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Research Note

On Derived Groups of Division Ring Extensions

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Let D be a division ring and K a proper division subring of D. Let D' and K' denote the derived groups of $D^* = D \setminus \{0\}$ and $K^* = K \setminus \{0\}$, moreover $M(D)$ and $M(K)$ denote the set of multiplicative commutators of D^* and K^*, respectively. Then, the main results obtained here are as follows:

1. If D' is radical over K', then D is a field.
2. If the center of D is uncountable and $M(D)$ is radical over $M(K)$, then D is a field.

INTRODUCTION

Let D be a division ring and K a proper subring of D. Faith proves in [1] (see also [2]) that if D is radical over K, then D is a field (commutative division ring). In [1], it is also shown that if D is a division algebra over a field F with a proper subalgebra K and if for each $d \in D$ there exist nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ (depending on d) such that for each $a \in F(d)$ a polynomial $f_a(x) = x^n - x^{n+1}P(x)$ can be found in the subring of $F[x]$ generated by x and $\alpha_1, \alpha_2, ..., \alpha_r$ satisfying $f_a(a) \in K$, then D is a field. In this paper, these results are generalized to the derived groups (commutator subgroups) D' and K' of the multiplicative groups D^* and K^*, respectively, as Mahdavi-Hezavehi [3] generalizes the Kaplansky's theorem (cf. [4,5]) for the derived group D'. Then, these results are generalized for the multiplicative commutator elements of divsion rings D and K with the uncountable center F of D, as Chiba [6] proved the Herstein's conjecture (cf. [7,8]) for the division rings with uncountable centers (for a special case of this conjecture see [9]). Related results and the origin of the conjecture may be cited in [10–12]. There also exists conjectures about torsion normal subgroups which are somehow related to this paper (see [13–15]).

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Let D be an arbitrary division ring and S any subset of D. If $a, b \in D^* = D \setminus \{0\}$, then $aba^{-1}b^{-1}$ is a multiplicative commutator of D^*. The set of all multiplicative commutators of D^* is denoted by $M(D)$ and the derived subgroup of D^* is denoted by D'. An element $x \in D$ is radical over $S \subseteq D$, if $x^{n(x)} \in S$ for some positive integer $n(x)$, depending on x. Furthermore, $M(D)$ (respectively D') is radical over $S \subseteq D$, if each element of $M(D)$ (respectively D') is radical over S. The center of D is denoted by $Z(D)$.

RADICAL DIVISION RING EXTENSIONS

In this section we begin our study with the following result obtained from [3,16].

Lemma A

Let D be a division ring with center F. If D' is radical over F, then D is a field [3, Lemma 2].

From Cartan-Brauer-Hua theorem, the following result is obtained. This is used throughout this paper (cf. [5,17,18]).

Corollary B

Let D be a noncommutative division ring and K a proper division subring of D, then, there exists $x \in M(D)$, such that $x \notin K$. In particular, $D' \not\subset K$.

The main result of this section is the following theorem which generalizes Theorem B of [1]. A result of [3] and some of the techniques utilized in [1] are used in the proof.

Theorem 1

Let D be a division ring and K a proper division subring of D. If the derived group D' of D^* is radical over K, then D is a field.

Proof

If D' is trivial, there is nothing to prove. So, it may be assumed that $D' \neq \{1\}$. First, it is claimed that for any $a, b \in D'$, $a \in K$ and $b \notin K$, there exists a positive integer n such that $ba^n = a^n b$. If this is not the case, consider the elements $x_1 = (b+a)^{-1}a(b+a)$ and $x_2 = (b+1)^{-1}a(b+1)$ which are in D'. Since D' is radical over K, there exists a positive integer n such that $x_1^n = (b+a)^{-1}a^n(b+a)$ and $x_2^n = (b+1)^{-1}a^n(b+1)$ are in K. Now, a simple calculation shows that $b(x_1^n - x_2^n) = a^n(a-1) + x_2^n - ax_1^n \in K$. If $x_1^n - x_2^n \neq 0$, then $b \in K$ which is a contradiction. If $x_1^n - x_2^n = 0$, then $(a-1)a^n = (a-1)x$, where $x = x_1^n = x_2^n$. However, $a \neq 1$. Hence, $x = a^n$ and $(b+1)a^n = a^n(b+1)$; therefore, $ba^n = a^n b$, which is a contradiction. Consequently the claim is established. Now, it is further shown that for every pair $x, y \in D'$, there exists some positive integer m such that $y^m x = xy^m$. To demonstrate this, let $x, y \in K$ and choose a such that $a \in D'$ and $a \notin K$ (this is possible as a result of Corollary B). Hence, $ax \in D'$ and $ax \notin K$. Let r and s be positive integers such that $y^r a = ay^r$ and $y^s ax = axy^s$. This implies that $y^{rs}x = xy^{rs}$. In this case, m is set equal to rs. Finally, the case $y \notin K$ is reduced to the cases just considered since $y^t \in K$ for suitable values of t. To complete the proof, let $D_0 = F(x,y)$ denote the division subalgebra generated by x, y and F, where $F = Z(D)$ is the center of D. Let $a \in D_0'$ be any element, since $D_0' \subseteq D'$, then $a^n y = ya^n$ and $a^n x = xa^n$, for suitable values of n. This shows that a^n commutes with every element of D_0, i.e., $a^n \in Z(D_0)$, the center of D_0. Therefore, D_0' is radical over the center of D_0 and then by Lemma A, D_0 is a field. Since x and y were arbitrary elements of D'. This implies that D' is commutative. Thus, from [5] D is a field and this completes the proof.

When D' is radical over K', then D' is also radical over K. Therefore, using Theorem 1, the following are obtained.

Corollary 1

Let D be a division ring and K a proper division subring of D. If D' is radical over K', then D is a field.

Also, as an immediate consequence of Theorem 1, the following is obtained.

Corollary 2

Let D be a division ring and K be a proper division subring of D. If K' has a finite index in D', then D is a field.

In [19], Nakayama proves that if D is a division ring with a center F and given r nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ such that for each element $d \in D$, there exist r positive integers $n_1(d), n_2(d), ..., n_r(d)$ with $n_1(d) < n_i(d)$ and $d^{n_1(d)}\alpha_1 + d^{n_2(d)}\alpha_2 + ... + d^{n_r(d)}\alpha_r \in F$, then D is a field. This result holds for the derived group D'. To see this, the following proposition is needed which may be viewed as a generalization of Lemmas 1 and 2 in [19].

Proposition

Let D be a division ring with center F. Assume that either $Char F = 0$ or F is non-algebraic over its prime subfield and let $F \subset K \subset D$ be an algebraic proper extension of F which is not purely inseparable over F. Then, the following conclusions are drawn:

1. There exists a pair of distinct valuations on K which agree on F such that $v_1 \neq v_2$ on $K \cap D'$.
2. There can not exist a natural number r and a set of r non-zero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ such that for each element $a \in K \cap D'$, there exist r positive integers $n_1(a), n_2(a), ..., n_r(a)$ satisfying:
   (a) $n_1(a) < n_i(a)$, $i = 2, 3, ..., r$,
   (b) $\Sigma_1^r a^{n_i(a)}\alpha_i \in F$.

Proof

1. By Lemma 1 of [19], it is known that there exists a pair of distinct valuations v_1, v_2 on K which coincide on F. Now, assume on the contrary that $v_1 = v_2$ on $K \cap D'$. From Lemma 1 of [3], it is known that for each element $a \in K$, there exists a natural number $n(a)$ such that $a^{n(a)} = \lambda c_a$, where $\lambda \in F$ and $c_a \in K \cap D'$. Thus, it is obtained that $v_1(a^{n(a)}) = v_1(\lambda) + v_1(c_a) = v_2(\lambda) + v_2(c_a) = v_2(\lambda c_a) = v_2(a^{n(a)})$. Since the value group is not torsion it is concluded that $v_1 = v_2$ on K which is impossible.
2. From part 1, it is known that $K \cap D'$ is not central. Now, using part 1, the same proof as given in [19] for Lemma 2 may be applied to $K \cap D'$ to complete the proof.

Theorem 2

Let D be a division ring with center F. Let r be a natural number and $\alpha_1, \alpha_2, ..., \alpha_r \in F$ be r fixed non-zero elements in F. For each element a in the derived group D', assume that there exist r positive integers $n_1(a), n_2(a), ..., n_r(a)$ such that:

(a) $n_1(a) < n_i(a)$, $i = 2, 3, ..., r$,

(b) $\Sigma_1^r a^{n_i(a)}\alpha_i \in F$.

Then it is obtained that $D = F$.

Proof

Assume that $D \neq F$. From assumption (b), it is seen that each element of D' is algebraic over F. Corollary

4 of [3] implies that there exists an element $a \in D'$ not in F which is separable over F. Let $K = F(a)$. Furthermore, $Char F = p > 0$ and F is algebraic over its prime subfield, by part 2 of Proposition. Therefore, it may be concluded that each element of D' is algebraic over the prime subfield. However, this is a contradiction by virtue of Corollary 5 of [3] and thus the result follows.

Now the following theorem which generalizes the above stated result is proven.

Theorem 3

Let D be a division ring with center F and K a proper division subring of D. Suppose that, for each $d' \in D'$, there exist r nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ such that for every $a \in F(d')$, the positive integers $n_1(a), n_2(a), ..., n_r(a)$ exist with $n_1(a) < n_i(a)$ and $a^{n_1(a)}\alpha_1 + a^{n_2(a)}\alpha_2 + ... + a^{n_r(a)}\alpha_r \in K$. Then, D is a field.

Proof

Let $d' \in D'$ be any element and $E = F(d') \cap K$. If $d' \notin K$, then E is a proper subfield of $F(d')$. Therefore, from Lemma 2 of [19], $F(d')$ is purely inseparable over E or algebraic over its prime subfield (finite field). In either case, d' is radical over K and hence D' is radical over K. Thus, from Theorem 1, D is a field.

The following result may also be considered as a generalization of Theorem 1 of [1] for the derived group D'.

Theorem 4

Let D be a division ring over a field F and K a proper division subring of D with $K \neq D$. Assume that for each $d' \in D'$, there exist nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ (depending on d') such that for each $a \in F(d')$, a polynomial $f_a(x) = x^n - x^{n+1}P(x)$ can be found in the subring of $F[x]$ generated by x and $\alpha_1, \alpha_2, ..., \alpha_r$ satisfying $f_a(a) \in K$, then D is a field.

Proof

From [20, Hilfssatz 2] and using the same techniques as in the proof of Theorem 3, the result follows.

DIVISION RINGS WITH UNCOUNTABLE CENTERS

In this section, the study begins with the following result from [6] which proves the Herstein's conjecture (cf. [7,8]) for a division ring with an uncountable center.

Lemma C

Let D be a division ring with uncountable center F, $a \in D$ such that $(axa^{-1}x^{-1})^{n(x)} \in F$ for all $x \in D^*$, where $n(x)$ is a positive integer depending on x. Then, D is a field.

If $M(D)$ is radical over $Z(D)$, then hypothesis of Lemma C holds. Therefore, the following corollary is obtained.

Corollary D

If D is a division ring with an uncountable center F, in which $M(D)$ is radical over F, then D is a field.

The following lemma is needed to prove the main result of this section.

Lemma

Let D be a division ring with uncountable center F and K a proper division subring of D. Assume that $M(D)$ is radical over K. If for every $a \in K$, which is a power of an element of $M(D)$ and every $b, c \in M(D)$ with $bc \notin K$, there exists a positive integer n such that $a^n bc = bca^n$, then D is a field.

Proof

First, it is shown that for every pair $x, y \in M(D)$, there exist a positive integer m such that $y^m x = xy^m$. If $x \notin K$ and $y \in K$, from the hypothesis a positive integer m exists such that $y^m x = xy^m$. Let $x, y \in K$, $z \in M(D)$ is chosen such that $z \notin K$. Then, $zx \notin K$. Let r and s be positive integers such that $y^r z = zy^r$ and $y^s zx = zxy^s$. Then, $y^{rs}x = xy^{rs}$. Now let m be equal to rs. The case $y \notin K$ is reduced to the cases just decided since $y^t \in K$ for some t. Next, it is shown that D is a field. Let $D_0 = F(x, y)$ be the division subalgebra generated by x, y and $F = Z(D)$. Let also $w \in M(D_0)$. Then, a positive integer n exists such that $w^n y = yw^n$ and $w^n x = xw^n$. This shows that w^n commutes with every elements of D_0, i.e., $w^n \in Z(D_0)$. Therefore, from Corollary D, D_0 is a field. Since x and y are arbitrary elements of $M(D)$ it is concluded that $M(D)$ is commutative. Now, each element of the derived group D' is a product of the elements of $M(D)$ which itself is radical over K. Thus, D' is radical over K since $M(D)$ is commutative. Therefore, from Theorem 1, it is concluded that D is a field and the proof is completed.

It is now time to prove another generalization of Theorem B of [1] for division rings with uncountable centers.

Theorem 5

Let D be a division ring with uncountable center F. If $M(D)$ is radical over a proper division subring, K, then D is a field.

Proof

Suppose D is not a field. Then, from the Lemma, there exist $a \in K$ and $b, c \in M(D)$ with $bc \notin K$, such that a is a power of an element of $M(D)$ and $a^n bc \neq bca^n$ for all $n \geq 1$. Now, the same technique as used in the proof of Theorem 1 and the last part of the Lemma may be applied to elements $(bc + a)^{-1}a(bc + a)$ and

$(bc+1)^{-1}a(bc+1)$ to deduce that D is commutative and so the result follows.

When $M(D)$ is radical over $M(K)$, then it is radical over K. Therefore, from Theorem 5 the following corollary is obtained.

Corollary 3

Let D be a division ring with an uncountable center and K a proper division subring of D. If $M(D)$ is radical over $M(K)$, then D is a field.

Now, the following theorems whose proofs are similar to those of the previous section are stated.

Theorem 6

Let D be a division ring with uncountable center F and K a proper division subring of D. Let for each $d \in M(D)$, there exist r nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ such that for every $a \in F(d)$, the positive integers $n_1(a), n_2(a), ..., n_r(a)$ exist with $n_1(a) < n_i(a)$ and $a^{n_1(a)}\alpha_1 + a^{n_2(a)}\alpha_2 + ... + a^{n_r(a)}\alpha_r \in K$. Then, D is a field.

Theorem 7

Let D be a division ring with an uncountable center F and K a proper division subring of D with $K \neq D$. Assume that for each element $d \in M(D)$, there exist nonzero elements $\alpha_1, \alpha_2, ..., \alpha_r \in F$ (depending on d) such that for each $a \in F(d)$, a polynomial $f_a(x) = x^n - x^{n+1}P(x)$ can be found in the subring of $F[x]$ which is generated by x and $\alpha_1, \alpha_2, ..., \alpha_r$ satisfying $f_a(a) \in K$. Then, D is a field.

Remark 1

In [1], it is proved that if a division ring D is radical over a proper subring, then D is a field. In this case, the subring is a division subring, however, if D' or $M(D)$ is considered instead of D, this result does not hold. For example, let D be the division ring of rational quaternions. It is well known that the 2-adic valuation of Q (rational numbers) extends to D (cf. [21]). It is clear that the valuation ring B of D contains D' and $M(D)$, so D' and $M(D)$ are radical over B, but neither B is a division ring nor D is a field.

Remark 2

In Theorems 3 and 6, if it is assumed that $\alpha_1, \alpha_2, ..., \alpha_r \in Z(D') = Z(D) \cap D'$ (cf. [16]), then the results also hold.

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