Scientia Iranica, Vol. 4, Nos. 1&2, pp 1–7

SCIENTIA
IRANICA

Periodic Solution of a Certain Class of Nonlinear Fourth Order Differential Equation

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In this paper the fourth order differential equation is considered:

$$x^{(4)} + f_3(x'')x''' + f_2(x')x'' + f_1(x)x' + g(t,x,x',x',x''') = p(t) \ , \qquad (1)$$

where f_i is continuous and the functions g and p are continuous and ω-periodic in t. Here Leray-Schauder principle, as suggested by Güssefeldt [1], is used to prove the existence of nontrivial periodic solutions of Equation 1.

INTRODUCTION

In [2,3], Ezeilo considered the following differential equation:

$$x''' + \psi(x')x'' + \phi(x)x' + \theta(x,x',x'') = p(t) \ ,$$

where, under certain conditions on the nonlinear terms ψ, φ_1 and θ, Ezeilo proved the ultimate boundedness of the solutions. Later Reissig [4] proved that these conditions were also sufficient to show the oscillatory character of these solutions provided p is periodic. Here, under similar conditions, the existance of an ω-periodic solution of Equation 1 is proved.

Furthermore, the differential Equation 1 is studied and the following equations are defined:

$$P(t) = \int_0^t p(s)ds \ , \ F_i(x_i) = \int_0^{x_1} f_i(\xi)d\xi \ ,$$

$$i = 1,2,3.$$

THEOREM 1

Assume:

i) $F_3(x_3)\,\mathrm{sgn}\,x_3 \to +\infty$ as $|x_3| \to \infty$,

ii) $|g(t,x_1,x_2,x_3,x_4)| \leq K$ for all the values of independent variables and:

$$g(t,x_1,x_2,x_3,x_4)\,\mathrm{sgn}\,x_1 \geq 0 \text{ for } |x_1| \geq h,$$

iii) $|F_2(x_2) - bx_2| \leq M$ ($b > 0, a$ constant),

iv) $|P(t)| \leq m$, for all t ($\int_0^\omega p(t)dt = 0$).

Then, there exists at least one ω-periodic solution for Equation 1.

Proof

Using the following notations, let $\mu \in [0,1]$

$$f_i(x_i,\mu) = \frac{1}{2-\mu}f_i(x_i) - \frac{1-\mu}{2-\mu}f_i(-x_i),$$

$$i = 1,2,3$$

$$F_i(x_i,\mu) = \int_0^{x_i} f_i(\xi,\mu)d\xi$$

$$= \frac{1}{2-\mu}F_i(x_i) - \frac{1-\mu}{2-\mu}F_i(-x_i),$$

$$i = 1,2,3 \ .$$

$$g(t,x_1,x_2,x_3,x_4,\mu) = \mu g(t,x_1,x_2,x_3,x_4)$$

$$+ (1-\mu)K\frac{x_1}{1+|x_1|} \ .$$

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It follows that:

$$F_i(x_i,0)=\frac{1}{2}[F_i(x_i)-F_i(-x_i)]=-F(-x_i,0)$$

$$F_i(x_i,1)=F_i(x_i),\quad i=1,2,3$$

$$g(t,x_1,x_2,x_3,x_4,0)=K\frac{x_1}{1+|x_1|}\ \text{(odd)}$$

Evidently the following relations hold:

i)′ $F_3(x_3,\mu)\operatorname{sgn} x_3\to\infty$ as $|x_3|\to\infty$ uniformly with respect to $\mu\in[0,1]$,

ii)′ $|g(t,x_1,x_2,x_3,x_4,\mu)|\le\mu|g(t,x_1,x_2,x_3,x_4)| + x(1-\mu)\le K,\mu\in[0,1]$,

$$g(t,x_1,x_2,x_3,\mu)\operatorname{sgn} x_1\ge(1-\mu)K\frac{|x_1|}{1+|x_1|} > 0,\quad |x_1|\ge h,\mu\in[0,1]\ .$$

iii)′ $|F_2(x_2,\mu)-bx_2|=$

$$\left|\frac{F_2(x_2)}{2-\mu}-\frac{1-\mu}{2-\mu}F_2(-x_2) -\frac{bx_2}{2-\mu}+b\frac{1-\mu}{2-\mu}(-x_2)\right| \le\left|\frac{F_2(x_2)-bx_2}{2-\mu}\right| +\frac{1-\mu}{2-\mu}\left|F_2(-x_2)-b(-x_2)\right|\le M\ .$$

Next, the following first order system depending on the parameter μ is considered.

$$\begin{aligned} x_1'&=x_2\ ,\\ x_2'&=x_3\ ,\\ x_3'&=x_4+[bx_2-F_2(x_2,\mu)]\\ &\quad -F_1(x_1,\mu)-F_3(x_3,\mu)+\mu p(t)\ ,\\ x_4'&=-cx_1-bx_3+[cx_1-g(t,x_1,x_2,x_3,x_4,\mu)], \end{aligned}\tag{2}$$

where the constants c and b are such that $c>\frac{b^2}{4}$. In the vector form of:

$$x'=Ax+f(t,x,\mu)\ ,\quad A=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-c&0&-b&0\end{pmatrix},$$

$$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\quad f=\begin{pmatrix}0\\0\\bx_2-F_2-F_1-F_3+\mu p\\cx_1-g\end{pmatrix}.\tag{3}$$

Since the characteristic equation:

$$\det(\lambda I-A)=\lambda^4+b\lambda^2+c=0\ ,$$

has no purely imaginary roots, the homogeneous linear system:

$$x'=Ax\ ,\tag{4}$$

admits only the trivial ω-periodic solution $x(t)\equiv 0$ [5].

System of Equation 2 (or 3) is equivalent to the fourth order differential equation:

$$x^{(4)}+f_3(x'',\mu)x'''+f_2(x',\mu)x''+f_1(x,\mu)x' +g(t,x,x',x'',x''',\mu)=\mu p(t),\tag{5}$$

which is identical to Equation 1 for $\mu=1$.

For $\mu=0$, the nonlinear terms F_1,F_2,F_3 and g are odd with respect to x_1,x_2,x_3, that is:

$$f(t,-x,0)=-f(t,x,0)\ ,$$

where $f(t,x,\mu)$ is the nonlinear perturbation term in Equation 3. This term is a continuous function of:

$$(t,x,\mu)\in[0,\omega]\times\mathbb{R}^4\times[0,1]\ .$$

For $t\in[0,\omega]$ the ω-periodic solution $x(t)$ of Equation 3 can be represented as the continuous solution of the integral equation:

$$x(t)=T(x(t),\mu)=\int_0^\omega G(t-\tau)\,f(\tau,x(\tau),\mu)\,dt\ ,\tag{6}$$

or shortly as:

$$V_\mu(x)=x(t)-T(x(t),\mu)=0\ .$$

The Green matrix $G(s), s\in[-\omega,\omega]$, is continuous for $s\ne 0$ and it fulfils the jump condition:

$$G(0^+)-G(0^-)=E\ .$$

Let B be Banach space, $B=\{x(t)\in C[0,\omega]: x(0)=x(\omega)\}$ normed by $\|x\|=\sup|x(t)|$, $t\in[0,\omega]$. Then the operator T generates a continuous and compact mapping of the normed product space $B\times[0,1]$ into the Banach space B. The periodic solution of Equation 3 correspond to the fixed points of T or to the null vector of V_μ.

Let $S_R=\{x(t):\ \|x\|=R\}$, $B_R=\{x(t):\ \|x\|<R\}$. On the sphere S_R, the vector field V_μ ($V_\mu x = x-T(x,\mu)$) is studied for every fixed $\mu\in[0,1]$. If there is no null vector on S_R, the vector fields V_0 and V_1 are called homotopic. The rotations of homotopic vector fields are identical. Since V_0 is an odd vector

field, its rotation is an odd number. The rotation of V_0 on the sphere S_R being different from zero, the ball B_R contains at least one null vector of V_0. This result is also true of all vector fields $V_\mu(\mu \in [0,1])$. Consequently for every μ, the system of Equation 3 admits at least one ω-periodic solution $x(t)$ with norm $\|x\| < R$ [6,7].

If it could be shown that the solutions of integral Equation 6 are a priori bounded, ($\|x\| < R, R$ a uniform bound) then only two alternative possibilities exist:

a. For $\mu = 1$ there is at least one solution $x(t)$ of the integral equation with norm R,

b. Such a solution does not occur and the vector field $V_\mu(0 \le \mu \le 1)$ has no null vector on the sphere S_R, but at least one null vector within B_R.

In any case, the theorem is proved and, therefore, the differential Equation 1 has a periodic solution bounded by R.

To obtain a priori estimate, a periodic solution of the System 2 is considered. The components $x_3(t)$ and $x_4(t)$ are studied with the aid of positive auxiliary function:

$$W(x_3, x_4) = \frac{1}{2}(bx_3^2 + x_4^2) - K(|x_3| + |x_4| - |\,|x_3| - |x_4|\,|) \operatorname{sgn}(x_3 \cdot x_4) + 2\frac{b+1}{b}K^2 .$$

It should be noted that $W(x_3, x_4)$ tends to infinity as $|x_3| + |x_4| \to \infty$.

To evaluate the total derivative, W', the following cases are considered.

case 1, $x_3 \cdot x_4 > 0$

If $|x_3| < |x_4|$, then:

$$W = \frac{1}{2}(bx_3^2 + x_4^2) - 2K|x_3| + 2\frac{b+1}{b}K^2 ,$$

$$W' = bx_3x_3' + x_4x_4' - 2Kx_3' \operatorname{sgn} x_3 .$$

Substitute from Equation 2 in the expression for W':

$$\begin{aligned} W' = &-bx_3F_1 + bx_3(bx_2 - F_2) - bx_3F_3 \\ &+ bx_3\mu p - x_4 g - 2Kx_4 \operatorname{sgn} x_3 \\ &+ 2KF_1 \operatorname{sgn} x_3 - 2k(bx_2 - F_2) \operatorname{sgn} x_3 \\ &+ 2KF_3 \operatorname{sgn} x_3 - 2K\mu p \operatorname{sgn} x_3 , \end{aligned}$$

or:

$$\begin{aligned} W' \le\, & b|x_3|\{M_1 + M_2 + m\} + |x_4|(|g| - 2K) \\ &+ 2K\{M_1 + M_2 + m\} \\ &+ 2K|F_3| < |x_3|\{b(M_1 + M_2 + m) - K\} \\ &+ 2K\{M_1 + M_2 + M_3 + m\} . \end{aligned}$$

Choose $K > b(M_1 + M_2 + m)$. Then $w' < 0$ for:

$$|x_3| > \frac{2K(M_1 + M_2 + m)}{K - b(M_1 + M_2 + m)} . \tag{7}$$

Now if $|x_4| < |x_3|$:

$$W = \frac{1}{2}(bx_3^2 + x_4^2) - 2K|x_4| + 2\frac{b+1}{b}K^2 ,$$

$$W' = bx_3x_3' + x_4x_4' - 2Kx_3' \operatorname{sgn} x_3 .$$

Substituting from Equation 2,

$$\begin{aligned} W' = &-bx_3F_1 + bx_3(bx_2 - F_2) + b\mu x_3 p \\ &- bx_3F_3 - x_4 g + 2Kbx_3 + 2Kg \operatorname{sgn} x_4 , \end{aligned}$$

$$\begin{aligned} W' \le\, & b|x_3|\{M_1 + M_2 + m\} + |x_4|\,|g| \\ &+ 2Kb|x_3| + 2K^2 - bx_3F_3 . \end{aligned}$$

By (i)$'$ given α there exists β such that $|F_3| > \alpha$ if $|x_3| > p$. Hence:

$$\begin{aligned} W' \le &- |x_3| \\ &\{b\alpha - [b(M_1 + M_2 + m + 2K) + K]\} \\ &+ 2K^2 . \end{aligned}$$

If $b\alpha > K + 5(M_1 + M_2 + m) + 2K)$, then $W' < 0$ for:

$$|x_3| > \max \left\{\beta, \frac{2K^2}{\alpha b - [K + b(M_1 + M_2 + m + 2K)]}\right\} .$$

case 2, $x_3 \cdot x_4 < 0$

Here again first assume that $|x_3| < |x_4|$. Then:

$$\begin{aligned} W &= \frac{1}{2}(bx_3^2 + x_4^2) + 2K|x_3| + 2\frac{b+1}{b}K^2 \\ &= bx_3x_3' + x_4x_4' + 2Kx_3' \operatorname{sgn} x_3 . \end{aligned}$$

Substituting from Equation 2:

$$\begin{aligned} W' \le\, & b|x_3|\{M_1 + M_2 + m\} + |x_4|K - 2K|x_4| \\ &+ 2K(M_1 + M_2 + 2m) , \end{aligned}$$

or:

$$W' \le b|x_3|\{M_1+M_2+m\} - K|x_3| + 2K(M_1+M_2+2m) \ .$$

Choose $K > b(M_1+M_2+m)$, then $W' < 0$ for:

$$|x_3| > \frac{2K(M_1+M_2+m)}{K-b(M_1+M_2+2m)} \ . \quad (8)$$

Next assume $|x_3| > |x_4|$. Then:

$$W = \frac{1}{2}(bx_3^2+x_4^2) - 2K|x_3| + 2\frac{b+1}{b}K^2$$

$$\begin{aligned} W' &= bx_3x_3' + x_4x_4' + 2Kx_3' \operatorname{sgn} x_3 \\ &= bx_3(bx_2-F_2) - bx_3F_1 - bx_3F_3 + b\mu x_3 P \\ &\quad - x_4 g - 2Kbx_3 \operatorname{sgn} x_4 - 2Kg \operatorname{sgn} x_4 \\ &\quad + 2Kg \operatorname{sgn} x_3 \ . \end{aligned}$$

Finally the following is obtained:

$$W' \le b|x_3|(M_1+M_2+m) + |x_4|K + 2bK|x_3| + 2K^2 - b|x_3|\,|F_3| \ .$$

Again for $\alpha > 0$ there exists $\beta > 0$ such that $|F_3| > \alpha$ if $|x_3| > \beta$. Therefore:

$$W' \le |x_3|\{b(M_1+M_2+m+2K) + K - b\alpha\} + 2K^2$$

If $b\alpha > K + b[M_1+M_2+m+2K]$, then $W' < 0$ for:

$$|x_3| > \max \left\{\beta, \frac{2K^2}{b\alpha - [K+b(M_1+M_2+m+2K)]}\right\} . \quad (9)$$

Therefore W', the total derivative, by virtue of Equation 2 satisfies $W' < 0$ if $\max\{|x_3| - H_3, |x_4| - H_4\} \ge 0$, where H_3 and H_4 are given by the right sides of Inequalities 7–10.

By Lyapunov second method [6,8,9] the following estimates are obtained:

$$|x_3(t)| < B_3, \quad |x_4(t)| < B_4 \ ,$$

where the bounds B_3 and B_4 are determined by b, m, M_1, M_2, K and the properties of F_3 which do not depend on μ.

Next, $x_2(t)$ and $x_1(t)$ are estimated. Assume $x'(t_0) = 0$, then for $t_0 \le t \le t_0 + \omega$:

$$x'(t) = \int_{t_0}^{t} x''(\tau)d\tau \ ,$$

$$|x'(t)| \le B_3\omega = B_2, \quad \text{or } |x_2(t)| \le B_2 \ .$$

To find a bound for $x_1(t)$, Equation 1 is integrated from 0 to ω to get:

$$\int_0^{\omega} g(t, x(t), x'(t), x''(t_0))dt = 0 \ .$$

This equation contradicts relation (ii)$'$ if $|x_1(t)| \ge h$ for all t. Consequently,

$$|x_1(\tau)| < h \quad \text{for some } \tau \in (0,\omega) \ .$$

On the interval $\tau \le t \le \tau + \omega$, $|x_1(t) - x_1(\tau)| = |t-\tau|\,|x_2(\tau+\theta(t-\tau))|$ hence, $|x_1(t)| \le h + \omega^2 B_3 = B_1$. By virtue of the periodicity of $x(t)$, this relation holds for all t.

Denoting $R = \sqrt{B_1^2+B_2^2+B_3^2+B_4^2}$, it is noted that the alternative possibilities (a) and (b) on which the proof is based, must exist.

THEOREM 2

Consider the differential equation:

$$x^{(4)} + a_3x''' + a_2x'' + a_1x' + a_0x + g(t, x, x', x'', x''') = p(t) \ , \quad (10)$$

where the constants a_0, a_2, a_3 are assumed to be positive, $a_1 \le 0$. The functions p and g are continuous and p is ω-periodic. Now if:

i) $|g(t, x_1, x_2, x_3, x_4)| \le M$ for all values of independent variables,

ii) $g(t, x_1, x_2, x_3, x_4) \operatorname{sgn} x_1 \ge 0$ for $|x_1| > h$ ($h > 0$, a constant),

iii) $|P(t)| \le m$, $\int_0^{\omega} p(t)dt = 0$,

then differential Equation 10 has an ω-periodic solution.

Proof

Consider the following differential equation:

$$x^{(4)} + a_2x'' + a_0x + \mu(a_3x''' + a_1x') = \mu\{p(t) - g(t, x, x', x'', x''')\} \ , \quad (11)$$

where $\mu \in [0,1]$. For $\mu = 0$ a homogeneous differential equation, $x^{(4)} + a_2x'' + a_0x = 0$, is obtained. Clearly this equation has no nontrivial ω-periodic solution provided that $a_0 > \frac{a_2^2}{4}$ is chosen.

To get the required a-priori bounds, both sides of Equation 11 are multiplied by x''' and are integrated from 0 to ω:

$$a_3 \int_0^{\omega} [x'''(t)]^2 dt - a_1 \int_0^{\omega} [x''(t)]^2 dt = \mu \int_0^{\omega} [p(t) - g(t, x(t), x'(t), x''(t), x'''(t)]x'''(t)\, dt \ .$$

Therefore, the following is obtained:

$$a_3\int_0^\omega x'''^2 dt - a_1\int_0^\omega x''^2 dt$$
$$\le \int_0^\omega |p(t) - g(t,x_1,x_2,x_3,x_4)|x''' dt,$$

and using Schwartz inequality yields to,

$$\int_0^\omega [x'''(t)]^2 dt \le D_3 = \frac{\omega}{a_3}(m+M)^2 \ .$$

Applying Wirtinger's inequality yields to,

$$\int_0^\omega [x''(t)]^2 dt \le \left(\frac{\omega}{2\pi}\right)^2 \int_0^\omega [x'''(t)]^2 dt \le D_2$$
$$= \left(\frac{\omega}{2\pi}\right)^2 D_3 \ ,$$
$$\int_0^\omega [x'(t)]^2 dt \le \left(\frac{\omega}{2\pi}\right)^2 \int_0^\omega [x''(t)]^2 dt \le D_1$$
$$= \left(\frac{\omega}{2\pi}\right)^2 D_2 \ .$$

By Rolle's theorem, there exist points t_1 and $t_2 \in (0,\omega)$ such that $x''(t_2) = x'(t_1) = 0$ and hence,

$$x''(t) = \int_{t_2}^t x'''(t)dt, \quad x'(t) = \int_{t_1}^t x''(t)dt.$$

Applying Wirtinger's inequality again yields to,

$$|x''(t)| \le \int_0^\omega |x'''(t)|dt \le \left\{\int_0^\omega |x'''(t)|^2 dt\right\}^{1/2} \omega^{1/2}$$
$$\le (\omega D_3)^{1/2} = D''$$

$$|x'(t)| \le \int_0^\omega |x''(t)|dt \le \left\{\int_0^\omega |x''(t)|^2 dt\right\}^{1/2} \omega^{1/2}$$
$$\le (\omega D_2)^{1/2} = D' \ .$$

To get an estimate on $x'''(t)$, Equation 11 is integrated from t_0 to t,

$$x'''(t) = -a_2[x'(t) - x'(t_0)] - a_0\int_{t_0}^t x(t)dt$$
$$- \mu a_3[x''(t) - x''(t_0)]$$
$$- \mu a_1[x(t) - x(t_0)]$$
$$+ \mu\int_{t_0}^t b(t)dt - \mu\int_{t_0}^t g(t,x,x',x'',x''')dt \ .$$

It has been assumed that $x'''(t_0) = 0$ for $t_0 \in (0,\omega)$, by virtue of periodicity of $x'''(t)$. Then, the following is obtained:

$$|x'''(t)| \le 2a_2D' + a_0\omega D + 2a_3D'' + 2a_1D$$
$$+ (m+M)\omega = D''' \ .$$

Subsequently, $x(t)$ is substituted in Equation 11 and is integrated from 0 to ω,

$$a_0\int_0^\omega x(t)dt =$$
$$-\mu\int_0^\omega g(t,x(t),x'(t),x''(t),x'''(t))dt \ .$$

Now, suppose $|x(t)| \ge h$ for all t. Multiplying the above equation by $\operatorname{sgn} x$ a contradiction by assumption (ii) is set, hence $|x(\tau)| < h$ for some $\tau \in (0,\omega)$.

For $t \in [\tau, \tau+\omega]$, $|x(t) - x(\tau)| = (t-\tau)|x'(\tau + \theta(t-h)|, 0 < \theta < 1|x(t)| \le h + D' = D$. By virtue of periodic character of $x(t)$ the above estimate must hold for all t.

Denoting:

$$R = \sqrt{(D''')^2 + (D'')^2 + (D')^2 + D^2} \ ,$$

again the alternative statements (a) and (b), on which the proof of Theorem 1 is based, must hold.

THEOREM 3

If the condition $a_1 \le 0$ is dropped, then, under the assumptions of Theorem 2, it can still be proved that Equation 11 possesses an ω-periodic solution.

Proof

Again, assuming $x(t)$ is periodic, Equation 11 is multiplied by x''' and is integrated from 0 to ω:

$$a_3\int_0^\omega x'''^2 dt - a_1\int_0^\omega x''^2 dt =$$
$$\int_0^\omega [p(t) - g(t,x,x',x'',x''')]x''' dt \ . \qquad (12)$$

Let a_k''', b_k''' and $a_{k'}'', b_k''$ be, respectively, the coefficients of the Fourier series for $x'''(t)$ and $x''(t)$. Then,

$$a_k''' = \frac{2}{\omega}\int_0^\omega x'''(t)\cos\frac{2\pi k}{\omega}dt$$
$$= \frac{4\pi k}{\omega^2}\int_0^\omega x''(t)\sin\frac{2\pi k}{\omega}dt$$
$$= \frac{2\pi k}{\omega}b_k'' \ ,$$

$$b_k''' = \frac{2}{\omega}\int_0^\omega x'''(t)\sin\frac{2\pi k}{\omega}dt$$
$$= -\frac{4\pi k}{\omega^2}\int_0^\omega x''(t)\cos\frac{2\pi k}{\omega}dt$$
$$= -\frac{2\pi k}{\omega}a_k'' \,.$$

Applying Parseval's equalities, yields to:

$$\int_0^\omega (x''')^2 dt = \frac{\omega}{2}\sum_{k=1}\left(a_k'''^2 + b_k'''^2\right)$$
$$\geq \frac{4\pi^2}{\omega^2}\,\frac{\omega}{2}\sum a_k''^2 + b_k''^2 \,,$$

or:

$$\int_0^\omega x'''^2 dt \geq \frac{4\pi^2}{\omega^2}\int_0^\omega x''^2 dt \,.$$

Substitutions in Equation 12, yields to:

$$\left(a_3 - \frac{a_1\omega^2}{4\pi^2}\right)\int_0^\omega x'''^2 dt$$
$$\leq a_3\int_0^\omega x'''^2 dt - a_1\int_0^\omega x''^2 dt$$
$$\leq (m+M)\sqrt{\omega}\left\{\int_0^\omega x'''^2 dt\right\}^{1/2} \,,$$

or:

$$\left\{\int_0^\omega x'''^2 dt\right\}^{1/2} \leq \frac{4\pi^2(m+M)\sqrt{\omega}}{4\pi^2 a_2 - a_1\omega^2} = D_3^* \,. \tag{13}$$

Using this result, the proof can be completed by following the same steps as in Theorem 2.

EXAMPLE

Consider a mass-spring-dashpot system shown in Figure 1. One side of the mass m_1 is connected to the dashpot c and a nonlinear spring in parallel. The other side of the mass m_1 is connected to the mass m_2 through a linear spring k_1. The mass m_2 is connected to the linear spring k_2. The system is rigidly fixed at both ends. In addition, an ω-periodic force $h(t)$ acts on m_1 in horizontal direction. It is assumed that coulomb's friction μ exists between the masses m_1 and m_2 and the supporting floor. The force displacements relationship of the spring k is assumed to be given by $F = \phi(\xi)$.

Differential equations for the masses m_2 and m_1 are:

$$m_2 x'' + k_1(x-\xi) + k_2 x + \mu m_2 g\,\mathrm{sgn}(x') = 0 \,,$$
$$m_1\xi'' + \phi(\xi) + c\xi' + k_1(\xi - x)$$
$$+ \mu m_1 g\,\mathrm{sgn}(\xi') = h(t) \,.$$

The following differential equation for the displacement, x, of the mass m_2 is obtained:

$$x^{(4)} + a_3 x''' + a_2 x'' + a_1 x' + a_0 x$$
$$+ f(x, x', x'', x''') = h(t) \,, \tag{14}$$

where $a_0 \frac{k_2}{(m_1 m_2)}$, $a_1 = c\frac{k_1+k_2}{m_1 m_2}$, $a_2 = \left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_1 m_2}\right)$ and $a_3 = \frac{c}{m_1}$. Also,

$$f(x, x', x'', x''') = \frac{k_1}{m_1 m_2}$$
$$\{\phi(\xi) + \mu m_2 g\,\mathrm{sgn}(x') + \mu m_1 g\,\mathrm{sgn}(\xi')\} \,.$$

The displacement ξ of m_1 is related to x by:

$$\xi = \frac{k_1+k_2}{k_1}x + \frac{m_2}{k_1}x'' + \frac{\mu m_2 g}{k_1}\,\mathrm{sgn}(x') \,. \tag{15}$$

Applying Theorem 3, the masses m_1, m_2 and spring constants k_1 and k_2 must satisfy the following inequality:

$$\frac{4k_2}{m_1 m_2} > \left(\frac{k_1}{m_1} + \frac{k_1+k_2}{m_1 m_2}\right)^2 \,.$$

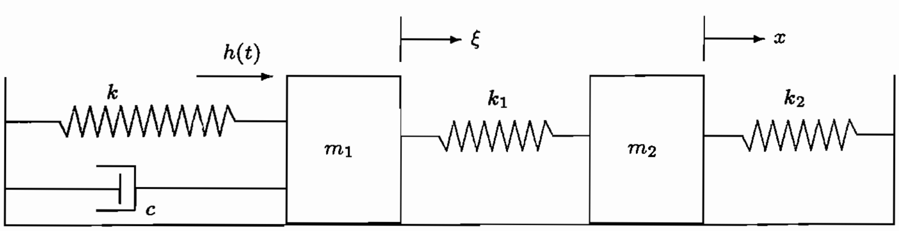

Figure 1. Mass-spring-dashpot system.

Next, following inequality is considered:

$$f(x,x',x'',x''')\,\mathrm{sgn}\,x > 0\ .$$

The following inequality must hold:

$$\phi(\xi)\,\mathrm{sgn}\,x + \mu g(m_2\,\mathrm{sgn}\,x' + m_1\,\mathrm{sgn}\,\xi')\,\mathrm{sgn}\,x > 0.$$

Assuming the following force-displacement relationship for the nonlinear spring k:

$$\phi(\xi) = \xi + \alpha\xi^3\ , \quad \alpha > 0, \tag{16}$$

the following inequality is obtained:

$$\xi\,\mathrm{sgn}\,x + \alpha\xi^3\,\mathrm{sgn}\,x + \mu g(m_2\,\mathrm{sgn}\,x' + m_1\,\mathrm{sgn}\,\xi')\,\mathrm{sgn}\,x > 0\ .$$

Clearly the above inequality is satisfied if:

$$\xi\,\mathrm{sgn}\,x + \mu g(m_2\,\mathrm{sgn}\,x' + m_1\,\mathrm{sgn}\,\xi')\,\mathrm{sgn}\,x > 0\ .$$

or:

$$\xi\,\mathrm{sgn}\,x - \mu g(m_1 + m_2) > 0\ .$$

Using Equation 15, the following inequality is obtained:

$$\frac{k_1+k_2}{k_1}|x| - \frac{m_2}{k_1}|x''| - \frac{\mu m_2 g}{k_1} - \mu g(m_1+m_2) > 0\ .$$

Therefore, if h is chosen as follows:

$$h = \frac{k_1}{k_1+k_2}\left\{\frac{m_2}{k_1}(\omega D_3^*)^{1/2} + \mu g(m_1 + m_2 + \frac{m_2}{k_1}\right\}\ , \tag{17}$$

where D_3^* is given by Statement 13, then $|x| > h$ implies $f(x,x',x'',x''')\,\mathrm{sgn}\,x > 0$.

Next from Equation 15,

$$|\xi| \le \frac{k_1+k_2}{k_1}|x| + \frac{m_2}{k_1}|x''| + \frac{\mu m_2 g}{k_1}\ ,$$

or:

$$|\xi| \le \frac{k_1+k_2}{k_1}\left[h + \frac{\omega}{2\pi}(\omega D_3^*)^{1/2}\right] + \frac{m_2}{k_1}(\omega D_3^*)^{1/2} + \frac{\mu m_2 g}{k_1}\ .$$

$|\xi| < \Delta$ is obtained, where:

$$\Delta = \frac{k_1+k_2}{k_1}h + \left(\frac{k_1+k_2}{k_1}\,\frac{\omega}{2\pi} + \frac{m_2}{k_1}\right)(\omega D_3^*)^{1/2} + \frac{\mu m_2 g}{k_1}\ ,$$

and h is given by Equation 17.

Hence $|f(x,x',x'',x''')| \le M$ is obtained where $M = \Delta + \alpha\Delta^3 + \mu g(m_1+m_2)$. It follows that differential Equation 15 has at least an ω-periodic solution.

CONCLUSION

Using the Leray-Schauder principle, it has been shown that, Equation 1 under conditions set in Theorem 1 possesses an w-periodic solution. The results obtained were used to show the existence of periodic solution for the particular case of Equation 10. These considerations were, then, applied to the mechanical system (Figure 1) to show its periodic behavior.

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Scientia Iranica, Vol. 4, Nos. 1&2, pp 8–11

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Galerkin Approximations for a Semilinear Stochastic Integral Equation

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In this paper, the Galerkin method is used to approximate the solution of the H-valued integral equation:

$$X_t = \int_{-\infty}^{t} U(t-s) f_s(X_s) ds + V_t,$$

where H is a real separable Hilbert space. $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A, such that A^{-1} is compact and f is of monotone type and is bounded by a polynomial. Furthermore, V_t is a cadlag adapted process.

INTRODUCTION

Let H be a real separable Hilbert space with norm $\| \ \|$ and inner product $\langle \ , \ \rangle$. Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a complete stochastic basis with a right continuous filtration and $\{W_t, t \geq 0\}$ is an H-valued cylindrical Brownian motion with respect to $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Consider the stochastic semilinear equation:

$$dX_t = AX_t dt + f_t(X_t)\, dt + dW_t \ , \qquad (1)$$

where A is a closed, self-adjoint, negative definite, unbounded operator such that A^{-1} is nuclear. A mild solution of Equation 1 with initial condition, $X(0) = X_0$, is the solution of the integral equation:

$$X_t = U(t,0)X_0 + \int_0^t U(t-s) f_s(X_s) ds + \int_0^t U(t-s) dW_s \ , \qquad (2)$$

where $U(t)$ is the semigroup generated by A.

Marcus [1] has proved that when f is independent of t as well as ω and uniformly Lipschitz, then the solution of Equation 2 is asymptotically stationary. To prove this, Marcus studied the following integral equation:

$$X_t = \int_{-\infty}^t U(t-s) f_s(X_s) ds + \int_{-\infty}^t U(t-s) dW_s, \qquad (3)$$

where the parameter set of the processes is extended to the whole real line. This gave the motivation for studying the existence of the solution of a more general equation:

$$X_t = \int_{-\infty}^t U(t-s) f_s(X_s) ds + V_t, \qquad (4)$$

where $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A such that A^{-1} is compact, f is of monotone type, bounded by a polynomial and V_t is a cadlag adapted process. In [2], the existence and the uniqueness of the solution to Equation 4 is proved. In this paper it is proven that finite dimensional Galerkin approximations converge strongly to the solution of Equation 4. In [3] this result is used to prove the stationarity of the solution of Equation 4. Results of this paper are presented in [4] without proof.

PRELIMINARIES

A Semilinear Evolution Equation

Let g be an H-valued function defined on a set $D(g) \subset H$. Recall that g is monotone if, for each pair, $x, y \in D(g)$,

$$\langle g(x) - g(y), x - y \rangle \ \geq 0 \ .$$

We say g is bounded if there exists an increasing continuous function ψ on $[0, \infty)$ such that $\|g(x)\| \leq \psi(\|x\|), \forall x \in D(g)$. g is demi-continuous if, whenever

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(x_n) is a sequence in $D(g)$ which converges strongly to a point $x \in D(g)$, then $g(x_n)$ converges weakly to $g(x)$.

Consider the following integral equation:

$$X_t = \int_0^t U(t-s)f(X_s)ds + V_t \ , \qquad (5)$$

where f, V and the generator A of the semigroup U satisfy the following hypothesis.

***Hypothesis* 1**

a) $U(t)$ is a semigroup generated by a strictly negative definite, self-adjoint unbounded operator A such that A^{-1} is compact. Then there is $\lambda > 0$ such that $\|U(t)\| \leq e^{-\lambda t}$.

b) Let $\varphi(t) = K(1+t^p)$ for some $p > 0$, $K > 0$. $-f$ is a monotone demi-continuous mapping from H to H such that $\|f(x)\| \leq \varphi(\|x\|)$ for all $x \in H$.

c) Let $r = 2p^2$. V_t is cadlag adapted process such that $\sup_{t\in R} E\{\|V_t\|^r\} < \infty$.

***Proposition* 1**

Suppose that f , V, A and U satisfy Hypothesis 1. Then Equation 5 has a unique adapted cadlag (continuous, if V_t is continuous) solution. Furthermore,

$$\|X(t)\| \leq \|V(t)\| + \int_0^t e^{(\lambda)(t-s)}\|f(s, V(s))\|ds \ . \qquad (6)$$

For the proof see [5].

The Stability of the Solution

***Proposition* 2**

Let f^1 and f^2 be two mappings satisfying Hypothesis 1 bounded by functions φ_1 and φ_2 respectively.

Suppose V^1, V^2, U^1 and U^2 satisfy Hypothesis 1. Let $X^i(t)$, $i = 1, 2$ be the solution of the integral equations:

$$X^i(t) = \int_0^t U^i(t-s)f^i_s(X^i(s))ds + V^i(t). \qquad (7)$$

Define $v^3(t)$ and $\bar{I}$ as:

$$V^3(t) := \int_0^t (U^2(t-s) - U^1(t-s))f^2(X^1(s))ds,$$

$$\bar{I} := 4\int_0^T e^{-2\lambda s}\|f^2(X^2(s)) - f^1(X^1(s))\|^2 ds \ .$$

Then the following is obtained:

$$\begin{aligned}\|X^2(t) - X^1(t)\|^2 &\leq 4\|V^2(t) - V^1(t)\|^2 \\ &+ 4\|V^3(t)\|^2 \\ &+ \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^2(s) - V^1(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \bar{I}\left(\int_0^t e^{-2\lambda s}\|V^3(s)\|^2 ds\right)^{\frac{1}{2}} \\ &+ \int_0^t e^{-2\lambda s}\|f^2(X^1(s)) - f^1(X^1(s))\|^2 ds.\end{aligned} \qquad (8)$$

For proof see [6].

EXAMPLES

A Semilinear Stochastic Evolution Equation

The existence and uniqueness of the solution of the integral Equation 2 have been studied in [7]. Marcus assumed that f is independent of $\omega \in \Omega$ and $t \in S$ and that there are $M > 0$, and $p \geq 1$ for which:

$$\langle f(u) - f(v), u - v \rangle \ \leq -M\|u - v\|^p \ ,$$

and:

$$\|f(u)\| \leq C(1 + \|u\|^{p-1}).$$

Marcus proved that this integral equation has a unique solution in $L^p(\Omega, L^p(S, H))$.

As a consequence of Proposition 1, this result can be extended to a more general f and the existence of a strong solution of Equation 2 which is continuous instead of merely being in $L^p(\Omega, L^p(S, H))$ can be shown.

The Ornstein-Uhlenbeck process $V_t = \int_0^t U(t-s)dW(s)$ has been well-studied e.g., in [8] where they show that V_t has a continuous version. Therefore, Equation 2 can be rewritten as:

$$X_t = \int_0^t U(t-s)f_s(X_s)ds + V_t,$$

where V_t is an adapted continuous process. Then, by Proposition 1, the Equation 2 has a unique continuous adapted solution.

A Semilinear Stochastic Partial Differential Equation

Let D be a bounded domain with a smooth boundary in R^d. Let $-A$ be a uniformly strongly elliptic second order differential operator with smooth coefficients on D. Let B be the operator $B = d(x)D_N + e(x)$, where D_N is the normal derivative on ∂D, and d and e are in

$C^\infty(\partial D)$. Let A (with the boundary condition $Bf \equiv 0$) be self-adjoint.

Consider the initial-boundary-value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f_t(u) + \dot{W} & \text{on} \quad D \times [0,\infty), \\ Bu = 0 & \text{on} \quad \partial D \times [0,\infty), \\ u(0,x) = 0 & \text{on} \quad D, \end{cases} \tag{9}$$

where $\dot{W} = \dot{W}(t,x)$ is a white noise in space–time for the definition and properties of white noise (see [9]), and f_t is a non-linear function that will be defined below. Let $p > \frac{d}{2}$. W can be considered as a Brownian motion $\tilde{W}_t$ on the Sobolev space H_{-p} (see [9] Chapter 4, page 4.11). There is a complete orthonormal basis $\{e_k\}$ for H_p.

The operator A (plus boundary conditions) has eigenvalues $\{\lambda_k\}$ with respect to $\{e_k\}$, i.e., $Ae_k = \lambda_k e_k$, $\forall k$. The eigenvalues satisfy $\Sigma_j(1+\lambda_j^{-p}) < \infty$ if $p > \frac{d}{2}$ (see [9] Chapter 4, page 4.9) Then $[A^{-1}]^p$ is nuclear and $-A$ generates a contraction semigroup $U(t) \equiv e^{-tA}$. This semigroup satisfies Hypothesis 1.

Now consider the initial-boundary-value Problem 9 as a semilinear stochastic evolution equation:

$$du_t + Au_t dt = f_t(u_t)dt + d\tilde{W}_t \ , \tag{10}$$

with initial condition $u(0) = 0$, where $f : S \times \Omega \times H_{-p} \to H_{-p}$ satisfies Hypothesis 1(b) relative to the separable Hilbert space $H = H_{-p}$. The mild solution of Equation 10 (which is also a mild solution of Problem 9) can be defined, to be the solution of:

$$u_t = \int_0^t U(t-s) f_s(u_s) ds + \int_0^t U(t-s) d\tilde{W}_s. \tag{11}$$

Since $\tilde{W}_t$ is a continuous local martingale on the separable Hilbert space H_{-p}, then $\int_0^t U(t-s)d\tilde{W}_s$ has an adapted continuous version (see for example [10]). If the following is defined

$$V_t := \int_0^t U(t-s) d\tilde{W}_s,$$

then by Proposition 1, Equation 11 has a unique continuous solution with values in H_{-p}.

A SEMILINEAR INTEGRAL EQUATION ON THE WHOLE REAL LINE

Let us reduce the integral Equation 4 to the following integral equation:

$$X_t = \int_{-\infty}^t U(t-s) f(X_s + V_s) ds \ . \tag{12}$$

The following theorem translates Proposition 1 to the case when parameter set of the process is the whole real line.

***Theorem* 1**

If A, f and V satisfy Hypothesis 1, then the integral Equation 12 has a unique continuous solution X such that:

$$\|X_t\| \le \int_{-\infty}^t e^{-\lambda(t-s)} \varphi(\|V_s\|) ds \ , \tag{13}$$

$$E\{\|X_t\|\} \le \frac{1}{\lambda} \sup_{s \in R} E\{\varphi(\|V_s\|)\} := K_1 \ . \tag{14}$$

For proof see [2].

Galerkin Approximations

Let $U(t)$ be a semigroup generated by a strictly negative definite closed unbounded self-adjoint operator A such that A^{-1} is compact. Then there is a complete orthonormal basis (ϕ_n) and eigenvalues $0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots$ with $\lambda_n \to \infty$, such that $A\phi_n = -\lambda_n \phi_n$.

Let H_n be the subspace of H generated by $\{\phi_0, \phi_1, \dots, \phi_{n-1}\}$ and let J_n be the projection operator on H_n.

Define:

$$f_n = J_n f, \quad V_n(t) = J_n V(t),$$

$$U_n(t) = J_n V(t) J_n \ ,$$

and define $X_n(t)$ and $X(t)$ as solutions of:

$$X_n(t) = \int_{-\infty}^t U_n(t-s) f_n(X_n(s)) ds + V_n(t) \ , \tag{15}$$

and:

$$X(t) = \int_{-\infty}^t U(t-s) f(X(s)) ds + V(t) \ . \tag{16}$$

Now the following theorem can be proved.

***Theorem* 2**

If A, U, f and V satisfy Hypothesis 1, then one has:

$$E(\|X_n(t) - X(t)\|) \to 0.$$

Proof

Define:

$$X_n^k(t) = \int_{-k}^t U_n(t-s) f_n(X_n^k(s)) ds + V_n(t),$$

$$X^k(t) = \int_{-k}^t U(t-s) f(X^k(s)) ds + V(t),$$

and:

$$\bar{V}_{n,k}(t) = \int_{-k}^t (U_n(t-s) - U(t-s) f(X^k(s)) ds.$$

By Proposition 2, the following is obtained:

$$\|X_n^k(t) - X^k(t)\|^2 \le 4\|V_n(t) - V(t)\|^2 + 4\|\bar{V}_{n,k}(t)\|^2 + \bar{I}\left(\int_{-k}^t e^{2\lambda_0 s}\|V_n(s) - V(s)\|^2 ds\right)^{\frac{1}{2}} + \bar{I}\left(\int_{-k}^t e^{2\lambda_0 s}\|\bar{V}_{n,k}(s)\|^2 ds\right)^{\frac{1}{2}} + \int_{-k}^t e^{2\lambda_0 s}\|f_n(X(s)) - f(X(s))\|^2 ds. \quad (17)$$

Taking expectations and using the Schwartz inequality and Fubini's theorem, Statement 17 implies that:

$$E\{\|X_n^k(t)-X^k(t)\|^2\} \le 4E\{\|V_n(t)-V(t)\|^2\} + 4E\{\|\bar{V}_{n,k}(t)\|^2\} + (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^t e^{2\lambda_0 s}E(\|V_n(s)-V(s)\|^2)ds\right)^{\frac{1}{2}} + (E\{\bar{I}^2\})^{\frac{1}{2}}\left(\int_{-\infty}^t e^{2\lambda_0 s}E(\|\bar{V}_{n,k}(s)\|^2)ds\right)^{\frac{1}{2}} + \int_{-\infty}^t e^{2\lambda_0 s}E(\|f_n(X(s))-f(X(s))\|^2)ds. \quad (18)$$

It is first shown that:

$$E\{\|X_n^k(t) - X^k(t)\|^2\} \to 0 \quad \text{uniformly in } k \ . \quad (19)$$

Since $V_n = J_n V$ and $f_n = J_n f$, the first, third and 5th term of the right hand side of Statement 18 converge to zero. Then to prove Statement 19 it is enough to show that $E(\|\bar{V}_{n,k}(t)\|^2)$ converges to zero uniformly in k and $t \in (-\infty, T]$.

By using $\|f(x)\| \le C(1 + \|x\|^p)$ and Statement 6, it is shown that:

$$\sup_{t\in R} E(\|V(t)\|^{2p}) < \infty \ ,$$

and, using Fubini's theorem, one has:

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \le \sup_{t\in R} E\{1 + \|V(t)\|^p\} \int_{-\infty}^0 \|U(-s) - U_n(-s)\|_L^2 ds \ .$$

Since:

$$\|U(-s) - U_n(-s)\|_L \to 0 \text{ for } s < 0 \text{ and}$$

$$\|U(-s) - U_n(-s)\|_L \le e^{2\lambda_0 s} \ ,$$

then by the dominated convergence theorem:

$$\sup_{t\in R} E(\|\bar{V}_{n,k}(t)\|^2) \to 0 \quad \text{uniformly in } k \ .$$

Then:

$$E(\|X_{n,k}(t) - X^k(t)\|^2) \to 0 \text{ uniformly in } k \ .$$

By the proof of Theorem 1 (see [2]), then $E(\|X_{n,k}(t) - X_n(t)\|) \to 0$ as $k \to \infty$, hence $E(\|X^k(t) - X(t)\|) \to 0$ and it is obtained that $E(\|X_n(t) - X(t)\|) \to 0$. Q.E.D.

ACKNOWLEDGMENT

This research is partially supported by Sharif University of Technology and Institute for Studies in Theoretical Physics and Mathematics.

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