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Research Note

An Augmented Galerkin Method for Singular Integral Equations with Hilbert Kernel

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In this paper a Fourier series expansion method is described for a class of singular integral equations with Hilbert kernel and constant coefficients. Furthermore, a number of numerical examples are given showing that Galerkin method works well in practice.

INTRODUCTION

In recent papers, Delves [1] and others [2,3] described a Chebyshev series method for the numerical solution of integral equations with non-singular kernels or some particular singular kernels, for example Green's function kernel, logarithmic and Cauchy kernels and so on.

Here, a numerical solution of singular integral equation with Hilbert kernel of the following form is considered:

$$a\varphi(x) + \frac{b}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\cot\frac{t-x}{2}\,dt + \mu\int_{-\pi}^{\pi}k(t,x)\varphi(t)\,dt = f(x)\ , \qquad (1)$$

$$-\pi \le x \le \pi\ ,$$

where a, b and μ are real constants, with $b \neq 0$, $k(t,x)$ and $f(x)$ are real periodic functions of t and x with period 2π and are assumed to be the known L_2-functions. $\varphi(x)$ is the unknown L_2-function with period 2π .

A theoretical consideration of the existence and convergence theorems is described in [4,5].

As is known the set:

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \dots\right\},$$

is an orthonormal and complete basis for $L_2[-\pi,\pi]$. Therefore, every square integrable function is completely determined (expect for its value at a finite number of points) by its Fourier series, whether this series converges or not. The Fourier series of a continuous, piecewise smooth function $f(x)$ (with period 2π) converges to $f(x)$ absolutely and uniformly [6].

The expansion method approximates φ by φ_N, where:

$$\varphi(x) \simeq \varphi_N(x) = \frac{1}{2}a_0 + \sum_{i=1}^{N}(a_i\cos ix + b_i\sin ix)\ , \qquad (2)$$

and with above assumptions, $\varphi_N(x)$ converges to $\varphi(x)$ in the mean, i.e.,

$$\lim_{N\to\infty}\int_{-\pi}^{\pi}\left[\varphi(x) - \varphi_N(x)\right]^2 dx = 0.$$

Obviously, κ the index of Equation 1 is zero. Let:

$$\theta = arg(a - ib),$$

be the characteristic number of Equation 1. If $\kappa = 0$, then there are two cases, $[\theta]_\pi \neq \frac{\pi}{2}$ or $[\theta]_\pi = \frac{\pi}{2}$, where the notation $[x]_\pi$ denotes the number congruent to x in $[0,\pi)$ for the modulus π [4].

Suppose $f(x)$ and $k(t,x)$ are continuously differentiable. If $[\theta]_\pi \neq \frac{\pi}{2}$ and μ is not an eigenvalue of $k(t,x)$, then Equation 1 has a unique solution which can be obtained using Galerkin equations (otherwise Equation 1 has an infinity of solutions). If $[\theta]_\pi = \frac{\pi}{2}$, when $a = 0$ and Equation 1 is a first kind integral equation, under the constraint condition,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x) - \mu\int_{-\pi}^{\pi}k(t,x)\varphi(t)\,dt\right]dx = 0\ ,$$

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then Equation 1 has an infinity of solutions but if the following additional condition is imposed (unisolving condition),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\,dt = C ,$$

where C is a given real constant, then Equation 1 has a unique solution [4].

The last condition imposes:

$$a_0 = 2C ,$$

to the Galerkin equations.

In the next section, the following formulae are used,

$$\cos ix = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cot\frac{t-x}{2}\sin it\,dt ,$$

$$\sin ix = \frac{-1}{2\pi}\int_{-\pi}^{\pi}\cot\frac{t-x}{2}\cos it\,dt ,$$

in the sense of Cauchy principal integral (see [7]) to obtain Galerkin equations.

THE AUGMENTED GALERKIN ALGORITHM

The Galerkin equations for the coefficients a_i and b_i in Equation 2 are:

$$\overline{\mathbf{A}}\mathbf{X} = \overline{\mathbf{F}} , \qquad (3)$$

where, for $i = 0, 1, \ldots, N$,

$$\overline{\mathbf{A}}_{ij} = \eta_j\Big[a\pi\delta_{ij} + \mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}k(t,x)\cos ix\cos jt\,dt\,dx\Big] ,$$

$$j = 0, 1, \ldots, N ,$$

and:

$$\overline{\mathbf{A}}_{i(N+j)} = b\pi\delta_{ij} + \mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}k(t,x)\cos ix\sin jt\,dt\,dx ,$$

$$j = 1, \ldots, N ,$$

and for $i = 1, \ldots, N$,

$$\overline{\mathbf{A}}_{(N+i)j} = \eta_j\Big[-b\pi\delta_{ij} + \mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}k(t,x)\sin ix\cos jt\,dt\,dx\Big] ,$$

$$j = 0, 1, \ldots, N ,$$

and:

$$\overline{\mathbf{A}}_{(N+i)(N+j)} = a\pi\delta_{ij} + \mu\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}k(t,x)\sin ix\sin jt\,dt\,dx ,$$

$$j = 1, \ldots, N .$$

For $i = 0, 1, \ldots, N$,

$$\overline{\mathbf{F}}_i = \int_{-\pi}^{\pi} f(x)\cos ix\,dx ,$$

and for $i = 1, \ldots, N$,

$$\overline{\mathbf{F}}_{(N+i)} = \int_{-\pi}^{\pi} f(x)\sin ix\,dx ,$$

and:

$$\mathbf{X} = [a_0, a_1, \ldots, a_N, b_1, b_2, \ldots, b_N]^t ,$$

where:

$$\eta_j = \begin{cases}\frac{1}{2} & j = 0\\ 1 & j \neq 0 ,\end{cases}$$

and δ_{ij} is Kronecker delta. When $a = 0$ (in the second case), only a_0 is set equal to $2C$.

The augmented Galerkin scheme of [2] is used to find a solution for Equation 3.

The assumption that Equation 1 has an L_2-solution, implies that the representation:

$$\varphi(x) = \frac{1}{2}a_0 + \sum_{i=1}^{\infty}(a_i\cos ix + b_i\sin ix) ,$$

is convergent in L_2 space and as a result:

$$\sum_{i=0}^{\infty}a_i^2 < \infty , \qquad \sum_{i=1}^{\infty}b_i^2 < \infty .$$

Therefore, the following constraints are imposed:

$$|a_i| \leq \delta_i = C_f\hat{i}^{\,-r} , \; i = 0, 1, \ldots$$

$$|b_i| \leq \delta_i = C_f\hat{i}^{\,-r} , \; i = 1, \ldots$$

with $C_f > 0$ and $r > \frac{1}{2}$ to Equation 3 for having an L_2 solution, where $\hat{i} = \max(1, i)$ [2]. In other words, when the mentioned constraints are satisfied, it is expected that a_i's and b_i's belong to l_2 and hence the obtained solution belongs to L_2. The constants C_f and r play the role of regularization parameters and some strategies are discussed in [2] to determine suitable values for them. Here, Strategy 1 described in [2] is used, in which:

$$C_f = \lambda\|\overline{\mathbf{F}}\|_\infty \big/ \|\overline{\mathbf{A}}\|_\infty ,$$

where λ must be set heuristically, say $2 \leq \lambda \leq 10$ (it can be proved that $\lambda \geq 1$).

COMPUTATIONAL DETAILS

To compute integrals in Equation 3, m-panel Gauss-Kronrod Quadrature rule with t-points is used. Hence,

$$\int_{-\pi}^{\pi} f(x)TR(ix)\,dx \simeq (-1)^i \frac{\pi}{m}$$

$$\sum_{p=1}^{m}\sum_{s=1}^{t} w_s f\left(\frac{\pi}{m}y_s^p - \pi\right) TR\left(\frac{i\pi}{m}y_s^p\right),$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} k(z,x)TR(ix)TR'(jz)\,dz\,dx$$

$$\simeq (-1)^{i+j}\frac{\pi^2}{m^2}\sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{s=1}^{t}\sum_{t=1}^{t}$$

$$k\left(\frac{\pi}{m}y_s^p - \pi, \frac{\pi}{m}y_t^q - \pi\right)$$

$$TR\left(\frac{i\pi}{m}y_t^q\right) TR'\left(\frac{j\pi}{m}y_s^p\right),$$

where:

$$y_j^i = x_j - 1 + 2i\ ,$$

$TR(x)$ and $TR'(x)$ may be $\sin(x)$ or $\cos(x)$ and w_s and x_s are weights and nodes for t-point Gauss-Kronrod quadrature rule, respectively.

To solve Equation 3, the augmented Galerkin scheme of [2] is considered:

$$\text{Minimize } \|\mathbf{AX} - \mathbf{F}\| \qquad (4)$$

Subject to

$$|a_i| \le \delta_i = C_f \hat{i}^{\,-r}\ ,$$

$$|b_j| \le \delta_j = C_f \hat{j}^{\,-r}\ ,$$

$$i = 0, 1, \ldots, N\ ,$$

$$j = 1, 2, \ldots, N\ ,$$

where $\mathbf{A}$ and $\mathbf{F}$ are numerical approximations to $\overline{\mathbf{A}}$ and $\overline{\mathbf{F}}$.

NUMERICAL EXAMPLES AND RESULTS

Here, a set of three examples are considered. All computations were carried out on an IBM-PC using C language and long double precision. Computed errors are defined as follows:

$$\|E_N\|_2 = \sqrt{\frac{\sum_{i=1}^{99}\{\varphi(s_i) - \varphi_N(s_i)\}^2}{99}}\ ,$$

$$\|E_N\|_\infty = \operatorname*{Max}_{1\le i\le 99} |\varphi(s_i) - \varphi_N(s_i)|\ ,$$

where $s_i = -\pi + i\pi/50$.

Example 1

For $-\pi < x < \pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$-\int_{-\pi}^{\pi}\left(\frac{t+x}{2}\right)^2\varphi(t)\,dt = \pi - sin(x),$$

with solution $\varphi(x) = cos(x)$.

Here, $f(x)$ and $k(x,y)$ are analytic functions and the Fourièr series of them are known and $\varphi(x)$ is in C^∞, therefore, the regularization parameter, r, can take any value.

Example 2

For $-\pi < x < \pi$,

$$\varphi(x) + \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$+\int_{-\pi}^{\pi}\frac{t+x}{2}\varphi(t)\,dt = \frac{\pi^2}{3}$$

$$-4\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\cos(ix) - \sin(ix)}{i^2}\ ,$$

with solution $\varphi(x) = x^2$. All functions have the same behavior as in Example 1 and r can take any value.

Example 3

For $-\pi < x < \pi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\cot\frac{t-x}{2}\,dt$$

$$+\int_{-\pi}^{\pi}\sin(t)\sin(x)\varphi(t)\,dt$$

$$= 12\left(\pi\sin(x) + \sum_{i=1}^{\infty}(-1)^{i+1}\frac{\cos(ix)}{i^3}\right),$$

with solution $\varphi(x) = x(\pi - x)(\pi + x)$ and r can take any value.

Results for the above examples are presented in Tables 1–3. Tables give the accuracy, $\|E_N\|_2$, obtained by the augmented Galerkin algorithm for $m = 1$ or $m = 2$ (number of panels in integration) and by different values for r and λ in examples with $t = 15$.

Table 1. Results for Example 1.

$r=5,\lambda=4$					$r=2,\lambda=10$		
$m=1$			$m=2$		$m=2$		
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	3.68E-13	2.14E-13	3.75E-16	2.42E-16	2	3.75E-16	2.42E-16
3	3.50E-9	1.97E-9	1.23E-15	6.50E-16	3	1.23E-15	6.50E-16
4	1.50E-6	8.91E-7	3.33E-13	1.98E-13	4	3.33E-13	1.98E-13
5	1.15E-4	7.13E-5	3.33E-13	1.98E-13	5	3.33E-13	1.98E-13
6	2.60E-4	1.36E-4	3.05E-9	1.89E-9	6	3.05E-9	1.89E-9
7	3.31E-4	1.45E-4	3.05E-9	1.89E-9	7	3.05E-9	1.89E-9
8	3.66E-4	1.47E-4	1.37E-6	8.72E-7	8	1.37E-6	8.72E-7
9	3.59E-4	1.38E-4	1.37E-6	8.72E-7	9	1.37E-6	8.72E-7
10	1.51E-1	1.06E-1	1.42E-5	8.97E-6	10	1.04E-4	7.03E-5

$t=15$

Table 2. Results for Example 2.

$r=1,\lambda=10$					$r=2,\lambda=10$		
$m=1$			$m=2$		$m=1$		
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	1.20	3.70E-1	1.20	3.70E-1	2	1.20	3.70E-1
3	7.68E-1	2.21E-1	7.68E-1	2.21E-1	3	7.68E-1	2.21E-1
4	5.26E-1	1.49E-1	5.26E-1	1.49E-1	4	5.26E-1	1.49E-1
5	3.74E-1	1.08E-1	3.74E-1	1.08E-1	5	3.74E-1	1.08E-1
6	2.83E-1	8.22E-2	2.70E-1	8.18E-2	6	2.83E-1	8.22E-2
7	2.83E-1	1.63E-1	1.97E-1	6.45E-2	7	2.83E-1	1.63E-1
8	1.64	9.94E-1	1.59E-1	5.23E-2	8	5.10E-1	2.62E-1
9	3.47	1.68	1.45E-1	4.34E-2	9	7.37E-1	3.13E-1
10	4.38	1.86	1.33E-1	3.67E-2	10	7.47E-1	2.88E-1

$t=15$

Table 3. Results for Example 3.

$r=1,\lambda=10$					$r=2,\lambda=10$		
$m=1$			$m=2$		$m=2$		
N	$\|E_N\|_\infty$	$\|E_N\|_2$	$\|E_N\|_\infty$	$\|E_N\|_2$	N	$\|E_N\|_\infty$	$\|E_N\|_2$
2	7.12E-1	3.53E-1	7.12E-1	3.53E-1	2	7.12E-1	3.53E-1
3	3.70E-1	1.59E-1	3.70E-1	1.59E-1	3	3.70E-1	1.59E-1
4	2.26E-1	8.62E-2	2.25E-1	8.62E-2	4	2.25E-1	8.62E-2
5	1.59E-1	5.39E-2	1.48E-1	5.27E-2	5	1.48E-1	5.27E-2
6	2.75E-1	1.61E-1	1.08E-1	3.49E-2	6	1.08E-1	3.49E-2
7	2.14	1.35	8.13E-2	2.45E-2	7	8.13E-2	2.45E-2
8	1.13E+1	6.59	6.15E-2	1.79E-2	8	6.15E-2	1.79E-2
9	3.57E+1	1.97E+1	4.73E-2	1.36E-2	9	4.73E-2	1.36E-2
10	3.75E+1	2.03E+1	4.02E-2	1.06E-2	10	4.02E-2	1.06E-2

$t=15$

CONCLUSIONS

From the above results, it is concluded that the augmented Galerkin method allows an almost routine solution of Hilbert integral equations. In Example 1, the exact coefficients $a_0, a_2, a_3, \ldots$ and $b_1, b_2, \ldots$ are zero and the method presented for small values of N is very accurate. But for Example 2, $a_0, a_1, \ldots$ and for Example 3, $b_1, b_2, \cdots$ are non-zero and hence a large value for N must be chosen. Table 4 shows $\|E_N\|_2$ for exact Fourier coefficients of $\varphi(x)$ in Examples 2 and 3. Table 4 shows that the

Table 4. $||E_N||_2$ for exact Fourier coefficients of $\varphi(x)$.

N	Example 2	Example 3
2	0.370239	0.353477
3	0.221158	0.158695
4	0.148709	0.086192
5	0.107584	0.052678
6	0.081816	0.034878
7	0.064531	0.024464
8	0.052347	0.017922
9	0.043430	0.013581
10	0.036709	0.010575
12	0.027448	0.006824
14	0.021553	0.004685
16	0.017602	0.003364
18	0.014845	0.002497
20	0.012852	0.001899

method presented here works well in practice, these numbers are the lower bound of obtained $||E_N||_2$ in Tables 2 and 3.

It is important to note that the elements of matrix in Equation 3 do not tend to zero as $N \to \infty$ and therefore, direct methods or iterative methods cannot be used to solve Equation 3, but solving Equation 4 is independent of N under some mild conditions that are valid here [2].

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