Entropy generation analysis in peristaltic flow of magneto-nanoparticles suspended in water under second order slip conditions

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Abstract: Here consideration is given to the peristalsis of magneto- nanoparticles suspended in water. Explicitly $Fe_3O_4$ – water nanofluid is utilized for two-dimensional flow in channel considered to be symmetric with complaint walls. Uniform magnetic field is applied. Temperature equation is arranged for viscous dissipation. Second order velocity and thermal slip conditions are utilized. Small Grashof number leads to perturbation solution. Examination of entropy generation is also carried out in this study. Maxwell and Hamilton-Crosser models are used. Analysis is based on the comparative study of these two models representing the cylindrical and spherical shaped particles. Graphs for velocity, temperature, entropy generation and Bejan numbers are plotted under the influence of sundry variables. Trapping is observed via plotting streamlines.

Keywords: $Fe_3O_4$ – water nanofluid; Peristalsis; Mixed convection; Second order velocity and thermal slip conditions; Entropy generation; MHD.

1. Introduction
Nanotechnology has attracted the attention of recent researchers due to its ample applications in industrial, biomedical and engineering fields. Nanotechnology has many advancement in our daily life. Through use of nanoparticles we can remove the harmful and dangerous viruses and bacteria from water. These nanoparticles are also used in improving the vehicle fuel efficiency and corrosion resistance by making the vehicle parts from nanocomposite materials, in sports goods to make it stronger and light weight etc. The term nanofluid was first used by Choi [1]. Nano material consists of material with nano size ($10^{-9}$ of a meter). These small size particles are very useful. Nanofluids are made with the help of these particles which are mixture of base fluid usually water, oil, ethylene glycol etc. and nanoparticles. Nanoparticles have different shaped (spherical, cylindrical, tube, blade, bricks etc). Different types of nanoparticles include the oxides, carbides, nanotubes etc. Different two phase models have been utilized by different researchers in the literature. In two phase models for the nanofluids in which the thermo-physical characteristics of nanomaterial and base liquid are separately defined (Maxwell [2] and Hamilton Crosser's [3]). So these type of models have the advantage to investigate the different types of nanofluids by using the different types of nanoparticles and based fluids and make a comparison among them to know which one is better in performance. But in other single phase model like Buongiorno model [4] the thermo-physical properties of each fluid is not specific. So one can analyze the fluid in general by

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considering the prominent effects of nanofluid i.e. Brownian motion and thermophoresis effects. The nanofluid has enhanced thermo-physical properties due to which it has many advancements in engineering, biomedical and industrial applications. Because of such usage and fame scientists are becoming curious about the use of nanofluids. So nanofluid became a hot topic of the era. Mahanthesh et. al [5] investigated the two phase model for nanofluid by utilizing the SWCNT and MWCNT when flow is due to the rotating disk. Hsiao [6] examined the problem for stretching sheet and he accounted the micropolar nanofluid. Makinde et. al [7] discussed the flow caused by rotating disk in presence of aluminum and titanium alloy nanoparticles. Some other literature regarding the nanofluids is seen through the refs. [8-13] which shed some more light on the utility of nanofluids.

In physiology the bulk fluid is moved through the peristaltic phenomenon. In this mechanism a progressive wave moves with channel walls which forced the fluid to flow. This phenomenon is greatly present in human body. The transportation of food from esophagus, movement of chyme, transportation of urine to bladder, transport of lymph in lymphatic vessels etc. obviously involves peristalsis. Besides the physiological processes, the peristalsis is significant in industrial processes and in biomedical applications. Modern devices like dialysis machine, open heart bypass machine etc operate under peristalsis. This principle is useful for transfer of corrosive fluid because it avoid the contamination of fluid with machinery. Peristalsis is employed in area for sanitary fluid transport. Many engineering devices like hose pump, roller pump, finger pump etc. are working on the same principle. Because of such involvement of peristalsis in physiology, biomedical and industrial applications this topic is regarded quite important. Thus many works on peristalsis are done after the pioneering research of Latham [14]. Afterward the research on peristalsis is sizeable (see few studies [15-27]) in this direction. Peristaltic transport of nanofluids can be consulted by the attempts [28-36].

It is well known that several aspects (like chemical reaction, Joule heating etc) in thermodynamic system change its entropy. Basically the entropy of the system is defines as the measure of disorder in system. Bejan [37] initially works on entropy analysis. Existing literature witnesses that little information is available for peristaltic transport of fluid in presence of entropy (see refs. [38-40]). Main objective here is to venture further in this regime. Thus we consider the peristalsis of nanofluid subject to entropy generation. Mixed convection is also considered. \( \text{Fe}_3\text{O}_4 \) with water (as based fluid) is used as nanofluid. There is no study found where second order slip conditions has been carried out for both cases of velocity and temperature. This study fills this void. Specifically the Hamilton- Crosser and Maxwell models in comparative study are used for the cylindrical and spherical shaped particles respectively. Resulting problems employing lubrication approach are solved by regular perturbation method. Grashof number is adopted as the perturbation quantity. The velocity, temperature, entropy generation and Bejan numbers are sketched and examined. Trapping phenomenon is studied by graphical illustration of streamlines.

## 2. Flow configuration

Peristaltic flow of an incompressible nanofluid composing of \( \text{Fe}_3\text{O}_4 \) and water is considered. The channel (with width \( 2d \)) is considered symmetric. Flexible walls channel placed at the positions \( y = \pm \eta \) where the left and right walls are denoted by \( - \) and \( + \) respectively (see Fig. 1). The rectangular coordinates system is settled such as the \( x \)-axis lies in the direction of channel length where the position of the \( y \) -axis is in the direction perpendicular to the \( x \)-axis. The temperature of the walls is maintained at \( T_0 \). Contribution due to constant applied magnetic field is taken into
account. Induced magnetic and electric fields effects are omitted. Mixed convection and viscous dissipation are studied. Sinusoidal wave have wavelength \( \lambda \), amplitude \( a \) and speed \( c \). The shape of wave is defined by equation given below:

\[
y = \pm \eta(x,t) = \pm \left[ d + a \sin \frac{2\pi}{\lambda} (x - ct) \right].
\]

Figure 1

The equations for the considered flow configuration are:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

\[
\rho_{\text{eff}} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) u = -\frac{\partial p}{\partial x} + \mu_{\text{eff}} \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] - \sigma_{\text{eff}} B_0^2 u + g (\rho \beta)_{\text{eff}} (T - T_0),
\]

\[
\rho_{\text{eff}} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) v = -\frac{\partial p}{\partial y} - \sigma_{\text{eff}} B_0^2 v + \mu_{\text{eff}} \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right],
\]

\[
(\rho C)^{\text{eff}} \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) T = \mu_{\text{eff}} \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2 \left( \frac{\partial v}{\partial y} \right)^2 \right] + K_{\text{eff}} \left[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right].
\]

The quantities used in the above mentioned equations are defined as: \( u \) and \( v \) for components of velocity in the \( x \) and \( y \) directions, \( \rho_{\text{eff}} \) for effective density, \( p \) the pressure, \( \mu_{\text{eff}} \) the effective viscosity, \( g \) the acceleration due to gravity, \( \sigma_{\text{eff}} \) the effective thermal conductivity, \( (\rho \beta)_{\text{eff}} \), \( K_{\text{eff}} \) and \( (\rho C)^{\text{eff}} \) for effective thermal expansion, the effective thermal conductivity of nanofluids and the effective heat capacity respectively. \( T \) is used to define temperature whereas \( t \) for time.

The notations \( \rho_{\text{eff}} \), \( (\rho \beta)_{\text{eff}} \), \( (\rho C)^{\text{eff}} \), \( \mu_{\text{eff}} \), \( \sigma_{\text{eff}} \) and \( K_{\text{eff}} \) for the two phase models are:
\[(\rho \beta)_{\text{eff}} = (1 - \phi) \rho_f \beta_f + \phi \rho_p \beta_p, \quad \rho_{\text{eff}} = (1 - \phi) \rho_f + \phi \rho_p,\]
\[
(\rho C)_{\text{eff}} = (1 - \phi)(\rho C)_f + \phi(\rho C)_p, \quad \mu_{\text{eff}} = \frac{\mu_f}{(1 - \phi)^{2.5}},
\]
\[
\frac{\sigma_{\text{eff}}}{\sigma_f} = 1 + \frac{3\phi}{(\sigma_f + 2)(\sigma_f - 1)},
\]
\[
\frac{K_{\text{eff}}}{K_f} = \frac{K_p + 2K_f - 2\phi(K_f - K_p)}{K_p + 2K_f + \phi(K_f - K_p)} \quad \text{for Maxwell's model and}
\]
\[
\frac{K_{\text{eff}}}{K_f} = \frac{K_p + (n-1)K_f - (n-1)\phi(K_f - K_p)}{K_p + (n-1)K_f + \phi(K_f - K_p)} \quad \text{for Hamilton-Crosser's model, (6)}
\]
in which the symbols \( f \) and \( p \) in the subscript are used to represent the fluid and nanoparticles whereas \( \phi \) depicts volume fraction of nanoparticles. In this study two models of effective thermal conductivity are used in above equation. The Hamilton-Crosser model is used for the cylindrical shaped particles for \( n = 6 \) whereas Maxwell model is used for spherical shaped particles. Here \( n \) represents the shape of the nanoparticles. It is defined by \( 3/\Psi \) where \( \Psi \) depicts the sphericity of nanomaterials. Value \( \Psi = 0.5 \) is used for cylindrical shaped material whereas \( \Psi = 1 \) for spherical shaped material.

Thermophysical properties of base liquid and nanoparticle is mentioned below in Table 1.

**Table 1**

<table>
<thead>
<tr>
<th>Dimensionless parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^* = \frac{x}{\lambda} ), ( u^* = \frac{u}{c} ), ( y^* = \frac{y}{d} ), ( \eta^* = \frac{\eta}{d} ), ( v^* = \frac{v}{c} ), ( t^* = \frac{ct}{\lambda} ),</td>
</tr>
<tr>
<td>( p^* = \frac{d^2 p}{c \lambda \mu_f} ), ( \theta = \frac{T - T_0}{T_0} ), ( \text{Pr} = \frac{\mu_f C_f}{K_f} ), ( \text{Re} = \frac{\rho_f c d}{\mu_f} ),</td>
</tr>
<tr>
<td>( Ec = \frac{c^2}{C_f T_0} ), ( Br = \text{Pr Ec} ), ( M = \frac{\sigma_f B_0 d}{\sqrt{\mu_f}} ), ( Gr = \frac{g \rho_f \beta_f T_0 d^2}{c \mu_f} ),</td>
</tr>
<tr>
<td>( u = \frac{\partial \psi}{\partial y}, \quad v = -\delta \frac{\partial \psi}{\partial x} ).</td>
</tr>
</tbody>
</table>

Here \( \text{Pr}, \ \text{Re}, \ \text{Br}, \ \text{Ec}, \ M \) and \( Gr \) denote the Prandtl, Reynolds, Brinkman, Eckert, Hartman and Grashof numbers respectively.

After implication of lubrication approach yields
\[
\frac{\partial p}{\partial x} = \frac{1}{(1 - \phi)^{2.5}} \frac{\partial^2 \psi}{\partial y^2} + GrA_0 \theta - M^2 A_1 \frac{\partial \psi}{\partial y}, \quad (8)
\]
\[
\frac{\partial p}{\partial y} = 0, \quad (9)
\]
\[ K_1 \frac{\partial^2 \theta}{\partial y^2} + \frac{Br}{(1 - \phi)^{2.5}} \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2 = 0, \]  
(10)

\[ A_1 = 1 + \frac{3(\frac{\sigma}{\sigma_y} - 1)\phi}{(\frac{\sigma}{\sigma_y} + 2) - (\frac{\sigma}{\sigma_y} - 1)\phi}, \quad A_3 = 1 - \phi + \phi \left( \frac{\rho\beta_f}{\rho\beta} \right), \]

\[ K_1 = \frac{K_p + 2K_f - 2\phi(K_f - K_p)}{K_p + 2K_f + \phi(K_f - K_p)} \quad \text{for Maxwell's model} \]

\[ K_1 = \frac{K_p + (n-1)K_f - (n-1)\phi(K_f - K_p)}{K_p + (n-1)K_f + \phi(K_f - K_p)} \quad \text{for Hamilton-Crosser's model}. \]

Boundary conditions becomes:

\[ \frac{\partial \psi}{\partial y} \pm \beta_1 \frac{\partial^2 \psi}{\partial y^2} \pm \beta_2 \frac{\partial^3 \psi}{\partial y^3} = 0, \quad \theta \pm \gamma_1 \frac{\partial \theta}{\partial y} \pm \gamma_2 \frac{\partial^2 \theta}{\partial y^2} = 0, \quad \text{at } y = \pm \eta, \]

(12)

\[ \left[ E_1 \frac{\partial^3}{\partial x^3} + E_2 \frac{\partial^3}{\partial x \partial t^2} + E_3 \frac{\partial^2}{\partial t \partial x} \right] \eta = \frac{1}{(1 - \phi)^{2.5}} \left( \frac{\partial^3 \psi}{\partial y^3} + GrA_3 \theta - M^2 A_1 \frac{\partial \psi}{\partial y} \right), \quad \text{at } y = \pm \eta. \]

(13)

Here velocity and temperature slip parameters in dimensionless form is denoted by \( \beta_1, \beta_2 \) and \( \gamma_1, \gamma_2 \) respectively.

### 2.1. Entropy analysis

Viscous dissipation is represented by

\[ \Phi = \mu_{eff} \left[ \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + 2 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right]. \]

(14)

Expression for volumetric entropy generation in dimensional form is

\[ S_{gen}^v = \frac{K_{eff}}{T_m^2} \left( \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right) + \Phi. \]

(15)

Dimensionless form becomes:

\[ N_s = \frac{S_{gen}^v}{S_G} = K \left( \frac{\partial \theta}{\partial y} \right)^2 + \frac{Br}{\Lambda (1 - \phi)^{2.5}} \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2, \]

(16)

\[ S_G^v = \frac{K_f T_0^2}{T_m^2 d^2}, \quad \Lambda = \frac{T_d}{T_m}. \]

(17)

Bejan number is:

\[ Be = \frac{N_{s_{cond}}}{N_{s_{cond}} + N_{s_{visc}}}. \]

(18)

Here Eq. (15) contains two parts. \( N_{s_{cond}} \) is the entropy generation due to temperature difference whereas \( N_{s_{visc}} \) is the entropy generation due to viscous dissipation.
3. Solution methodology
We adopted the perturbation technique for the solution. We choose the small Grashof number as perturbation parameter. The corresponding systems and their solutions are:

3.1. Zeroth order systems

\[
\frac{1}{(1-\phi)^{2.5}} \frac{\partial^4 \psi_0}{\partial y^4} - M^2 A_1 \frac{\partial^2 \psi_0}{\partial y^2} = 0,
\]

\[ (19) \]

\[
K_1 \frac{\partial^2 \theta_0}{\partial y^2} + \frac{Br}{(1-\phi)^{2.5}} \left( \frac{\partial^2 \psi_0}{\partial y^2} \right)^2 = 0,
\]

\[ (20) \]

\[
\frac{\partial \psi_0}{\partial y} \pm \beta_1 \frac{\partial^2 \psi_0}{\partial y^2} \pm \beta_2 \frac{\partial^3 \psi_0}{\partial y^3} = 0, \quad \text{at} \quad y = \pm \eta,
\]

\[ (21) \]

\[
\left[ E_2 \frac{\partial^3}{\partial x^3} + E_2 \frac{\partial^3}{\partial x \partial y^2} + E_3 \frac{\partial^2}{\partial t \partial x} \right] \eta = \frac{1}{(1-\phi)^{2.5}} \frac{\partial^3 \psi_0}{\partial y^3} - M^2 A_1 \frac{\partial \psi_0}{\partial y}, \quad \text{at} \quad y = \pm \eta,
\]

\[ (22) \]

\[
\theta_0 \pm \gamma_1 \frac{\partial \theta_0}{\partial y} \pm \gamma_2 \frac{\partial^2 \theta_0}{\partial y^2} = 0, \quad \text{at} \quad y = \pm \eta.
\]

\[ (23) \]

The corresponding stream function and temperature solutions are

\[
\psi_0 = \frac{A_0 e^{\frac{\psi_0}{M^2}} \left( e^{\frac{\psi_0}{M^2}} C_1 + C_2 \right)}{A_1 M^2} + C_3 + yC_4,
\]

\[ (24) \]

\[
\theta_0 = -\frac{1}{4A_1 K_1 M^2} A_0 Br \left( e^{\frac{\psi_0}{M^2}} (C_2^2 + C_1^2 e^{\frac{\psi_0}{M^2}}) \right) + F_1 + yF_2.
\]

\[ (25) \]

3.2. First order systems
Here we have

\[
\frac{1}{(1-\phi)^{2.5}} \frac{\partial^4 \psi_1}{\partial y^4} + A_1 \frac{\partial \theta_0}{\partial y} - M^2 A_1 \frac{\partial^2 \psi_1}{\partial y^2} = 0,
\]

\[ (26) \]

\[
K_1 \frac{\partial^2 \theta_1}{\partial y^2} + \frac{Br}{(1-\phi)^{2.5}} \left( \frac{2 \partial^2 \psi_0 \partial^2 \psi_1}{\partial y^2 \partial y^2} \right) = 0,
\]

\[ (27) \]

\[
\frac{\partial \psi_1}{\partial y} \pm \beta_1 \frac{\partial^2 \psi_1}{\partial y^2} \pm \beta_2 \frac{\partial^3 \psi_1}{\partial y^3} = 0, \quad \text{at} \quad y = \pm \eta,
\]

\[ (28) \]
\[
\frac{1}{(1-\phi)^{2.5}} \frac{\partial^3 \psi}{\partial y^3} + A_3 \theta_0 - M^2 A_1 \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad y = \pm \eta, \tag{29}
\]

\[
\theta_1 \pm \gamma_1 \frac{\partial \theta_1}{\partial y} \pm \gamma_2 \frac{\partial^2 \theta_1}{\partial y^2} = 0, \quad \text{at} \quad y = \pm \eta. \tag{30}
\]

The solution expressions are

\[
\psi_1 = -\frac{1}{24A_1^2 K_i M^3} \left( \frac{1}{\sqrt{A_1}} \right) (-12A_3 A_2 K_i M^3 y^2 + 8A_0 A_1 M^3 (A_3 Br C_2 y^3 - 3e^{\frac{B_1 C_2}{\sqrt{K_i}}}(e^{\frac{B_1 C_2}{\sqrt{K_i}}} B_1 + B_2)) + yB_4 + B_5), \tag{31}
\]

\[
\theta_1 = \frac{1}{3A_1^2 K_i M^3} A_0 Br \left( \frac{A_3 A_2 A_3 Br C_2 e^{\frac{B_1 C_2}{\sqrt{K_i}}}}{3\sqrt{A_1 M}} - \frac{A_3 A_2 A_3 Br C_2 e^{\frac{B_1 C_2}{\sqrt{K_i}}}}{3\sqrt{A_1 M}} \right)

\frac{9}{2} A_0 A_2 B_2 C e^{\frac{B_1 C_2}{\sqrt{K_i}}} K_i M^2 \right) - \frac{9}{2} A_0 A_2 B_2 C e^{\frac{B_1 C_2}{\sqrt{K_i}}} K_i M^2

9A_2 (B_2 C_2 + B_2 C_2) K_i M^2 y^2 - 3\sqrt{A_1 A_2 C_2 e^{\frac{B_1 C_2}{\sqrt{K_i}}}}

\left( \sqrt{A_0} (-25A_1^2 Br C_2 + 6\sqrt{A_1 A_2 K_i M}) \right)

\frac{25A_1^2 Br C_2 + 6\sqrt{A_1 A_2 K_i M}}{\sqrt{A_1 M}}

3\sqrt{A_1 A_2 C_2 e^{\frac{B_1 C_2}{\sqrt{K_i}}}} \cdot \left( \sqrt{A_0} (25A_1^2 Br C_2 + 6\sqrt{A_1 A_2 K_i M}) \right)

-12A_3 Br C_2 y + G_1 + yG_2. \tag{32}
\]

Here constants Bi’s, Ci’s, Gi’s and Fi’s can be computed via MATHEMATICA.

### 4. Discussion

This portion is devoted to the analysis of velocity, temperature, entropy generation, Bejan numbers and stream lines. Each quantity is analyzed in different subsections.

#### 4.1. Analysis of velocity

In this subsection behavior of velocity is discussed under the influence of different important parameters. Fig. 2 represents impact of nanoparticle volume fraction for velocity profile. This graphs shows the decreasing behavior which is related to the fact that by increasing the quantity of nanoparticles (as \( \phi = 0.01, 0.03, 0.05, 0.07 \)) resistance to the fluid increases so fluid velocity decays. Here the values for H-C’s model are greater than the Maxwell’s model. Fig. 3 has been plotted against Hartman number. It elucidates that velocity has decreasing behavior for larger Hartman number \( (M = 2, 3, 4, 5) \). As the Lorentz force acts as the resistive force. Grashof
number behavior can be notified through Fig. 4. Here we see increment in velocity profile by enhancing Grashof number \( Gr = 0.1 - 0.7 \). This is due to increase in buoyancy forces which facilitate the flow. Velocity profile for wall parameters can be observed through Fig. 5. The results illustrate that the velocity has the increasing behavior for elastance parameters \( E_1 = 0.01, 0.02 \) and \( E_2 = 0.02, 0.04 \) whereas decreasing behavior is observed for the damping parameter \( E_3 = 0.01, 0.02 \). Obviously elastance parameters provide less resistance so velocity increases whereas as damping resist the flow. Slip parameters result is demonstrated through Figs. 6 and 7. Here we have observed that the velocity profile shows enhancement when we enhance the slip parameters \( \beta_1 = 0.1, 0.3, 0.5, 0.7 \) and \( \beta_2 = -0.1, -0.3, -0.5, -0.7 \). We also noticed that this behavior is more prominent for second order slip parameter than the first order. Further the velocity profile is noted higher for case of Hamilton-Crosser's than the Maxwell's model.

4.2. Analysis of temperature

In this subsection the temperature profile for different pertinent parameters are displayed. Fig. 8 provides graphs for \( \phi = 0.01, 0.03, 0.05, 0.07 \) versus temperature distribution. This graph represents that the temperature profile is decays for \( \phi \). As increase in \( \phi \) enhances the thermal conductivity and cooling capabilities as well. Moreover the temperature is higher for Maxwell's model rather than H-C model. Fig. 9 plots for the results of Hartman number on \( \theta \). This Fig. demonstrates the decreasing behavior of temperature where Hartman number increases from \( 2 - 5 \). Increment is seen in temperature profile by varying the values of Grashof number (as \( 0.1 - 0.7 \)) (see Fig. 10). As \( Gr \) increasing the velocity so the mean kinetic energy of the particles. Hence increase in temperature. Wall parameters impact is elucidated through Fig. 11. It shows the similar behavior as in case of velocity profile when we varies the parameters as \( E_1 = 0.01, 0.02 \), \( E_2 = 0.02, 0.04 \) and \( E_3 = 0.01, 0.02 \). The reasons can be linked to velocity. First and second order thermal slip parameters outcome is seen through Figs. 12 and 13. For first order thermal slip as we take the values \( \gamma_1 = 0.01, 0.03, 0.05, 0.07 \) the temperature increases throughout the channel whereas for second order thermal slip parameter ranges from \( \gamma_2 = -0.01 \) to \(-0.07 \) the temperature increases near the center. A comparative study reveals that the temperature remains higher for spherical shaped particles than cylindrical shaped ones.

4.3. Analysis of entropy generation and Bejan number
This subsection consists of entropy generation and Bejan number for different embedded parameters. To explain the impact of $\phi$ on entropy generation Fig. 14 is sketched. The results display that the entropy generation decreases with larger values of $\phi$ i.e. 0.01, 0.03, 0.05, 0.07. It is due to decrease in temperature for larger nanoparticles volume fraction as entropy of system is directly linked with temperature. Fig. 15 portrayed the results for Hartman number. Through larger values of Hartman number (as 1.0, 1.5, 2.0, 2.5) the entropy generation decreases. Grashof number has increasing impact on $Ns$ as $Gr$ takes the values between (0.1–0.7) (see Fig. 16). The result in this case is qualitatively similar to temperature. Entropy generation enhances when the ratio of $Br$ to $\Lambda$ enlarges (0.1–0.7) (see Fig. 17). To notify the influence of wall parameters Fig. 18 is sketched. Entropy generation is increasing function for $E_1 (=0.01, 0.02)$ and $E_2 (=0.02, 0.04)$ whereas it is decreasing function of $E_3 (=0.01, 0.02)$. 

For all cases the values for H-C model is greater than Maxwell's model. For the behavior of Bejan numbers on pertinent parameters Figs. 19-23 are drawn. Fig. 19 displays the nanoparticle volume fraction impact on Be. The inverse relation is seen between Bejan number and nanoparticle volume fraction i.e. increment in $\phi$ (0.1–0.7) decreases Bejan number. For Hartman number as varies between 1.0–2.5 decreased is noticed (see Fig. 20). Fig. 21 is drawn for results of Grashof number (0.1–0.7) versus Bejan number. This Fig. portrayed that the direct relation is seen between Bejan and Grashof numbers. Bejan number enhances via enhancement in ratio of $Br$ to $\Lambda$ as 0.1–0.7 (see Fig. 22). The wall parameters results are revealed by Fig. 23. An enhancement is seen for larger elastance parameters $E_1 (=0.01, 0.02)$ and $E_2 (=0.02, 0.04)$ whereas decay is observed for the case of larger damping parameter $E_3 (=0.01, 0.02)$. Moreover, in all cases the values of Hamilton-Crosser’s model are less than Maxwell's model.

4.4. Streamlines
The streamlines are plotted for description of trapping. Fig. 24 (a) and (b) displayed the impact of Hartman number for Maxwell model whereas Fig. 24 (c) and (d) portrayed the influence for Hamilton Crosser model. For both cases the size of trapped bolus increases with larger values of Hartman number ($M = 1.0, 2.0$). Figs. 25 and 26 (a)–(d) are sketched for behavior of first and second order slip parameters. These streamlines indicate that trapped bolus size enhances via enhancement in first order slip as (0.01, 0.03) and second order slip parameter as (−0.01, −0.03). Walls parameters impact for Maxwell model can be observed via Fig. 27 (a)–(d). However Fig. 27 (e)–(h) are for Hamilton-Crosser model. Both models show same behavior for these parameters i.e. trapped bolus size increases for $E_1 (=0.7, 0.9)$ and $E_2 (=0.4, 0.6)$ whereas decrease is noticed for $E_3 (=0.2, 0.5)$. 

Figure 14
Figure 15
Figure 16
Figure 17
Figure 18
Figure 19
Figure 20
Figure 21
Figure 22
Figure 23

4.4. Streamlines
The streamlines are plotted for description of trapping. Fig. 24 (a) and (b) displayed the impact of Hartman number for Maxwell model whereas Fig. 24 (c) and (d) portrayed the influence for Hamilton Crosser model. For both cases the size of trapped bolus increases with larger values of Hartman number ($M = 1.0, 2.0$). Figs. 25 and 26 (a)–(d) are sketched for behavior of first and second order slip parameters. These streamlines indicate that trapped bolus size enhances via enhancement in first order slip as (0.01, 0.03) and second order slip parameter as (−0.01, −0.03). Walls parameters impact for Maxwell model can be observed via Fig. 27 (a)–(d). However Fig. 27 (e)–(h) are for Hamilton-Crosser model. Both models show same behavior for these parameters i.e. trapped bolus size increases for $E_1 (=0.7, 0.9)$ and $E_2 (=0.4, 0.6)$ whereas decrease is noticed for $E_3 (=0.2, 0.5)$. 

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Figure 14
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Figure 22
Figure 23
5. Conclusions
Some main observations of this attempt are:

- Enhancement in velocity is seen for both first order and second order velocity slip in both models whereas reduction is observed for case of nanoparticle volume fraction.
- The values in Hamilton-Crosser model remain higher than Maxwell's model especially near the channel center for velocity profile.
- Grashof and Hartman numbers for velocity have opposite effect.
- Results obtained indicate that quantities in Maxwell's model exceeds than Hamilton-Crosser model in case of temperature.
- Enhancement is observed in entropy generation number for larger $Br\Lambda^{-1}$ and Grashof number. Moreover inverse behavior of entropy generation number is obtained for the case of Hartman number and nanoparticle volume fraction.
- Bolus sizes increases in trapping phenomenon for the case of both first and second order velocity slip parameters.
- Bolus sizes reduces for $E_3$ and it enhances for $E_1$ and $E_2$ in both models.

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References


Captions List:

Fig. 1. Flow geometry
Fig. 2. $u$ via $\phi$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $M=1.0$, $\varepsilon=0.2$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 3. $u$ via $M$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $\phi=0.01$ $Br=3.0$ $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 4. $u$ via $Gr$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 5. $u$ via $E_{i}$, $E_{2}$, $E_{3}$ when $t=0.1$, $x=0.2$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 6. $u$ via $\beta_{1}$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 7. $u$ via $\beta_{2}$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 8. $\theta$ via $\phi$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 9. $\theta$ via $M$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 10. $\theta$ via $Gr$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 11. $\theta$ via $E_{i}$, $E_{2}$, $E_{3}$ when $t=0.1$, $x=0.2$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 12. $\gamma_{1}$ via $\gamma_{1}$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 13. $\gamma_{2}$ via $\gamma_{2}$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $\phi=0.1$, $Br=3.0$, $Gr=0.03$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$.

Fig. 14. $Ns$ via $\phi$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$, $Br\Lambda^{-1}=1.0$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 15. $Ns$ via $M$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $\phi=0.1$, $Br\Lambda^{-1}=1.0$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 16. $Ns$ via $Gr$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $\phi=0.1$, $M=1.0$, $Br\Lambda^{-1}=1.0$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 17. $Ns$ via $Br\Lambda^{-1}$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $\phi=0.1$, $M=1.0$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 18. $Ns$ via $E_{i}$, $E_{2}$, $E_{3}$ when $t=0.1$, $x=0.2$, $\varepsilon=0.2$, $\phi=0.1$, $M=1.0$, $Br\Lambda^{-1}=1.0$, $Gr=0.03$, $Br=3.0$, $\beta_{1}=0.01$, $\beta_{2}=-0.01$, $\gamma_{1}=0.01$, $\gamma_{2}=-0.01$.

Fig. 19. $Be$ via $\phi$ when $t=0.1$, $E_{3}=0.01$, $x=0.2$, $E_{2}=0.01$, $E_{i}=0.02$, $\varepsilon=0.2$, $M=1.0$. 

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$Br\Lambda^{-1} = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$.

**Fig. 20.** $Be$ via $M$ when $t = 0.1$, $E_1 = 0.01$, $x = 0.2$, $E_2 = 0.01$, $E_3 = 0.02$, $\varepsilon = 0.2$, $\phi = 0.1$, $Br\Lambda^{-1} = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$.

**Fig. 21.** $Be$ via $Gr$ when $t = 0.1$, $E_2 = 0.01$, $x = 0.2$, $E_2 = 0.01$, $E_3 = 0.02$, $\varepsilon = 0.2$, $\phi = 0.1$, $M = 1.0$, $Br\Lambda^{-1} = 1.0$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$.

**Fig. 22.** $Be$ via $Br\Lambda^{-1}$ when $t = 0.1$, $E_3 = 0.01$, $x = 0.2$, $E_2 = 0.01$, $E_1 = 0.02$, $\varepsilon = 0.2$, $\phi = 0.1$, $M = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$.

**Fig. 23.** $Be$ via $E_1$, $E_2$, $E_3$ when $t = 0.1$, $x = 0.2$, $\varepsilon = 0.2$, $\phi = 0.1$, $M = 1.0$, $Br\Lambda^{-1} = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$.

**Fig. 24.** $\psi$ via $M$ for Maxwell model when $E_3 = 0.01$, $t = 0$, $E_2 = 0.01$, $\varepsilon = 0.2$, $E_4 = 0.02$, $\phi = 0.1$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$. (a) $M = 1.0$. (b) $M = 2.0$.

**Fig. 25.** $\psi$ via $M$ for Hamilton-Boxer model when $E_3 = 0.01$, $t = 0$, $E_2 = 0.01$, $\varepsilon = 0.2$, $E_4 = 0.02$, $\phi = 0.1$, $M = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$. (a) $\beta_i = 0.01$. (b) $\beta_i = 0.03$.

**Fig. 26.** $\psi$ via $M$ for Hamilton-Boxer model when $E_3 = 0.01$, $t = 0$, $E_2 = 0.01$, $\varepsilon = 0.2$, $E_4 = 0.02$, $\phi = 0.1$, $M = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$. (a) $\beta_2 = -0.01$. (b) $\beta_2 = 0.03$.

**Fig. 27.** $\psi$ via $E_1$, $E_2$, $E_3$ for Maxwell model when $t = 0$, $\varepsilon = 0.2$, $\phi = 0.1$, $M = 1.0$, $Gr = 0.03$, $Br = 3.0$, $\beta_1 = 0.01$, $\beta_2 = -0.01$, $\gamma_1 = 0.01$, $\gamma_2 = -0.01$. (a) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.2$. (b) $E_1 = 0.9$, $E_2 = 0.4$, $E_3 = 0.2$. (c) $E_1 = 0.7$, $E_2 = 0.6$, $E_3 = 0.2$. (d) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.5$. (e) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.2$. (f) $E_1 = 0.9$, $E_2 = 0.4$, $E_3 = 0.2$. (g) $E_1 = 0.7$, $E_2 = 0.6$, $E_3 = 0.2$. (h) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.5$. (i) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.2$. (j) $E_1 = 0.7$, $E_2 = 0.6$, $E_3 = 0.2$. (k) $E_1 = 0.7$, $E_2 = 0.4$, $E_3 = 0.5$.
Table 1: Thermophysical parameters of water and nanoparticles [13].

Figures and Table:
Fig. 10

\( \theta \), \( \text{Gr} = 0.1, 0.3, 0.5, 0.7 \)

Fig. 11

\( \theta \), \( E_1, E_2, E_3 = 0.01, 0.02, 0.01, 0.02, 0.01, 0.01, 0.04, 0.01 \)

Fig. 12

\( \gamma_1 = 0.01, 0.03, 0.05, 0.07 \)
Fig. 13

\[ \gamma_2 = -0.01, -0.03, -0.05, 0.07 \]

\[ \phi = 0.01, 0.03, 0.05, 0.07 \]

Fig. 14

\[ M = 1.0, 1.5, 2.0, 2.5 \]

Fig. 15
Fig. 19

\[ Be \]
\[ \phi = 0.1, 0.3, 0.5, 0.7 \]

Fig. 20

\[ Be \]
\[ M = 1.0, 1.5, 2.0, 2.5 \]

Fig. 21

\[ Be \]
\[ Gr = 0.1, 0.3, 0.5, 0.7 \]
Fig. 22

- Plot with lines labeled "Maxwell" and "H-C"
- BrA^{-1} = 0.1, 0.3, 0.5, 0.7
- Y-axis: Be
- X-axis: "Y"

Fig. 23

- Plot with lines labeled "Maxwell" and "H-C"
- E_1, E_2, E_3 = 0.01, 0.02, 0.01
- E_1, E_2, E_3 = 0.02, 0.02, 0.01
- E_1, E_2, E_3 = 0.01, 0.04, 0.01
- E_1, E_2, E_3 = 0.01, 0.02, 0.04
- Y-axis: Be
- X-axis: "Y"

Fig. 24:

- Four subplots labeled (a) and (b)
- Color scales ranging from light to dark purple
Fig. 26: (a), (b), (c), (d)

Fig. 27: (a), (b), (c), (d)
Fig. 27: (e), (f), (g), (h)

Table 1

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Biography of authors:

- **Tasawar Hayat** is a Pakistani leading mathematician who has done pioneering research contributions in the area of fluid mechanics. Due to his a lot of achievement he was awarded by many awards. He is currently serving with the honor of National Distinguished Professor in Quaid-I-Azam university, Pakistan.
- **Sadaf Nawaz** is Ph.D student of mathematics at Quaid-i-Azam university, Pakistan. Her research interests are fluid mechanics, peristalsis, nanofluid, non-linear flow problems, entropy analysis and heat transfer.
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