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Galerkin Approximations for a Semilinear Stochastic Integral Equation

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In this paper, the Galerkin method is used to approximate the solution of the H-valued integral equation:

$$X_t = \int_{-\infty}^{t} U(t-s) f_s(X_s) ds + V_t,$$

where H is a real separable Hilbert space. $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A, such that A^{-1} is compact and f is of monotone type and is bounded by a polynomial. Furthermore, V_t is a cadlag adapted process.

INTRODUCTION

Let H be a real separable Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete stochastic basis with a right continuous filtration and $\{W_t, t \geq 0\}$ is an H-valued cylindrical Brownian motion with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Consider the stochastic semilinear equation:

$$dX_t = AX_t dt + f_t(X_t)\, dt + dW_t \ , \tag{1}$$

where A is a closed, self-adjoint, negative definite, unbounded operator such that A^{-1} is nuclear. A mild solution of Equation 1 with initial condition, $X(0) = X_0$, is the solution of the integral equation:

$$X_t = U(t,0)X_0 + \int_0^t U(t-s) f_s(X_s) ds + \int_0^t U(t-s) dW_s \ , \tag{2}$$

where $U(t)$ is the semigroup generated by A.

Marcus [1] has proved that when f is independent of t as well as ω and uniformly Lipschitz, then the solution of Equation 2 is asymptotically stationary. To prove this, Marcus studied the following integral equation:

$$X_t = \int_{-\infty}^t U(t-s) f_s(X_s) ds + \int_{-\infty}^t U(t-s) dW_s, \tag{3}$$

where the parameter set of the processes is extended to the whole real line. This gave the motivation for studying the existence of the solution of a more general equation:

$$X_t = \int_{-\infty}^t U(t-s) f_s(X_s) ds + V_t, \tag{4}$$

where $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A such that A^{-1} is compact, f is of monotone type, bounded by a polynomial and V_t is a cadlag adapted process. In [2], the existence and the uniqueness of the solution to Equation 4 is proved. In this paper it is proven that finite dimensional Galerkin approximations converge strongly to the solution of Equation 4. In [3] this result is used to prove the stationarity of the solution of Equation 4. Results of this paper are presented in [4] without proof.

PRELIMINARIES

A Semilinear Evolution Equation

Let g be an H-valued function defined on a set $D(g) \subset H$. Recall that g is monotone if, for each pair, $x, y \in D(g)$,

$$\langle g(x) - g(y), x - y \rangle \ \geq 0 \ .$$

We say g is bounded if there exists an increasing continuous function ψ on $[0, \infty)$ such that $\|g(x)\| \leq \psi(\|x\|), \forall x \in D(g)$. g is demi-continuous if, whenever

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(x_n) is a sequence in $D(g)$ which converges strongly to a point $x \in D(g)$, then $g(x_n)$ converges weakly to $g(x)$.

Consider the following integral equation:

$$X_t = \int_0^t U(t-s)f(X_s)ds + V_t \ , \qquad (5)$$

where f, V and the generator A of the semigroup U satisfy the following hypothesis.

***Hypothesis* 1**

a) $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A such that A^{-1} is compact. Then there is $\lambda > 0$ such that $\|U(t)\| \leq e^{-\lambda t}$.

b) Let $\varphi(t) = K(1+t^p)$ for some $p > 0$, $K > 0$. $-f$ is a monotone demi-continuous mapping from H to H such that $\|f(x)\| \leq \varphi(\|x\|)$ for all $x \in H$.

c) Let $r = 2p^2$. V_t is cadlag adapted process such that $\sup_{t \in R} E\{\|V_t\|^r\} < \infty$.

***Proposition* 1**

Suppose that f , V, A and U satisfy Hypothesis 1. Then Equation 5 has a unique adapted cadlag (continuous, if V_t is continuous) solution. Furthermore,

$$|X(t)\| \leq \|V(t)\| + \int_0^t e^{(\lambda)(t-s)}\|f(s, V(s))\|ds \ . \qquad (6)$$

For the proof see [5].

The Stability of the Solution

***Proposition* 2**

Let f^1 and f^2 be two mappings satisfying Hypothesis 1 bounded by functions φ_1 and φ_2 respectively.

Suppose V^1, V^2, U^1 and U^2 satisfy Hypothesis 1. Let $X^i(t)$, $i = 1, 2$ be the solution of the integral equations:

$$X^i(t) = \int_0^t U^i(t-s)f^i_s(X^i(s))ds + V^i(t). \qquad (7)$$

Define $v^3(t)$ and $\bar{I}$ as:

$$V^3(t) := \int_0^t (U^2(t-s) - U^1(t-s))f^2(X^1(s))ds,$$

$$\bar{I} := 4\int_0^T e^{-2\lambda s}\|f^2(X^2(s)) - f^1(X^1(s))\|^2 ds \ .$$

Then the following is obtained:

$$\begin{aligned}\|X^2(t) - X^1(t)\|^2 &\leq 4\|V^2(t) - V^1(t)\|^2 \\ &+ 4\|V^3(t)\|^2 \\ &+ \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^3(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds.\end{aligned} \qquad (8)$$

For proof see [6].

EXAMPLES

A Semilinear Stochastic Evolution Equation

The existence and uniqueness of the solution of the integral Equation 2 have been studied in [7]. Marcus assumed that f is independent of $\omega \in \Omega$ and $t \in S$ and that there are $M > 0$, and $p \geq 1$ for which:

$$\langle f(u) - f(v), u - v \rangle \leq -M\|u - v\|^p \ ,$$

and:

$$\|f(u)\| \leq C(1 + \|u\|^{p-1}).$$

Marcus proved that this integral equation has a unique solution in $L^p(\Omega, L^p(S, H))$.

As a consequence of Proposition 1, this result can be extended to a more general f and the existence of a strong solution of Equation 2 which is continuous instead of merely being in $L^p(\Omega, L^p(S, H))$ can be shown.

The Ornstein-Uhlenbeck process $V_t = \int_0^t U(t-s)dW(s)$ has been well-studied e.g., in [8] where they show that V_t has a continuous version. Therefore, Equation 2 can be rewritten as:

$$X_t = \int_0^t U(t-s)f_s(X_s)ds + V_t,$$

where V_t is an adapted continuous process. Then, by Proposition 1, the Equation 2 has a unique continuous adapted solution.

A Semilinear Stochastic Partial Differential Equation

Let D be a bounded domain with a smooth boundary in R^d. Let $-A$ be a uniformly strongly elliptic second order differential operator with smooth coefficients on D. Let B be the operator $B = d(x)D_N + e(x)$, where D_N is the normal derivative on ∂D, and d and e are in

$C^\infty(\partial D)$. Let A (with the boundary condition $Bf \equiv 0$) be self-adjoint.

Consider the initial-boundary-value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f_t(u) + \dot{W} & \text{on} \quad D \times [0,\infty), \\ Bu = 0 & \text{on} \quad \partial D \times [0,\infty), \\ u(0,x) = 0 & \text{on} \quad D, \end{cases} \tag{9}$$

where $\dot{W} = \dot{W}(t,x)$ is a white noise in space–time for the definition and properties of white noise (see [9]), and f_t is a non-linear function that will be defined below. Let $p > \frac{d}{2}$. W can be considered as a Brownian motion $\tilde{W}_t$ on the Sobolev space H_{-p} (see [9] Chapter 4, page 4.11). There is a complete orthonormal basis $\{e_k\}$ for H_p.

The operator A (plus boundary conditions) has eigenvalues $\{\lambda_k\}$ with respect to $\{e_k\}$, i.e., $Ae_k = \lambda_k e_k$, $\forall k$. The eigenvalues satisfy $\Sigma_j(1+\lambda_j^{-p}) < \infty$ if $p > \frac{d}{2}$ (see [9] Chapter 4, page 4.9) Then $[A^{-1}]^p$ is nuclear and $-A$ generates a contraction semigroup $U(t) \equiv e^{-tA}$. This semigroup satisfies Hypothesis 1.

Now consider the initial-boundary-value Problem 9 as a semilinear stochastic evolution equation:

$$du_t + Au_t dt = f_t(u_t)dt + d\tilde{W}_t \ , \tag{10}$$

with initial condition $u(0) = 0$, where $f : S \times \Omega \times H_{-p} \to H_{-p}$ satisfies Hypothesis 1(b) relative to the separable Hilbert space $H = H_{-p}$. The mild solution of Equation 10 (which is also a mild solution of Problem 9) can be defined, to be the solution of:

$$u_t = \int_0^t U(t-s)f_s(u_s)ds + \int_0^t U(t-s)d\tilde{W}_s. \tag{11}$$

Since $\tilde{W}_t$ is a continuous local martingale on the separable Hilbert space H_{-p}, then $\int_0^t U(t-s)d\tilde{W}_s$ has an adapted continuous version (see for example [10]). If the following is defined

$$V_t := \int_0^t U(t-s)d\tilde{W}_s,$$

then by Proposition 1, Equation 11 has a unique continuous solution with values in H_{-p}.

A SEMILINEAR INTEGRAL EQUATION ON THE WHOLE REAL LINE

Let us reduce the integral Equation 4 to the following integral equation:

$$X_t = \int_{-\infty}^t U(t-s)f(X_s + V_s)ds \ . \tag{12}$$

The following theorem translates Proposition 1 to the case when parameter set of the process is the whole real line.

Theorem 1

If A, f and V satisfy Hypothesis 1, then the integral Equation 12 has a unique continuous solution X such that:

$$\|X_t\| \le \int_{-\infty}^t e^{-\lambda(t-s)}\varphi(\|V_s\|)ds \ , \tag{13}$$

$$E\{\|X_t\|\} \le \frac{1}{\lambda}\sup_{s\in R} E\{\varphi(\|V_s\|)\} := K_1 \ . \tag{14}$$

For proof see [2].

Galerkin Approximations

Let $U(t)$ be a semigroup generated by a strictly negative definite closed unbounded self-adjoint operator A such that A^{-1} is compact. Then there is a complete orthonormal basis (ϕ_n) and eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with $\lambda_n \to \infty$, such that $A\phi_n = -\lambda_n\phi_n$.

Let H_n be the subspace of H generated by $\{\phi_0, \phi_1, \ldots, \phi_{n-1}\}$ and let J_n be the projection operator on H_n.

Define:

$$f_n = J_n f, \quad V_n(t) = J_n V(t),$$

$$U_n(t) = J_n V(t) J_n \ ,$$

and define $X_n(t)$ and $X(t)$ as solutions of:

$$X_n(t) = \int_{-\infty}^t U_n(t-s)f_n(X_n(s))ds + V_n(t) \ , \tag{15}$$

and:

$$X(t) = \int_{-\infty}^t U(t-s)f(X(s))ds + V(t) \ . \tag{16}$$

Now the following theorem can be proved.

Theorem 2

If A, U, f and V satisfy Hypothesis 1, then one has:

$$E(\|X_n(t) - X(t)\|) \to 0.$$

Proof

Define:

$$X_n^k(t) = \int_{-k}^t U_n(t-s)f_n(X_n^k(s))ds + V_n(t),$$

$$X^k(t) = \int_{-k}^t U(t-s)f(X^k(s))ds + V(t),$$

and:

$$\bar{V}_{n,k}(t) = \int_{-k}^t (U_n(t-s) - U(t-s)f(X^k(s))ds.$$

By Proposition 2, the following is obtained:

$$\|X_n^k(t) - X^k(t)\|^2 \leq 4\|V_n(t) - V(t)\|^2 + 4\|\bar{V}_{n,k}(t)\|^2 + \bar{I}\left(\int_{-k}^{t} e^{2\lambda_0 s}\|V_n(s) - V(s)\|^2 ds\right)^{\frac{1}{2}} + \bar{I}\left(\int_{-k}^{t} e^{2\lambda_0 s}\|\bar{V}_{n,k}(s)\|^2 ds\right)^{\frac{1}{2}} + \int_{-k}^{t} e^{2\lambda_0 s}\|f_n(X(s)) - f(X(s))\|^2 ds. \quad (17)$$

Taking expectations and using the Schwartz inequality and Fubini's theorem, Statement 17 implies that:

$$E\{\|X_n^k(t)-X^k(t)\|^2\} \leq 4E\{\|V_n(t)-V(t)\|^2\} + 4E\{\|\bar{V}_{n,k}(t)\|^2\} + (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^{t} e^{2\lambda_0 s}E(\|V_n(s)-V(s)\|^2)ds\right)^{\frac{1}{2}} + (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^{t} e^{2\lambda_0 s}E(\|\bar{V}_{n,k}(s)\|^2)ds\right)^{\frac{1}{2}} + \int_{-\infty}^{t} e^{2\lambda_0 s}E(\|f_n(X(s))-f(X(s))\|^2)ds. \quad (18)$$

It is first shown that:

$$E\{\|X_n^k(t) - X^k(t)\|^2\} \to 0 \quad \text{uniformly in } k . \quad (19)$$

Since $V_n = J_n V$ and $f_n = J_n f$, the first, third and 5th term of the right hand side of Statement 18 converge to zero. Then to prove Statement 19 it is enough to show that $E(\|\bar{V}_{n,k}(t)\|^2)$ converges to zero uniformly in k and $t \in (-\infty, T]$.

By using $\|f(x)\| \leq C(1 + \|x\|^p)$ and Statement 6, it is shown that:

$$\sup_{t\in R} E(\|V(t)\|^{2p}) < \infty ,$$

and, using Fubini's theorem, one has:

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \leq \sup_{t\in R} E\{1 + \|V(t)\|^p\} \int_{-\infty}^{0} \|U(-s) - U_n(-s)\|_L^2 ds .$$

Since:

$$\|U(-s) - U_n(-s)\|_L \to 0 \text{ for } s < 0 \text{ and}$$

$$\|U(-s) - U_n(-s)\|_L \leq e^{2\lambda_0 s} ,$$

then by the dominated convergence theorem:

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \to 0 \quad \text{uniformly in } k .$$

Then:

$$E(\|X_{n,k}(t) - X^k(t)\|^2) \to 0 \text{ uniformly in } k .$$

By the proof of Theorem 1 (see [2]), then $E(\|X_{n,k}(t) - X_n(t)\|) \to 0$ as $k \to \infty$, hence $E(\|X^k(t) - X(t)\|) \to 0$ and it is obtained that $E(\|X_n(t) - X(t)\|) \to 0$. Q.E.D.

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