MSM^{(\delta)}(q_1, q_2, \ldots, q_n) \leq \max\{q_i\}$.

### III. Some MSM Operators for LZNs

In this part, according to the operational laws of LZNs presented by Wang et al. [28], we propose the linguistic Z-Numbers MSM (LZ MSM) operator and linguistic Z-Numbers weight MSM (LZW MSM) operator, and then we will investigate several characters and special cases.

#### 3.1 LZ MSM operator

**Definition 8.** Let $z_i (i = 1, 2, \ldots, n)$ be a set of LZNs, and $\delta = 1, \ldots, n$, then LZ MSM operator is a mapping $LZ MSM : \Phi^\delta \to \Phi$ which can be defined as:

$$LZ MSM^{(\delta)}(z_1, z_2, \ldots, z_n) = \left( \bigoplus_{i = 1}^{n} \bigotimes_{i \in \{1, \ldots, \delta\}} \frac{z_i}{C_{\delta}^i} \right)^{\|v^\delta\|} \left( \prod_{i = 1}^{n} \frac{1}{C_{\delta}^i} \right)^{\|v^\delta\|},$$

(20)

where $\Phi$ is a collection of all LZNs, $C_{\delta}^i$ is the binomial coefficient and $(i, i_2, \ldots, i_\delta)$ traverses all the $\delta$-tuple combinations of $(1, 2, \ldots, n)$.

Based on operation rules of LZNs, we can deduce the result given as theorem 2.

**Theorem 2.** Suppose $z_i = (A_{j(i)}, B_{j(i)}) (i = 1, 2, \ldots, n)$ is a collection of LZNs and $\delta = 1, 2, \ldots, n$, then the result aggregated from (20) is also a LZN:

$$LZ MSM^{(\delta)}(z_1, z_2, \ldots, z_n) =$$

$$\left( \psi^{-1} \left( \sum_{i = 1}^{n} \prod_{j = 1}^{\delta} \frac{v^\delta(A_{j(i)})}{C_{\delta}^i} \right) \phi^{-1} \left( \sum_{i = 1}^{n} \prod_{j = 1}^{\delta} v^\delta(A_{j(i)}) \phi^{-1}(B_{j(i)}) \right) \right)^{\|v^\delta\|}. $$

(21)
Proof.

We first calculate $\hat{\otimes}_{j \in \mathcal{E}} z_{i}$, and get

$$\hat{\otimes}_{j \in \mathcal{E}} z_{i} = \left( \varPsi \prod_{j \in \mathcal{E}} \varPsi(A_{i,j}) \right) h^{-1} \prod_{j \in \mathcal{E}} h^{j}(B_{i,j}) \right),$$

and

$$\otimes_{j \in \mathcal{E}} \left( \hat{\otimes}_{j \in \mathcal{E}} z_{i} \right) =$$

$$\left( \varPsi \prod_{j \in \mathcal{E}} \varPsi(A_{i,j}) \right) h^{-1} \left( \sum_{j \in \mathcal{E}} \prod_{j \in \mathcal{E}} h^{j}(B_{i,j}) \right).$$

Then we get $\frac{1}{C_{z}} \left( \otimes_{j \in \mathcal{E}} \left( \hat{\otimes}_{j \in \mathcal{E}} z_{i} \right) \right) =$

$$\left( \varPsi \prod_{j \in \mathcal{E}} \varPsi(A_{i,j}) \right) h^{-1} \left( \sum_{j \in \mathcal{E}} \prod_{j \in \mathcal{E}} h^{j}(B_{i,j}) \right).$$

Therefore, we have

$$\left( \otimes_{j \in \mathcal{E}} \left( \hat{\otimes}_{j \in \mathcal{E}} z_{i} \right) \right)^{\otimes_{j \in \mathcal{E}}} =$$

$$\left( \varPsi \prod_{j \in \mathcal{E}} \varPsi(A_{i,j}) \right) h^{-1} \left( \sum_{j \in \mathcal{E}} \prod_{j \in \mathcal{E}} h^{j}(B_{i,j}) \right),$$

so the theorem 2 is proved.

Next, several properties and special cases of the LZMSM will be explored.

**Theorem 3** (Idempotency). Suppose $z_{i} = (A_{i}, B_{i})$ $(i = 1, 2, \ldots, n)$ is a collection of LZNs, if $z_{i} = z_{i}$ then $LZMSM^{(i)}(z_{1}, z_{2}, \ldots, z_{n}) = z_{i} = (A_{i}, B_{i})$ (22)

**Proof.**

Since $z_{i} = z_{i} = (A_{i0}, B_{i0})$ $(i = 1, 2, \ldots, n)$, we have

$$\prod_{j \in \mathcal{E}} h^{j}(A_{i,j}) = h^{j}(A_{i0})^{j},$$

$$\prod_{j \in \mathcal{E}} h^{j}(A_{i,j}) = h^{j}(B_{i,j}) = h^{j}(B_{i0})^{j},$$

$$\sum_{j \in \mathcal{E}} h^{j}(A_{i,j}) = C_{z}^{j} h^{j}(A_{i0})^{j},$$

So, we can get $LZMSM^{(i)}(z_{1}, z_{2}, \ldots, z_{n}) =$
\[
\left( v^{-1} \left( \frac{\sum \prod_{i=1, j 
eq 0}^{\delta} \psi'(A_{i(j)})}{C_{\alpha}} \right) \right)^z
\]

\[
h^{-1} \left( \frac{\sum \prod_{i=1, j 
eq 0}^{\delta} \psi'(A_{i(j)}) \cdot h'(B_{i(j)})}{\sum \prod_{i=1, j 
eq 0}^{\delta} \psi'(A_{i(j)})} \right)^z
\]

\[
= (A_{i(0)}', B_{i(0)}') = z_i.
\]

**Theorem 4 (Commutativity).** Suppose \( z = (A_{i(0)}, B_{i(0)}) \) \((i = 1, 2, \ldots, n)\) is a set of LZNs, and \( z' = (A_{i(0)}', B_{i(0)}') \) be any permutation of \( z \), then

\[
\text{LZMSM}^{(3)}(z, z_2, \ldots, z_n) = \text{LZMSM}^{(3)}(z', z_2', \ldots, z_n')
\]

**(23)**

**Proof.**

According to Eq. (20), we obtain

\[
\text{LZMSM}^{(3)}(z, z_2, \ldots, z_n) = \left( \frac{\sum_{i=1, j 
eq 0}^{\delta} \psi(A_{i(j)})}{C_{\alpha}} \right)^{\delta z}
\]

\[
\text{LZMSM}^{(3)}(z', z_2', \ldots, z_n') = \left( \frac{\sum_{i=1, j 
eq 0}^{\delta} \psi(A_{i(j)}')}{C_{\alpha}} \right)^{\delta z'}
\]

Since \( z' = (A_{i(0)}', B_{i(0)}') \) be any permutation of \( z \), then

\[
\left( \frac{\sum_{i=1, j 
eq 0}^{\delta} \psi(A_{i(j)})}{C_{\alpha}} \right)^{\delta z} = \left( \frac{\sum_{i=1, j 
eq 0}^{\delta} \psi(A_{i(j)}')}{C_{\alpha}} \right)^{\delta z'}.
\]

Thus, \( \text{LZMSM}^{(3)}(z, z_2, \ldots, z_n) = \text{LZMSM}^{(3)}(z', z_2', \ldots, z_n') \).

Now, we will explore several special cases about LZMSM by adjusting values of \( \delta \).

1. When \( \delta = 1 \), according to LZMSM operator from Eq. (21), we have

\[
\text{LZMSM}^{(3)}(z, z_2, \ldots, z_n) =
\]

\[
\psi^{-1} \left( \frac{\sum_{i=1}^{\delta} \psi(A_{i(1)})}{C_1} \right)^{\delta z'} h^{-1} \left( \frac{\sum_{i=1}^{\delta} \psi(A_{i(1)}) \cdot h'(B_{i(1)})}{\sum_{i=1}^{\delta} \psi(A_{i(1)})} \right)^{\delta z'}
\]

\[
= \psi^{-1} \left( \frac{1}{n} \sum_{i=1}^{\delta} \psi(A_{i(1)}) \right)^{\delta z'} h^{-1} \left( \frac{\sum_{i=1}^{\delta} \psi(A_{i(1)}) \cdot h'(B_{i(1)})}{\sum_{i=1}^{\delta} \psi(A_{i(1)})} \right)^{\delta z'} (let \ i = j)
\]

7
\[
\psi_{\gamma(i)}(A_{\gamma(i)}) = \left( \sum_{\alpha=1}^{n} \frac{\psi'(A_{\alpha})}{c_{\gamma(i)}} \right)^{2} \left( \sum_{\beta=1}^{n} \frac{\psi'(A_{\beta})}{c_{\gamma(i)}} \right), \quad (24)
\]

(2) When \( \delta = 2 \), the LZMSM operator may become the following formula.

\[
LZMSM^{(2)}(z_{i}, z_{j}, \ldots, z_{n}) = \left\{ \psi^{-1}\left( \frac{2}{n(n-1)} \times \sum_{i \neq j}^{n} \psi'(A_{i}) \cdot \psi'(A_{j}) \right) \right\}^{2}.
\]

\[
LZMSM^{(2)}(z_{i}, z_{j}, \ldots, z_{n}) = \left\{ \frac{1}{2} \left( \sum_{i \neq j}^{n} \psi'(A_{i}) \cdot \psi'(B_{ij}) \cdot \psi'(A_{j}) \cdot \psi'(B_{ij}) \right) \right\}^{2}.
\]

(3) When \( \delta = n \), on the basis of Eq. (21), we can obtain

\[
LZMSM^{(n)}(z_{i}, z_{j}, \ldots, z_{n}) = \left\{ \psi^{-1}\left( \frac{1}{n(n-1)} \sum_{i \neq j}^{n} \psi'(A_{i}) \cdot \psi'(A_{j}) \right) \right\}^{2}.
\]

\[
LZMSM^{(n)}(z_{i}, z_{j}, \ldots, z_{n}) = \left\{ \frac{1}{2} \left( \sum_{i \neq j}^{n} \psi'(A_{i}) \cdot \psi'(B_{ij}) \cdot \psi'(A_{j}) \cdot \psi'(B_{ij}) \right) \right\}^{2}.
\]

3.2 LZWMSM operator

We realize LZMSM operator fails to think about the weights of attributes which are very important. In some practical cases, particularly in MADM problems, the weights have a significant impact on decision making results. For overcoming the limitation of LZMSM operator, we develop the LZWMSM operator in following.

**Definition 9.** Let \( \delta = 1, 2, \ldots, n \) and \( z_{i}(i = 1, 2, \ldots, n) \) is a set of LZNs, \( w = (w_{1}, \ldots, w_{n})^{T} \) is the weight vector of \( z_{i} \). Then LZWMSM operator is a mapping \( LZWMSM : \Phi^{\gamma} \rightarrow \Phi \), which is given as:
\[
LZWMSM^{(d)}(\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n) = \left( \bigoplus_{\mathcal{I} \subseteq \{1, \ldots, n\}, |\mathcal{I}| = d} \left( \bigotimes_{j \in \mathcal{I}} w_j \overline{z}_j \right) \right)^{\bPhi} \frac{\binom{d}{i}}{\binom{n}{i}}, \tag{27}
\]

where \( \Phi \) is the set of all LZNs, \( \binom{d}{i} \) is the binomial coefficient and \((i_1, i_2, \ldots, i_n)\) traverses all the \( \bPhi \)-tuple combination of \((1, 2, \ldots, n)\).

According to the operation rules of the LZNs, the aggregation result from we Eq. (27) can be deduced given as follows.

**Theorem 5.** Let \( d = 1, 2, \ldots , n \), and \( \overline{z}_i = (A_{(i)})^i, B_{(i)}) \) \((i = 1, 2, \ldots , n)\) is a collection of LZNs, then the result aggregated from Eq. (27) is also a LZN. We have

\[
LZWMSM^{(d)}(\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n) =
\left( \left( \sum_{\mathcal{I} \subseteq \{1, \ldots, n\}, |\mathcal{I}| = d} \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right)^{\bPsi} \right)^{\bPhi} \frac{\binom{d}{i}}{\binom{n}{i}}.
\tag{28}
\]

**Proof.**
We first calculate \( w_j \overline{z}_j \) and get

\[
w_j \overline{z}_j = \left( \psi^{i+1} \left( \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right) \right) \text{ and } w_j \overline{z}_j = \left( \psi^{i+1} \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right) \psi^{i+1} \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right).
\]

Then we get

\[
\bigoplus_{\mathcal{I} \subseteq \{1, \ldots, n\}, |\mathcal{I}| = d} \left( \bigotimes_{j \in \mathcal{I}} w_j \overline{z}_j \right) = \left( \sum_{\mathcal{I} \subseteq \{1, \ldots, n\}, |\mathcal{I}| = d} \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right)^{\bPsi} \cdot \frac{\binom{d}{i}}{\binom{n}{i}}.
\]

\[
\left( \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right)^{\bPhi} \frac{\binom{d}{i}}{\binom{n}{i}}.
\]

\[
\left( \prod_{j \in \mathcal{I}} w_j \overline{z}_j \left( A_{(j)} \right) \right)^{\bPhi} \frac{\binom{d}{i}}{\binom{n}{i}}.
\]
\[
\begin{align*}
\psi^{-1} \left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} w_{i,j} \left( A_{i,j} \right)}{C_{i}} \right), \\
h^{-1} \left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \cdot h \left( B_{i,j} \right) \right)}{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \right)} \right)
\end{align*}
\]

Finally, we obtain

\[
\begin{align*}
\left( \bigoplus_{i=1}^{n} \bigotimes_{j=1}^{i} w_{i,j} \right)_{i} \bigotimes_{j=1}^{i} C_{i} = \\
\left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \right)}{C_{i}} \right)^{1/p}, \\
h^{-1} \left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \cdot h \left( B_{i,j} \right) \right)}{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \right)} \right)^{1/p}.
\end{align*}
\]

and it is also a LZN. So, the theorem 5 is proved.

By taking different values of parameter \( \delta \), we can explore several special cases of LZWMSM operator.

(1) When \( \delta = 1 \), Eq. (28) will become the following formula:

\[
\text{LZWMSM}^{(1)} (z_1, z_2, \ldots, z_n) = \\
\psi^{-1} \left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} w_{i,j} \left( A_{i,j} \right)}{C_{i}} \right)^{1/p}, \quad h^{-1} \left( \frac{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \cdot h \left( B_{i,j} \right) \right)}{\sum_{i=1}^{n} \prod_{j=1}^{i} \left( w_{i,j} \left( A_{i,j} \right) \right)} \right)^{1/p}
\]

\[
= \left( \frac{\sum_{i=1}^{n} w_{i,j} \left( A_{i,j} \right)}{n} \right)^{1/p}, \\
h^{-1} \left( \frac{\sum_{i=1}^{n} \left( w_{i,j} \left( A_{i,i} \right) \cdot h \left( B_{i,i} \right) \right)}{\sum_{i=1}^{n} w_{i,j} \left( A_{i,i} \right)} \right) \quad \text{(let } i = j) \\
= \psi^{-1} \left( \frac{\sum_{i=1}^{n} w_{i,j} \left( A_{i,i} \right)}{n} \right), \\
h^{-1} \left( \frac{\sum_{i=1}^{n} w_{i,j} \left( A_{i,i} \right) \cdot h \left( B_{i,i} \right)}{\sum_{i=1}^{n} w_{i,j} \left( A_{i,j} \right)} \right) \quad \text{(29)}
\]

(2) When \( \delta = 2 \), Eq. (28) will become the following formula:
LZWMSM$^{(2)}(z_1, z_2, \ldots, z_n) =$

\[
\psi^{-1}\left(\frac{2}{n(n-1)} \times \frac{1}{2} \sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot w_i \psi^r(A_{(i)})\right)^\frac{1}{2},
\]

\[
h^{-1}\left(\frac{1}{2} \cdot \left(\sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot h'(B_{(i)}) \cdot w_i \psi^r(A_{(i)}) \cdot h'(B_{(i)})\right)^\frac{1}{2}\right)
\]

\[
\frac{1}{2} \times \left(\sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot w_i \psi^r(A_{(i)})\right)
\]

\[
= \psi^{-1}\left(\frac{1}{n(n-1)} \times \sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot \psi^r(A_{(i)})\right)^\frac{1}{2}.
\]

\[
h^{-1}\left(\frac{\sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot h'(B_{(i)}) \cdot \psi^r(A_{(i)}) \cdot h'(B_{(i)})}{\sum_{i=1}^{n} w_i \psi^r(A_{(i)}) \cdot \psi^r(A_{(i)})}\right)^\frac{1}{2}.
\]  (30)

(3) When $\delta = n$, Eq. (28) will become the following formula:

LZWMSM$^{(n)}(z_1, z_2, \ldots, z_n) =$

\[
\psi^{-1}\left(\frac{\prod_{i=1}^{n} w_i \psi^r(A_{(i)})}{C_n}\right)^\frac{1}{2}.
\]

\[
h^{-1}\left(\sum_{i=1}^{n} \prod_{j=1}^{n} w_j \psi^r(A_{(j)}) \cdot h'(B_{(j)})\right)^\frac{1}{2}\left(\text{let } i = j\right)
\]

\[
= \psi^{-1}\left(\prod_{j=1}^{n} w_j \psi^r(A_{(j)})\right)^\frac{1}{2} \cdot h^{-1}\left(\prod_{j=1}^{n} \left(\prod_{j=1}^{n} w_j \psi^r(A_{(j)}) \cdot h'(B_{(j)})\right)^\frac{1}{2}\right)
\]

\[
= \psi^{-1}\left(\prod_{j=1}^{n} w_j \psi^r(A_{(j)})\right)^\frac{1}{2} \cdot h^{-1}\left(\prod_{j=1}^{n} h'(B_{(j)})\right)^\frac{1}{2}.
\]  (31)

IV. A MADM METHOD ON THE BASIS OF LZWMSM OPERATOR

In this part, the LZWMSM operator is utilize to handle the MADM problems. Suppose the set of alternatives is $\{\delta_1, \delta_2, \ldots, \delta_n\}$, and the set of attributes is $\{c_1, c_2, \ldots, c_n\}$ with the weight vector $w = (w_1, \ldots, w_n)^T$ which
satisfies \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 (j = 1,2,\ldots,n) \). Suppose \( Z = [z_{ij}]_{mn} \) is the decision matrix of MADM problems, and \( \bar{z}_{ij} = (A_{i(t)}, B_{j(t)}) (i = 1,2,\ldots,m; j = 1,2,\ldots,n) \) is the assessment information given by the decision maker (DM) in regard to alternative \( \delta_i \) for attribute \( c_j \), and is expressed by LZNs, where \( A_{i(t)} \in \{A_1, A_2, A_3, A_4, A_5\} \) and \( B_{j(t)} \in \{B_1, B_2, B_3, B_4, B_5\} \). Then, our aim is to obtain the ranking result alternatives and choose the best or most suitable one(s).

Next, we will propose a MADM method based on LZWMSM operator and give its rational decision-making process.

**Step 1.** Standardize the evaluation information of attributes.

In general, the attributes can be divided into two types, i.e., cost attributes and benefit attributes. In order to eliminate the impact of different types of attribute evaluation information on decision results, we need transform them to the same type. Generally, we are used to transforming the cost type to benefit one. Suppose \( \bar{z}_{ij} = (A_{i(t)}, B_{j(t)}) \) is cost type of evaluation information, and we can standardize it to benefit type as (the standardized evaluation information is also represented by \( \bar{z}_{ij} \)):

\[
\bar{z}_{ij} = (A_{i(-t)}, B_{j(t)})
\]

**Step 2.** Utilize the LZWMSM operator to aggregate the decision-making information of all attributes to a comprehensive evaluation value of each alternative.

\[
r_{i} = \text{LZWMSM}^{(0)}(Z_{i1}, Z_{i2}, \ldots, Z_{in})
\]

**Step 3.** Compute the score function \( LS(r_{i}) \) of the comprehensive values \( r_{i} \), and then rank all alternatives \( \{\delta_1, \delta_2, \ldots, \delta_n\} \).

**Step 4.** Rank all the alternatives and choose the most suitable one(s).

Rank all the alternatives \( \{\delta_1, \delta_2, \ldots, \delta_n\} \) and choose the most suitable one(s) by calculating the score function \( LS(r_{i}) \).

**Step 5.** End.

**V. AN ILLUSTRATIVE EXAMPLE**

For the sake of showing the utilization of the approach proposed in section 4, we will give an example (Adapted from [28]) related to the apps evaluation for five medical inquiry apps \( \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\} \). The DM \( D \) evaluates all the alternatives based on the following four attributes: application platform \( (c_1) \), user experience \( (c_2) \), visual foreground \( (c_3) \) and network background \( (c_4) \). The DM uses the two LTSs \( A = \{A_1, A_2, A_3, A_4, A_5\} = \{\text{very poor, poor, slightly poor, fair, slightly good, good, very good}\} \) and \( B = \{B_1, B_2, B_3, B_4\} = \{\text{uncertain, slightly uncertain, medium, slightly sure, sure}\} \) to give the evaluation value \( \bar{z}_{ij} = (A_{i(t)}, B_{j(t)}) (i = 1,2,3,4,5; j = 1,2,3,4) \). The decision-making information given by DM about alternatives \( \delta_i \) with respect to attribute \( c_j \) can be obtained from the LT in \( A \) and the reliability about the decision-making information can be obtained from LTS in \( B \). Then all of the evaluation results constitute a decision matrix \( R = [r_{ij}]_{5 \times 4} \) listed in Table 1, where \( r_{ij} \) is represented by LZN \( \left( A_{i(t)}, B_{j(t)} \right) \). \( w = (0.2,0.25,0.25,0.3)^T \) is the weight vector of \( c_j \). Our aim is to choose the most suitable app for our daily life.

| Table 1 Linguistic Z-numbers decision matrix R given by D |

**5.1 Decision-making steps**

Next, the process is given for getting the most suitable alternative(s).
**Step 1.** Standardize the evaluation information of attributes.

All of the attributes are benefit type, so we can omit the normalization.

**Step 2.** Utilize the LZWMSM operator to aggregate the decision-making information of all attributes to a comprehensive evaluation value of each alternative. (suppose $\delta = 2$, $\varphi(x_i) = F(x_i)$ and $h'(x_i) = F_1(x_i)$), and get

$r_i = (2.058,3.164)$, $r_2 = (1.869,2.729)$, $r_3 = (1.867,2.619)$, $r_4 = (1.780,3.181)$, $r_5 = (1.653,3.201)$.

**Step 3.** Compute $LS(r_i)$ of the comprehensive values $r_i$, and obtain

$L S(r) = 0.1317$, $LS(r_2) = 0.1320$, $LS(r_3) = 0.1441$, $LS(r_4) = 0.1320$.

**Step 4.** Rank all the alternatives and choose the most suitable one.

On the basis of the score functions $LS(r_i)$, we can rank the alternatives $\{\delta_i, \delta_2, \delta_3, \delta_4, \delta_5\}$ shown as follows.

$\delta_4 > \delta_1 > \delta_3 > \delta_2 > \delta_5$.

So, alternative $\delta_4$ will be chosen.

**Step 5.** End.

### 5.2 The analysis of the influence about the values of the parameter $\delta$ on final ranking results of the illustrate example

For discussing the influence of parameter $\delta$ on the evaluation results, we will change different values of parameter $\delta$ in calculating process. The ranking results are given in Table 2.

<table>
<thead>
<tr>
<th>Table 2 Ranking results by using different parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>According to Table 2, we can easily get the best choice is always $\delta_4$ and the ranking results by taking different values of $\delta$ are a little different. This is because when $\delta = 1$, the LZWMSM operator does not take account of the interrelationship among different attributes. For the same alternative, the score function $LS(r_i)$ becomes smaller as the value of parameter $\delta$ increases. Qin and Liu [43] indicated that parameter $\delta$ can be regarded as the risk preference of DM. In different realistic MADM problems, DMS can select suitable values of $\delta$ based on different risk preference. The risk preference DM can choose a larger value of $\delta$ and instead, a smaller value can be chosen. In the application, we generally use the round function $[\cdot]$ to obtain the $\delta$ value as $\left[\frac{n}{2}\right]$, where $n$ is the number of aggregated elements. Qin and Liu [43] explained that when $\delta = \left[\frac{n}{2}\right]$, the DM remains neutral which may be rational.</td>
</tr>
</tbody>
</table>

### 5.3 Certified of the validity

For proving the validity of this proposed approach, we will use the method presented by Wang et al. [28] (It is worth noting that the attribute weights in this example are deterministic and additive, i.e. $\sum_{j=1}^{5}w_j = 1$) to deal with the same example. To make the results more convincing, we will use the same linguistic scaled model with the method presented by Wang et al. [28] to deal with the evaluation information $A_{(0)}$ of LZNs as:

$F_1(x_i) = \rho_x = \begin{cases} \frac{1}{a^\frac{x}{2} - a^{-\frac{x}{2}}} \left( g = 0,1,2,\ldots,T \right) \\ \frac{2a^\frac{1}{2} - 2}{a^\frac{x}{2} + a^{-\frac{x}{2}} - 2} \left( g = \frac{T}{2} + 1,\frac{T}{2} + 2,\ldots,T \right) \end{cases}$

and then we can get the result by LZWMSM operators. Table 3 gives the ranking results about the different two methods.

<table>
<thead>
<tr>
<th>Table 3 Ranking results by using Wang’s method and method based on LZWMSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Based on Table 3, we can obtain the ranking result of Wang et al. [28] is same with our method when $\delta = 1$, because the approach introduced by Wang et al. [28] and our method ($\delta = 1$) don’t take account of the interrelationship between different arguments, which can prove that our presented method is effective. Besides, the ranking results of the two approaches are</td>
</tr>
</tbody>
</table>
different when $\hat{\delta} = 3$. The reason is that Wang’s method proposed in [28] is used the extended TODIM approach and cannot think about the interrelationship between different multi-attributes. However, the method based on LZWMSM operator can take account of interrelationship about multi-attributes by defined different value of $\hat{\delta}$. In real MADM problems, there are more or less connections between different attributes, so it is not accurate to consider independently in many cases. For instance, when we choose the way of travel, we may consider the two factors of weather and traffic conditions, however, snowing days may cause traffic congestion. So, the method by utilizing LZWMSM operator has more widely applications.

### 5.4 Further comparison analysis

From the above analysis, we have proved the availability of our proposed method based on LZWMSM operator. Next, for better illustrating the superiority of our proposed method, we will deal with the example in 5.1 by using a method proposed by Qiao et al. [44]. The finally comparison result is shown in Table 4.

<table>
<thead>
<tr>
<th>Table 4 Ranking results by using Qiao’s method and method based on LZWMSM</th>
</tr>
</thead>
</table>

According to table 4, we obtain that when $\hat{\delta} = 1$, the optimal solution obtained by Qiao’s method [44] and our method is the same. However, the ranking results of the two methods are slightly different. The reason is because when the expert evaluates the attribute by using qualitative linguistic form, Qiao’s method [44] needs to convert the linguistic information into triangular fuzzy numbers (TFNs) in the calculation process, which will result in a certain degree of information loss. Our proposed method and that proposed by Wang et al. [28] both use LSFs to directly process the LZNs evaluation information, which can reduce the degree of information loss. Therefore, the method proposed in this paper may get more practical results.

In short, the MADM method according to LZWMSM operator may be more convenient and flexible compared with some existing methods, such as [6] which are so complex based on the traditional operation laws. Moreover, our method is more general because of considering the interrelationship among multi-attributes based on different cases.

Through the above comparison analysis, we can summarize the advantages of our approach in the following ways:

1. We use operations presented by Wang et al. [28] and choose different LSFs to calculate the result, which are more easily and flexible compared with methods in [44]. Besides, the method in [44] need to convert linguistic assessment information to TFNs for calculation which may cause loss of original information. We omit the intermediate conversion steps which can reduce the loss of the original data, so that our results may be more realistic.

2. Compared with the method in [28], We use the MSM operator that can take account of interrelationship between different numbers of attributes and also can reflect the attitude of the DMs, so that we can select different value of $\hat{\delta}$ according to different actual scenarios. Obviously, in real life, the method by utilizing LZWMSM operator may be more general and has a wider range of applications than Wang’s method.

3. As an extension sub-class of Z-number, LZN proposed in [28] uses the LVs to represent Z-number’s two components that can combine the advantages of Z-numbers and LTs. In some specific MADM problems, many questions are emergency and fuzzy. Compared with quantitative evaluation, DM is more likely to give some simple qualitative evaluations, so that LZNs are more flexible and practical in that situation.

Through the comparisons and analysis of the above sections, the proposed method on the basis of the LZWMSM operator may be more general than some existing other approaches for aggregating LZNs.

### VI. Conclusions

We extend MSM operator to handle LZNs by using the new operations introduced by Wang et al. [28] which are easier and more flexible than the traditional operations by combining with the LSFs. Then we propose the LZM operator and LZWMSM operator, and explore several properties of them. Moreover, several special cases are also investigated, and a MADM method is given by using LZWMSM operator. Compared with some existing methods, our presented approach may be more general and flexible. The significant advantage is that our MADM method can think about the interrelationship between different numbers of arguments which have the flexibility by considering different values of parameter $\hat{\delta}$. Besides, the method based on LZWMSM operator uses LZNs for evaluation, which can not only consider the reliability about the constraint $A_{l}^{(v)}$, but also be more flexible in many specific environments.

In future research, we should expand the Z-number more deeply, such as exploring the more rational operations,
uncertainty of \( A_{E(i)} \) and \( B_{E(i)} \) in LZNs. Besides, we will make better use of LZNs to handle realistic MADM problems by combination with some operators [45-48].

ACKNOWLEDGMENT

This paper is supported by the National Natural Science Foundation of China (Nos. 71771140, 71471172), 文化名家暨“四个一批”人才项目(Project of cultural masters and “the four kinds of a batch” talents), the Special Funds of Taishan Scholars Project of Shandong Province (No. ts201511045). The authors also would like to express appreciation to the anonymous reviewers and Editors for their very helpful comments that improved the paper.

COMPLIANCE WITH ETHICAL STANDARDS

1) Disclosure of potential conflicts of interest

We declare that we do have no commercial or associative interests that represent a conflict of interests in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work.

2) Research involving human participants and/or animals

This article does not contain any studies with human participants or animals performed by any of the authors.

REFERENCES


Table 1 Linguistic Z-numbers decision matrix $R$ given by $D$

<table>
<thead>
<tr>
<th>$\delta_i$</th>
<th>$(A_i, B_i)$</th>
<th>$(A_i, B_i)$</th>
<th>$(A_i, B_i)$</th>
<th>$(A_i, B_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_1$</td>
<td>$(A_1, B_1)$</td>
<td>$(A_1, B_1)$</td>
<td>$(A_1, B_1)$</td>
<td>$(A_1, B_1)$</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>$(A_2, B_2)$</td>
<td>$(A_2, B_2)$</td>
<td>$(A_2, B_2)$</td>
<td>$(A_2, B_2)$</td>
</tr>
<tr>
<td>$\delta_3$</td>
<td>$(A_3, B_3)$</td>
<td>$(A_3, B_3)$</td>
<td>$(A_3, B_3)$</td>
<td>$(A_3, B_3)$</td>
</tr>
<tr>
<td>$\delta_4$</td>
<td>$(A_4, B_4)$</td>
<td>$(A_4, B_4)$</td>
<td>$(A_4, B_4)$</td>
<td>$(A_4, B_4)$</td>
</tr>
</tbody>
</table>

Table 2 Ranking results by using different parameter values

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>Score function $LS(\xi_i)$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 1$</td>
<td>$LS(x_1) = 0.1757, LS(x_2) = 0.1346$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.1276, LS(x_4) = 0.1455$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.1331$</td>
<td></td>
</tr>
<tr>
<td>$\delta = 2$</td>
<td>$LS(x_1) = 0.1738, LS(x_2) = 0.1317$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.1262, LS(x_4) = 0.1441$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.1320$</td>
<td></td>
</tr>
<tr>
<td>$\delta = 3$</td>
<td>$LS(x_1) = 0.1719, LS(x_2) = 0.1284$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.1247, LS(x_4) = 0.1426$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.1309$</td>
<td></td>
</tr>
<tr>
<td>$\delta = 4$</td>
<td>$LS(x_1) = 0.1698, LS(x_2) = 0.1252$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.1230, LS(x_4) = 0.1412$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.1297$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 Ranking results by using Wang’s method and method based on LZWMSM

<table>
<thead>
<tr>
<th>Method</th>
<th>Score values $LS$ or Priority weight $P$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method by Wang et al. [28] based on $Z - TODIM$</td>
<td>$\zeta(x_1) = 1, \zeta(x_2) = 0.350$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(x_3) = 0, \zeta(x_4) = 0.558$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\zeta(x_5) = 0.262$</td>
<td></td>
</tr>
<tr>
<td>LZWMSM ($\delta = 1$)</td>
<td>$LS(x_1) = 0.147, LS(x_2) = 0.119$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.112, LS(x_4) = 0.124$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.118$</td>
<td></td>
</tr>
<tr>
<td>LZWMSM ($\delta = 3$)</td>
<td>$LS(x_1) = 0.143, LS(x_2) = 0.113$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$LS(x_3) = 0.109, LS(x_4) = 0.121$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$LS(x_5) = 0.118$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4 Ranking results by using Qiao’s method and method based on LZWMSM

<table>
<thead>
<tr>
<th>Method</th>
<th>Score values $LS$ or Priority weight $P$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method by Qiao et al. [44]</td>
<td>$\zeta(x_1) = 1.148, \zeta(x_2) = -0.572$</td>
<td>$\delta_1 \succ \delta_i \succ \delta_2 \succ \delta_3$</td>
</tr>
<tr>
<td></td>
<td>$\zeta(x_3) = -0.248, \zeta(x_4) = 0.089$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\zeta(x_5) = -0.417$</td>
<td></td>
</tr>
</tbody>
</table>
LZWMSM (\(\tilde{\delta} = 1\))

<table>
<thead>
<tr>
<th></th>
<th>(LS(\tau_1) = 0.1757)</th>
<th>(LS(\tau_2) = 0.1346)</th>
<th>(\delta_1 \succ \delta_4 \succ \delta_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(LS(\tau_3) = 0.1276)</td>
<td>(LS(\tau_4) = 0.1455)</td>
<td>(\delta_3 \succ \delta_4 \succ \delta_5)</td>
</tr>
<tr>
<td></td>
<td>(LS(\tau_5) = 0.1331)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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