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On the Convergence of a Time Discretization Scheme for the Navier-Stokes Equations

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A linearized version of the implicit Euler scheme is considered for the approximation of the solutions to the Navier-stokes equations in an n-dimensional domain. The rates of convergence in the H^1 and L^2 norms are established.

INTRODUCTION

In this paper, the concern is the discretization in time of the Navier-Stokes equations in a bounded n-dimensional domain:

$$\partial_t u(t,x) - \Delta u(t,x) + \nabla p(t,x) + (u\cdot\nabla)u(t,x) = 0,\ x\in\Omega,\ t>0, \quad (1)$$

$$\operatorname{div} u(t,x) = 0, \quad x\in\Omega, \quad t>0,$$

$$u(t,x) = 0, \quad x\in\partial\Omega, \quad t>0,$$

$$u(0,x) = u_0(x), \quad x\in\Omega. \quad (2)$$

Here, $u(t,x) = (u_1(t,x), u_2(t,x), ..., u_n(t,x))$ is the velocity, $p(t,x)$ is the pressure, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$ and u_0 is the initial velocity field.

As in [1–3], Equations 1 and 2 were cast as an evolution equation in the appropriate Hilbert space:

$$V = \{v = (v_1, v_2, ..., v_n) : v_1, v_2, ..., v_n \in C_0^\infty(\Omega), \quad \operatorname{div} v = 0\}.$$

H is the closure of V in $L^{2,2}(\Omega)$ (the space of R^n-valued functions, each component of which is in $L^2(\Omega)$) equipped with the inner product:

$$(u,v) = \int_\Omega \sum_{i=1}^n u_i(x)v_i(x)\,dx,$$

and the induced norm $\|u\| = (u,u)^{1/2}$.

$\overline{V}$ is the closure of V in $H_0^{1,2}(\Omega)$ (the Sobolev space of R^n-valued functions, each component of which is in $H_0^1(\Omega)$) equipped with the inner product:

$$(u,v)_1 = \int_\Omega \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,dx,$$

and the induced norm $\|u\|_1 = (u,u)_1^{1/2}$.

Similarly, the spaces $H^{s,2}(\Omega)$ and the norms $\|\cdot\|_s$ are defined in terms of the standard Sobolev spaces.

$P : L^{2,2}(\Omega) \to H$ denotes the orthogonal projection and defines the Stokes operator $A : D(A) \subset H \to H$, $D(A) = \overline{V}\cap H^{2,2}(\Omega)$ by $Au = -P\Delta u$, $u\in D(A)$.

Within this framework, Equations 1 and 2 are expressed as the evolution equation in H : $u(t)\in D(A)$, $t\geq 0$ and:

$$\frac{d}{dt}u(t) + Au(t) + B(u(t),u(t)) = 0, \quad t>0, \quad u(0) = u_0, \quad (3)$$

where $B(u,v) = P(u\cdot\nabla)v$.

The application of a linearized version of the implicit Euler scheme to Equation 3 determines the sequence $u_{k,n}\in D(A)$, $n = 0,1,2,\ldots$, such that:

$$\overline{\partial}_t u_{k,n} + Au_{k,n} + B(u_{k,n-1}, u_{k,n}) = 0, \quad n = 1,2,\ldots, \quad u_{k,0} = u_0, \quad (4)$$

where $k>0$ is the time step and:

$$\overline{\partial}_t u_{k,n} = \frac{u_{k,n} - u_{k,n-1}}{k}.$$

Now the following results will be established:

Theorem 1

If $u_0\in D(A)$ and $t = nk$:

$$\|u_{k,n} - u(t)\|_1 \leq \frac{Ce^{-\delta t}}{t^{1/2}}k, \quad (5)$$

for $k<k_0$, where C, δ and k_0 are positive constants depending on the data u_0 and Ω only.

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Theorem 2

If $u_0 \in D(A)$ and $t = nk$:

$$\|u_{k,n} - u(t)\| \leq Ce^{-\delta t}k \ , \tag{6}$$

for $k < k_0$, where C, δ and k_0 are positive constants depending on the data u_0 and Ω only.

Here, and in the sequel, C, δ and k_0 will denote possibly different constants which depend only on the data. This convention renders the proofs of results, such as the above theorems, more readable. In any case, the interested reader should have no difficulty in tracing the dependence of the various constants on the data.

The above results are parallel to those pertaining to the approximation by the implicit Euler scheme of the analytic semigroup generated by the positive-definite self-adjoint operator A, as discussed, for example, in [4,5].

The convergence of the scheme described by Equation 4 has been discussed by Girault and Raviart [6]. They have established the L^2-convergence of the scheme in terms of the smoothness properties of the solution. In the n-dimensional case the results presented here are directly in terms of the data and it is also possible to assert convergence in the H^1-norm. This approach, unlike that of [6], is based on Fujita-Kato's approach to the Navier-Stokes equations [1,2] and has been inspired by Okamoto's papers [7,8] on the spatial discretization of Equation 3. In fact, these results complement Okamoto's results. Fully discrete schemes have not been considered since the technicalities, which are considerable, vary depending on the spatial discretization schemes that are utilized and may obscure the essential goal of the paper, i.e., the demonstration of the convergence of the linearized implicit Euler scheme (Equation 4) at the predicted rate for $u_0 \in D(A)$.

It is relatively easier to establish the rate of convergence of a particular scheme by assuming the actual solution to be sufficiently regular. However, as is emphasized by Heywood and Rannacher [9,10] and also discussed by Rautmann [11] and Temam [12,3], such regularity assumptions may entail global compatibility conditions which are not met or which are not verifiable, in general.

In the next section certain a priori estimates will be obtained and the proofs of the theorems will be presented.

SOME A PRIORI ESTIMATES

A priori bounds on $\|u_{k,n}\|$, $\|A^{1/4}u_{k,n}\|$ and $\|A^{1/2}u_{k,n}\|$, $n = 1, 2, \ldots$, parallel to the a priori bounds established by Okamoto for spatial discretization are first established [8]. In establishing the a priori bounds and the actual convergence proof, it will be necessary to appeal frequently to the results of Fujita and Kato [1,2], Fujita and Morimoto [13], Temam [14,3] and Foias and Temam [15] with regard to the fractional powers of the Stokes operator A and properties of the trilinear form $b(u,v,w) = (B(u,v),w)$. The a priori estimates that have been established by Okamoto [8] for the actual solution are essential as well. The reader will notice the parallels between the presented treatment of time-discretization and Okamoto's treatment of spatial discretization.

Lemma 1

If $\{u_{k,n}\}_{n=0}^{\infty}$ is the solution of the linearized implicit Euler scheme, Equation 4, the following a priori estimates are valid:

$$\|u_{k,n}\|^2 + 2\sum_{j=1}^{n} \|A^{1/2}u_{k,j}\|^2 k \leq \|u_0\|^2, \quad n = 1, 2, \ldots \ , \tag{7}$$

$$\|A^{1/4}u_{k,n}\| \leq C(\|A^{1/4}u_0\|, \Omega)e^{-\delta t}, \tag{8}$$

$$\|A^{1/2}u_{k,n}\| \leq C(\|A^{1/2}u_0\|, \Omega)e^{-\delta t}, \quad 0 < k < k_0 \ , \tag{9}$$

where, $t = nk$, C, δ and k_0 are positive constants which depend on the data u_0 and Ω.

Proof

The inner product of Equation 4 with $u_{k,n}$ is formed and the following is obtained:

$$(\overline{\partial}_t u_{k,n}, u_{k,n}) + (Au_{k,n}, u_{k,n}) + (B(u_{k,n-1}, u_{k,n}), u_{k,n}) = 0. \tag{10}$$

Since,

$$(B(u_{k,n-1}, u_{k,n}), u_{k,n}) = b(u_{k,n-1}, u_{k,n}, u_{k,n}) = 0 \tag{11}$$

as in [14, p 163] and:

$$(\overline{\partial}_t u_{k,n}, u_{k,n}) = \frac{1}{2}\overline{\partial}_t\|u_{k,n}\|^2 + \frac{k}{2}\|\overline{\partial}_t u_{k,n}\|^2 \tag{12}$$

as in [5, p 157], Equation 10 yields:

$$\frac{1}{2}\overline{\partial}_t\|u_{k,n}\|^2 + \|A^{1/2}u_{k,n}\|^2 \leq 0 \tag{13}$$

so that:

$$\|u_{k,n}\|^2 + 2\sum_{j=1}^{n} \|A^{1/2}u_{k,j}\|^2 k \leq \|u_0\|^2,$$

i.e., Statement 7 is established.

In order to establish the a priori bound, Statement 8, on $\|A^{1/4}u_{k,n}\|$, the inner product of Equation 4 with $A^{1/2}u_{k,n}$ is formed and the following is obtained:

$$(\overline{\partial}_t u_{k,n}, A^{1/2}u_{k;n}) + (Au_{k,n}, A^{1/2}u_{k,n})$$
$$+ (B(u_{k,n-1}, u_{k,n}), A^{1/2}u_{k,n}) = 0 \ ,$$

and:

$$\frac{1}{2}\overline{\partial}_t\|A^{1/4}u_{k,n}\|^2 + \|A^{3/4}u_{k,n}\|^2$$
$$+ (A^{-1/4}B(u_{k,n-1}, u_{k,n}), A^{3/4}u_{k,n}) \leq 0 \ ,$$

so that:

$$\frac{1}{2}\overline{\partial}_t\|A^{1/4}u_{k,n}\|^2 + \|A^{3/4}u_{k,n}\|^2 \leq$$
$$\frac{1}{2}\|A^{-1/4}B(u_{k,n-1}, u_{k,n})\|^2 + \frac{1}{2}\|A^{3/4}u_{k,n}\|^2,$$

and:

$$\overline{\partial}_t\|A^{1/4}u_{k,n}\|^2 + \|A^{3/4}u_{k,n}\|^2 \leq$$
$$\|A^{-1/4}B(u_{k,n-1}, u_{k,n})\|^2 \ . \tag{14}$$

The proof of Lemma 3 in [2] is easily modified to show that:

$$\|A^{-1/4}B(v, w)\| \leq C\|A^{1/4}v\| \ \|A^{1/2}w\|. \tag{15}$$

It is also known that:

$$\|A^{1/4}v\| \leq C\|A^{3/4}v\|,$$

so that, Statement 14 yields:

$$\|A^{1/4}u_{k,n}\|^2 + C\sum_{j=1}^{n}\|A^{1/4}u_{k,j}\|^2 k \leq$$
$$C\sum_{j=1}^{n}\|A^{1/2}u_{k,j}\|^2 \ \|A^{1/4}u_{k,j-1}\|^2 k + \|A^{1/4}u_0\|^2,$$

which can be written as:

$$\|A^{1/4}u_{k,n}\|^2 \leq C\sum_{j=1}^{n}(\|A^{1/2}u_{k,j}\|^2 - 1)$$
$$\|A^{1/4}u_{k,j-1}\|^2 k + \|A^{1/4}u_0\|^2. \tag{16}$$

A discrete Gronwall type result, as presented for example by Jerome [16, p 53], is directly applicable to Statement 16 and yields:

$$\|A^{1/4}u_{k,n}\|^2 \leq C\|A^{1/4}u_0\|^2(1 + \|A^{1/2}u_{k,n}\|^2 k)$$
$$\exp(\sum_{j=1}^{n}\|A^{1/2}u_{k,j}\|^2 k - \delta t) \tag{17}$$

where $t = nk$. Considering Statement 7, it is possible to obtain the following from Statement 17:

$$\|A^{1/4}u_{k,n}\|^2 \leq C\|A^{1/4}u_0\|^2(1 + \|u_0\|^2)$$
$$\exp(\|u_0\|^2 - \delta t),$$

which readily yields Statement 8.

Now, the a priori bound, Statement 9, on $\|A^{1/2}u_{k,n}\|$ will be established. From Equation 4,

$$u_{k,n} = E_k u_{k,n-1} - E_k B(u_{k,n-1}, u_{k,n})k \ , \tag{18}$$

where $E_k = (I + kA)^{-1}$, I denoting the identity. Repeated use of Equation 18 results in:

$$u_{k,n} = E_k^n u_0 - \sum_{j=1}^{n} E_k^{n-j+1}B(u_{k,j-1}, u_{k,j})k \ , \tag{19}$$

so that:

$$A^{1/2}u_{k,n} = E_k^n A^{1/2}u_0$$
$$- \sum_{j=1}^{n} A^{3/4}E_k^{n-j+1}A^{-1/4}B(u_{k,j-1}, u_{k,j})k. \tag{20}$$

Since A is positive-definite self-adjoint, the inequality:

$$\|E_k^n A^{1/2}u_0\| \leq Ce^{-\delta t}\|A^{1/2}u_0\| \quad (t = nk), \tag{21}$$

is obtained via spectral representation as in [5, Ch.7]. Even though the exponential decay factor is not present in the statement of the results in [5], the required modification of the proofs is straightforward and does not warrant a lengthy exposition here.

For $j = n$ in the sum appearing in Equation 20,

$$\|A^{3/4}E_k A^{-1/4}B(u_{k,n-1}, u_{k,n})\|k \leq$$
$$\frac{C}{k^{3/4}}\|A^{-1/4}B(u_{k,n-1}, u_{k,n})\|k \ , \tag{22}$$

as in [5, Ch.7]. Statements 8, 15 and 22 yield:

$$\|A^{1/2}E_k B(u_{k,n-1}, u_{k,n})\|k \leq$$
$$C\|A^{1/4}u_{k,n-1}\| \ \|A^{1/2}u_{k,n}\|k^{1/4} \leq$$
$$C\|A^{1/4}u_0\|k^{1/4}\|A^{1/2}u_{k,n}\|. \tag{23}$$

From Statements 8, 20, 21 and 23, the following is obtained:

$$\|A^{1/2}u_{k,n}\| \leq Ce^{-\delta nk}\|A^{1/2}u_0\|$$
$$+ Ck\sum_{j=1}^{n-1}\|A^{3/4}E_k^{n-j+1}A^{-1/4}B(u_{k,j-1}, u_{k,j})\|, \tag{24}$$

if k is sufficiently small, say, $k < k_0$.

Now, as a result of Statement 8:

$$\|A^{3/4}E_k^{n-j+1}A^{-1/4}B(u_{k,j-1},u_{k,j})\| \le \frac{Ce^{-(n-j+1)k\delta}}{(nk-(j-1)k)^{3/4}}\|A^{1/4}u_{k,j-1}\|\,\|A^{1/2}u_{k,j}\| \le \frac{C(\|A^{1/4}u_0\|,\Omega)}{(nk-(j-1)k)^{3/4}}e^{-nk\delta}e^{jk\delta}\|A^{1/2}u_{k,j}\|. \quad (25)$$

From Statements 24 and 25 it is possible to obtain:

$$e^{nk\delta}\|A^{1/2}u_{k,n}\| \le C\|A^{1/2}u_0\| + C(\|A^{1/4}u_0\|,\Omega)\sum_{j=1}^{n-1}\frac{e^{jk\delta}\|A^{1/2}u_{k,j}\|k}{(nk-(j-1)k)^{3/4}}. \quad (26)$$

$\varphi(s)$ is defined as $e^{jk\delta}\|A^{1/2}u_{k,j}\|$, $s \in [(j-1)k, jk)$. From Statement 26, the following is obtained:

$$\varphi(t) \le C\|A^{1/2}u_0\| + C(\|A^{1/4}u_0\|,\Omega)\int_0^t \frac{\varphi(s)}{(t-s)^{3/4}}\,ds. \quad (27)$$

As in Lemma 6.5 of [8], the above inequality leads to:

$$\varphi(t) \le C\|A^{1/2}u_0\|\exp\{C\beta^4 t\}, \quad (28)$$

where β depends on the data also. Furthermore, Statement 28 yields:

$$\|A^{1/2}u_{k,n}\| \le C(\|A^{1/2}u_0\|,\Omega)\exp\{C\beta^4 nk - \delta nk\}. \quad (29)$$

Then, the a priori estimate, Statement 9, on $\|A^{1/2}u_{k,n}\|$ is obtained by Okamoto's argument [8, proof of Prop. 6.6] from the above inequality.

Aside from the estimates on the solution of Equation 3 that may be referred to, the following estimate will be needed.

Lemma 2

If $u_0 \in D(A)$,

$$\|A^{1/2}D_t u(t)\| \le \frac{C(\|Au_0\|,\Omega)}{t^{1/2}}e^{-\delta t}, \quad t > 0\,. \quad (30)$$

Proof

By differentiating Equation 3 and setting $v(t) = D_t u(t)$, the following is obtained:

$$D_t v(t) + Au(t) + B(v(t),u(t)) + B(u(t),v(t)) = 0, \quad t > 0\,, \quad (31)$$
$$v(0) = -Au_0 - B(u_0,u_0)\,.$$

As in [15, Section 1],

$$\|B(u_0,u_0)\| \le C\|u_0\|_1\|u_0\|_2. \quad (32)$$

As a consequence of the results of the fractional powers of the Stokes operator A [13] and the regularity results concerning the Stokes problem as in [14, Ch.1], the following inequality can be obtained from Statement 32:

$$\|B(u_0,u_0)\| \le C\|A^{1/2}u_0\|\,\|Au_0\| \le C(\|Au_0\|,\Omega). \quad (33)$$

From Equation 31:

$$v(t) = e^{-tA}v_0 - \int_0^t e^{-(t-s)A}[B(v(s),u(s)) + B(u(s),v(s))]ds, \quad (34)$$

so that, by using Statement 33:

$$\|A^{1/2}v(t)\| \le \frac{C(\|Au_0\|,\Omega)}{t^{1/2}}e^{-\delta t} + \int_0^t \|A^{3/4}e^{-(t-s)A}A^{-1/4}[B(v(s),u(s)) + B(u(s),v(s))]\|ds. \quad (35)$$

From Statement 15,

$$\|A^{-1/4}[B(v(s),u(s)) + B(u(s),v(s))]\| \le C\|A^{1/2}u(s)\|\,\|A^{1/2}v(s)\| \le C(\|A^{1/2}u_0\|,\Omega)\,\|A^{1/2}v(s)\|, \quad (36)$$

using Okamoto's estimate on $\|A^{1/2}u(s)\|$ [8, Prop. 6.1]. Furthermore, Statements 35 and 36 lead to:

$$e^{\delta t}\|A^{1/2}v(t)\| \le C(\|Au_0\|,\Omega)\left(t^{-1/2} + \int_0^t \frac{e^{\delta s}\|A^{1/2}v(s)\|}{(t-s)^{3/4}}\,ds\right). \quad (37)$$

Again, as in [8], Statement 37 yields Estimate 30. Now, it is time to establish the error estimates.

THE ERROR ESTIMATES

The H^1-convergence result is restated.

Theorem 1

If $u_0 \in D(A)$ and $t = nk,\ n = 1, 2, \ldots,$

$$\|u_{k,n} - u(t)\|_1 \leq \frac{C\|Au_0\|, \Omega)e^{-\delta t}}{t^{1/2}} k. \tag{38}$$

Proof

It is known that:

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} B(u(s), u(s))ds, \quad t = nk\ , \tag{39}$$

$$u_{k,n} = E_k^n u_0 - \sum_{j=1}^n E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k, \tag{40}$$

as stated in the previous section. As in [5],

$$\|A^{1/2}(E_k^n - e^{-tA})u_0\| \leq \frac{C(\|Au_0\|, \Omega)e^{-\delta t}}{t^{1/2}} k, \tag{41}$$

where, as always, $\delta > 0$ also depends on the data. Thus,

$$\|A^{1/2}(u(t) - u_{k,n})\| \leq \frac{Ce^{-\delta t}}{t^{1/2}} k + \|A^{1/2} \int_0^t e^{-(t-s)A} B(u(s), u(s))ds - A^{1/2} \sum_{j=1}^n E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k\|. \tag{42}$$

Also, it is written that:

$$\int_0^t e^{-(t-s)A} B(u(s), u(s))ds - \sum_{j=1}^n E_k^{n-j+1} B(u_{k,j-1}, u_{k,j})k = \left[\int_0^t e^{-(t-s)A} B(u(s), u(s))ds - \sum_{j=1}^n E_k^{n-j+1} B(u_{j-1}, u_j)k\right] + \left[\sum_{j=1}^n E_k^{n-j+1} \big(B(u_{j-1}, u_j) - B(u_{k,j-1}, u_{k,j})\big)k\right], \tag{43}$$

where u_j denotes $u(jk)$.

The last expression of Statement 43 is written as:

$$\sum_{j=1}^n E_k^{n-j+1} (B(u_{j-1}, u_j) - B(u_{k,j-1}, u_{k,j}))k = \sum_{j=1}^n E_k^{n-j+1} B(u_{j-1} - u_{k,j-1}, u_j)k + \sum_{j=1}^n E_k^{n-j+1} B(u_{k,j-1}, u_j - u_{k,j})k = \sum_{j=1}^{n-1} E_k^{n-j} B(u_j - u_{k,j}, u_{j+1})k + \sum_{j=1}^{n-1} E_k^{n-j+1} B(u_{k,j-1}, u_j - u_{k,j})k + E_k B(u_{k,n-1}, u_n - u_{k,n})k\ , \tag{44}$$

since $u_{k,0} = u_0$.

As stated before,

$$\|A^{1/2} E_k B(u_{k,n-1}, u_n - u_{k,n})\|k = \|A^{3/4} E_k A^{-1/4} B(u_{k,n-1}, u_n - u_{k,n})\|k \leq \frac{C}{k^{3/4}} \|A^{1/4} u_{k,n-1}\| \cdot \|A^{1/2}(u_n - u_{k,n})\|k \leq Ck^{1/4} \|A^{1/2}(u_n - u_{k,n})\|, \tag{45}$$

which is a result of Estimate 8 on $\|A^{1/4}u_{k,n}\|$.

Next, the following is considered:

$$\sum_{j=1}^{n-1} \|A^{1/2} E_k^{n-j} B(u_j - u_{k,j}, u_{j+1})\|k = \sum_{j=1}^{n-1} \|A^{3/4} E_k^{n-j} A^{-1/4} B(u_j - u_{k,j}, u_{j+1})\|k \leq Ce^{-\delta t} \sum_{j=1}^{n-1} \frac{e^{jk\delta}}{(nk - jk)^{3/4}} \|A^{1/4}(u_j - u_{k,j})\| \|A^{1/2} u_{j+1}\|k \leq Ce^{-\delta t} \sum_{j=1}^{n-1} \frac{e^{jk\delta}}{(nk - jk)^{3/4}} \|A^{1/2}(u_j - u_{k,j})\|k, \tag{46}$$

due to Estimate 9 on $\|A^{1/2}u_n\|$. Similarly,

$$\sum_{j=1}^{n-1}\|A^{1/2}E_k^{n-j+1}B(u_{k,j-1},u_j-u_{k,j})\|k\le$$

$$Ce^{-\delta t}\sum_{j=1}^{n-1}\frac{e^{jk\delta}}{(nk-jk)^{3/4}}\|A^{1/2}(u_j-u_{k,j})\|k. \tag{47}$$

From Statements 42–47, the following is obtained for sufficiently small values of k,

$$e^{\delta t}\|A^{1/2}(u_n-u_{k,n})\|\le$$

$$\frac{Ck}{t^{1/2}}+C\sum_{j=1}^{n-1}\frac{e^{\delta jk}}{(t-jk)^{3/4}}\|A^{1/2}(u_j-u_{k,j})\|k$$

$$+e^{\delta t}\|A^{1/2}\int_0^t e^{-(t-s)A}B(u(s),u(s))ds$$

$$-A^{1/2}\sum_{j=1}^{n}E_k^{n-j+1}B(u_{j-1},u_j)k\|. \tag{48}$$

Therefore, as stated before, the theorem will be established once it is shown that:

$$\left\|A^{1/2}\int_0^t e^{-(t-s)A}B(u(s),u(s))ds - A^{1/2}\sum_{j=1}^{n}E_k^{n-j+1}B(u_{j-1},u_j)k\right\|\le\frac{Ce^{-\delta t}}{t^{1/2}}k. \tag{49}$$

Next, it is written that:

$$\int_0^t e^{-(t-s)A}B(u(s),u(s))ds$$

$$-\sum_{j=1}^{n}E_k^{n-j+1}B(u_{j-1},u_j)k=$$

$$\int_0^t e^{-(t-s)A}(B(u(s),u(s))-B(u(t),u(t)))ds$$

$$-\sum_{j=1}^{n}E_k^{n-j+1}(B(u_{j-1},u_j)-B(u(t),u(t)))k$$

$$+\left[\int_0^t e^{-(t-s)A}-\sum_{j=1}^{n}E_k^{n-j+1}k\right]B\big(u(t),u(t)\big). \tag{50}$$

Now, the last line of Equation 50 will be dealt with. As stated in [17, p 489],

$$\int_0^t e^{-(t-s)A}ds=(I-e^{-tA})A^{-1}. \tag{51}$$

It is also easily verified that:

$$\sum_{j=1}^{n}E_k^{n-j+1}k=(I-E_k^n)A^{-1}. \tag{52}$$

From Equations 51 and 52,

$$\left\|A^{1/2}\left[\int_0^t e^{-(t-s)A}-\sum_{j=1}^{n}E_k^{n-j+1}k\right]B(u(t),u(t))\right\|=$$

$$\|A^{1/2}(E_k^n-e^{-tA})A^{-1}B(u(t),u(t))\|\le$$

$$\frac{Ce^{-\delta t}}{t^{1/2}}\|B(u(t),u(t))\|k\le$$

$$\frac{Ce^{-\delta t}}{t^{1/2}}\|Au(t)\|\ \|A^{1/2}u(t)\|k\le$$

$$\frac{C(\|Au_0\|,\Omega)}{t^{1/2}}e^{-\delta t}, \tag{53}$$

as a result of the error estimates on the approximation of e^{-tA}, as in [5] and the a priori estimates established by Okamoto [8]. Clearly, Statement 33 has been used as well. From Statements 50 and 53, Statement 49 will be established once the following is estimated:

$$\left\|A^{\frac12}\int_0^t e^{-(t-s)A}\Big(B\big(u(s),u(s)\big)-B\big(u(t),u(t)\big)\Big)ds - \sum_{j=1}^{n}E_k^{n-j+1}\Big(B(u_{j-1},u_j)-B\big(u(t),u(t)\big)\Big)k\right\|.$$

To this end, the following is written:

$$\int_0^t e^{-(t-s)A}(B(u(s),u(s))-B(u(t),u(t)))ds$$

$$-\sum_{j=1}^{n}E_k^{n-j+1}\Big(B\big(u_{j-1},u_j\big)-B\big(u(t),u(t)\big)\Big)k=$$

$$\sum_{j=1}^{n}\int_{(j-1)k}^{jk}\left[e^{-(t-s)A}-e^{-(t-(j-1)k)A}\right]$$

$$\Big(B\big(u(s),u(s)\big)-B\big(u(t),u(t)\big)\Big)ds$$

$$+\sum_{j=1}^{n}\int_{(j-1)k}^{jk}e^{-(t-(j-1)k)A}$$

$$\Big(B\big(u(s),u(s)\big)-B\big(u_{j-1},u_j\big)\Big)ds$$

$$+\sum_{j=1}^{n}\left[e^{-(t-(j-1)k)A}-E_k^{n-(j-1)}\right]$$

$$\Big(B\big(u_{j-1}.u_j\big)-B\big(u(t),u(t)\big)\Big)k=$$

$I_1 + I_2 + I_3$. (54)

Now, it will be established that:

$$\|A^{1/2}I_\ell\| \leq \frac{Ce^{-\delta t}}{t^{1/2}}k, \quad \ell = 1,2,3, \tag{55}$$

and this will conclude the proof.

In order to estimate $\|A^{1/2}I_1\|$, it is first noted that:

$$e^{-(t-s)A} - e^{-(t-(j-1)k)A} =$$
$$e^{-(t-s)A} - e^{-(t-s)A}e^{-(s-(j-1)k)A} =$$
$$e^{-(t-s)A}[I - e^{-(s-(j-1)k)A}],$$

for $s \in [(j-1)k, jk)$. Therefore,

$$\|A^{1/2}[e^{-(t-s)A} - e^{-(t-(j-1)k)A}]G\| =$$
$$\|A^{3/2}e^{-(t-s)A}A^{-1}(I - e^{-(s-(j-1)k)A})G\| \leq$$
$$\frac{Ce^{-\delta(t-s)}}{(t-s)^{3/2}}\|G\|k, \quad s \in [(j-1)k, jk), \tag{56}$$

as in [18, proof of Theorem 1].

As in the proof of Lemma 2,

$$\|B(u(s),u(s)) - B(u(t),u(t))\| \leq$$
$$C\|Au(t)\|\;\|A^{1/2}(u(s)-u(t))\| \leq$$
$$C(\|Au_0\|,\Omega)e^{-\delta s}\frac{(t-s)}{s^{1/2}}, \tag{57}$$

using Okamoto's estimate on $\|Au(t)\|$ [8, Prop 6.3] and Lemma 2. Statements 54, 56 and 57 yield:

$$\|A^{1/2}I_1\| \leq Ce^{-\delta t}k\sum_{j=1}^{n}\int_{(j-1)k}^{jk}\frac{1}{(t-s)^{1/2}s^{1/2}}ds$$
$$= Ce^{-\delta t}k\int_0^t \frac{1}{(t-s)^{1/2}s^{1/2}}ds \leq Ce^{-\delta t}k. \tag{58}$$

In order to estimate $\|A^{1/2}I_2\|$ it is written that:

$$I_2 = \int_0^k e^{-tA}B(u(s),u(s))ds - e^{-tA}B(u_0,u_1)k$$
$$+\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u(s)-u_{j-1},u(s))ds$$
$$+\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}B(u_{j-1},u(s)-u_j)ds. \tag{59}$$

It is first observed that:

$$\|A^{1/2}e^{-tA}B(u(s),u(s))\| \leq$$
$$\frac{Ce^{-\delta t}}{t^{1/2}}\|A^{1/2}u(s)\|\;\|Au(s)\| \leq$$
$$\frac{C(\|Au_0\|,\Omega)e^{-\delta t}}{t^{1/2}},$$

so that:

$$\|A^{1/2}\int_0^k e^{-tA}B(u(s),u(s))ds\| \leq \frac{Ce^{-\delta t}}{t^{1/2}}k. \tag{60}$$

Similarly,

$$\|A^{1/2}B(u_0,u_1)\|k \leq \frac{Ce^{-\delta t}}{t^{1/2}}k. \tag{61}$$

Using Statements 59–61, the result of Statement 55 for $\ell = 1$ will be established if such an estimate is proven for the remaining terms of Equation 59. Treating the following statement will be sufficient,

$$\left\|A^{1/2}\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}\right.$$
$$\left. B\big(u(s)-u_{j-1},u(s)\big)ds\right\|,$$

since the last term is treated in a similar manner. For $j = 2,3,\ldots,n,\; s \in [(j-1)k, jk)$,

$$\|A^{1/2}e^{-(t-(j-1)k)}B(u(s)-u_{j-1},u(s))\| \leq$$
$$\frac{Ce^{-\delta t}e^{\delta(j-1)k}}{(t-(j-1)k)^{1/2}}\|Au(s)\|$$
$$\|A^{1/2}(u(s)-u_{j-1})\| \leq$$
$$\frac{Ce^{-\delta t}k}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}}, \tag{62}$$

again due to [8] and Lemma 2.

From Statement 62, the following is obtained:

$$\left\|A^{1/2}\sum_{j=2}^{n}\int_{(j-1)k}^{jk} e^{-(t-(j-1)k)A}\right.$$
$$\left. B\big(u(s)-u_{j-1},u(s)\big)ds\right\| \leq$$
$$Ce^{-\delta t}k\Big(\sum_{j=2}^{n}\frac{k}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}}\Big) \leq$$
$$Ce^{-\delta t}k\int_0^t\frac{1}{(t-s)^{1/2}s^{1/2}}ds \leq Ce^{-\delta t}k. \tag{63}$$

The last line of Equation 59 is treated similarly and only the task of estimating $\|A^{1/2}I_3\|$ remains. It is written that:

$$I_3 = [e^{-tA} - E_k^n](B(u_0,u_1) - B(u(t),u(t)))k + \sum_{j=2}^{n}\left[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}\right] B\big(u_{j-1} - u(t), u_j\big)k + \sum_{j=2}^{n}\left[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}\right] B\big(u(t), u_j - u(t)\big)k. \quad (64)$$

Each term of the first line of the above equation is treated in a similar manner. For example,

$$\|A^{1/2}E_kB(u_0,u_1)\|k \le \frac{Ce^{-\delta t}}{t^{1/2}}\|B(u_0,u_1)\|k \le \frac{C(\|Au_0\|,\Omega)e^{-\delta t}}{t^{1/2}}k.$$

Now,

$$\sum_{j=2}^{n}\left\|A^{1/2}\left[e^{-(t-(j-1)k)A} - E_k^{n-(j-1)}\right] B\big(u_{j-1} - u(t), u_j\big)k\right\| \le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\|B(u_{j-1}-u(t),u_j)\|k \le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\|A^{1/2}(u_{j-1}-u(t))\| \|Au_j\|k \le \sum_{j=2}^{n}\frac{Ce^{-\delta(t-(j-1)k)}k}{(t-(j-1)k)^{3/2}}\cdot\frac{(t-(j-1)k)}{((j-1)k)^{1/2}}e^{-\delta jk}k = Ce^{-\delta t}k\Big(\sum_{j=2}^{n}\frac{1}{(t-(j-1)k)^{1/2}((j-1)k)^{1/2}}k\Big) \le Ce^{-\delta t}k, \quad (65)$$

again due to [5,8] and Lemma 2.

The last line of Equation 64 is handled similarly and Statement 55 is established for I_3 as well. As anticipated earlier, the proof of Theorem 1 is concluded, since $\|A^{1/2}(u(t)-u_{k,n})\|$ is equivalent to $\|u(t)-u_{k,n}\|_1$, as in [13].

Theorem 2 on L^2-convergence is now restated and the proof is merely sketched, since it parallels very closely the proof of Theorem 2. The reader should have no difficulty in supplying the details.

Theorem 2

If $u_0 \in D(A)$ and $t = nk$,

$$\|u_{k,n} - u(t)\| \le Ce^{-\delta t}k\ ,$$

for $k < k_0$, where C, δ and k_0 are positive constants depending on the data u_0 and Ω only.

Proof

As before, the starting point is Equations 39 and 40. It is first observed that:

$$\|(e^{-tA} - E_k^n)u_0\| \le C(\|Au_0\|,\Omega)e^{-\delta t}k \quad (66)$$

as in [5]. Then, taking note of Equations 43 and 44, parallel to Statement 45, it is noted that:

$$\|E_kB(u_{k,n-1}, u_n - u_{k,n})\|k = \|A^{1/2}E_kA^{-1/2}B(u_{k,n-1}, u_n - u_{k,n})\|k \le \frac{C}{k^{1/2}}\|A^{-1/2}B(u_{k,n-1}, u_n - u_{k,n})\|k. \quad (67)$$

Furthermore, it is claimed:

$$\|A^{-1/2}B(u_{k,n-1}, u_n - u_{k,n})\| \le C\|u_{k,n-1} - u_{n-1})\|_1\ \|u_{k,n} - u_n\|_1 + C\|u_{n-1}\|_2\ \|u_{k,n} - u_n\|. \quad (68)$$

Indeed,

$$b(u_{k,n-1}, u_n - u_{k,n}, A^{-1/2}v) = b(u_{k,n-1} - u_{n-1}, u_n - u_{k,n}, A^{-1/2}v) + b(u_{n-1}, u_n - u_{k,n}, A^{-1/2}v),$$

and, due to [15, Sec. 1] and [13],

$$\left|b(u_{k,n-1} - u_{n-1}, u_n - u_{k,n}, A^{-1/2}v)\right| \le C\|u_{k,n-1} - u_{n-1}\|_1\ \|u_n - u_{k,n}\|_1\ \|A^{-1/2}v\|_1 \le C\|u_{k,n-1} - u_{n-1}\|_1\ \|u_n - u_{k,n}\|_1\ \|v\|,$$

$$|b(u_{n-1}, u_n - u_{k,n}, A^{-1/2}v)| = |b(u_{n-1}, A^{-1/2}v, u_n - u_{k,n})| \le C\|u_{n-1}\|_2\ \|A^{-1/2}v\|_1\ \|u_n - u_{k,n}\| \le C\|u_{n-1}\|_2\ \|u_n - u_{k,n}\|\ \|v\|,$$

so that Statement 68 is valid.

Theorem 1 and Statement 67 yield:

$$\|E_k B(u_{k,n-1}, u_n - u_{k,n})\|k \leq$$

$$Ce^{-\delta t}k^{1/2}(\frac{k^2}{t} + \|u_{k,n} - u_n\|) \leq$$

$$Ce^{-\delta t}k + Ce^{-\delta t}k^{1/2}\|u_{k,n} - u_n\|, \quad (69)$$

for $0 < k < k_0$, say. Statement 69 is the counterpart of Statement 45. By using Statement 68 and similar estimates, the counterparts of the other steps of the proof of Theorem 1 are easily established. Now, merely a sample will be provided:

$$\sum_{j=1}^{n-1} \|E_k^{n-j} B(u_j - u_{k,j}, u_{j+1})\|k =$$

$$\sum_{j=1}^{n-1} \|A^{1/2} E_k^{n-j} A^{-1/2} B(u_j - u_{k,j}, u_{j+1})\|k \leq$$

$$\sum_{j=1}^{n-1} \frac{Ce^{-(n-j)k\delta}}{(nk - jk)^{1/2}} \|A^{-\frac{1}{2}} B(u_j - u_{k,j}, u_{j+1})\|k \leq$$

$$\sum_{j=1}^{n-1} \frac{Ce^{-(n-j)k\delta}}{(nk - jk)^{1/2}} \|Au_{j+1}\| \; \|u_j - u_{k,j}\|k \leq$$

$$Ce^{-\delta t} \sum_{j=1}^{n-1} \frac{1}{(nk - jk)^{1/2}} \|u_j - u_{k,j}\|k. \quad (70)$$

This is the counterpart of Statement 46.

In this manner one establishes that

$$\|u_{k,n} - u(t)\| \leq Ce^{-\delta t}\Bigg(k + \sum_{j=1}^{n-1} \frac{e^{\delta jk}}{(t - jk)^{1/2}} \|u_{k,j} - u_j\|k\Bigg),$$

if k is sufficiently small and, therefore, the L^2-convergence result follows.

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