Scientia Iranica, Vol. 4, Nos. 1&2, pp 12–18

SCIENTIA
IRANICA

Regularized Cosine Functions and Polynomials of Group Generators

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Let iA_j $(1 \leq j \leq n)$ be commuting generators of bounded strongly continuous groups and $P(A) = \sum_{|\alpha| \leq m} a_\alpha A^\alpha$ $(A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n})$. It is proven that $\overline{P(A)}$ generates an exponentially bounded regularized cosine function and an l-times integrated cosine function under suitable conditions on the polynomial $P(\xi)$. Then, these results are applied to the partial differential operators $P(D)$ $(iA_j = \partial/\partial x_j,\ 1 \leq j \leq n)$ on spaces like $L^p(\mathbf{R}^n)\,(1 \leq p < \infty), C_0(\mathbf{R}^n)$ and $BUC(\mathbf{R}^n)$.

INTRODUCTION

Many physical problems, including initial-value partial differential equations, may be modelled as an abstract Cauchy problem,

$$\frac{d}{dt}u(t,x) = Au(t,x)\,(t \geq 0) \quad u(0,x) = x\ , \qquad (1)$$

where A is a linear operator on a Banach space X and $t \mapsto u(t,x) \in C([0,\infty).X)$. The problem is well-posed, that is, depends continuously on the initial data x if, and only if, A generates a strongly continuous semigroup. Moreover, it is known that many partial differential operators, such as the Schrödinger operator $i\Delta$ on $L^p(\mathbf{R}^n)(p \neq 2)$, cannot be treated by strongly continuous semigroups (see [1]).

Recently, there has been extensive development and application of two generalizations of strongly continuous semigroups, known as regularized semigroups [1,2] and integrated semigroups [3-6], that deal with Equation 1 when it is ill-posed. Intuitively, if $(Jf)(t) \equiv \int_0^t f(s)ds$, then the strongly continuous semigroup generated by A is e^{tA}, the C-regularized semigroup is $e^{tA}C$ and the n-times integrated semigroup is $J^n(e^{tA})$.

One may similarly deal with a second-order abstract Cauchy problem:

$$\begin{cases} \dfrac{d^2}{dt^2}u(t,x_1,x_2) = Au(t,x_1,x_2)\ (t \geq 0), \\ u(0,x_1,x_2) = x_1, \quad \dfrac{d}{dt}u(0,x_1,x_2) = x_2, \end{cases} \qquad (2)$$

with a cosine family when Equation 2 is well-posed. Cosine families have received less attention than strongly continuous semigroups, partly because Equation 2 may be re-written as a first-order abstract Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t,x) = \begin{bmatrix} 0 & 1 \\ A & 0 \end{bmatrix} u(t,x)\ (t \geq 0), \\ u(0,x) = x, \end{cases}$$

so, the semigroups generated by $\left[\begin{smallmatrix} 0 & 1 \\ A & 0 \end{smallmatrix}\right]$ may be discussed instead. However, this matrix reduction is not always successful; it is sometimes necessary to leave Equation 2 as a second-order problem and sometimes it is simpler to work with a cosine family generated by A rather than a semigroup generated by $\left[\begin{smallmatrix} 0 & 1 \\ A & 0 \end{smallmatrix}\right]$. Similarly, many partial differential operators such as the Laplace operator Δ in $L^p(\mathbf{R}^n)$ with maximal distributional domain do not generate a strongly continuous cosine function unless $n = 1, 1 \leq p < \infty$ or $n \geq 2$ and $p = 2$ (see [7]).

Thus, it is rational to deal with the obvious second-order analogues of regularized or integrated semigroups and regularized or integrated cosine families. Intuitively, one replaces exponentials by a cosine family: a cosine family generated by A is $\cosh(t\sqrt{A})$, a C-regularized cosine family is $\cosh(t\sqrt{A})C$ and an n-times integrated cosine family is $J^n(\cosh(t\sqrt{A}))$.

In this paper, regularized and integrated cosine families generated by operators of the form $P(A_1, \cdots, A_n)$ are considered, where P is a polynomial and $iA_1, \cdots, iA_n$ generate commuting bounded strongly continuous groups. A special case of such an operator is a constant-coefficient partial differential operator in the usual function spaces $L^p(\mathbf{R}^n)$ $(1 \leq p < \infty)$, $C_0(\mathbf{R}^n)$, etc. Regularized and

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integrated semigroups generating such operators, have been thoroughly studied in [1,8–10]. For cosine families, proof follows the same idea: natural functional calculus is used for $A \equiv (A_1, \cdots, A_n)$:

$$f(A) \equiv \int_{\mathbf{R}^n} e^{i(s \cdot A)} \mathcal{F} f(s)\, ds \ , \tag{3}$$

where $\mathcal{F}$ is the Fourier transform and $f(s) \equiv \cosh(t\sqrt{s})g(s)$ is chosen for a regularized cosine family, $J^n(t \mapsto \cosh(t\sqrt{s}))$ for an n-times integrated cosine family, rather than $e^{ts}g(s)$ for a regularized semigroup and $J^n(t \mapsto e^{ts})$ for an n-times integrated semigroup.

Of course it is much more difficult to work with the function $s \mapsto \cosh(t\sqrt{s})$ rather than $s \mapsto e^{ts}$ Since many derivatives must be estimated for the appropriate Fourier multiplier techniques. It is particularly difficult to get sharp results, that is, to minimize the order of integration or maximize the range of the regularized operator C.

In this paper, it is assumed that the polynomials $P(\xi)$ with constant coefficients are real valued and bounded. It is proven that $\overline{P(A)}$ on X generates a regularized or l-times integrated cosine function depending on $P(\xi)$ being elliptic or non-elliptic. Then, these results are applied to partial differential operators on the spaces $L^p(\mathbf{R}^n)(1 \leq p < \infty), BUC(\mathbf{R}^n)$ and $C_0(\mathbf{R}^n)$ and the same results are obtained, which improve the related results given in [4,5]. Here, the main aim is to extend the known results analogous to the one proved in [9].

PRELIMINARIES

In this paper, all operators are linear. Let X be a Banach space with norm $\|\cdot\|$. $B(X)$ denotes the set of all bounded linear operators from X into itself. If A is an operator, $\mathcal{D}(A)$ will be written for the domain of A, $\mathcal{R}(A)$ for its range, $\rho(A)$ for its resolvent set and $R(\lambda, A)(\lambda \in \rho(A))$ for its resolvent. For a polynomial $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha (\xi \in \mathbf{R}^n)$ with real constant coefficients, $P(A) = \sum_{|\alpha| \leq m} a_\alpha A^\alpha$ is defined with a maximal domain, where $A^\alpha = A_1^{\alpha_1} \cdots A_n^{\alpha_n}$ for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}_0^n = \mathbf{N} \cup \{0\}$. From [1,9], it is known that $P(A)$ is closable and cannot be directly defined with Statement 3, although it may be indirectly defined. Finally, $\mathcal{S}$ will denote the space of rapidly decreasing functions on $\mathbf{R}^n$.

Definition 1

Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous family in $B(X)$.

a) If there exists an injective C in $B(X)$ and constants $M, \omega \geq 0$ such that $2C(s)C(t) = C(s+t)C + C(|s-t|)C$ for $t, s \geq 0$, $C(0) = C$ and $\|C(t)\| \leq M e^{\omega t}$ for $t \geq 0$, then $\{C(t)\}_{t \geq 0}$ is called an exponentially bounded C-regularized cosine function (C-regularized cosine function, for short). Its generator is defined by $Ax = C^{-1} \lim_{h \to 0} 2h^{-2}(C(h)x - Cx)$ with maximal domain, i.e., $\mathcal{D}(A) = \{x \in X;$ the limit exists and is in $\mathcal{R}(C)\}$.

b) If there exists a linear operator A on X, $l \in \mathbf{N}_0^n$ and constants $M, \omega \geq 0$ such that:

$$R(\lambda^2, A)x = \lambda^{l-1} \int_0^\infty e^{-\lambda t} C(t)x\, dt \ ,$$

for all $x \in X$ and $\lambda > \omega$,

and $\|C(t)\| \leq M e^{\omega t} (t \geq 0)$, then $\{C(t)\}_{t \geq 0}$ is called an exponentially bounded l-times integrated cosine function (l-times integrated cosine function, for short) and A is its generator.

c) $\{C(t)\}_{t \geq 0}$ is said to be norm-continuous if $C(\cdot) \in C([0, \infty), B(X))$.

Basic material on C-regularized or l-times integrated cosine functions may be found in [4,11–15].

Remark 1

If A generates an r-times integrated cosine function $\{C_r(t)\}_{t \geq 0}$ for some $r \in \mathbf{N}_0^n$, then A generates an $l+r$-times integrated cosine function $\{C(t)\}_{t \geq 0}$ given by:

$$C(t) = \int_0^t \frac{(t-s)^{l-1}}{(l-1)!} C_r(t)\, dt \ , \quad \text{for } t \geq 0.$$

Definition 2

Let $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ $(\xi \in \mathbf{R}^n)$ with the principal part $P_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha$ $(\xi \in \mathbf{R}^n)$. Then, $P(\xi)$ is elliptic if $P_m(\xi) = 0$ which implies that $\xi = 0$.

Remark 2

If $P(\xi)$ is elliptic, then there exist constants $M, L > 0$ such that:

$$|P(\xi| \geq M|\xi|^m, \quad \text{for } \ |\xi| \geq L \ , \tag{4}$$

and m is even if $n \geq 2$ (see [16]).

Lemma 1

Let $C \in B(X)$ be injective and A a linear operator in X. Then A is the generator of a C-regularized cosine function $\{C(t)\}_{t \geq 0}$ if, and only if, the following conditions hold [11,15]:

a) A is closed and $A = C^{-1}AC$;

b) There exists a $\omega \geq 0$ such that $\lambda^2 - A$ is injective and $\mathcal{R}(C) \subset \mathcal{R}(\lambda^2 - A)$ for $\lambda > \omega$;

c) There exists an $M \geq 0$ such that $\|C(t)\| \leq M e^{\omega t}$ for $t \geq 0$ and:

$$R(\lambda^2, A)^{-1}Cx = \lambda^{-1} \int_0^\infty e^{-\lambda t} C(t)x\, dt \ ,$$

for $x \in X$ and $\lambda > \omega$. (5)

The following lemma will play an important role in the proofs presented here (see [1,5,9]).

Lemma 2

a) $f \to f(A)$ is an algebric homomorphism from $\mathcal{S}$ into $B(X)$ and there exists $M \geq 0$, such that:

$$\|f(A)\| \leq M\|\mathcal{F}f\|_{L^1(\mathbf{R}^n)}\,, \text{ for any } f \in \mathcal{S}\,.$$

b) Let $E = \{x(\phi) \equiv \int_{\mathbf{R}^n} \phi(s)e^{i(s,A)}xds, \phi \in \mathcal{S}$ and $x \in X\}$. Then, $E \subset \mathcal{D}(A^\infty) = \cap_{k\in N_0^n}\mathcal{D}(A^k)$, $\overline{E} = X$, $\overline{P(A)|_E} = \overline{P(A)}$ and:

$$\begin{cases} P(A)x(\phi) = x(P(iD)\phi)\,, \\ \quad \text{for } \phi \in \mathcal{S} \text{ and } x \in X\,, \\ P(A)x(\phi) = x(\phi)P(A)\,, \\ \quad \text{for } \phi \in \mathcal{S} \text{ and } x \in \mathcal{D}(P(A)). \end{cases} \tag{6}$$

c) Bernstein's theorem: Let $n/2 < j \in N$, then $H^j(\mathbf{R}^n) \hookrightarrow \mathcal{F}L^1$ and there exists a constant $M > 0$ such that:

$$\|\mathcal{F}f\|_{L^1} \leq M\,\|f\|_{L^2}^{1-\frac{n}{2j}} \sum_{|k|=j} \|D^k f\|_{L^2}^{n/2j}\,,$$

for $f \in H^j(\mathbf{R}^n)$.

Lemma 3

Let $\lambda \in \rho(A) \neq \emptyset$. Then, A is the generator of a $2n$-times integrated cosine function $C_{2n}(\cdot)$ if and only if, for $\lambda > \omega$, A is the generator of a $R(\lambda^2, A)^n$-regularized cosine function $C(\cdot)$. $C(\cdot)$ and $C_{2n}(\cdot)$ are related to each other in the following way: $C(t) = D^{2n}C_{2n}(t)R(\lambda^2, A)^n$ and

$$C_{2n}(t) = (\lambda^2 - A)^n \frac{1}{(2n-1)!}\int_0^t (t-u)^{2n-1}C(u)xdu$$

for all $x \in X$ and $t \geq 0$ (see [11,15]).

MAIN RESULTS

For convenience, in this section, M will denote a general constant independent of t and ξ. Now, the main results are presented.

Theorem 1

Let $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ be a real valued elliptic polynomial, $\omega = \sup\{P(\xi);\ \xi \in \mathbf{R}^n\} < \infty$, $\omega' > \omega$ and $l > n/4$ for $l \in \mathbf{N}$. Then, $\overline{P(A)}$ generates a norm-continuous C-regularized cosine function $\{C(t)\}_{t\geq 0}$ such that:

$$\|C(t)\| \leq M(1+t^n)e^{\omega_1 t}, \quad \text{for } t \geq 0, \tag{7}$$

where $C = (\omega' - P)^{-l}(A)$ and $\omega_1^2 = \max(0, \omega)$.

Proof

For fixed $\omega' > \omega$ and $l > n/4$ ($l \in \mathbf{N}$), since $P(\xi)$ is elliptic, there exist constants $M, L > 1$ such that:

$$|\omega' - P(\xi)| \geq M|\xi|^m, \quad \text{for } |\xi| \geq L. \tag{8}$$

Again, since $P(\xi)$ is real valued and bounded,

$$0 \leq Re\sqrt{P(\xi)} \leq \{\max(0,\omega)\}^{1/2},$$
$$\text{for all } \xi \in \mathbf{R}^n\,. \tag{9}$$

Let $f_t(\xi) = (\omega' - P(\xi))^{-l}\cosh(t\sqrt{P(\xi)})$ for $t \geq 0$ and $l \in \mathbf{N}$. Let $t \geq 0$ and $|k| \leq [\frac{n}{2}] + 1$ ($k \in \mathbf{N}_0^n$). By induction on $|k|$ and Statement 8, it can be seen that:

$$|D^k(\omega' - P(\xi))^{-l}| \leq M|\xi|^{-lm-|k|}\,, \tag{10}$$

for $|\xi| \geq L$ and:

$$|D^k U(t\sqrt{P(\xi)})| = \sum_{j=1}^{|k|} t^j U(t\sqrt{P(\xi)})Q_j$$
$$(\xi)(P(\xi))^{-|k|+j/2}, \quad \text{for } P(\xi) \neq 0, \tag{11}$$

where $Q_j(\xi)$ is a polynomial of degree $\leq (m-1)|k|$ and $U(\cdot)$ denotes the function $\cosh(\cdot)$ or $\sinh(\cdot)$. Thus by Remark 2 and Equation 11,

$$|D^k\cosh(t\sqrt{P(\xi)})| \leq M(1+t^{|k|})e^{\omega_1 t}|\xi|^{(m/2-1)|k|}, \tag{12}$$

for $t \geq 0$ and $|\xi| \geq L$. By Leibniz's formula,

$$D^k f_t(\xi) = \sum_{k_1+k_2=k} \binom{k}{k_1} D^{k_1}\left((\omega' - P(\xi))^{-l}\right)$$
$$D^{k_2}\left(\cosh(t\sqrt{P(\xi)})\right).$$

It follows from Statements 10 and 12 that:

$$|D^k f_t(\xi)| \leq M(1+t^{|k|})e^{\omega_1 t}|\xi|^{m(|k|/2-l)-|k|}\,,$$

for $|\xi| \geq L$ and $t \geq 0$.

Let $|k| \leq l$. Then, for $|\xi| \geq L$ and $t \geq 0$,

$$|D^k f_t(\xi)| \leq M(1+t^{|k|})e^{\omega_1 t}|\xi|^{-l}\,. \tag{13}$$

On the other hand, since $\cosh(t\sqrt{P}) = \sum_{j=0}^\infty t^{2j}P^j/(2j)!$ for $t \geq 0$, the following is obtained:

$$D^k\cosh(t\sqrt{P(\xi)}) = \sum_{j=0}^{|k|}\sum_{i=0}^{\infty} \frac{t^{2j}(i+j)!}{(2i+2j)!i!}\lambda^{2i}Q_j(\xi)\,, \tag{14}$$

for $t \geq 0$, $\xi \in \mathbf{R}^n$ and $k \in \mathbf{N}_0^n$, where $\lambda = t\sqrt{P(\xi)}$ and $Q_j(\xi)$ is a polynomial of degree$\leq \max(0, jm - |k|)$. Thus, if $|\lambda| \leq 1$ from Equation 14 and $|\lambda| > 1$ from Equation 11:

$$|D^k \cosh(t\sqrt{P(\xi)})| \leq M \sum_{j=0}^{|k|} t^{2j} \leq M(1 + t^{2|k|}), \tag{15}$$

for $|\xi| \leq L$ and $t \geq 0$. Note that $|\omega' - P(\xi)| > \omega' - \omega > 0$. Combining Statements 10 and 15, it is concluded that:

$$|D^k f_t(\xi)| \leq M(1 + t^{2|k|}),$$

$$\text{for } |\xi| \leq L \text{ and } t \geq 0. \tag{16}$$

Consequently, for all $|k| \leq l$, Statements 13 and 16 imply that:

$$\|D^k f_t\|_{L^2} \leq M(1 + t^{2|k|})e^{\omega_1 t} \quad \text{for } t \geq 0.$$

Then, by Bernstein's theorem (see Lemma 2(c)), it can be seen that $f_t(\xi) \in H^l(\mathbf{R}^n)$ and

$$\|\mathcal{F} f_t\|_{L_1} \leq M(1 + t^n)e^{\omega_1 t}, \quad \text{for } t \geq 0. \tag{17}$$

Now $C(t) = f_t(A)$ is defined for $t \geq 0$. Consequently, from Lemma 2(a) one realizes that $\|C(t)\| \leq M(1 + t^n)e^{\omega_1 t}$ for $t \geq 0$, $(C(t + s) + C(|t - s|))C = 2C(t)C(s)$ for $t, s \geq 0$ and $C(0) = C = (\omega' - P)^{-l}(A)$.

To see the norm continuity of $\{C(t)\}_{t \geq 0}$, for $t, t + h \geq 0$ and $|\xi| \geq L$, the following is derived from Statements 10 and 11:

$$|D^k(f_{t+h}(\xi) - f_t(\xi))| \leq M \sum_{j=0}^{|k|} |(t+h)^j U((t+h)$$

$$\sqrt{P(\xi)}) - t^j U(t\sqrt{P(\xi)})| \cdot |\xi|^{m(\frac{|k|}{2} - l) - |k|}$$

$$\to 0 \text{ as } h \to 0 \text{ for } |\xi| \geq L \text{ and } k \in \mathbf{N}_0^n.$$

Using Bernstein's theorem and Lebesgue's dominated convergence theorem, it is determind that $\lim_{h \to 0} \|\mathcal{F}(f_{t+h} - f_t)\|_{L^1} = 0$ and, therefore, the claim follows from Lemma 2(a).

It is now proven that $\overline{P(A)} = C^{-1}P(A)C$. For any $x(\phi) \in E$, using Statement 3 the following is obtained:

$$C(t)x(\phi) = \int_{\mathbf{R}^n} \mathcal{F}(f_t \mathcal{F}^{-1}\phi)(\xi) e^{i(\xi, A)} x d\xi$$

$$= x(\mathcal{F}(f_t \mathcal{F}^{-1}\phi)) \subseteq E. \tag{18}$$

Thus, by properties of the Fourier transform and Equation 6 it is obtained that:

$$\begin{aligned} P(A)C(t)x(\phi) &= x(P(iD)\mathcal{F}(f_t \mathcal{F}^{-1}\phi)) \\ &= x(\mathcal{F}(f_t \mathcal{F}^{-1}(P(iD)\phi))) \\ &= C(t)x(P(iD)\phi) \\ &= C(t)P(A)x(\phi), \end{aligned} \tag{19}$$

and:

$$P(A)C(t)x = C(t)P(A)x, \quad \text{for } x \in \mathcal{D}(P(A)), \tag{20}$$

which implies that $P(A) \subseteq C^{-1}P(A)C$ and therefore, $\overline{P(A)} \subseteq C^{-1}P(A)C$. On the other hand, from Statement 10 it can be easily proven that $\mathcal{F}(\omega' - P)^{-1} \in L^1$ for $\omega' > \omega$. Let $C_1 = (\omega' - P)^{-1}(A)$, it follows from Equation 6 that $(\omega' - P(A))C_1 x = C_1(\omega' - \overline{P(A)})x = x$ for $x \in E$. Thus, by Lemma 2(b) $\omega' \in \rho(\overline{P(A)})$ and $R(\omega', \overline{P(A)}) = C_1$. In fact, it has been proven that the resolvent set of $\overline{P(A)}$ is nonempty, if $P(\xi)$ is elliptic polynomial and so $C = (\omega' - \overline{P(A)})^{-l}$. It is immediate that $C^{-1}P(A)C \subseteq \overline{P(A)}$ and, therefore, the claim stated here holds.

Finally, it is shown that $\overline{P(A)}$ is the generator of $\{C(t)\}_{t \geq 0}$. To this end, $L_\lambda \in B(X)$ is defined by:

$$L_\lambda x = \lambda^{-1} \int_0^\infty e^{-\lambda t} C(t)x \, dt,$$

$$\text{for } x \in X \text{ and } \lambda > \omega. \tag{21}$$

Let $x(\phi) \in E$ and $x \in X$. Then, by Equations 18 and 21 and the fact that $\overline{P(A)}$ is closed, it can be seen that $L_\lambda x(\phi) \in \mathcal{D}(\overline{P(A)})$ and:

$$\overline{P(A)} L_\lambda x(\phi) = L_\lambda \overline{P(A)} x(\phi), \quad \text{for } \lambda > \omega. \tag{22}$$

Next, from Statements 13 and 16 it can be easily concluded that:

$$|D^k(f_t \mathcal{F}^{-1}\phi)(\xi)| \leq \sum_{j=0}^{|k|} |\phi_j(\xi)|(1 + t^{2j})e^{\omega_1 t},$$

$$\text{for } t \geq 0,$$

where $\phi_j \in \mathcal{S}(0 \leq j \leq |k|)$. Then, it is obtained that:

$$\|D^k(f_t \mathcal{F}^{-1}\phi)\|_{L^2} \leq M(1 + t^{2j})e^{\omega_1 t},$$

$$\text{for } t \geq 0,$$

which implies from Berstein's theorem that $\|\mathcal{F}(f_t \mathcal{F}^{-1}\phi)\|_{L^1} \leq M(1 + t^n)e^{\omega_1 t}$ for $t \geq 0$. A direct computation shows that $\|f_t \mathcal{F}^{-1}\phi\|_{L^1} \leq M e^{\omega_1 t}$, for

$t \geq 0$. Thus, combining Equations 19 and 21 and Fubini's theorem, the following equations are obtained:

$$L_\lambda x(\phi) = \lambda^{-1}\int_0^\infty e^{-\lambda t}(\int_{\mathbf{R}^n} \mathcal{F}(f_t\phi)e^{i(\xi,A)}x\,d\xi)dt$$
$$= \int_{\mathbf{R}^n} \mathcal{F}(\lambda^{-1}\int_0^\infty e^{-\lambda t} f_t dt \mathcal{F}^{-1}\phi)(\xi)e^{i(\xi,A)}x\,d\xi$$
$$= \{(\lambda^2 - P(\cdot))^{-1}(\omega' - P(\cdot))^{-l}\mathcal{F}^{-1}\phi\}(A)\ ,$$

which implies that $L_\lambda E \subset E$ $(\lambda > \omega)$. Note that $A_j (1 \leq j \leq n)$ is closed and commuting. Hence, for $\lambda > \omega$ and $x \in X$,

$$L_\lambda(\lambda^2 - P(A))x(\phi) = L_\lambda x((\lambda^2 - P(iD))\phi)$$
$$= \{(\lambda^2 - P(\cdot))^{-1}(\omega' - P(\cdot))^{-l}\mathcal{F}^{-1}$$
$$((\lambda^2 - P(iD))\phi)\}(A)$$
$$= \{(\omega' - P(\cdot))^{-l}\mathcal{F}^{-1}\phi\}(A) = Cx(\phi)\ . \tag{23}$$

Consequently, it follows from Equations 22 and 23 and Lemma 2(b) that:

$$\begin{cases} (\lambda^2 - \overline{P(A)})L_\lambda x = Cx\ , \\ \quad \text{for } x \in X \text{ and } \lambda > \omega; \\ L_\lambda(\lambda^2 - \overline{P(A)})x = Cx\ , \\ \quad \text{for } x \in \mathcal{D}(\overline{P(A)}) \text{ and } \lambda > \omega\ . \end{cases}$$

This implies that $L_\lambda x = R(\lambda^2, \overline{P(A)})Cx$ for $\lambda > \omega$ and $x \in X$. Thus, the claims herein follow from Lemma 1.

Combining Theorem 1 with Lemma 3, the following theorem is presented for integrated cosine functions.

Theorem 2

Let $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ be a real valued elliptic polynomial, $\omega = \sup\{P(\xi);\ \xi \in \mathbf{R}^n\} < \infty$ and $l > n/2$ for $l \in \mathbf{N}$. Then, $\overline{P(A)}$ generates a norm-continuous l-times integrated cosine function $\{C(t)\}_{t\geq 0}$ such that $\|C(t)\| \leq M(1 + t^{l+n})e^{\omega_1 t}$ for $t \geq 0$, where $\omega_1^2 = \max(0, \omega)$.

Furthermore, for the case of non-elliptic polynomial the following theorem is presented.

Theorem 3

Let $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ be a real valued polynomial, $\omega = \sup\{P(\xi);\ \xi \in \mathbf{R}^n\} < \infty$, and $l > n/2$ for $l \in \mathbf{N}$. Then, $\overline{P(A)}$ generates a norm-continuous C-regularized cosine function $\{C(t)\}_{t\geq 0}$ such that $\|C(t)\| \leq M(1 + t^n)e^{\omega_1 t}$ for $t \geq 0$, where $C = (1 + |A|^2)^{-ml/2}$ is defined as the fractional power.

Proof

Let $g_t(\xi) = (1 + |\xi|^2)^{-ml/2}\cosh(t\sqrt{P(\xi)})$ for $l > n/2, t \geq 0$ and $\xi \in \mathbf{R}^n$. Then, for $k \in \mathbf{N}_0^n$, a direct calculation shows that:

$$|D^k(1+|\xi|^2)^{-ml/2}| \leq M|\xi|^{-|k|-ml}, \quad \text{for } |\xi| \geq L. \tag{24}$$

Thus, by Statements 12 and 24 and Leibniz's formula the following is obtained:

$$|D^k g_t(\xi)| \leq M(1 + t^{|k|})e^{\omega_1 t}|\xi|^{m(\frac{|k|}{2} - l) - |k|},$$
$$\text{for } |\xi| \geq L \text{ and } t \geq 0.$$

The rest of the proof may be carried out as in the corresponding part of Theorem 1.

In the following, some applications of the presented results are given in concrete spaces. Let X be one of the following spaces $L^p(\mathbf{R}^n)(1 \leq p < \infty)$, $C_0(\mathbf{R}^n) = \{f \in C(\mathbf{R}^n); lim_{x\to\infty} f(x) = 0\}$ and $BUC(\mathbf{R}^n) = \{f \in C(\mathbf{R}^n); f$ is bounded and uniformly continuous $\}$ with the sup-norm. The differential operator $\sum_{|\alpha \leq m} a_\alpha D^\alpha$ is denoted by $P(D)$ associated with the constant coefficient symbol $P(i\xi) = \sum_{|\alpha \leq m} a_\alpha (i\xi)^\alpha$ and the set:

$$\begin{cases} \mathcal{D}(P(D)) = \{f \in X;\ \mathcal{F}^{-1}(P\mathcal{F}f) \in X\}, \\ P(D)f = \mathcal{F}^{-1}(P\mathcal{F}f)\ , \quad \text{for } f \in \mathcal{D}(P(D)). \end{cases}$$

Obviously, $P(D)$ is closed and densely defined on these spaces. In addition, it is known that $iD_j = \partial/\partial x_j (1 \leq j \leq n)$ (with distributional domain) is the generator of the translation group:

$$(T_j(t)f)(x_1, \ldots, x_j, \ldots, x_n) =$$
$$f(x_1, \ldots, x_{j-1}, x_j + t, x_{j+1}, \ldots, x_n)\ ,$$
$$\text{for } t \in \mathbf{R}\ ,$$

on X. This group is strongly continuous for $p < \infty$. All results in Theorems 1-3 are applied to $P(D)$ on X.

Theorem 4

Let $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ be a real valued polynomial, $\omega \equiv \sup\{P(\xi); \xi \in \mathbf{R}^n\} < \infty$, $\omega' > \omega$, $\omega_1^2 = \max(0, \omega)$, $n_X = n|\frac{1}{2} - \frac{1}{p}|$ for $X = L^p(\mathbf{R}^n)$ $(1 \leq p < \infty)$ and $n_X = n/2$ for other X. Then, the following assertions hold.

a) If $P(\xi)$ is elliptic and $l > n_X/2$ for $l \in \mathbf{N}$, then $P(D)$ generates a norm-continuous C-regularized cosine function $\{C(t)\}_{t\geq 0}$ on X, such that $\|C(t)\| \leq M(1 + t^{n_X})e^{\omega_1 t}$ for $t \geq 0$.

b) If $P(\xi)$ is elliptic and $l > n_X$ for $l \in \mathbf{N}$, then $P(D)$ generates a norm-continuous l-times integrated cosine function $\{C(t)\}_{t\geq 0}$ on X, such that $\|C(t)\| \leq M(1+t^{l+n_X})e^{\omega_1 t}$ for $t \geq 0$.

c) If $l > n_X$, then $P(D)$ generates a norm-continuous C-regularized cosine function $\{C(t)\}_{t\geq 0}$ on X, such that $\|C(t)\| \leq M(1+t^{n_X})e^{\omega_1 t}$ for $t \geq 0$, where $C = (1-\Delta)^{-ml/2}$.

Proof

Note that $\rho(P(D)) \neq \emptyset$ for the elliptic polynomial $P(\xi)$. Then, the assertions (a) and (c) follow Theorems 1-3 on $X = L^1(\mathbf{R}^n), C_0(\mathbf{R}^n)$ and $BUC(\mathbf{R}^n)$. If $X = L^p$ $(1 < p < \infty)$, using the Riesz-Thorin convexity theorem and a similar method in [5,10], it may be proven that the claims stated hold. Here the detailed proofs are omitted.

Remark 3

In the case of integrated cosine functions, Theorem 4(b) is well-known for $X = L^p(\mathbf{R}^n)(1 < p < \infty)$ (see Theorem 6.5 in [4]). However, Theorem 6.6 in [4] required $m > \frac{n}{2}$ and $l > \frac{n}{2} + 2$ $(l \in \mathbf{N})$ on $L^1(\mathbf{R}^n), C_0(\mathbf{R}^n)$ and $BUC(\mathbf{R}^n)$. Theorem 6.5 in [5] supposes that $P(\xi) = -a^2(\xi)$, where $a(\xi)$ is a real valued elliptic polynomial on $\mathbf{R}^n$. Obviously, Theorem 4(b) has improved the results of [4,5]. Furthermore, other results in Theorem 4 are new. In particular, the Laplace operators Δ generate an α-times integrated cosine function on $L^p(\mathbf{R}^n)(1 < p < \infty)$ or $C_0(\mathbf{R}^n)$, $BUC(\mathbf{R}^n)$ for $\alpha \geq (n-1)|\frac{1}{p} - \frac{1}{2}|$ or $\alpha \geq (n-1)/2$ $(\alpha \in R^+)$ (see [5,17]).

Theorem 3 may be immediately applied to the second order inhomogeneous Cauchy problem:

$$\begin{cases} u_{tt}(t) = \overline{P(A)}u(t) + g(t)\ , & \text{for } t \geq 0 \\ u(0) = x\ , \quad u_t(0) = y\ , \end{cases} \tag{25}$$

where $g \in C(R^+, X)$ and $R^+ = [0, \infty)$.

Let Y_s $(s \geq 0)$ denote $\mathcal{D}((1+|A|^2)^{s/2})$, a Banach space with the graph norm $\|\cdot\|_s = \|(1+|A|^2)^{-s/2}\cdot\|$. For a regularized cosine function $\{C(t)\}_{t\geq 0}$, $S(t)x = \int_0^t C(s)x\,ds$ for $t \geq 0$ and $x \in X$. From Theorem 3.1 in [12], it is known that the connection between regularized cosine functions and Equation 25 is given by the following fact: Let $\overline{P(A)}$ be the generator of a C-regularized cosine function $\{C(t)\}_{t\geq 0}$. Then, for a given $x, y \in X$, $v(t) \in \mathcal{R}(C)$ $(t \geq 0)$ and $C^{-1}v(\cdot) \in C^2(R^+, X)$, where:

$$v(t) = C(t)x + S(t)y + \int_0^t S(t-s)g(s)\,ds,$$

for $t \geq 0$,

$u(t) \equiv C^{-1}v(t)$ is the unique solution of Equation 25. Thus, from Theorem 3, the following result is obtained.

Theorem 5

Let $P(\xi)$ be as in Theorem 3. If $l \in \mathbf{N}, l > n_X$ and $g \in C(R^+, Y_{m(l+1/2)})$, then for every pair $(x,y) \in Y_{m(l+1)} \times Y_{m(l+1/2)}$, Equation 25 has a unique solution $u \in C^2(R^+, X)$ such that:

$$\|u\| \leq M(1+t^{n_X})e^{\omega_1 t}(\|x\|_{m(l+1)} + t\|y\|_{m(l+1)}$$
$$+ t^2 \sup_{0\leq s\leq t} \|g(s)\|_{m(l+1)}).$$

Proof

Let $\{C(t)\}_{t\geq 0}$ be the C-regularized cosine function generated by $\overline{P(A)}$ in Theorem 3, $C = (1+|A|^2)^{-ml/2}$ and $\|C(t)\| \leq M(1+t^{n_X})e^{\omega_1 t}$ for $t \geq 0$.

For $t \in R^+$,

$$u(t) = C(t)(1+|A|^2)^{ml/2}x + S(t)(1+|A|^2)^{ml/2}y$$
$$+ \int_0^t S(t-s)(1+|A|^2)^{ml/2}g(s)\,ds\ . \tag{26}$$

Using similar proof to that of Theorem 4.6 in [9], it is obtained that $u \in C^2(R^+, X)$, therefore, Equation 25 holds. To prove the uniqueness, let $v \in C^2(R^+, X)$ be any solution of Equation 25. Then, it follows that:

$$\frac{d}{ds}S(t-s)v'(s) = -C(t-s)v'(s)$$
$$+ S(t-s)\left(\overline{P(A)}v(s) + f(s)\right)$$

$$\frac{d}{ds}C(t-s)v(s) = -S(t-s)\overline{P(A)}v(s)$$
$$+ C(t-s)v'(s)\ .$$

Thus, $\frac{d}{ds}(S(t-s)v'(s) + C(t-s)v(s)) = S(t-s)f(s)$. Integrating this equation from 0 to t, the following statement is obtained:

$$Cv(s) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)\,ds\ , \tag{27}$$

where $C = (1+|A|^2)^{-ml/2}$ as fractional powers is injective. It follows from Equations 26 and 27 that $u(t) \equiv v(t)$ for $t \in R^+$.

Let $W^{\alpha,X}(\mathbf{R}^n)$ $(\alpha \geq 0)$ be the completion of $\mathcal{S}$ under the norm $\|u\|_{\alpha,X} \equiv \|u\|_X + \|\mathcal{F}^{-1}((1+|\xi|^2)^{\alpha/2}\mathcal{F}u)\|_X$ for $f \in \mathcal{S}$. When $X = L^p$ $(1 \leq p < \infty)$, $W^{\alpha,p} \equiv W^{\alpha,X}$ is called Bessel-potential space. In particular, $W^{\alpha,X}$ is the usual Sobolev space if $\alpha \in \mathbf{N}_0$ and $1 < p < \infty$ (see [10]).

Naturally, Theorem 5 can be applied to the following initial value problem:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \sum_{|\alpha|\le m} a_\alpha D^\alpha u(t,x) + f(t,x) \ , \\ \quad \text{for } (t,x) \in R^+ \times \mathbf{R}^n \ , \\ u(0,x) = \varphi(x) \ , \quad u_t(0,x) = \psi(x) \ , \\ \quad \text{for } x \in \mathbf{R}^n \ , \end{cases} \tag{28}$$

on the concrete space X.

Theorem 6

Suppose $\omega \equiv \{P(\xi); \xi \in \mathbf{R}^n\} < \infty$ and $l > n_X$. If $g \in C(R^+, W^{m(l+1/2),X})$, then for every pair $(x,y) \in W^{m(l+1),X} \times W^{m(l+1/2),X}$, Equation 28 has a unique solution, $u \in C^2(R^+, X)$, such that:

$$\|u(t,\cdot)\| \le M(1+t^{n_X})e^{\omega_1 t}(\|\varphi\|_{m(l+1),X} + t\|\psi\|_{m(l+1),X} + t^2 \sup_{0\le s\le t} \|g(s)\|_{m(l+1),X}).$$

Example 1

Consider Klein-Gordon equation (cf. [18, P_{132}]),

$$\begin{cases} u_{tt} = a\Delta u - bu \qquad (a,b>0) \\ u(0,x) = \phi \ , \quad u_t(0,x) = \psi \end{cases} \tag{29}$$

on $L^p(\mathbf{R}^n)$. Then, by Theorem 6, for every pair $(\varphi,\psi) \in W^{\alpha,p}(\mathbf{R}^3) \times W^{\alpha,p}(\mathbf{R}^3)$, where $\alpha > 2n_X$, Equation 29 has a unique solution, $u \in C^2(R^+, L^p(\mathbf{R}^3)) \cap C(R^+, W^{2,p}(\mathbf{R}^3))$.

ACKNOWLEDGMENT

This project is supported by the Natural Science Foundation of China. The author would like to thank Professor Ralph deLaubenfels for his kind help.

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