Multiple attribute decision making based on Muirhead mean operators under 2-tuple linguistic Pythagorean fuzzy environment

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Abstract: In this paper, we extend the Muirhead mean (MM) operator and dual MM (DMM) operator with 2-tuple linguistic Pythagorean fuzzy numbers (2TLPFNs) to define the 2-tuple linguistic Pythagorean fuzzy MM (2TLPFMM) operator, 2-tuple linguistic Pythagorean fuzzy weighted MM (2TLPFWMM) operator, 2-tuple linguistic Pythagorean fuzzy DMM (2TLPFDMM) operator and 2-tuple linguistic Pythagorean fuzzy weighted DMM (2TLPFNWDMM) operator. Based on these proposed operators, two methods are developed to deal with the multiple attribute decision making (MADM) problems with 2TLPFNs and the validity and advantages of the proposed method are analyzed by comparison with some existing approaches. The methods proposed in this paper can effectively handle the MADM problems with 2TLPFNs. Finally, an example for green supplier selection is given to show the proposed methods.

Keywords: multiple attribute decision making (MADM); Pythagorean fuzzy numbers(PFNs)； 2-tuple linguistic Pythagorean fuzzy set (2TLPFSs); 2TLPFMM operator; 2TLPFDMM operator; green supplier selection

1. Introduction

Recently, Pythagorean fuzzy set (PFS) [1-2] has been proposed with the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1. Zhang and Xu[3]designed the TOPSIS for MADM with PFNs. Peng and Yang[4] defined the superiority and inferiority ranking model to cope with multiple attribute group decision making (MAGDM) with PFNs. Beliakov and James [5] investigated the “averaging” under PFNs. Reformat and Yager [6] handled the recommender system under PFNs. Gou et al.[7] researched the properties of continuous PFNs. Garg[8] developed the generalized Einstein operations with PFNs. Zeng et al. [9] defined the hybrid model to solve the MADM with PFNs. Garg[10] studied an accuracy function with IVPFNs. Ren et al.[11] extended TODIM to solve the MADM with PFNs. Wei & Lu[12] extended MSM (Maclaurin symmetric mean) operator[13] with PFNs. Wei[14] developed some interaction operators under PFNs. Wu & Wei[15]proposed Hamacher operators with PFNs. Wei & Lu[16] defined some Hamacher operators under dual hesitant PFNs. Lu et al.[17] proposed some Hamacher operators with hesitant PFNs. Wei et al.[18] gave the Pythagorean hesitant fuzzy hamacher operators. Gao et al.[19] proposed the interaction operators under PFNs in MADM. Garg [20] defined some generalized geometric interactive operators based on Einstein operations with PFNs. Wei & Wei [21] defined the similarity measures of PFNs based on cosine function. Wei & Lu[22] developed the power operators with PFNs in MADM. Garg [23] proposed a new decision-making model with probabilistic information and immediate probabilities in order to aggregate the PFNs. Liang [24] gave the Bonferroni mean operators under PFNs. Garg

[25] proposed a novel correlation coefficients between PFNs. Wang et al. [26] defined the generalized Dice similarity measures to deal with MAGDM with PFNs. Tang et al. [27] defined some MM operators for green suppliers selection with interval-valued PFNs (IVPFNs). Khan et al.[28] extended TOPSIS method on the basis of Choquet integral with IVPFNs. Wan et al.[29] used the mathematical programming method to solve MAGDM with PPFNs. Garg [30] defined some exponential operations for IVPFNs. Garg [31] defined the improved accuracy function of IVPFNs. Garg [32] proposed the improved score function of IVPFNs based on the TOPSIS method.

However, all the above methods and models are not useful to depict the truth-membership degree and falsity-membership degree information of an element to a set by 2-tuple linguistic variables according to the given linguistic term sets, which can reflect the decision maker's confidence level [33-37]. In order to overcome this issue, Deng et al. [38] proposed the 2-tuple linguistic Pythagorean fuzzy set (2TLPFSs) to solve this issue on the basis of the PFS [1-2] and 2-tuple linguistic sets [39-40]. Deng et al. [41] proposed some Hamy mean operators with 2TLPFNs. And Muirhead mean (MM) operator [42] is a useful to depict interrelationships among any number by a variable vector. Therefore, the MM operator can give a robust and flexible mechanism to aggregate information in MADM. Because the 2TLPFNs can easily describe the fuzzy and uncertain information, and the MM can depict interrelationships among any number by a variable vector, thus, it is very necessary to extend the MM operator to deal with the 2TLPFNs.

The purpose of this work is to extend the MM operator to 2TLPFNs to study MADM problems more effectively. Thus the main contribution of this paper is: (1) the MADM problems are investigated with 2TLPFNs; (2) some MM operator and dual MM operator are proposed with 2TLPFNs and some properties of these operators are analyzed; (3) some novel algorithms are proposed to solve MADM problems based on these operators with 2TLPFNs; (4) a numerical case for green supplier selection is given to illustrate the advantages of the new method.

For the sake of clarity, the rest of this research is organized as follows. In section 2, the concept of 2TLPFSs is proposed. In Section 3, some MM operators with 2TLPFNs are defined. In Section 4, an example is given for green supplier selection. Section 5 concludes this paper.

2. Preliminaries

The concept of 2-tuple linguistic sets (2TLSs), Pythagorean fuzzy sets (PFSs) and 2TLPFSs are introduced in this section.

2.1. 2TLSs

Definition 1[39-40]. Let $S=\{s_i \mid i=0,1,\ldots,t\}$ be a linguistic term set with odd cardinality. s_i denoted the possible value in a linguistic variable, and S can be depicted as:

$$S=\left\{\begin{matrix} s_0=extremely\ poor, s_1=very\ poor, s_2=poor, s_3=medium, \\ s_4=good, s_5=very\ good, s_6=extremely\ good. \end{matrix}\right\}$$

2.2. PFSs

Let X be a space of points (objects) with a generic element in the fixed set X, denoted by x. Pythagorean fuzzy sets (PFSs) A in X is shown as following [1-2]:

$$A=\left\{\left\langle x,\mathrm{u}_A(x),v_A(x)\right\rangle \mid x\in X\right\} \tag{1}$$

which $u_A(x)$ and $v_A(x)$ denotes the membership and non-membership degrees, which satisfies $u_A(x):X\to[0,1], v_A(x):X\to[0,1]$ and $(u_A(x))^2+(v_A(x))^2\le 1$.

2.3. 2TLPFSs

Deng et al. [38] gave the definition of 2-tuple linguistic Pythagorean fuzzy set (2TLPFSs).

Definition 2[38]. Assume that $P=\{p_0,p_1,\dots,p_t\}$ is a 2TLSs with odd cardinality $t+1$. If $p=\{(s_\phi,\varphi),(s_\theta,\vartheta)\}$ is defined for $(s_\phi,\varphi),(s_\theta,\vartheta)\in P$ *and* $\varphi,\vartheta\in[0,t]$, where (s_ϕ,φ) *and* (s_θ,ϑ) depict independently the truth degree, indeterminacy degree, and falsity degree by 2TLSs, then the definition of 2TLPFSs is defined as follows:

$$p_j=\left\{\left(s_{\phi_j},\varphi_j\right),\left(s_{\theta_j},\vartheta_j\right)\right\} \tag{2}$$

where $0\le\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\le t, 0\le\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\le t$, and $0\le\left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2+\left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2\le t^2$.

Then, the score and accuracy function of 2TLPFNs is given as follows:

Definition 3[38]. Let $p_1=\left\{\left(s_{\phi_1},\varphi_1\right),\left(s_{\theta_1},\vartheta_1\right)\right\}$ be a 2TLPFNs in P. Then the score and accuracy functions of p_1 are defined as follows:

$$S(p_1)=\Delta\left\{\frac{t}{2}\left(1+\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\right)^2-\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\right)^2\right)\right\}, S(p_1)\in[0,t] \tag{3}$$

$$H(p_1)=\Delta\left\{t\left(\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\right)^2\right)\right\}, H(p_1)\in[0,t]. \tag{4}$$

Furthermore, Deng et al. [38] proposed the comparison laws between 2TLPFNs:

Definition 4[38]. Let $p_1=\left\{\left(s_{\phi_1},\varphi_1\right),\left(s_{\theta_1},\vartheta_1\right)\right\}$ and $p_2=\left\{\left(s_{\phi_2},\varphi_2\right),\left(s_{\theta_2},\vartheta_2\right)\right\}$ be two 2TLPNs, then

(1) if $S(p_1)<S(p_2)$, then $p_1<p_2$;

(2) if $S(p_1)>S(p_2)$, then $p_1>p_2$;

(3) if $S(p_1)=S(p_2), H(p_1)<H(p_2)$, then $p_1<p_2$;

(4) if $S(p_1)=S(p_2), H(p_1)>H(p_2)$, then $p_1>p_2$;

(5) if $S(p_1)=S(p_2), H(p_1)=H(p_2)$, then $p_1=p_2$.

Then, Deng et al. [38] defined some new operations on the 2TLPFNs.

Definition 5[38]. Let $p_1 = \left\{\left(s_{\phi_1}, \varphi_1\right), \left(s_{\theta_1}, \vartheta_1\right)\right\}$ and $p_2 = \left\{\left(s_{\phi_2}, \varphi_2\right), \left(s_{\theta_2}, \vartheta_2\right)\right\}$ be two 2TLPFNs, then

(1) $$p_1 \oplus p_2 = \left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_2},\varphi_2\right)}{t}\right)^2\right)}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\bullet\frac{\Delta^{-1}\left(s_{\theta_2},\vartheta_2\right)}{t}\right)\right)\right\};$$

(2) $$p_1 \otimes p_2 = \left\{\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\bullet\frac{\Delta^{-1}\left(s_{\phi_2},\varphi_2\right)}{t}\right)\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_2},\vartheta_2\right)}{t}\right)^2\right)}\right)\right\};$$

(3) $$\lambda p_1 = \left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\right)^2\right)^\lambda}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\right)^\lambda\right)\right\};$$

(4) $$\left(p_1\right)^\lambda = \left\{\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_1},\varphi_1\right)}{t}\right)^\lambda\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_1},\vartheta_1\right)}{t}\right)^2\right)^\lambda}\right)\right\}.$$

2.4. MM operators

Muirhead [42] proposed the Muirhead mean (MM) operator.

Definition 6[42]. Let $a_j\,(j = 1, 2, \ldots, n)$ be a set of nonnegative real numbers, and $P = (p_1, p_2, \ldots, p_n) \in R^n$ be a vector of parameters. If

$$\mathrm{MM}^{\mathrm{P}}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n!}\sum_{\sigma\in S_n}\prod_{j=1}^{n} a_{\sigma(j)}^{p_j}\right)^{\frac{1}{\sum_{j=1}^{n} p_j}} \tag{5}$$

where $\sigma(j)(j = 1, 2, \ldots, n)$ is any a permutation of $\{1, 2, \ldots, n\}$ and S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

3. Some MM operators with 2TLPFNs

3.1 The 2TLPFMM operator

This section proposes some MM operators and dual MM operators with 2TLPFNs.

Definition 7. Let $p_j = \left\{\left(s_{\phi_j}, \varphi_j\right), \left(s_{\theta_j}, \vartheta_j\right)\right\}$ be a group of 2TLPFNs. The 2-tuple linguistic Pythagorean fuzzy MM (2TLPFMM) operator is:

$$2\mathrm{TLPFMM}^{\lambda}(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\underset{j=1}{\overset{n}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \tag{6}$$

Theorem 1. Let $p_j = \left\{\left(s_{\phi_j}, \varphi_j\right), \left(s_{\theta_j}, \vartheta_j\right)\right\}$ be a group of 2TLPFNs. The fused value by using 2TLPFMM operators is also a 2TLPFN where

$$
\begin{aligned}
&2\text{TLPFMM}^{\lambda}(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!} \left(\underset{\sigma \in S_n}{\oplus} \left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j} \right) \right) \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \\
&= \left\{ \begin{array}{l}
\Delta\left(t \left(\sqrt{1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right) \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right), \\
\Delta\left(t \sqrt{1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right)
\end{array} \right\}
\end{aligned} \tag{7}
$$

Proof:

$$
p_{\sigma(j)}^{\lambda_j} = \left\{ \Delta\left(t \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{\lambda_j} \right), \Delta\left(t \sqrt{1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j}} \right) \right\} \tag{8}
$$

Thus,

$$
\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j} = \left\{ \Delta\left(\text{t} \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{\lambda_j} \right), \Delta\left(t \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j}} \right) \right\} \tag{9}
$$

Thereafter,

$$
\underset{\sigma \in S_n}{\oplus} \left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j} \right) = \left\{ \begin{array}{l}
\Delta\left(t \sqrt{1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right)} \right), \\
\Delta\left(t \sqrt{\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j} \right)} \right)
\end{array} \right\} \tag{10}
$$

Furthermore,

$$\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)=\left\{\begin{array}{l}\Delta\left(t\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right),\\ \Delta\left(t\left(\sqrt{\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)}\right)^{\frac{1}{n!}}\right)\end{array}\right\} \tag{11}$$

Therefore,

$$\begin{aligned}&2\text{TLPFMM}^{\lambda}(p_1,p_2,\ldots,p_n)=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\\ &=\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\end{array}\right\}\end{aligned} \tag{12}$$

Hence, (7) is kept.

Then we prove that (7) is a 2TLPFN. So, we shall prove these two conditions: ① $0\le\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\le t, 0\le\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\le t$; ② $0\le\left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2+\left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2\le t^2$.

Let

$$\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}=\left(\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}=\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

Proof. ①Since $0 \le \frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \le 1$, we get

$$0 \le \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j} \le 1 \ \ and \ \ 0 \le 1 - \prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j} \le 1 \tag{13}$$

Then,

$$0 \le \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}} \le 1 \tag{14}$$

$$0 \le \left(\sqrt{1\text{-}\left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \le 1 \tag{15}$$

That means $0 \le \Delta^{-1}\left(s_{\phi_j}, \varphi_j\right) \le t$, so ① is kept. Similarly, we can have $0 \le \Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right) \le t$.

② Since $\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2 \le 1$, we have the following inequality

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2$$

$$= \left(1\text{-}\left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} + \left(1 - \left(1 - \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$$\le \left(1\text{-}\left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1\text{-}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} + \left(1 - \left(1 - \left(\prod_{\sigma \in S_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)\right)^{\frac{1}{\mathrm{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)$$

$=1$ i.e.,

$$0 \le \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^2 \le 1.$$

That means $0 \le \left(\Delta^{-1}\left(s_{\phi_j}, \varphi_j\right)\right)^2 + \left(\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)\right)^2 \le t^2$, so ②is maintained.

Example 1. Let $\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\}$, and $\{(s_5,0.3),(s_3,\text{-}0.2)\}$ be three 2TLPFNs, and λ=(0.2,0.3,0.5) then according to (7), we have

$$\text{2TLPFMM}^{(0.2,0.3,0.5)}\begin{pmatrix}\{(s_3,0.4),(s_2,-0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{pmatrix}$$

$$=\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{\text{n!}}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\end{array}\right\}$$

$$
=\left\{\begin{array}{l}
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}\left(1-\left(\frac{3.4}{6}\right)^{0.4}\times\left(\frac{2.3}{6}\right)^{0.6}\times\left(\frac{5.3}{6}\right)^{1}\right)\times\left(1-\left(\frac{3.4}{6}\right)^{0.4}\times\left(\frac{5.3}{6}\right)^{0.6}\times\left(\frac{2.3}{6}\right)^{1}\right)\\
\times\left(1-\left(\frac{2.3}{6}\right)^{0.4}\times\left(\frac{3.4}{6}\right)^{0.6}\times\left(\frac{5.3}{6}\right)^{1}\right)\times\left(1-\left(\frac{2.3}{6}\right)^{0.4}\times\left(\frac{5.3}{6}\right)^{0.6}\times\left(\frac{3.4}{6}\right)^{1}\right)\\
\times\left(1-\left(\frac{5.3}{6}\right)^{0.4}\times\left(\frac{3.4}{6}\right)^{0.6}\times\left(\frac{2.3}{6}\right)^{1}\right)\times\left(1-\left(\frac{5.3}{6}\right)^{0.4}\times\left(\frac{2.3}{6}\right)^{0.6}\times\left(\frac{3.4}{6}\right)^{1}\right)\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right),\\
\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{0.5}\right)\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.5}}}\right),
\end{array}\right\}
$$

$$=\left\{\left(\mathrm{s}_3,0.5036\right),\left(\mathrm{s}_5,0.5351\right)\right\}$$

Then we shall discuss some properties of 2TLPFMM operator.

Property 1. (Idempotency) If $p_j=\left\{\left(s_{\phi_j},\varphi_j\right),\left(s_{\theta_j},\vartheta_j\right)\right\}(j=1,2,\ldots,n)$ are equal, then

$$\text{2TLPFNMM}^{\lambda}\left(\mathrm{p}_1,p_2,\cdots,p_n\right)=p \tag{16}$$

Proof. Since $p_{\sigma(j)}=p=\left\{\left(s_{\phi_j},\varphi_j\right),\left(s_{\theta_j},\vartheta_j\right)\right\}$, then

$$2TLPFMM^{\lambda}(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!} \left(\underset{\sigma \in S_n}{\oplus} \left(\overset{n}{\underset{j=1}{\otimes}} p_{\sigma(j)}^{\lambda_j} \right) \right) \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}$$

$$= \left\{ \begin{array}{l} \Delta\left(t \left(\sqrt{1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right) \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right), \\ \Delta\left(t \sqrt{1 - \left(1 - \left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \Delta\left(t \left(\sqrt{1 - \left(\left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \right)^{2\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right), \\ \Delta\left(t \sqrt{1 - \left(1 - \left(\left(\prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t} \right)^{2} \right)^{\lambda_j} \right) \right)^{\frac{1}{n!}} \right)^{n!} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right) \end{array} \right\}$$

$$= \Delta\left\{ \Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right), \Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right) \right\} = p$$

Property 2. (Monotonicity) Let $p_{x_i} = \left\{ \left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right) \right\} (i = 1, 2, \ldots, n)$ and $p_{y_i} = \left\{ \left(s_{\phi_{y_i}}, \varphi_{y_i}\right), \left(s_{\theta_{y_i}}, \vartheta_{y_i}\right) \right\} (i = 1, 2, \ldots, n)$ be two sets of 2TLPFNs. If $\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right) \leq \Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right),\ and\ \Delta^{-1}\left(s_{\theta_{x_i}}, \vartheta_{x_i}\right) \geq \Delta^{-1}\left(s_{\theta_{y_i}}, \vartheta_{y_i}\right)$ hold for all i, then

$$2TLPFMM^{\lambda}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_n}\right) \leq 2TLPFMM^{\lambda}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_n}\right) \tag{17}$$

Proof.

Let $2TLPFMM^{\lambda}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_n}\right) = \left\{ \left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right) \right\} (i = 1, 2, \ldots, n)$ and $2TLPFMM^{\lambda}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_n}\right) = \left\{ \left(s_{\phi_{y_i}}, \varphi_{y_i}\right), \left(s_{\theta_{y_i}}, \vartheta_{y_i}\right) \right\} (i = 1, 2, \ldots, n)$, given that $\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right) \leq \Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)$, we can obtain

$$\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j} \le \prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j} \tag{18}$$

$$1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j} \ge 1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j} \tag{19}$$

Thereafter,

$$1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}} \le 1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}} \tag{20}$$

Furthermore,

$$t\left(\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \le t\left(\sqrt{1\text{-}\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \tag{21}$$

That means $\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)\le\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)$. Similarly, we can have $\Delta^{-1}\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\ge\Delta^{-1}\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)$.

If $\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)<\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)$ and $\Delta^{-1}\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\ge\Delta^{-1}\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)$

$$\text{2TLPNMM}^{\lambda}\left(p_{x_1},p_{x_2},\cdots,p_{x_n}\right)<\text{2TLPNMM}^{\lambda}\left(p_{y_1},p_{y_2},\cdots,p_{y_n}\right)$$

If $\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)=\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)$ and *and* $\Delta^{-1}\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)=\Delta^{-1}\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)$

$$\text{2TLPNMM}^{\lambda}\left(p_{x_1},p_{x_2},\cdots,p_{x_n}\right)=\text{2TLPNMM}^{\lambda}\left(p_{y_1},p_{y_2},\cdots,p_{y_n}\right)$$

Thus, property 2 is right.

Property 3. (Boundedness) Let $p_i=\left\{\left(s_{\phi_i},\varphi_i\right),\left(s_{\theta_i},\vartheta_i\right)\right\}\left(i=1,2,\ldots,n\right)$ be a group of 2TLPFNs. If $p^{+}=\left(\max_i\left(S_{\phi_i},\varphi_i\right),\min_i\left(S_{\theta_i},\vartheta_i\right)\right)$ and $p^{-}=\left(\min_i\left(S_{\phi_i},\varphi_i\right),\max_i\left(S_{\theta_i},\vartheta_i\right)\right)$ then

$$p^{-}\le\text{2TLPFMM}^{\lambda}\left(p_1,p_2,\cdots,p_n\right)\le p^{+} \tag{22}$$

From property 1,

$$\text{2TLPFMM}^{\lambda}\left(p_1^{-},p_2^{-},\cdots,p_n^{-}\right)=p^{-}$$
$$\text{2TLPFMM}^{\lambda}\left(p_1^{+},p_2^{+},\cdots,p_n^{+}\right)=p^{+}$$

From property 2,

$$p^{-}\le\text{2TLPFMM}^{\lambda}\left(p_1,p_2,\cdots,p_n\right)\le p^{+}$$

3.2 The 2TLPFWMM operator

In real MADM, it's very important to pay attention to attribute weights. the 2-tuple linguistic Pythagorean number weighted MM (2TLPFWMM) operator is defined in this section.

Definition 8. Let $p_{x_i} = \left\{\left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right)\right\}(i = 1, 2, \ldots, n)$ be a group of 2TLPFNs with their weight vector be $w_i = (w_1, w_2, \ldots, w_n)^T$, and satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ be a vector of parameters. If

$$\text{2TLPFWMM}_{nw}^{\lambda}(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!}\left(\underset{\sigma \in S_n}{\oplus}\left(\underset{j=1}{\overset{n}{\otimes}}\left(nw_{\sigma(j)} p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \tag{23}$$

where $\sigma(j)(j = 1, 2, \ldots, n)$ is any a permutation of $\{1, 2, \ldots, n\}$ and S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

Theorem 2. Let $p_{x_i} = \left\{\left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right)\right\}(i = 1, 2, \ldots, n)$ be a group of 2TLPFNs. The fused value by 2TLPFWMM operators is also a 2TLPFN where

$$\begin{aligned}
&\text{2TLPFWMM}_{nw}^{\lambda}(p_1, p_2, \ldots, p_n) \\
&= \left(\frac{1}{n!}\left(\underset{\sigma \in S_n}{\oplus}\left(\underset{j=1}{\overset{n}{\otimes}}\left(nw_{\sigma(j)} p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right), \\
\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma \in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)
\end{array}\right\}
\end{aligned} \tag{24}$$

Proof:

$$nw_{\sigma(j)} p_{\sigma(j)} = \left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}}\right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{nw_{\sigma(j)}}\right)\right\} \tag{25}$$

Thus,

$$\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}=\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right),\\ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\end{array}\right\} \tag{26}$$

Therefore,

$$\underset{j=1}{\overset{n}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}=\left\{\begin{array}{l}\Delta\left(t\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right),\\ \Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\end{array}\right\} \tag{27}$$

Thereafter,

$$\begin{aligned}&\underset{\sigma\in S_n}{\oplus}\left(\underset{j=1}{\overset{n}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\\ &=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)}\right)\\ \Delta\left(t\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)\end{array}\right\}\end{aligned} \tag{28}$$

Furthermore,

$$\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)$$

$$=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right),\\ \Delta\left(t\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right)\end{array}\right\} \quad (29)$$

Therefore,

$$2\text{TLPFWMM}_{nw}^{\lambda}(p_1,p_2,\ldots,p_n)$$

$$=\left(\frac{1}{n!}\left(\underset{\sigma\in S_n}{\oplus}\left(\overset{n}{\underset{j=1}{\otimes}}\left(nw_{\sigma(j)}p_{\sigma(j)}\right)^{\lambda_j}\right)\right)\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$==\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\end{array}\right\} \quad (30)$$

Hence, (24) is kept.

Then we shall prove that (24) is a 2TLPFN.

① $0\le\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\le t, 0\le\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\le t$

② $0\le\left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2+\left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2\le t^2$

Proof. Let

$$\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}=\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}=\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

① Since $0\le\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\le 1$, we have

$$0\le\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\le 1\ \ and\ \ 0\le\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\le 1 \tag{31}$$

Then

$$0\le 1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\le 1 \tag{32}$$

$$0\le\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\le 1 \tag{33}$$

$$0\le\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\le 1 \tag{34}$$

That means $0\le\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\le t$, so ① is kept, similarly, we can have $0\le\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\le t$.

②Since $\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\le 1$, we can have the following inequality

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2$$

$$=\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$+\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

$$\leq\left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$ i.e.,

$$+\left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}=1$$

$$0\leq\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\leq 1.$$

That means $0\leq\left(\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)\right)^2+\left(\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)\right)^2\leq t^2$, so ② is maintained.

Example 2. Let $\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\}$, and $\{(s_5,0.3),(s_3,\text{-}0.2)\}$ be three 2TLPFNs, and λ=(0.2,0.3,0.5) , w=(0.4,0.2,0.4) then according to (24), we have

$$2TLPFWMM_{(0.4,0.2,0.4)}^{(0.2,0.3,0.5)}\begin{pmatrix}\{(s_3,0.4),(s_2,\text{-}0.3)\},\\ \{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,\text{-}0.2)\}.\end{pmatrix}$$

$$=\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\\ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\end{array}\right\}$$

$$
=\left\{\begin{array}{l}
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)
\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right),\\
\Delta\left(6\times\left(\sqrt{1-\left(1-\left(\begin{array}{l}
\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{1.2}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{1.7}{6}\right)^{2.4}\right)^{0.5}\right)
\end{array}\right)^{\frac{1}{3!}}\right)}\right)^{\frac{1}{0.2+0.3+0.5}}\right),
\end{array}\right\}
$$

$$
=\left\{\left(s_3,0.3821\right),\left(s_4,0.2002\right)\right\}
$$

Then we shall analysis some properties of 2TLPFWMM operator.

Property 4. (Monotonicity) Let $p_{x_i}=\left\{\left(s_{\phi_{x_i}},\varphi_{x_i}\right),\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\right\}(i=1,2,\ldots,n)$ and $p_{y_i}=\left\{\left(s_{\phi_{y_i}},\varphi_{y_i}\right),\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)\right\}(i=1,2,\ldots,n)$ be two sets of 2TLPFNs. If $\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)\le\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right),\ and\ \Delta^{-1}\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\ge\Delta^{-1}\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)$ hold for all i, then

$$2\text{TLPFWMM}_{nw}^{\lambda}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_n}\right) \le 2\text{TLPFWMM}_{nw}^{\lambda}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_n}\right) \tag{35}$$

The proof is similar to 2TLPFMM.

Property 5. (Boundedness) Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}\left(i = 1, 2, \ldots, n\right)$ be a set of 2TLPFNs. If $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$, then

$$p^- \le 2\text{TLPFWMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) \le p^+ \tag{36}$$

From theorem 2, we get

$$2\text{TLPFWMM}_{nw}^{\lambda}\left(p_1^-, p_2^-, \cdots, p_n^-\right)$$
$$= \left\{\begin{array}{l} \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right), \\ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right) \end{array}\right\} \tag{37}$$

$$2\text{TLPFWMM}_{nw}^{\lambda}\left(p_1^+, p_2^+, \cdots, p_n^+\right)$$
$$= \left\{\begin{array}{l} \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right), \\ \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right) \end{array}\right\} \tag{38}$$

From property 4, we get

$$p^- \le 2\text{TLPFWMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) \le p^+ \tag{39}$$

It’s obvious that 2TLPFWMM operator lacks the property of idempotency.

3.3 The 2TLPFDMM operator

Qin and Liu [43] gave the dual MM (DMM) operator.

Definition 9[43]. Let $a_i (i = 1, 2, \ldots, n)$ be a group of nonnegative real numbers, and $P = (p_1, p_2, \ldots, p_n) \in R^n$ be a vector of parameters. If

$$\mathrm{DMM}^{P}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} p_j} \left(\prod_{\sigma \in S_n} \sum_{j=1}^{n} p_j a_{\sigma(j)} \right)^{\frac{1}{n!}} \tag{40}$$

where $\sigma(j)(j = 1, 2, \ldots, n)$ is any a permutation of $\{1, 2, \ldots, n\}$ and S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

Wang et al. [44] defined some Picture fuzzy dual Muirhead mean operators for evaluating the financial investment risk. Hong et al. [45] proposed some hesitant fuzzy dual Muirhead mean operators in MADM. In this section, we propose the DMM operator for 2TLPNs as follows.

Definition 10. Let $p_{x_i} = \left\{ \left(s_{\phi_{x_i}}, \varphi_{x_i} \right), \left(s_{\theta_{x_i}}, \vartheta_{x_i} \right) \right\} (i = 1, 2, \ldots, n)$ be a group of 2TLPFNs and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ be a vector of parameters. If

$$\mathrm{2TLPFDMM}^{\lambda}(p_1, p_2, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left(\underset{\sigma \in S_n}{\otimes} \left(\overset{n}{\underset{j=1}{\oplus}} \left(\lambda_j p_{\sigma(j)} \right) \right) \right)^{\frac{1}{n!}} \tag{41}$$

Then we named $\mathrm{2TLPFDMM}^{\lambda}$ the 2-tuple linguistic Pythagorean fuzzy DMM (2TLPFDMM) operator, where $\sigma(j)(j = 1, 2, \ldots, n)$ is any a permutation of $\{1, 2, \ldots, n\}$ and S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

Theorem 3. Let $p_{x_i} = \left\{ \left(s_{\phi_{x_i}}, \varphi_{x_i} \right), \left(s_{\theta_{x_i}}, \vartheta_{x_i} \right) \right\} (i = 1, 2, \ldots, n)$ be a set of 2TLPFNs. The fused value by using 2TLPFDMM operators is also a 2TLPN where

$$\begin{aligned} &\mathrm{2TLPFDMM}^{\lambda}(p_1, p_2, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n} \lambda_j} \left(\underset{\sigma \in S_n}{\otimes} \left(\overset{n}{\underset{j=1}{\oplus}} \left(\lambda_j p_{\sigma(j)} \right) \right) \right)^{\frac{1}{n!}} \\ &= \left\{ \begin{array}{l} \Delta\left(t \sqrt{1 - \left(1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)} \right)}{t} \right)^2 \right)^{\lambda_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}}} \right), \\ \Delta\left(t \left(\sqrt{1 - \prod_{\sigma \in S_n} \left(1 - \prod_{j=1}^{n} \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)} \right)}{t} \right)^{2\lambda_j} \right)^{\frac{1}{n!}}} \right)^{\frac{1}{\sum_{j=1}^{n} \lambda_j}} \right) \end{array} \right\} \end{aligned} \tag{42}$$

Proof:

$$\lambda_j p_{\sigma(j)} = \left\{ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}} \right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{\lambda_j} \right) \right\} \tag{43}$$

Thus,

$$\begin{aligned} &\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}\right) \\ &= \left\{ \Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}} \right), \Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{\lambda_j} \right) \right\} \end{aligned} \tag{44}$$

Therefore,

$$\mathop{\otimes}_{\sigma\in S_n}\left(\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}\right)\right) = \left\{ \begin{array}{l} \Delta\left(t\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}} \right), \\ \Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)} \right) \end{array} \right\} \tag{45}$$

Furthermore,

$$\begin{aligned} &\left(\mathop{\otimes}_{\sigma\in S_n}\left(\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}\right)\right)\right)^{\frac{1}{n!}} \\ &= \left\{ \begin{array}{l} \Delta\left(t\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}}\right)^{\frac{1}{n!}} \right), \\ \Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}} \right) \end{array} \right\} \end{aligned} \tag{46}$$

Therefore,

$$2TLPFDMM^{\lambda}(p_1, p_2, \ldots, p_n) = \frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}\right)\right)\right)^{\frac{1}{n!}}$$

$$= \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right), \\ \Delta\left(t\left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \end{array} \right\} \tag{47}$$

Thus, (42) is kept.

And we can prove that (42) is a 2TLPFN.

① $0 \le \Delta^{-1}\left(s_{\phi_j}, \varphi_j\right) \le t, 0 \le \Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right) \le t$; ② $0 \le \left(\Delta^{-1}\left(s_{\phi_j}, \varphi_j\right)\right)^2 + \left(\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)\right)^2 \le t^2$.

Let

$$\frac{\Delta^{-1}\left(s_{\phi_j}, \varphi_j\right)}{t} = \sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)}{t} = \left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

①Since $0 \le \frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t} \le 1$, we get

$$0 \le \left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2 \le 1 \;\; and \;\; 0 \le 1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j} \le 1 \tag{48}$$

$$0 \le 1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}} \le 1 \tag{49}$$

Then,

$$0 \le \sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \le 1 \tag{50}$$

That means $0 \le \Delta^{-1}\left(s_{\phi_j},\varphi_j\right) \le t$, so ① is maintained, similarly, we can have $0 \le \Delta^{-1}\left(s_{\theta_j},\vartheta_j\right) \le t$.

② Since $\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2 \le 1$, we have the following inequality

$$\begin{aligned}
&\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2 \\
&= \left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \\
&+ \left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\
&\le \left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right). \\
&+ \left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} = 1
\end{aligned}$$

That means $0 \le \left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2 + \left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2 \le t^2$, so ② is maintained.

Example 3. Let $\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\}$, and $\{(s_5,0.3),(s_3,-0.2)\}$ be three 2TLPFNs, and λ=(0.2,0.3,0.5), then according to (42), we have

$$
\begin{aligned}
&2\text{TLPFDMM}^{(0.2,0.3,0.5)}\left(\begin{array}{l}\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{array}\right)\\
&=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\\
\Delta\left(t\left(\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2\lambda_j}\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)
\end{array}\right\}
\end{aligned}
$$

$$
=\left\{\begin{array}{l}
\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}
\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^2\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^{0.5}\right)
\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.5}}}\right),\\
\Delta\left(6\times\left(\sqrt{1-\left(\begin{array}{l}
\left(1-\left(\frac{1.7}{6}\right)^{0.4}\times\left(\frac{1.2}{6}\right)^{0.6}\times\left(\frac{2.8}{6}\right)^{1}\right)\times\left(1-\left(\frac{1.7}{6}\right)^{0.4}\times\left(\frac{2.8}{6}\right)^{0.6}\times\left(\frac{1.2}{6}\right)^{1}\right)\\
\times\left(1-\left(\frac{1.2}{6}\right)^{0.4}\times\left(\frac{1.7}{6}\right)^{0.6}\times\left(\frac{2.8}{6}\right)^{1}\right)\times\left(1-\left(\frac{1.2}{6}\right)^{0.4}\times\left(\frac{2.8}{6}\right)^{0.6}\times\left(\frac{1.7}{6}\right)^{1}\right)\\
\times\left(1-\left(\frac{2.8}{6}\right)^{0.4}\times\left(\frac{1.7}{6}\right)^{0.6}\times\left(\frac{1.2}{6}\right)^{1}\right)\times\left(1-\left(\frac{2.8}{6}\right)^{0.4}\times\left(\frac{1.2}{6}\right)^{0.6}\times\left(\frac{1.7}{6}\right)^{1}\right)
\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.3+0.5}}\right)
\end{array}\right\}
$$

$$
=\{(s_4,0.1781),(s_1,0.8043)\}
$$

Similar to 2TLPFMM operator, we can have the properties as follows.

Property 6. (Idempotency) If $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}\,(i=1,2,\ldots,n)$ are equal, then

$$\text{2TLPFDMM}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) = p \tag{51}$$

Property 7. (Monotonicity) Let $p_{x_i} = \left\{\left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right)\right\}(i = 1, 2, \ldots, n)$ and $p_{y_i} = \left\{\left(s_{\phi_{y_i}}, \varphi_{y_i}\right), \left(s_{\theta_{y_i}}, \vartheta_{y_i}\right)\right\}(i = 1, 2, \ldots, n)$ be two sets of 2TLPFNs. If $\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right) \le \Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{\theta_{x_i}}, \vartheta_{x_i}\right) \ge \Delta^{-1}\left(s_{\theta_{y_i}}, \vartheta_{y_i}\right)$ hold for all i, then

$$\text{2TLPFDMM}^{\lambda}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_n}\right) \le \text{2TLPFDMM}^{\lambda}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_n}\right) \tag{52}$$

Property 8. (Boundedness) Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}(i = 1, 2, \ldots, n)$ be a set of 2TLPFNs. If $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ then

$$p^- \le \text{2TLPFDMM}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) \le p^+ \tag{53}$$

3.4 The 2TLPFWDMM operator

In real MADM, it's very important to consider attribute weights. Thus, this section will propose the 2-tuple linguistic Pythagorean number weighted DMM (2TLPFWDMM) operator.

Definition 11. Let $p_{x_i} = \left\{\left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right)\right\}(i = 1, 2, \ldots, n)$ be a set of 2TLPFNs with weight vector be $w_i = (w_1, w_2, \ldots, w_n)^T$, and satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in R^n$ be a vector of parameters. If

$$\text{2TLPFWDMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) = \frac{1}{\sum_{j=1}^{n} \lambda_j}\left(\underset{\sigma \in S_n}{\otimes}\left(\bigoplus_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}} \tag{54}$$

where $\sigma(j)(j = 1, 2, \ldots, n)$ is any a permutation of $\{1, 2, \ldots, n\}$ and S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

Theorem 4. Let $p_{x_i} = \left\{\left(s_{\phi_{x_i}}, \varphi_{x_i}\right), \left(s_{\theta_{x_i}}, \vartheta_{x_i}\right)\right\}(i = 1, 2, \ldots, n)$ be a set of 2TLPFNs. The fused value of 2TLPFWDMM operators is also a 2TLPFN where

$$
\begin{aligned}
&2\text{TLPFWDMM}_{nw}^{\lambda}\left(p_1,p_2,\cdots,p_n\right)\\
&=\frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right),\\
\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)
\end{array}\right\}
\end{aligned}
\tag{55}
$$

Proof:

$$
p_{\sigma(j)}^{nw_{\sigma(j)}}=\left\{\begin{array}{l}
\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{nw_{\sigma(j)}}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}}\right)
\end{array}\right\}
\tag{56}
$$

Then,

$$
\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\\
\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right)
\end{array}\right\}
\tag{57}
$$

Thus,

$$\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\\ \Delta\left(t\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}}\right)^{\lambda_j}\right)\end{array}\right\} \tag{58}$$

Therefore,

$$\begin{aligned}&\mathop{\otimes}_{\sigma\in S_n}\left(\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\\ &=\left\{\begin{array}{l}\Delta\left(t\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right),\\ \Delta\left(t\sqrt{1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)}\right)\end{array}\right\}\end{aligned} \tag{59}$$

Furthermore,

$$\begin{aligned}&\left(\mathop{\otimes}_{\sigma\in S_n}\left(\mathop{\oplus}_{j=1}^{n}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}\\ &=\left\{\begin{array}{l}\Delta\left(t\left(\prod_{\sigma\in S_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}}\right)^{\frac{1}{n!}}\right),\\ \Delta\left(t\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)\end{array}\right\}\end{aligned} \tag{60}$$

Therefore,

$$2\text{TLPFWDMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right)$$

$$= \frac{1}{\sum_{j=1}^{n}\lambda_j}\left(\underset{\sigma\in S_n}{\otimes}\left(\overset{n}{\underset{j=1}{\oplus}}\left(\lambda_j p_{\sigma(j)}^{nw_{\sigma(j)}}\right)\right)\right)^{\frac{1}{n!}}$$

$$= \left\{ \begin{array}{l} \Delta\left(t\left(\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right)\right), \\ \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \end{array} \right\} \tag{61}$$

Thus, (55) is kept.

Then we shall prove that (55) is a 2TLPFN.

① $0 \le \Delta^{-1}\left(s_{\phi_j},\varphi_j\right) \le t, 0 \le \Delta^{-1}\left(s_{\theta_j},\vartheta_j\right) \le t$ ② $0 \le \left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2 + \left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2 \le t^2$

Proof: Let

$$\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t} = \sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}$$

$$\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t} = \left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}$$

① Since $0 \le \frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t} \le 1$, we have

$$0 \le \left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j} \le 1 \ \textit{and} \ 0 \le 1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j} \le 1 \tag{62}$$

Then

$$0 \le \left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}} \le 1 \tag{63}$$

$$0 \le \sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \le \tag{64}$$

That means $0 \le \Delta^{-1}\left(s_{\phi_j},\varphi_j\right) \le t$, so ① is maintained, similarly, we can have $0 \le \Delta^{-1}\left(s_{\theta_j},\vartheta_j\right) \le t$.

② Since $\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2 \le 1$,we have the following inequality

$$\begin{aligned}
&\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2 \\
&= \left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right) \\
&+ \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \\
&\le \left(1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right). \\
&+ \left(1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^2\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} = 1
\end{aligned}$$

That means $0 \le \left(\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)\right)^2 + \left(\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)\right)^2 \le t^2$ .so ② is maintained.

Example 4. Let $\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\}$, and $\{(s_5,0.3),(s_3,-0.2)\}$ be three 2TLPNs, and λ=(0.2,0.3,0.5) , w=(0.4,0.2,0.4) then according to (55), we have

$$\text{2TLPFWDMM}_{(0.4,0.3,0.3)}^{(0.2,0.3,0.5)}\left(\begin{array}{l}\{(s_3,0.4),(s_2,\text{-}0.3)\},\{(s_2,0.3),(s_1,0.2)\},\\ \{(s_5,0.3),(s_3,-0.2)\}.\end{array}\right)$$

$$=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{\sigma(j)}},\varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}}\right),\\ \Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{\sigma(j)}},\vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}\right)\end{array}\right\}$$

$$
=\left(\begin{array}{l}
\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}
\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.2}\times\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.3}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(\frac{5.3}{6}\right)^{2.4}\right)^{0.2}\times\left(1-\left(\frac{2.3}{6}\right)^{1.2}\right)^{0.3}\times\left(1-\left(\frac{3.4}{6}\right)^{2.4}\right)^{0.5}\right)
\end{array}\right)^{\frac{1}{3!}}\right)}^{\frac{1}{0.2+0.3+0.5}}\right),\\
\Delta\left(6\times\sqrt{1-\left(\begin{array}{l}
\left(1-\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.2}\times\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.5}\right)\\
\times\left(1-\left(1-\left(1-\left(\frac{2.8}{6}\right)^{2}\right)^{1.2}\right)^{0.2}\times\left(1-\left(1-\left(\frac{1.2}{6}\right)^{2}\right)^{0.6}\right)^{0.3}\times\left(1-\left(1-\left(\frac{1.7}{6}\right)^{2}\right)^{1.2}\right)^{0.5}\right)
\end{array}\right)^{\frac{1}{3!}}}^{\frac{1}{0.2+0.3+0.5}}\right),
\end{array}\right)
$$

$$
=\left\{\left(s_4,0.1744\right),\left(s_1,0.1173\right)\right\}
$$

Then we analysis the properties of 2TLPFDWMM operator.

Property 9. (Monotonicity) Let $p_{x_i}=\left\{\left(s_{\phi_{x_i}},\varphi_{x_i}\right),\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\right\}(i=1,2,\ldots,n)$ and $p_{y_i}=\left\{\left(s_{\phi_{y_i}},\varphi_{y_i}\right),\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)\right\}(i=1,2,\ldots,n)$ be two sets of 2TLPFNs. If $\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)\leq\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right),\ and\ \Delta^{-1}\left(s_{\theta_{x_i}},\vartheta_{x_i}\right)\geq\Delta^{-1}\left(s_{\theta_{y_i}},\vartheta_{y_i}\right)$ hold for all i , then

$$\text{2TLPFWDMM}_{nw}^{\lambda}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_n}\right) \leq \text{2TLPFWDMM}_{nw}^{\lambda}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_n}\right) \tag{65}$$

Property 10. (Boundedness) Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}(i = 1, 2, \ldots, n)$ be a set of 2TLPNs. If $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ then

$$p^- \leq \text{2TLPFWDMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) \leq p^+ \tag{66}$$

From theorem 4,

$$\begin{aligned} &\text{2TLPFWDMM}_{nw}^{\lambda}\left(p_1^-, p_2^-, \cdots, p_n^-\right) \\ &= \left\{ \begin{aligned} &\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\min \Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \right), \\ &\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\max \Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \right) \end{aligned} \right\} \end{aligned} \tag{67}$$

$$\begin{aligned} &\text{2TLPFWDMM}_{nw}^{\lambda}\left(p_1^+, p_2^+, \cdots, p_n^+\right) \\ &= \left\{ \begin{aligned} &\Delta\left(t\sqrt{1-\left(1-\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\frac{\max \Delta^{-1}\left(s_{\phi_{\sigma(j)}}, \varphi_{\sigma(j)}\right)}{t}\right)^{2nw_{\sigma(j)}}\right)^{\lambda_j}\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}}} \right), \\ &\Delta\left(t\left(\sqrt{1-\left(\prod_{\sigma\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\min \Delta^{-1}\left(s_{\theta_{\sigma(j)}}, \vartheta_{\sigma(j)}\right)}{t}\right)^{2}\right)^{nw_{\sigma(j)}}\right)^{\lambda_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\lambda_j}} \right) \end{aligned} \right\} \end{aligned} \tag{68}$$

From property 9,

$$p^- \leq \text{2TLPFWDMM}_{nw}^{\lambda}\left(p_1, p_2, \cdots, p_n\right) \leq p^+ \tag{69}$$

4. Numerical example and comparative analysis

4.1. Numerical example

MADM is a process of ranking a finite set of alternatives with respect to a list of attributes. It has been extensively studied and also applied in various areas, such as human resource selection [46], transportation management [47], military affair [48], construction engineering project risk assessment[49-50], potential evaluation of emerging technology commercialization[51-56], strategic suppliers' selection[57]. With the rapid development of growing enterprise competition and economic globalization, the competition among modern enterprises has become the competition among the supply chains [58-62]. Therefore, supplier selection problem has paid great attention in practical production management and supply chain management theory. In actual supplier selection problems, there are a large number of uncertainties, fuzziness and risk in the whole supply chains [63-67]. These factors have great influence in actual assessment and selection. In this section we shall give an example to show green suppliers selection under 2TLPFNs. Suppose that there are five possible green suppliers $A_i\left(i=1,2,3,4,5\right)$ are assessed according to four attributes: ①G_1 is the environmental factors; ②G_2 is the product quality factor; ③G_3 is the price factors; ④G_4 is the delivery factor. These five green suppliers $A_i\left(i=1,2,3,4,5\right)$ are evaluated by 2TLPFNs under four attributes by three experts. The evaluating results are listed in Table 1-3. And the attribute weight vector is $\omega=\left(0.16,0.27,0.29,0.28\right)$ and the expert weight vector is $\omega=\left(0.2,0.6,0.2\right)$.

In the following, we use the developed methods and models to select green suppliers.

Step 1. In accordance with the 2TLPFNs $r_{ij}\left(i=1,2,3,4,5,\ j=1,2,3,4\right)$, we can fuse the 2TLPFNs with 2TLPFWAA (2TLPFWGA) operator to have the 2TLPFNs $A_i\left(i=1,2,3,4,5\right)$ of the green suppliers A_i. Then the calculating values are shown in Table 4.

Definition 12. Let $p_j=\left\{\left(s_{\phi_j},\varphi_j\right),\left(s_{\theta_j},\vartheta_j\right)\right\}\left(j=1,2,\ldots,n\right)$ be a group of 2TLPFNs with weight vector be $w_i=(w_1,w_2,\ldots,w_n)^T$, and satisfying $w_i\in\left[0,1\right]$ and $\sum_{i=1}^{n}w_i=1$, then we shall obtain

$$\begin{aligned}&\text{2TLPFWA}\left(p_1,p_2,\ldots,p_n\right)=\sum_{j=1}^{n}w_jp_j\\&=\left\{\Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_j},\varphi_j\right)}{t}\right)^2\right)^{w_j}}\right),\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{w_j}\right)\right\}\end{aligned}\qquad(70)$$

$$2\text{TLPFWG}\left(p_1, p_2, \dots, p_n\right) = \prod_{j=1}^{n}\left(l_j\right)^{w_j}$$

$$= \left\{ \Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{\phi_j}, \varphi_j\right)}{t} \right)^{w_j} \right), \Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)}{t} \right)^2\right)^{w_j}} \right) \right\} \quad (71)$$

Step 2. According to table 4, we shall fuse the 2TLPFNs r_{ij} by 2TLPFWMM (2TLPFWDMM) operator to have the overall 2TLPFNs $A_i\left(i=1,2,3,4,5\right)$. Let $P=(1,1,0,0)$, then the calculating results are listed in Table 5.

Step 3. According to the calculating results in Table 2, the score are listed in Table 6.

Step 4. According to Table 6, the ordering are listed in Table 7, the best green supplier is A_2.

4.2. Influence analysis of the parameter

In order to show the effects on the ranking results by altering parameters of P in the 2TLPFWMM (2TLPFWDMM) operators, the results are listed in Tables 8-9.

4.3. Comparative analysis

Then, we shall compare our methods with LPFWA operator and LPFWG operator [68]. The comparative results are depicted in Table 10.

From above, it can seem that LPFWA operator and LPFWG operator don't consider the relationship. The proposed 2TLPFWMM and 2TLPFWDMM operators consider the relationship among arguments being aggregated.

5. Conclusion

2-tuple linguistic Pythagorean fuzzy numbers (2TLPFNs) have applied the advantages of 2-tuple linguistic term sets and Pythagorean fuzzy numbers. They can flexibly express cognitive information as well as effectively characterize the reliability of information. Therefore, it is of great significance to study MADM methods with 2TLPFNs. For this paper, we investigate the MADM problems with 2TLPFNs. Then, we expand the MM operator and DMM operator to propose some MM operators with 2TPFNs. The main characteristic of these operators are analyzed. Then, we utilized the 2TLPFWMM and 2TLPFWDMM operators for MADM problems with 2TPFNs. Finally, a practical example with green supplier selection is used to show the developed methods. In the future works, the extension and application with 2TPFNs needs to be investigated in the other uncertain and fuzzy domains [69-72].

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All figures and tables:

Table 1. 2TLPFN decision matrix (R_1)

	G_1	G_2	G_3	G_4
A_1	$<(s_4,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_3,0), (s_4,0)>$	$<(s_3,0), (s_2,0)>$
A_2	$<(s_5,0), (s_1,0)>$	$<(s_5,0), (s_2,0)>$	$<(s_5,0), (s_3,0)>$	$<(s_5,0), (s_2,0)>$
A_3	$<(s_4,0), (s_3,0)>$	$<(s_2,0), (s_4,0)>$	$<(s_4,0), (s_3,0)>$	$<(s_5,0), (s_2,0)>$
A_4	$<(s_1,0), (s_4,0)>$	$<(s_2,0), (s_5,0)>$	$<(s_3,0), (s_4,0)>$	$<(s_1,0), (s_2,0)>$
A_5	$<(s_4,0), (s_2,0)>$	$<(s_4,0), (s_3,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_3,0), (s_2,0)>$

Table 2. 2TLPFN decision matrix (R_2)

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0), (s_2,0)>$	$<(s_2,0), (s_1,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_5,0), (s_2,0)>$
A_2	$<(s_5,0), (s_2,0)>$	$<(s_4,0), (s_2,0)>$	$<(s_4,0), (s_3,0)>$	$<(s_5,0), (s_2,0)>$
A_3	$<(s_3,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_2,0), (s_2,0)>$
A_4	$<(s_1,0), (s_4,0)>$	$<(s_2,0), (s_5,0)>$	$<(s_3,0), (s_4,0)>$	$<(s_2,0), (s_3,0)>$
A_5	$<(s_3,0), (s_2,0)>$	$<(s_4,0), (s_3,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_5,0), (s_2,0)>$

Table 3. 2TLPFN decision matrix (R_3)

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_2,0), (s_3,0)>$	$<(s_2,0), (s_3,0)>$
A_2	$<(s_5,0), (s_1,0)>$	$<(s_4,0), (s_2,0)>$	$<(s_4,0), (s_3,0)>$	$<(s_5,0), (s_3,0)>$
A_3	$<(s_3,0), (s_2,0)>$	$<(s_2,0), (s_2,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_1,0), (s_2,0)>$
A_4	$<(s_1,0), (s_4,0)>$	$<(s_2,0), (s_5,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_3,0), (s_2,0)>$
A_5	$<(s_4,0), (s_1,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_1,0), (s_3,0)>$

Table 4. The calculating results by the 2TLPFWA operator

	G_1	G_2
A_1	$<(s_3, 0.2516), (s_2, 0.0000)>$	$<(s_3, 0.0797), (s_1, 0.0000)>$
A_2	$<(s_5, 0.0000), (s_2, -0.4843)>$	$<(s_4, 0.2725), (s_2, 0.0000)>$
A_3	$<(s_3, 0.2516), (s_2, 0.1689)>$	$<(s_3, 0.4443), (s_2, -0.4843)>$
A_4	$<(s_1, 0.0000), (s_4, 0.0000)>$	$<(s_2, 0.0000), (s_5, 0.0000)>$
A_5	$<(s_3, 0.4719), (s_2, -0.2589)>$	$<(s_4, -0.3079), (s_3, -0.2337)>$
	G_3	G_4
A_1	$<(s_2, -0.1668), (s_2, 0.4915)>$	$<(s_4, 0.4387), (s_2, 0.1689)>$
A_2	$<(s_4, 0.2725), (s_3, 0.0000)>$	$<(s_5, 0.0000), (s_2, 0.1689)>$
A_3	$<(s_3, 0.2516), (s_1, 0.2457)>$	$<(s_3, 0.1123), (s_2, 0.0000)>$
A_4	$<(s_3, -0.2502), (s_3, 0.4822)>$	$<(s_2, 0.1236), (s_3, -0.4492)>$
A_5	$<(s_2, 0.1097), (s_2, -0.4843)>$	$<(s_4, -0.4054), (s_2, 0.1689)>$

Table 5. The calculating results by the 2TLPFWMM (2TLPFWDMM) operator

	2TLPFWMM	2TLPFWDMM
A_1	$<(s_3, 0.1613), (s_2, 0.0510)>$	$<(s_3, 0.3350), (s_2, -0.0559)>$
A_2	$<(s_5, -0.3914), (s_2, 0.2446)>$	$<(s_5, -0.3441), (s_2, 0.2356)>$
A_3	$<(s_3, 0.2479), (s_2, -0.1613)>$	$<(s_3, 0.3146), (s_2, -0.2980)>$
A_4	$<(s_2, 0.0874), (s_4, -0.1043)>$	$<(s_2, 0.0152), (s_4, -0.2420)>$
A_5	$<(s_3, 0.4362), (s_2, 0.1589)>$	$<(s_4, -0.4135), (s_2, 0.0710)>$

Table 6. The score of the green suppliers

	2TLPFWMM	2TLPFWDMM
A_1	$(s_3,0.4822)$	$(s_4,-0.3881)$
A_2	$(s_4,0.3501)$	$(s_4,0.3900)$
A_3	$(s_4,-0.4027)$	$(s_4, -0.3259)$
A_4	$(s_2,-0.0984)$	$(s_2,0.1615)$
A_5	$(s_3,-0.4044)$	$(s_4,-0.2855)$

Table 7. Ordering of the green suppliers

	Ordering
2TLPFWMM	$A_2>A_3>A_5>A_1>A_4$
2TLPFWDMM	$A_2>A_5>A_3>A_1>A_4$

Table 8. Ranking results with different parameters for 2TLPFWMM operator

P	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
(1,0,0,0)	$(s_4,-0.3067)$	$(s_4,0.3979)$	$(s_4, -0.3333)$	$(s_2, 0.3187)$	$(s_4, -0.2537)$	$A_2>A_5>A_1>A_3>A_4$
(1,1,0,0)	$(s_3,0.4822)$	$(s_4, 0.3501)$	$(s_4, -0.4027)$	$(s_2, 0.0984)$	$(s_4, -0.4044)$	$A_2>A_3>A_5>A_1>A_4$
(1,1,1,0)	$(s_3,0.4049)$	$(s_4, 0.3341)$	$(s_4, -0.4453)$	$(s_2,-0.0208)$	$(s_4, -0.4732)$	$A_2>A_3>A_5>A_1>A_4$
(1,1,1,1)	$(s_3,0.3470)$	$(s_4,0.3247)$	$(s_4,-0.4792)$	$(s_2,-0.1261)$	$(s_3, 0.4758)$	$A_2>A_3>A_5>A_1>A_4$
(2,2,2,2)	$(s_3,0.3470)$	$(s_4,0.3247)$	$(s_4,-0.4792)$	$(s_2,-0.1261)$	$(s_3, 0.4758)$	$A_2>A_3>A_5>A_1>A_4$
(2,0,0,0)	$(s_4,-0.1833)$	$(s_4,0.4123)$	$(s_4, -0.3159)$	$(s_2,0.4430)$	$(s_4, -0.1637)$	$A_2>A_5>A_1>A_3>A_4$
(3,0,0,0)	$(s_4,-0.0694)$	$(s_4, 0.4284)$	$(s_4, -0.2989)$	$(s_3,-0.4552)$	$(s_4, -0.0815)$	$A_2>A_1>A_5>A_3>A_4$

Table 9. Ranking results with different parameters for 2TLPFWDMM operator

P	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
(1,0,0,0)	$(s_3, 0.3898)$	$(s_4, 0.2906)$	$(s_4,-0.3602)$	$(s_2,-0.0033)$	$(s_4, -0.4952)$	$A_2>A_3>A_5>A_1>A_4$
(1,1,0,0)	$(s_4,-0.3881)$	$(s_4, 0.3900)$	$(s_4, -0.3259)$	$(s_2,-0.1615)$	$(s_4, -0.2855)$	$A_2>A_5>A_3>A_1>A_4$
(1,1,1,0)	$(s_4,-0.3047)$	$(s_4, 0.4535)$	$(s_4, -0.3066)$	$(s_2,-0.2167)$	$(s_4, -0.2249)$	$A_2>A_5>A_3>A_1>A_4$
(1,1,1,1)	$(s_4,-0.2488)$	$(s_5,-0.4833)$	$(s_4,-0.2916)$	$(s_2, 0.2570)$	$(s_4, -0.1859)$	$A_2>A_5>A_3>A_1>A_4$
(2,2,2,2)	$(s_4,-0.2488)$	$(s_5,-0.4833)$	$(s_4,-0.2916)$	$(s_2, 0.2570)$	$(s_4, -0.1859)$	$A_2>A_5>A_3>A_1>A_4$
(2,0,0,0)	$(s_3, 0.3041)$	$(s_4, 0.1821)$	$(s_4,-0.3813)$	$(s_2, -0.1017)$	$(s_3, 0.4148)$	$A_2>A_3>A_5>A_1>A_4$
(3,0,0,0,)	$(s_3,0.2311)$	$(s_4, 0.0847)$	$(s_4,-0.4027)$	$(s_2, -0.1978)$	$(s_3, 0.3292)$	$A_2>A_3>A_5>A_1>A_4$

Table 10. Order of the green suppliers

	Order
LPFWA operator[68]	$A_2 > A_5 > A_1 > A_3 > A_4$
LPFWG operator[68]	$A_2 > A_3 > A_5 > A_1 > A_4$

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